W203, Unit 3 Proof

Artem Lebedev

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1. Correlation between X and $aX + b$

$$Corr[aX + b, X] = \frac{Cov[aX + b, X]}{\sqrt{Var[aX + b]} * \sqrt{Var[X]}}$$

$$= \frac{a * Cov[X, X]}{|a| * \sqrt{Var[X]} * \sqrt{Var[X]}}$$

$$= \frac{a}{|a|} * Corr[X, X] =$$

$$\frac{a}{|a|} = \begin{cases} 1, a > 0 \\ -1, a < 0 \end{cases}$$

2 Heavy Tails

2.1 Finite expectation

I will rely on the solution of well known "Basel problem": $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

From the definition of expectation:

$$E[M] = \sum_{x=1}^{\infty} x * p_M(x) = \sum_{x=1}^{\infty} x * \frac{c}{x^3} = \frac{c * \pi^2}{6}$$

which is a finite value.

2.2 Infinite variance

$$V[M] = E[x^2] - E^2[x], \text{ almost definition}$$

$$V[M] = \sum_{x=1}^{\infty} x^2 * p_M(x) - \left(\frac{c * \pi^2}{6}\right)^2$$

Note that:

$$\sum_{x=1}^{\infty} x^2 * p_M(x) = \sum_{x=1}^{\infty} \frac{cx^2}{x^3} = \sum_{x=1}^{\infty} \frac{c}{x}$$

which is a sum of harmonic series, known to be divergent, infinetely large. Therefore $V[M]$ is difference of a infinte number and finite number, which is infinite.