MULTILINEAR FUNCTION SERIES IN CONDITIONALLY FREE PROBABILITY WITH AMALGAMATION

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ABSTRACT. As in the cases of freeness and monotonic independence, the notion of conditional freeness is meaningful when complex-valued states are replaced by positive conditional expectations. In this framework, the paper presents several positivity results, a version of the central limit theorem and an analogue of the conditionally free R-transform constructed by means of multilinear function series.

1. Introduction

The paper addresses a topic related to conditionally free (or, shortly, using the term from [2], c-free) probability. This notion was developed in the '90's (see [1], [2]) as an extension of freeness within the framework of *-algebras endowed with not one, but two states. Namely, given a family of unital algebras $\{\mathfrak{A}\}_{i\in\mathcal{I}}$, each \mathfrak{A}_i endowed with two expectations $\varphi_i, \psi_i : \mathfrak{A}_i \longrightarrow \mathbb{C}$, their c-free product is the triple $(\mathfrak{A}, \varphi, \psi)$, where:

- (i) $\mathfrak{A} = *_{i \in I} \mathfrak{A}_i$ is the free product of the algebras \mathfrak{A}_i .
- (ii) $\psi = *_{i \in \mathcal{I}} \psi_i$ and $\varphi = *_{(\psi_i)_{i \in \mathcal{I}}} \varphi_i$ are expectations given by the relations

 - (a) $\psi(a_1 \cdots a_n) = 0$ (b) $\varphi(a_1 \cdots a_n) = \varphi_{\varepsilon(1)}(a_1) \cdots \varphi_{\varepsilon(n)}(a_n)$

for all $a_j \in \mathfrak{A}_{\varepsilon(j)}, j = 1, \ldots, n$ such that $\psi_{\varepsilon(j)}(a_j) = 0$ and $\varepsilon(1) \neq \cdots \neq \varepsilon(n)$.

A key result is that if the \mathfrak{A}_i are *-algebras and φ_i, ψ_i are positive functionals, then φ and ψ are also positive.

In [6], the positivity of the free product maps φ, ψ is proved for the case when φ_1, φ_2 are positive conditional expectations in a common C*-subalgebra, but ψ_1, ψ_2 remain positive C-valued maps. A more general situation is indeed discussed (see Theorem 3, Section 6, from [6]), but the question if φ, ψ repositive for $\varphi_{1,2}, \psi_{1,2}$ arbitrary positive conditional expectations is left unanswered.

A first answer was given in [8], where we showed that for $\mathfrak A$ a *-algebra, the analogous construction with both φ and ψ valued in a C^* -subalgebra $\mathfrak B$ of $\mathfrak A$ still retains the positivity. The present paper further develops this result (see Theorem 2.3) and also demonstrates the use of multilinear function series in c-free setting.

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In [2] is constructed a c-free version of Voiculescu's R-transform, which we will call the cR -transform, with the property that ${}^cR_{X+Y} = {}^cR_X + {}^cR_Y$ if X and Y are c-free elements from the algebra $\mathfrak A$ relative to φ and ψ (i.e. the relations (a) and (b) from the definition of the c-free product hold true for the subalgebras generated by X and Y.)

The apparatus of multilinear function series is used in recent work of K. Dykema ([3] and [4]) to construct suitable analogues for the R and S-transforms in the framework of freeness with amalgamation. We will show that this construction is also appropriate for the cR -transform mentioned above. The techniques used differ from the ones of [3], the Fock space type construction being substituted by combinatorial techniques similar to [2] and [7]. Particularly, Theorems 3.3 and 3.6 contain new (shorter) proves of the results 6.1–6.13 from [3].

The paper is structured in four sections. In Section 2 are stated the basic definitions and are proved the main positivity results. Section 3 describes the construction and the basic property of the multilinear function series ^cR-transform and Section 4 treats the central limit theorem and a related positivity property.

2. Definitions and positivity results

Definition 2.1. Let $\{\mathfrak{A}_i\}_{i\in\mathcal{I}}$ be a family of algebras, all containing the subalgebra \mathfrak{B} . Suppose \mathfrak{D} is a subalgebra of \mathfrak{B} and $\Psi_i:\mathfrak{A}_i\longrightarrow\mathfrak{D}$ and $\Phi_i:\mathfrak{A}_i\longrightarrow\mathfrak{B}$ are conditional expectations, $i\in\mathcal{I}$. We say that the triple $(\mathfrak{A},\Phi,\Psi)=\star_{i\in I}(\mathfrak{A}_i,\Phi_i,\Psi_i)$ is the conditionally free product with amalgamation over $(\mathfrak{B},\mathfrak{D})$, or shortly, the *c-free product*, of the triples $(\mathfrak{A}_i,\Phi_i,\Psi_i)_{i\in I}$ if

- (1) \mathfrak{A} is the free product with amalgamation over \mathfrak{B} of the family $(\mathfrak{A}_i)_{i\in\mathcal{I}}$
- (2) $\Psi = *_{i \in \mathcal{I}} \Psi_i$ and $\Phi = *_{(\Psi_i)_{i \in \mathcal{I}}} \Phi_i$ are determined by the relations

$$\Psi(a_1 a_2 \dots a_n) = 0
\Phi(a_1 a_2 \dots a_n) = \Phi(a_1) \Phi(a_2) \dots \Phi(a_n),$$

for all $a_i \in \mathfrak{A}_{\varepsilon(i)}, \varepsilon(i) \in \mathcal{I}$ such that $\varepsilon(1) \neq \varepsilon(2) \neq \cdots \neq \varepsilon(n)$ and $\Psi_{\varepsilon(i)}(a_i) = 0$.

When $\mathfrak{D} = \mathbb{C}$, this definition reduces to the one given in [6]. When both \mathfrak{B} and \mathfrak{D} are equal to \mathbb{C} , this definition was given in [2].

When discussing positivity, we need a *-structure on our algebras. We will demand that \mathfrak{B} and \mathfrak{D} be C*-algebras, while \mathfrak{A}_i and \mathfrak{A} are only required to be *-algebras.

The following results are slightly modified versions of Lemma 6.4 and Theorem 6.5 from [8].

Lemma 2.2. Let \mathfrak{B} be a C^* -algebra and \mathfrak{A}_1 , \mathfrak{A}_2 be two *-algebra containing \mathfrak{B} as a *-subalgebra, endowed with positive conditional expectations $\Phi_j: \mathfrak{A}_j \longrightarrow \mathfrak{B}$, j = 1, 2. If $a_1, \ldots, a_n \in \mathfrak{A}_1$, $a_{n+1}, \ldots, a_{n+m} \in \mathfrak{A}_2$ and $A = (A_{i,j})_{i,j} \in M_{n+m}(\mathfrak{B})$ is the matrix with the entries

$$A_{i,j} = \begin{cases} \Phi_1(a_i^* a_j) & \text{if} \quad i, j \le n \\ \Phi_1(a_i^*) \Phi_2(a_j) & \text{if} \quad i \le n, j > n \\ \Phi_2(a_i^*) \Phi_1(a_j) & \text{if} \quad i > n, j \le n \\ \Phi_2(a_i^* a_j) & \text{if} \quad i, j > n \end{cases}$$

then A is positive.

