Robust Optimization over Networks Using Distributed Restarting of Accelerated Dynamics

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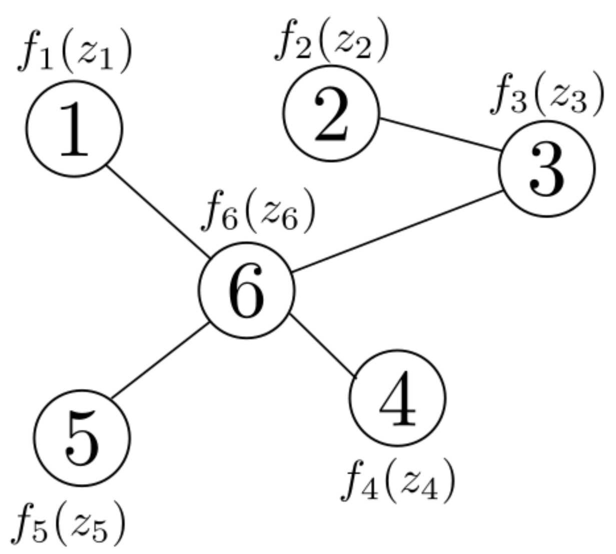
Consensus-Optimization Problem

$\mathcal{G} = (\mathcal{V},\ \mathcal{E})$



$$\min_{\mathbf{z}} F(\mathbf{z}) := \sum_{i=1}^{n} f_i(z_i)$$

$$\text{s.t. } \mathbf{Lz} = 0$$

$$\mathbf{L} = \mathcal{L} \otimes I_p$$

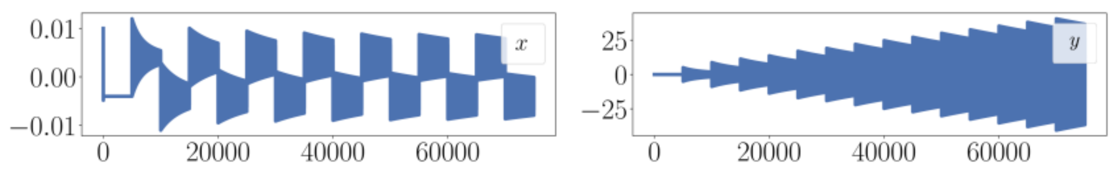
$$\mathbf{z} = \left[z_1^\top, z_2^\top, \cdots, z_n^\top\right]^\top$$

Objective and Challenges

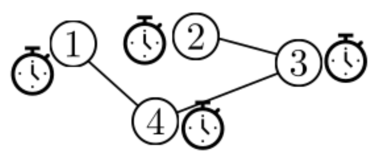
Fast convergence

$$\ddot{x} + \frac{3}{t}\dot{x} + \alpha\nabla f(x) = 0 \iff \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\frac{2}{t}(y-x)\\ -2\gamma t\nabla f(x_1)\end{pmatrix}$$

Robustness

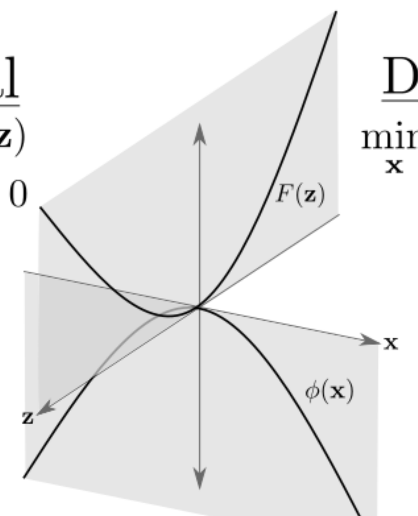


Fully Distributed



Mantain the sparcity of $\mathbf{L}$

Tools



$$\phi(\mathbf{x}) := \max_{\mathbf{z}} \left\{ \langle \mathbf{Lx}, \mathbf{z}\rangle - F(\mathbf{z}) \right\}$$

$$\nabla\phi(\mathbf{x}) = \mathbf{L}h(\mathbf{Lx})$$

$$h(\mathbf{u}) := \arg\max_{\mathbf{z}\in\mathbb{R}^{np}} \left\{ \langle \mathbf{u}, \mathbf{z}\rangle - F(\mathbf{z}) \right\}$$

Hybrid Dynamical Systems (HDS)

$$p \in C, \qquad \dot{p} = F(p)$$

$$p \in D, \qquad p^+ \in G(p)$$

$$\mathcal{H} = \{C, F, D, G\}$$

Hybrid Accelerated Restarting Distributed Dynamics (HARDD)