Proof. The vector space $\mathfrak{E} = \mathfrak{B} \oplus \ker(\Phi_1) \oplus \ker(\Phi_2)$ has a \mathfrak{B} -bimodule structure given by the algebraic operations on \mathfrak{A}_1 and \mathfrak{A}_2 . Consider the \mathfrak{B} -sesquilinear pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{E} \times \mathfrak{E} \longrightarrow \mathfrak{B}$$

determined by the relations:

$$\langle b_1, b_2 \rangle = b_1^* b_2, \text{ for } b_1, b_2 \in \mathfrak{B}$$

$$\langle u_j, v_j \rangle = \Phi_j(u_j^* v_j), \text{ for } u_j, v_j \in \ker(\Phi_j), j = 1, 2$$

$$\langle u_1, u_2 \rangle = \langle u_2, u_1 \rangle = 0 \text{ for } u_1 \in \ker(\Phi_1), \text{ and } u_2 \in \ker(\Phi_2).$$

$$\langle b, u_j \rangle = \langle u_j, b \rangle = 0 \text{ for all } b \in \mathfrak{B}, u_j \in \mathfrak{A}_j$$

With this notation, we have that $A_{i,j} = \langle a_i, a_j \rangle$, hence it suffices to show that $\langle a, a \rangle \ge 0$ for all $a \in \mathfrak{E}$.

Indeed, for an element $a = b + u_1 + u_2$ with $b \in \mathfrak{B}, u_j \in \ker(\Phi_j), j = 1, 2$, we have:

$$\langle a, a \rangle = \langle b + u_1 + u_2, b + u_1 + u_2 \rangle$$

$$= \langle b, b \rangle + \langle u_1, u_1 \rangle + \langle u_2, u_2 \rangle$$

$$= b^*b + \Phi_1(u_1^*u_1) + \Phi_2(u_2^*u_2)$$

$$\geq 0$$

Theorem 2.3. Let \mathfrak{B} be a C^* -algebra and \mathfrak{D} a C^* -subalgebra of \mathfrak{B} . Suppose that \mathfrak{A}_1 , \mathfrak{A}_2 are *-algebras containing \mathfrak{B} , each endowed with two positive conditional expectations $\Phi_j: \mathfrak{A}_j \longrightarrow \mathfrak{B}$, and $\Psi_j: \mathfrak{A}_j \longrightarrow \mathfrak{D}$, j = 1, 2 and consider the c-free product $(\mathfrak{A}, \Phi, \Psi) = *_{i=1,2}(\mathfrak{A}_i, \Phi_i, \Psi_i)$.

Then the maps Φ and Ψ are positive.

Proof. The positivity of Ψ is by now a classical result in the theory of free probability with amalgamation over a C^* -algebra (for example, see [9], Theorem 3.5.6). For the positivity of Φ we have to show that $\Phi(a^*a) \geq 0$ for any $a \in \mathfrak{A}$.

Any element of \mathfrak{A} can be written as

$$a = \sum_{k=1}^{N} s_{1,k} \dots s_{n(k),k},$$

where $s_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}$ $\varepsilon(1,k) \neq \varepsilon(2,k) \neq \cdots \neq \varepsilon(n(k),k)$. Writing

$$s_{(j,k)} = s_{(j,k)} - \Psi(s_{(j,k)}) + \Psi(s_{(j,k)})$$

and expanding the product, we can consider a of the form

$$a = d + \sum_{k=1}^{N} a_{1,k} \dots a_{n(k),k}$$

with $d \in \mathfrak{D} \subset \mathfrak{B}$ and $a_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}$ such that $\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0$ and $\varepsilon(1,k) \neq$ $\varepsilon(2,k) \neq \cdots \neq \varepsilon(n(k),k).$

Therefore

$$\Phi(a^*a) = \Phi\left(d^*d + d^*\left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right] + \left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right]^*d + \left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right]^*\left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right]\right).$$

Since Φ is a conditional expectation and $d \in \mathfrak{D} \subset \mathfrak{B}$, the above equality becomes

$$\Phi(a^*a) = d^*d + \sum_{k=1}^N d^*\Phi(a_{1,k} \dots a_{n(k),k}) + \sum_{k=1}^N \Phi(a_{n(k),k}^* \dots a_{1,k}^*)d$$

$$+ \sum_{k,l=1}^N \Phi(a_{n(k),k}^* \dots a_{1,k}^* a_{1,l} \dots a_{n(l),l}).$$

Using the definition of the conditionally free product with amalgamation over \mathfrak{B} and that $\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0$ for all j,k, one further has

$$\Phi(a^*a) = d^*d + \sum_{k=1}^N \Phi(d^*a_{1,k})\Phi(a_{2,k})\dots\Phi(a_{n(k),k})
+ \sum_{k=1}^N \Phi(a_{n(k),k})^*\dots\Phi(a_{2,k}^*)\Phi(a_{1,k}^*d)
+ \sum_{k,l=1}^N \left[\Phi(a_{n(k),k})^*\dots\Phi(a_{2,k}^*)\right]\Phi(a_{1,k}^*a_{1,l})\Phi(a_{2,l})\dots\Phi(a_{n(l),l})$$

that is

$$\Phi(a^*a) = d^*d + \sum_{k=1}^N \Phi(d^*a_{1,k}) \left[\Phi(a_{2,k}) \dots \Phi(a_{n(k),k}) \right]
+ \sum_{k=1}^N \left[\Phi(a_{2,k}) \dots \Phi(a_{n(k),k}) \right]^* \Phi(a_{1,k}^*d)
+ \sum_{k,l=1}^N \left[\Phi(a_{2,k}) \dots \Phi(a_{n(k),k}) \right]^* \Phi(a_{1,k}^*a_{1,l}) \left[\Phi(a_{2,l}) \dots \Phi(a_{n(l),l}) \right]$$

From Lemma 2.2, the matrix $S = (\Phi(a_{1,i}^* a_{1,j})_{i,j=1}^{N+1})$ is positive in $M_{N+1}(\mathfrak{B})$, therefore

$$S = T^*T$$
, for some $T \in M_{N+1}(\mathfrak{B})$.

Denote now $a_{1,N+1} = d$ and $v_k = \Phi(a_{2,k}) \dots \Phi(a_{n(k),k})$. The identity for $\Phi(a^*a)$ becomes:

$$\Phi(a^*a) = (v_1, \dots, v_N, 1)^*T^*T(v_1, \dots, v_N, 1)$$

 ≥ 0 , as claimed.

Theorem 2.4. Assume that $\mathcal{I} = \bigcup_{i \in \mathfrak{I}} \mathcal{I}_i$ is a partition of \mathcal{I} . Then:

$$*_{j \in \mathfrak{J}} (*_{i \in \mathcal{I}_j} (\mathfrak{A}_i, \Phi_i, \Psi_i)) = *_{i \in \mathcal{I}} (\mathfrak{A}_i, \Phi_i, \Psi_i)$$

Proof. The proof is identical to the proofs of similar results in [6] and [2]. Consider $a_i \in \mathfrak{A}_{\varepsilon(i)}, 1 \leq i \leq m$ such that $\varepsilon(1) \neq \varepsilon(2 \neq \cdots \neq \varepsilon(m))$ and $\Psi_{\varepsilon(i)}(a_i) = 0$. Let $1 = i_0 < i_1 < \cdots < i_k = m$ and $\mathfrak{J}_l = \{\varepsilon(i), i_{l-1} \leq i < i_l\}$. Since

$$(*_{j\in\mathfrak{J}_l}\Psi_j)\left((a_{i_l-1}\cdots a_{i_l})\right)=0,$$

it suffices to show that

$$\Phi(a_1 \cdots a_m) = \prod_{l=1}^k \left[(*_{(\Psi_j), j \in \mathfrak{J}_l} \Phi_j) (a_{i_l-1} \cdots a_{i_l}) \right].$$

But

$$\Phi(a_1 \cdots a_m) = \Phi_{\varepsilon(1)}(a_1) \cdots \Phi_{\varepsilon(m)}(a_m)$$

while, since $\Psi_{\varepsilon(i)}(a_i) = 0$,

$$(*_{(\Psi_j),j\in\mathfrak{J}_l}\Phi_j)(a_{i_l-1}\cdots a_{i_l})=\Phi_{i_l-1}(a_{i_l-1})\cdots\Phi_{i_l}(a_{i_l})$$

and the conclusion follows.

Definition 2.5. Let $\mathfrak A$ be an algebra (respectively a *-algebra), $\mathfrak B$ a subalgebra (*-subalgebra) of $\mathfrak A$ and $\mathfrak D$ a subalgebra (*-subalgebra) of $\mathfrak B$. Suppose $\mathfrak A$ is endowed with the conditional expectations $\Psi:\mathfrak A\longrightarrow\mathfrak D$ and $\Phi:\mathfrak A\longrightarrow\mathfrak D$.

- (i) The subalgebras (*-subalgebras) $(\mathfrak{A}_i)_{i\in\mathcal{I}}$ of \mathfrak{A} are said to be *c-free* with respect to (Φ, Ψ) if:
 - (a) $(\mathfrak{A}_i)_{i\in\mathcal{I}}$ are free with respect to Ψ .
 - (b) if $a_i \in \mathfrak{A}_{\varepsilon(i)}, 1 \leq i \leq m$, are such that $\varepsilon(1) \neq \cdots \neq \varepsilon(m)$ and $\Psi(a_i) = 0$, then $\Phi(a_1 \cdots a_m) = \Phi(a_1) \cdots \Phi(a_m)$.
- (ii) The elements $(X_i)_{i\in\mathcal{I}}$ of \mathfrak{A} are said to be c-free with respect to (Φ, Ψ) if the subalgebras (*-subalgebras) generated by \mathfrak{B} and X_i are c-free with respect to (Φ, Ψ) .

We will denote by $\mathfrak{B}\langle\xi\rangle$ the non-commutative algebra of polynomials in the symbol ξ and with coefficients from \mathfrak{B} (the coefficients do not commute with the symbol ξ). If \mathcal{I} is a family of indices, $\mathfrak{B}\langle\{\xi_i\}_{i\in\mathcal{I}}\rangle$ will denote the algebra of polynomials in the non-commuting variables $\{\xi\}_{i\in\mathcal{I}}$ and with coefficients from \mathfrak{B} . We will identify $\mathfrak{B}\langle\{\xi_i\}_{i\in\mathcal{I}}\rangle$ with the free product with amalgamation over \mathfrak{B} of the family $\{\mathfrak{B}\langle\xi_i\rangle\}_{i\in\mathcal{I}}$.

If $\mathfrak A$ is a *-algebra and $\mathfrak B$ is with the C^* -algebra, $\mathfrak B\langle \xi \rangle$ will also be considered with a *-algebra structure, by taking $\xi^* = \xi$. If X is a selfadjoint element from $\mathfrak A$, we define the conditional expectations

$$\Phi_X, \Psi_X : \mathfrak{B}\langle \xi \rangle \longrightarrow \mathfrak{B}$$

given by
$$\Phi_X(f(\xi)) = \Phi(f(X))$$
 and $\Psi_X(f(\xi)) = \Psi(f(X))$, for any $f(\xi) \in \mathfrak{B}\langle \xi \rangle$.

Corollary 2.6. Suppose that \mathfrak{A} is a *-algebra and X and Y are c-free selfadjoint elements of \mathfrak{A} such that the maps Φ_X, Ψ_X and Φ_Y, Ψ_Y are positive. Then the maps Φ_{X+Y} and Ψ_{X+Y} are also positive.

Proof. The positivity of Ψ_{X+Y} is an immediate consequence of the fact that X and Y are free with amalgamation over \mathfrak{B} with respect to Ψ . It remains to prove the positivity of Φ_{X+Y} .

Since the maps $\Phi_X: \mathfrak{B}\langle \xi_1 \rangle \longrightarrow \mathfrak{B}$ and $\Phi_Y: \mathfrak{B}\langle \xi_2 \rangle \longrightarrow \mathfrak{B}$ are positive, from Theorem 2.3 so is

$$\Phi_x *_{(\Psi_X, \Psi_Y)} \Phi_Y : \mathfrak{B}\langle \xi_1 \rangle *_{\mathfrak{B}} \mathfrak{B}\langle \xi_2 \rangle = \mathfrak{B}\langle \xi_1, \xi_2 \rangle \longrightarrow \mathfrak{B}$$

Remark also that

$$i_Z: \mathfrak{B}\langle \xi \rangle \ni f(\xi) \mapsto f(X+Y) \in \mathfrak{B}\langle \xi_1 \rangle *_{\mathfrak{B}} \mathfrak{B}\langle \xi_2 \rangle$$

is a positive **B**-functional.