Flows

$\mathbf{x}$: Dual variable

$\mathbf{y}$: Momentum

$\boldsymbol{\tau} = [\tau_1, \tau_2, \cdots, \tau_n]^\top$

$$\dot{\mathbf{p}} = \begin{pmatrix} 2\mathbf{D}(\boldsymbol{\tau}\otimes\mathbf{1}_p)^{-1}(\mathbf{y}-\mathbf{x}) \\ -2\gamma\Psi(\boldsymbol{\tau},\mathbf{x}) \\ \frac{1}{2}\mathbf{1}_n \end{pmatrix}$$

$$\Psi(\boldsymbol{\tau},\mathbf{x}) := \mathbf{LD}(\boldsymbol{\tau}\otimes\mathbf{1}_p)h(\mathbf{Lx})$$

$$C_A := \mathbb{R}^{np} \times \ker(\mathbf{L})^\perp \times [T_r, T_r + \Delta T]^n$$

- Nesterov Inspired Dynamics
- Sparsity of the communication matrix conserved
- One Timer per node

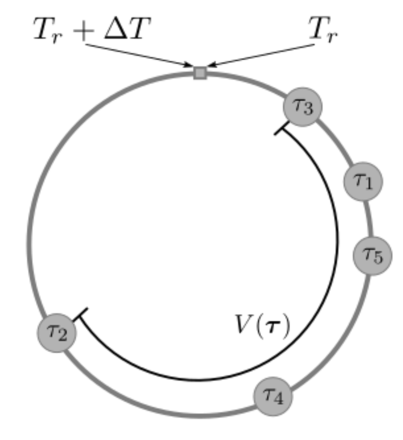
Jumps

$$\mathbf{p}^+ \in \begin{pmatrix} \{(1-q)\mathbf{x}+q\mathbf{s}\} \\ \{\mathbf{y}\} \\ \{\mathbf{g}\} \end{pmatrix}, \ (\mathbf{s},\mathbf{g}) \in \mathcal{T}(\boldsymbol{p})$$

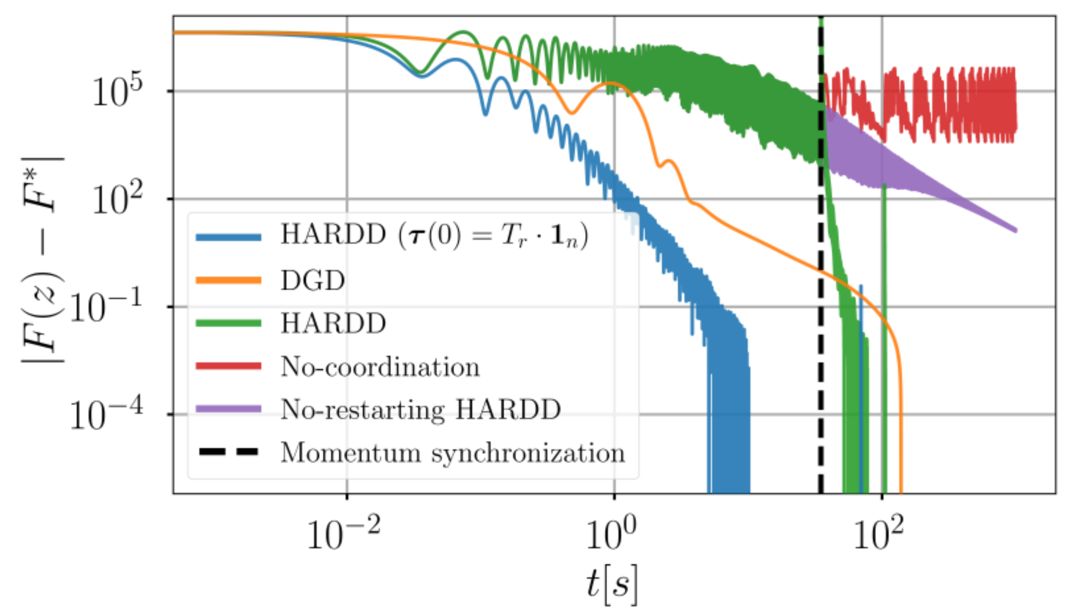
$$D_A := \mathbb{R}^{np} \times \ker(\mathbf{L})^\perp \times D_\tau$$

$$D_\tau := \left\{\boldsymbol{\tau} \in [T_r, T_r+\Delta T]^n : \max_{i\in\mathcal{V}} \tau_i = T_r + \Delta T\right\}$$

- Guarantees synchronization of timers
- For strongly convex functions resets dual variable $\mathbf{x}$ to the momentum $\mathbf{y}$



Stability and Robustness Certificates



Main Theorem

Under suitable conditions on the tunable parameters of the dynamics

- Set of minimizers of the cost function is Uniformly Globally Asymptotically Stable (UGAS)

$q = 0$

$\exists\lambda_0, c_0$

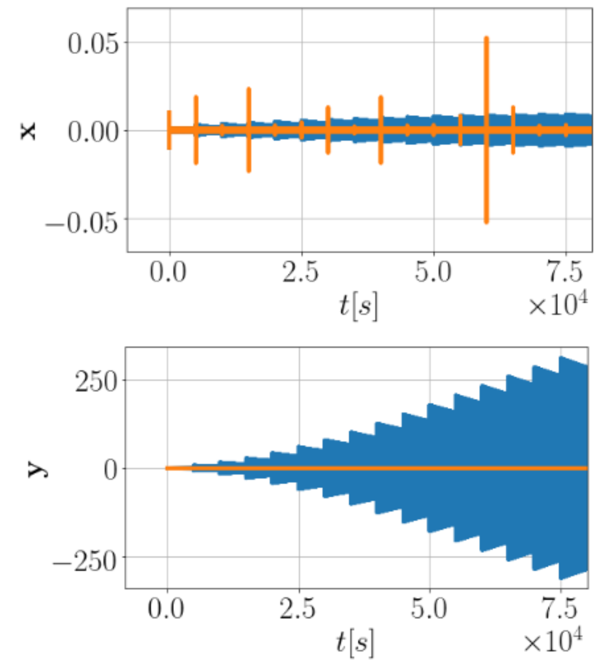
$F(\mathbf{z}(t,j)) - F^* \leq c_0 e^{-\lambda_0(t+j)}$

$q = 1$

If $\mathbf{x}(0,0) \in \ker(\mathbf{L})^\perp$

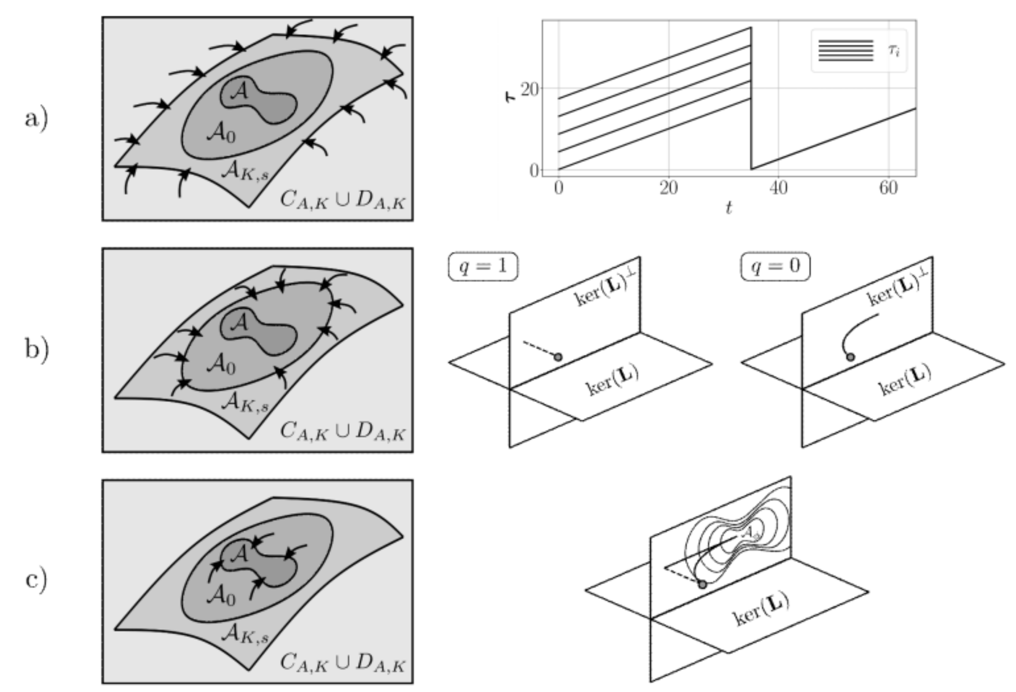
$F(\mathbf{z}(t,j)) - F^* \leq \dfrac{c_j}{\tau_i(t,j)}$

for all $i \in \mathcal{V}$



UGAS + Well-posed HDS gives positive margins of robustness with respect to arbitrarily small disturbances

Proof Sketch



Nested application of the Hybrid Reduction Principle to prove stability of set of minimizers by first proving UGAS of the set with with synchronized timers and the set of feasible solutions.

Contribution

- Formulation of the first robust and distributed restarting-based accelerated dynamics for the solution of the network consensus-optimization problem.

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