The conclusion follows from the fact that the c-freeness of X and Y is equivalent to

$$\Phi_{X+Y} = (\Phi_X *_{(\Psi_X, \Psi_Y)} \Phi_Y) \circ i_{X+Y}.$$

3. Multilinear function series and the ^cR-transform

Let \mathfrak{A} be a *-algebra containing the C^* -algebra \mathfrak{B} , endowed with a conditional expectation $\Psi: \mathfrak{A} \longrightarrow \mathfrak{B}$. If X is a selfadjoint element of \mathfrak{A} , then by the moment of order n of X we will understand the map

$$m_X^{(n)}$$
 : $\underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{n-1 \text{ times}} \longrightarrow \mathfrak{B}$

$$m_X^{(n)}(b_1,\ldots,b_{n-1}) = \Psi(Xb_1X\ldots Xb_{n-1}X)$$

If $\mathfrak{B} = \mathbb{C}$, then the moment-generating series of X

$$m_X(z) = \sum_{n=0}^{\infty} \Psi(X^n) z^n$$

encodes all the information about the moments of X. For $\mathfrak{B} \neq \mathbb{C}$, the straightforward generalization

$$\overline{m}_X(z) = \sum_{n=0}^{\infty} \Psi(X^n) z^n$$

generally fails to keep track of all the possible moments of X. A solution to this inconvenience was proposed in [3], namely the moment-generating multilinear function series of X. Before defining this notion, we will briefly recall the construction and several results on multilinear function series.

Let \mathfrak{B} be an algebra over a field K. We set \mathfrak{B} equal to \mathfrak{B} if \mathfrak{B} is unital and to the unitization of \mathfrak{B} otherwise. For $n \geq 1$, we denote by $\mathcal{L}_n(\mathfrak{B})$ the set of all K-multilinear maps

$$\omega_n: \underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{n \text{ times}} \longrightarrow \mathfrak{B}$$

A formal multilinear function series over \mathfrak{B} is a sequence $\omega = (\omega_0, \omega_1, \ldots),$ where $\omega_0 \in \mathfrak{B}$ and $\omega_n \in \mathcal{L}_n(\mathfrak{B})$ for $n \geq 1$. According to [3], the set of all multilinear function series over \mathfrak{B} will de denoted by $Mul[[\mathfrak{B}]]$.

For $\alpha, \beta \in Mul[[\mathfrak{B}]]$, the formal sum $\alpha + \beta$ and the formal product $\alpha\beta$ are the elements from $Mul[[\mathfrak{B}]]$ defined by:

$$(\alpha + \beta)_n(b_1, \dots, b_n) = \alpha_n(b_1, \dots, b_n) + \beta_n(b_1, \dots, b_n)$$
$$(\alpha \beta)_n(b_1, \dots, b_n) = \sum_{k=0}^n \alpha_k(b_1, \dots, b_k) \beta_{n-k}(b_{k+1}, \dots, b_n)$$

for any $b_1, \ldots, b_n \in \mathfrak{B}$.

If $\beta_0 = 0$, then the formal composition $\alpha \circ \beta \in Mul[[\mathfrak{B}]]$ is defined by

$$(\alpha \circ \beta)_0 = \alpha_0$$

and, for $n \geq 1$, by

$$(\alpha \circ \beta)_n(b_1, \dots, b_n) = \sum_{k=1}^n \sum_{k=1}^n \alpha_k \Big(\beta_{p_1}(b_1, \dots, b_{p_1}), \dots, \beta_{p_k}(b_{q_k+1}, \dots, b_{q_k+p_k}) \Big)$$

where the second summation is done over all k-tuples $p_1, \ldots, p_k \geq 1$ such that $p_1 + \dots + p_k = n \text{ and } q_i = p_1 + \dots + p_{i-1}.$

One can work with elements of $Mul[[\mathfrak{B}]]$ as if they were formal power series. The relevant properties are described in [3], Proposition 2.3 and Proposition 2.6. As in [3], we use 1, respectively I, to denote the identity elements of $Mul[[\mathfrak{B}]]$ relative to multiplication, respectively composition. In other words, 1 = (1, 0, 0, ...) and $I = (0, id_{\mathfrak{B}}, 0, 0, \dots)$. We will also use the fact that an element $\alpha \in Mul[\mathfrak{B}]$ has an inverse with respect to formal composition, denoted $\alpha^{(-1)}$, if and only if α has the form $(0, \alpha_1, \alpha_2, \dots)$ with α_1 an invertible element of $\mathcal{L}_1(\mathfrak{B})$.

Definition 3.1. With the above notation, the moment-generating multilinear function series \mathcal{M}_X of X is the element of $Mul[[\mathfrak{B}]]$ such that:

$$\mathcal{M}_{X,0} = \Psi(X)$$

 $\mathcal{M}_{X,n}(b_1,\ldots,b_n) = \Psi(Xb_1X\cdots Xb_nX).$

Given an element $\alpha \in Mul[[\mathfrak{B}]]$, the multilinear function series R_{α} is defined by the following equation (see [3], Def 6.1):

$$R_{\alpha} = \left((1 + \alpha I)^{-1} \right) \circ (I + I\alpha I)^{\langle -1 \rangle}. \tag{3.1}$$

A key property of R is that for any $X, Y \in \mathfrak{A}$ free over \mathfrak{B} , we have

$$R_{\mathcal{M}_{X+Y}} = R_{\mathcal{M}_{Y}} + R_{\mathcal{M}_{Y}}. (3.2)$$

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These relations were proved earlier in the particular case $\mathfrak{B} = \mathbb{C}$. One can also describe R_{α} by combinatorial means, via the recurrence relation

$$\alpha_n(b_1, \dots, b_n) = \sum_{k=0}^n \sum_{k=0} R_{\alpha,k} \Big([b_1 \alpha_{p(1)}(b_3, \dots, b_{i_1-2}) b_{i_1-1}], \dots \\ \dots, [b_{i(k-1)} \alpha_{p(k)}(b_{i(k-1)+1}, \dots, b_{i(k)-2}) b_{i(k)-1}] \Big) b_{i(k)} \alpha_{n-i_k}(b_{i_{k+1}}, \dots, b_n)$$

where the second summation is done over all $1 = i(0) < i(1) < \cdots < i(k) \le n$ and p(k) = i(k) - i(k-1) - 2.

Following an idea from [2], the above equation can be graphically illustrated by the picture:

In the case of scalar c-free probability, an analogue of the Voiculescu's R-transform is developed in [2]. In order to avoid confusions, we will denote it by cR .

The cR -transform has the property that it linearizes the c-free convolution of pairs of compactly supported measures. In particular, if X and Y are c-free elements from some algebra \mathfrak{A} , then

$${}^{c}R_{X+Y} = {}^{c}R_X + {}^{c}R_Y.$$

If the *-algebra $\mathfrak A$ is endowed with the $\mathbb C$ -valued states φ, ψ and X is a selfadjoint element of $\mathcal A$, then (see [2]), the coefficients $\{{}^cR_m\}_m \geq 0$ of cR_X are defined by the recurrence:

$$\varphi(X^{n}) = \sum_{k=1}^{n} \sum_{\substack{l(1),\dots,l(k) \geq 0 \\ l(1)+\dots+l(k)=n-k}} {}^{c}R_{k} \cdot \psi(X^{l(1)}) \cdots \psi(X^{l(k-1)}) \varphi(X^{l(k)})$$

equation that can be graphically illustrated by the picture, were the dark boxes stand for the application of φ and the light ones for the application of ψ :

The above considerations lead to the following definition:

Definition 3.2. Let $\beta, \gamma \in Mul[[\mathfrak{B}]]$. The multilinear function series ${}^{c}R_{\beta,\gamma}$ is the element of $Mul[[\mathfrak{B}]]$ defined by the recurrence relation

$$\beta_n(b_1,\ldots,b_n) = \sum_{k=0}^n \sum_{a=0}^n c R_{\beta,\gamma,k} \Big([b_1 \gamma_{p(1)}(b_3,\ldots,b_{i_1-2})b_{i_1-1}],\ldots \Big)$$

$$\dots, [b_{i(k-1)}\gamma_{p(k)}(b_{i(k-1)+1},\dots,b_{i(k)-2})b_{i(k)-1}])b_{i(k)}\beta_{n-i_k}(b_{i_{k+1}},\dots,b_n)$$

where the second summation is done over all $1 = i(0) < i(1) < \cdots < i(k) \le n$ and p(k) = i(k) - i(k-1) - 2.

The following analytical description of ${}^{c}R_{\beta,\gamma}$ also shows that it is unique and well-defined:

Theorem 3.3. For any $\beta, \gamma \in Mul[[\mathfrak{B}]]$,

$${}^{c}R_{\beta,\gamma} = \left[\beta(1+I\beta)^{-1}\right] \circ (I+I\gamma I)^{\langle -1\rangle} \tag{3.3}$$

Before proving the theorem, remark that the right-hand side of (3.3) is welldefined and unique, since $1 + I\gamma$ is invertible with respect to the formal multiplication, $I + I\beta I$ is invertible with respect to formal composition and its inverse has 0 as first component (see [3]). We will need the following auxiliary result:

Lemma 3.4. Let β be an element of $Mul[[\mathfrak{B}]]$ and I the identity element with respect to formal composition, $I = (0, id_{\mathfrak{B}}, 0, 0...)$.

(i) the multilinear function series $I\beta$ is given by:

$$(I\beta)_0 = 0$$

 $(I\beta)_n(b_1, \dots, b_n) = b_1\beta_{n-1}(b_2, \dots, b_n)$

(ii) the multilinear function series $I\beta I$ is given by

$$(I\beta I)_0 = 0$$

 $(I\beta I)_1(b_1) = 0$
 $(I\beta I)_n(b_1, \dots, b_n) = b_1\beta_{n-2}(b_2, \dots, b_{n-1})b_n$

Proof. Since $I = (0, id_{\mathfrak{B}}, 0, ...)$, one has:

$$(I\beta)_0 = I_0\beta_0 = 0.$$

If $n \geq 1$,

$$(I\beta)_{n}(b_{1},...,b_{n}) = \sum_{k=0}^{n} I_{k}(b_{1},...,b_{k})\beta_{n-k}(b_{k+1},...,b_{n})$$
$$= I_{1}(b_{1})\beta_{n-1}(b_{k+1},...,b_{n})$$
$$= b_{1}\beta_{n-1}(b_{k+1},...,b_{n}).$$

For $I\beta I$, the same computations give:

$$(I\beta I)_0 = (I\beta)_0 I_0 = 0$$

 $(I\beta I)_1 = (I\beta)_0 I_1(b_1) + (I\beta)_1(b_1) I_0$
 $= 0.$

If n > 2, one has:

$$(I\beta I)_n(b_1,\ldots,b_n) = \sum_{k=0}^n (I\beta)_k(b_1,\ldots,b_k) I_{n-k}(b_{k+1},\ldots,b_n)$$
$$= (I\beta)_{n-1}(b_1,\ldots,b_k) I_1(b_1)$$
$$= b_1\beta_{n-2}(b_2,\ldots,b_{n-1})b_n$$

Proof of the Theorem 3.3: Set $\sigma = I + I\beta I$. Then

$$({}^{c}R_{\beta,\gamma} \circ \sigma)_{n} (b_{1}, \dots, b_{n}) = \sum_{k=1}^{n} \sum_{\substack{p_{1}, \dots, p_{k} \geq 1 \\ p_{1} + \dots + p_{k} = n}} {}^{c}R_{\beta,\gamma,k} \Big(\sigma_{p_{1}}(b_{1}, \dots, b_{p_{1}}), \dots, \sigma_{p_{k}}(b_{n_{k}+1}, \dots, b_{n_{k}+n_{k}}) \Big)$$

where $q_i = p_1 + \dots + p_{i-1}$.

From Lemma (3.4)(ii), we have that

$$\sigma_n(b_1,\ldots,b_n) = (I + I\beta I)_n(b_1,\ldots,b_n)$$

therefore Definition 3.2 is equivalent to

$$\beta_n(b_1, \dots, b_n) = \sum_{k=0}^n ({}^cR_{\beta,\gamma} \circ (I + I\beta I)_k(b_1, \dots, b_k)) b_{k+1} \beta_{n-k-2}(b_{k+2}, \dots, b_n)$$

Considering now Lemma 3.4(i), the above relation becomes

$$\beta_n(b_1, \dots, b_n) = \sum_{k=0}^n ({}^cR_{\beta,\gamma} \circ (I + I\beta I)_k(b_1, \dots, b_k)) (I + I\beta)_{n-k}(b_{k+1}, \dots, b_n)$$

therefore

$$\beta = [{}^{c}R_{\beta,\gamma} \circ (I + I\gamma I)] (1 + I\beta)$$

which is equivalent to (3.3).

Remark 3.5. Up to a shift in the coefficients, equation (3.3) is similar to the result in the case $\mathfrak{B} = \mathbb{C}$ from [2], Theorem 5.1.

Let X be a selfadjoint element of \mathfrak{A} . If \mathfrak{A} is endowed with two \mathfrak{B} -valued conditional expectations Φ, Ψ , the element X will have two moment-generating multilinear function series, one with respect to Ψ , that we will denote by \mathcal{M}_X , and one with respect to Φ , denoted \mathfrak{M}_X . For brevity, we will use the notation cR_X for the multilinear function series ${}^cR_{\mathcal{M}_X,\mathfrak{M}_X}$.

Theorem 3.6. Let X and Y be two elements of \mathfrak{A} that are c-free with respect to the pair of conditional expectations (Φ, Ψ) . Then

$${}^cR_{X+Y} = {}^cR_X + {}^cR_Y$$

Proof. Let \mathcal{A} be an algebra containing \mathfrak{B} as a subalgebra and endowed with the conditional expectations $\Phi, \Psi : \mathcal{A} \longrightarrow \mathfrak{B}$. Consider the set $\mathcal{A}_0 = \mathcal{A} \setminus \mathfrak{B}$ (set difference). For $n \geq 1$ define the maps

$${}^{c}r: \underbrace{\mathcal{A}_{0} \times \cdots \times \mathcal{A}_{0}}_{n \text{ times}} \longrightarrow \mathfrak{B}$$

given by the recurrence formula:

$$\Phi(a_1 \cdots a_n) = \sum_{k=1}^n \sum_{\substack{l(1) < \cdots < l(k) \\ 1 < l(1), l(k) \le n}} {}^c r_k \Big(a_1 [\Psi(a_2 \cdots a_{l(1)-1})], \dots, \\ \dots, a_{l(k-1)} [\Psi(a_{l(k-1)+1} \cdots a_{l(k)})], a_{l(k)} [\Phi(a_{l(k)+1} \cdots a_n)] \Big)$$

Note that ${}^{c}r_{n}$ is well defined, and that, for any $b_{1}, \ldots, b_{n} \in \mathfrak{B}$,

$${}^{c}r_{n+1}(X, b_1 X, \dots, b_n X) = {}^{c}R_{X,n}(b_1, \dots, b_n).$$
 (3.4)

As in Section 2, consider $\mathfrak{B}\langle \xi_i \rangle$, the noncommutative algebras of polynomials in the symbols $\xi_i, i=1,2$ and with coefficients from \mathfrak{B} and the conditional expectations

$$\Phi_X, \Psi_X : \mathfrak{B}\langle \xi_1 \rangle \longrightarrow \mathfrak{B}$$

given by

$$\Phi_X(f(\xi_1)) = \Phi(f(X))
\Psi_X(f(\xi_1)) = \Psi(f(X))$$

and their analogues Φ_Y, Ψ_Y for $\mathfrak{B}\langle \xi_2 \rangle$.

On $\mathfrak{B}\langle \xi_1, \xi_2 \rangle$, identified to $\mathfrak{B}\langle \xi_1 \rangle *_{\mathfrak{B}} \mathfrak{B}\langle \xi_2 \rangle$, consider the conditional expectations Ψ_0, Φ_0, φ given by:

$$\Psi_{0} = \Psi_{X} * \Psi_{Y}
\Phi_{0}(f(\xi_{1}, \xi_{2})) = \Phi(f(X, Y))
\varphi(a_{1}a_{2} \dots a_{n}) = \sum_{k=1}^{n} \sum_{\substack{l(1) < \dots < l(k) \\ 1 < l(1), l(k) \le n}} \rho_{k} \Big(a_{1} [\Psi_{0}(a_{2} \dots a_{l(1)-1})], \dots,
\dots, a_{l(k-1)} [\Psi_{0}(a_{l(k-1)+1} \dots a_{l(k)_{1}})], a_{l(k)} [\varphi(a_{l(k)+1} \dots a_{n})] \Big)$$

where a_1, \ldots, a_n are elements of the set $\mathfrak{B}\langle \xi_1, \xi_2 \rangle_0 = \mathfrak{B}\langle \xi_1 \rangle \cup \mathfrak{B}\langle \xi_2 \rangle \setminus \mathfrak{B}$, and the maps

$$\rho_n: \underbrace{\mathfrak{B}\langle \xi_1, \xi_2 \rangle_0 \times \dots \mathfrak{B}\langle \xi_1, \xi_2 \rangle_0}_{n \text{ times}} \longrightarrow \mathfrak{B}$$

are given by:

$$\rho_n(a_1, \dots, a_n) = \begin{cases} {}^c r(a_1, \dots, a_n) & \text{if all } a_1, \dots, a_n \in \mathfrak{B}\langle \xi_1 \rangle \\ {}^c r(a_1, \dots, a_n) & \text{if all } a_1, \dots, a_n \in \mathfrak{B}\langle \xi_2 \rangle \\ 0 & \text{otherwise} \end{cases}.$$

We will show that $\varphi = \Phi_0$, in particular φ is also well-defined. Consider the element $a \in \mathfrak{B}\langle \xi_1, \xi_2 \rangle$ of the form $a = a_1 \cdots a_n$ with $a_j \in \mathfrak{B}\langle \xi_{\varepsilon(j)} \rangle$, such that $\varepsilon(1) \neq \varepsilon(2) \neq \cdots \neq \varepsilon(n)$ and $\Psi_0(a_j) = 0$. The computation of $\varphi(a_1 \cdots a_n)$ is done via the recurrence relation above. Because of the definition of ρ and the fact that $\Psi_0 = \Psi_X * \Psi_Y$, only the term with k = 1 contribute at the sum, i.e.

$$\varphi(a_1 \cdots a_n) = \varphi(a_1 \varphi(a_2 \cdots a_n))
= \varphi_{\varepsilon(1)}(a_1 \varphi(a_2 \cdots a_n))
= \varphi_{\varepsilon(1)}(a_1) \varphi(a_2 \cdots a_n)$$

and the identity between φ and Φ_0 follows by induction over n.

Since $\varphi = \Phi_0$, the maps ρ_n and cr_n are satisfying the same recurrence relation, hence

$$\rho_n(a_1,\ldots,a_n) = {}^c r(a_1,\ldots,a_n).$$

In particular

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$${}^{c}R_{X+Y,n}(b_{1},\ldots,b_{n}) = {}^{c}r_{n+1}((X+Y)b_{1}(X+Y)\ldots(X+Y)b_{n}(X+Y))$$

$$= \rho_{n+1}((X+Y)b_{1}(X+Y)\ldots(X+Y)b_{n}(X+Y))$$

$$= \rho_{n+1}((X)b_{1}(X)\ldots(X)b_{n}(X)) + \rho_{n+1}((Y)b_{1}(Y)\ldots(Y)b_{n}(Y))$$

$$= {}^{c}R_{X,n}(b_{1},\ldots,b_{n}) + {}^{c}R_{Y,n}(b_{1},\ldots,b_{n}).$$

4. Central limit theorem

Consider the ordered set $\langle n \rangle = \{1, 2, \dots, n\}$ and π a partition of $\langle n \rangle$ with blocks B_1, \dots, B_m :

$$\langle n \rangle = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_m.$$

The blocks B_p and B_q of π are said to be *crossing* if there exist i < j < k < l in $\langle n \rangle$ such that $i, k \in B_p$ and $j, l \in B_q$.

The partition π is said to be non-crossing if all pairs of distinct blocks of π are not crossing. We will denote by $NC_2(n)$ the set of all non-crossing partitions of $\langle n \rangle$ whose blocks contain exactly 2 elements and by $NC_{\leq s}(n)$ the set of all non-crossing partitions of $\langle n \rangle$ whose blocks contain at most s elements.

Let now γ be a non-crossing partition of $\langle n \rangle$ and B and C be two blocks of π . We say that B is interior to C if there exist two indices i < j in $\langle n \rangle$ such that $i, j \in C$ and $B \subset \{i+1, \ldots, j-1\}$. The block B is said to be *outer* if it is not interior to any other block of γ . In a non-crossing partition of $\langle n \rangle$, the block containing 1 is always outer.

Consider now an element X of \mathfrak{A} . Let π be a partition from $NC_2(n+1)$ (n= odd) and $B_1=(1,k)$ be the block of π containing 1. We define, by recurrence, the following expressions:

$$V_{\pi}(X, b_1, \dots, b_n) = \Psi(Xb_1V_{\pi|\{2,\dots,j-1\}}(X, b_2, \dots, b_{k-2})b_{k-1}X)b_k$$

$$V_{\pi|\{k+1,\dots,n+1\}}(X, b_{k+1}, \dots, b_n)$$

$$W_{\pi}(X, b_1, \dots, b_n) = \Phi(Xb_1V_{\pi|\{2,\dots,j-1\}}(X, b_2, \dots, b_{k-2})b_{k-1}X)b_k$$

$$W_{\pi|\{k+1,\dots,n+1\}}(X, b_{k+1}, \dots, b_n)$$

Theorem 4.1. (Central Limit Theorem) Let $(X_n)_{n\geq 1}$ be a sequence of c-free elements of $\mathfrak A$ such that:

- (1) all X_n have the same moment-generating multilinear function series, \mathfrak{M} with respect to Φ and M with respect to Ψ .
- (2) $\Psi(X_n) = \Phi(X_n) = 0.$

Set

$$S_N = \frac{X_1 + \dots + X_N}{\sqrt{N}},$$

Then:

(i)
$$\lim_{N\to\infty} {}^c R_{S_N} = (0, \mathfrak{M}_1(\cdot), 0, \dots)$$

- (ii) $\lim_{N\to\infty} R_{S_N} = (0, M_1(\cdot), 0, \dots)$
- (iii) there exist two conditional expectations $\nu: \mathfrak{B}\langle \xi \rangle \longrightarrow \mathfrak{B}$, depending only on $M_1(\cdot)$, and $\mu: \mathfrak{B}\langle \xi \rangle \longrightarrow \mathfrak{B}$, depending only on $M_1(\cdot)$ and $\mathfrak{M}_1(\cdot)$, such that

$$\lim_{N \to \infty} \Psi_{S_N} = \nu$$

$$\lim_{N \to \infty} \Phi_{S_N} = \mu$$

in the weak sense; in particular,

$$\nu(\xi b_1 \xi \dots b_n \xi) = \sum_{\pi \in NC_2(n)} V_{\pi}(X_1, b_1, \dots, b_n)
\mu(\xi b_1 \xi \dots b_n \xi) = \sum_{\pi \in NC_2(n)} W_{\pi}(X_1, b_1, \dots, b_n).$$

Proof. Let X be an element of \mathfrak{A} with the same moment generating series as X_j , $j \geq 1$. As shown in [3],

$$R_{S_N} = \sum_{k=1}^N R_{\frac{X_k}{\sqrt{N}}} = NR_{\frac{X}{\sqrt{N}}}.$$

Also, from Theorem 2.4 and Theorem 3.6, it follows that

$${}^{c}R_{S_{N}} = \sum_{k=1}^{N} {}^{c}R_{\frac{X_{k}}{\sqrt{N}}} = N^{c}R_{\frac{X}{\sqrt{N}}}.$$

Since R and ${}^{c}R$ are multilinear and $M_{0}=\mathfrak{M}_{0}=0$, we have that

$$\lim_{N \to \infty} {}^{c}R_{S_{N},n} = \lim_{N \to \infty} \frac{N}{N^{\frac{n+1}{2}}} {}^{c}R_{X,n}$$

$$= \begin{cases} 0 & \text{if } n \neq 1 \\ \mathfrak{M}_{1}(\cdot) & \text{if } n = 1 \end{cases}$$

and the similar relations for $R_{S_N,n}$, hence (i) and (ii) are proved.

For (iii) it suffices to check the relations for $\nu(\xi b_1 \xi \dots b_n \xi)$ and $\mu(\xi b_1 \xi \dots b_n \xi)$, which are a trivial corollary of (i), (ii), and the recurrence formulas that define R and cR .

Remark 4.2. For $\mathfrak{B} = \mathbb{C}$, the theorem is a weaker version of Theorem 4.3 from [2]. If Ψ is \mathbb{C} -valued, then the result is similar to Corollary 5.1 from [6]. Also, under the assumptions that for some $a, b \in \mathfrak{B}$ we have that:

$$\lim_{N \to \infty} N\Psi(X_1 \cdots X_N) = a$$

$$\lim_{N \to \infty} N\Psi(X_1 \cdots X_N) = b$$

the same techniques lead to a Poisson-type limit Theorem, similar to Corollary 2, Section 5 of [6].

In the following remaining pages we will describe the positivity of the limit functionals μ and ν in terms of Φ and Ψ . The central result is Corollary 4.4.

For simplicity, suppose that \mathfrak{B} is a unital *-algebra (otherwise, we can replace \mathfrak{B} by its unitisation). Consider the symbol ξ , the *-algebra $\mathfrak{B}\langle\xi\rangle$ of polynomials in ξ with coefficients from \mathfrak{B} , as defined before, and consider also the linear space $\mathfrak{B}\xi\mathfrak{B}$ generated by the set $\{b_1\xi b_2;\ b_1,b_2\in\mathfrak{B}\}$ with the \mathfrak{B} -bimodule structure given by

$$a_1b_1\xi b_2a_2 = (a_1b_1)\xi(b_2a_2)$$

for all $a_1, a_2, b_1, b_2 \in \mathfrak{B}$.

Lemma 4.3. For any positive \mathfrak{B} -sesquilinear pairing $\langle \cdot, \cdot \rangle$ on $\mathfrak{B}\xi\mathfrak{B}$ there exists a positive conditional expectation $\varphi : \mathfrak{B}\langle \xi \rangle \longrightarrow \mathfrak{B}$ such that for any $b_1, b_2 \in \mathfrak{B}$ one has that

$$\varphi(\xi b_1^* b_2 \xi) = \langle b_1 \xi, b_2 \xi \rangle$$

Proof. Without loss of generality, we can suppose that \mathfrak{B} is unital (otherwise we can replace \mathfrak{B} by its unitization).

Consider the Full Fock bimodule over $\mathfrak{B}\xi\mathfrak{B}$

$$\mathcal{F}\langle \xi \rangle = \mathfrak{B} \oplus \Big(\bigoplus_{n \geq 1} \underbrace{\mathfrak{B} \xi \mathfrak{B} \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} \mathfrak{B} \xi \mathfrak{B}}_{n \text{ times}} \Big)$$

with the pairing given by

$$\langle a,b\rangle = a^*b$$

$$\langle a_1\xi \otimes \cdots \otimes a_n\xi, b_1\xi \otimes \cdots \otimes b_m\xi \rangle = \delta_{m,n}\langle a_n\xi, \langle \ldots, \langle a_1\xi, b_1\xi \rangle b_2\xi \rangle, \ldots b_n\xi \rangle.$$

$$(a,a_j,b,b_j \in \mathfrak{B}, j=1,\ldots,n)$$

Note that the \mathfrak{B} -linear operators $A_1, A_2 : \mathcal{F}\langle \xi \rangle \longrightarrow \mathcal{F}\langle \xi \rangle$ described by the relations

$$A_1b = \xi b$$

$$A_1(a_1\xi \otimes \cdots \otimes a_n\xi b) = \xi \otimes a_1\xi \otimes \cdots \otimes a_n\xi b$$

$$A_2b = 0$$

$$A_2(a_1\xi \otimes \cdots \otimes a_n\xi b) = \langle \xi, a_1\xi \rangle a_2\xi \otimes \cdots \otimes a_n\xi b$$

are self-adjoint to each other, in the sense that

$$\langle A_1 \widetilde{\zeta}_1, \widetilde{\zeta}_2 \rangle = \langle \widetilde{\zeta}_1, A_2 \widetilde{\zeta}_2 \rangle$$

for any $\widetilde{\zeta}_1, \widetilde{\zeta}_2 \in \mathcal{F}\langle \xi \rangle$, therefore $S = A_1 + A_2$ is selfadjoint. Moreover, for any $a, b \in \mathfrak{B}$,

$$\langle 1, Sa^*bS1 \rangle = \langle aS1, bS1 \rangle$$

$$= \langle a(A_1 + A_2)1, b(A_1 + A_2)1 \rangle$$

$$= \langle a\mathcal{E}, b\mathcal{E} \rangle$$

and the conclusion follows by setting $\varphi(p(\xi)) = \langle 1, p(S)1 \rangle$ for all $p \in \mathfrak{B}\langle \xi \rangle$.

Corollary 4.4. The maps μ and ν from Theorem 4.1 are positive if and only if for any $b \in \mathfrak{B}$ one has that $\Phi(Xb^*bX) \geq 0$ and $\Psi(Xb^*bX) \geq 0$.

Proof. One implication is trivial, since, if ν and μ are positive, then

$$\Psi(Xb^*bX) = \nu(Xb^*bX) = \nu((bX)^*bX) \ge 0$$

and

$$\Phi(Xb^*bX) = \mu(Xb^*bX) = \mu((bX)^*bX) \ge 0.$$

Suppose now that $\Phi(Xb^*bX) \geq 0$ and $\Psi(Xb^*bX) \geq 0$ for all $b \in \mathfrak{B}$. We will use the same argument as in [9] and [8].

Consider the set of selfadjoint symbols $\{\xi_i\}_{i\geq 1}$. On each \mathfrak{B} -bimodule $\mathfrak{B}\xi_i\mathfrak{B}$ we have the positive \mathfrak{B} -sesquilinear pairings $\langle\cdot,\cdot\rangle_{\Phi}$ and $\langle\cdot,\cdot\rangle_{\Psi}$ determined by

$$\langle a\xi_i, b\xi_i \rangle_{\Phi} = \Phi(Xa^*bX)$$

$$\langle a\xi_i, b\xi_i \rangle_{\Psi} = \Psi(Xa^*bX).$$

As shown in Lemma 4.3, the above \mathfrak{B} -sesquilinear pairings determine positive conditional expectations $\varphi_1, \psi_i : \mathfrak{A}_i \longrightarrow \mathfrak{B}$, where $\mathfrak{A}_i = \mathfrak{B}\langle \xi_i \rangle$ be the *-algebras of polynomials in ξ with coefficients from $\mathfrak{B}, i \geq 1$.

For $\tau: \mathfrak{B}\langle \xi \rangle \longrightarrow \mathfrak{B}$ a conditional expectation, and $\lambda \geq 0$, note with $D_{\lambda}\tau$ the dilation with λ of τ , i.e.

$$D_{\lambda}\tau(\xi b_1\xi\cdots b_n\xi) = \lambda^{n+1}\tau(\xi b_1\xi\cdots b_n\xi)$$

Remark that if τ is positive, then $D_{\lambda}\tau$ is also positive.

With the notations above, consider, as in Definition 2.1, the conditionally free product $(\mathfrak{A}, \Phi, \Psi) = \star_{i \in \mathcal{I}} (\mathfrak{A}_i, \Phi_i, \Psi_i)$. The elements $\{\xi_i\}_{i \geq 1}$ are conditionally free in \mathfrak{A} , so Theorem 4.1 implies that:

$$\mu = \lim_{N \to \infty} \Phi_{\underline{\xi_1 + \dots + \xi_N}} = D_{\frac{1}{\sqrt{N}}} \Phi_{\xi_1 + \dots + \xi_N}$$

$$\nu = \lim_{N \to \infty} \Psi_{\underline{\xi_1 + \dots + \xi_N}} = D_{\frac{1}{\sqrt{N}}} \Psi_{\xi_1 + \dots + \xi_N}$$

$$= D_{\frac{1}{\sqrt{N}}} \left(*_{i=1}^N \Psi_{\xi_i} \right).$$

We have that $\star_{i=1}^N \Psi_{\xi_i} \geq 0$ since it is the free product of states (see, for example [9]), hence the positivity of ν .

Also, Theorem 2.4 and Corollary 2.6 imply that $\Phi_{\xi_1+\cdots+\xi_N} \geq 0$, therefore $\mu \geq 0$.

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