Setup

Numbering the cells from 1 to n, we define the random variable X_i to be the number of mines in cell i and we use a slight abuse of notation such that $P(X_i) := P(X_i \geq 1)$. Suppose there are r sets of equivalent cells (not including the outer group), then we denote a set of equivalent cells as G_j, for $1 \leq j \leq r$, for example a group of four neighbouring cells might be $G_1 = \{11, 12, 13, 14\}$. We also define the random variable representing the total number of mines in the jth group to be Y_j, which is given by $Y_j = \sum_{i \in G_j} X_i$, and we represent the distribution of mines across the groups with the vector $\mathbf{Y} = (Y_j)_{j=1}^r$. We also use the notation $X^{(j)}$ to denote the number of mines in a single cell within group j.

Given the situation of the board, there are a number of valid configurations for the mines in the unclicked cells, corresponding to vectors $\mathbf{y}^{(i)}$ with $P(\mathbf{Y} = \mathbf{y}^{(i)}) > 0$. We use C_i to denote the event that configuration i represents the actual hidden distribution of mines across these groups, which does not take into account the full board situation, so in general $\sum_i P(C_i) < 1$. We then have $P(\mathbf{Y} = \mathbf{y}^{(i)}) = P(C_i)/\sum_j P(C_j)$.

0.1 One mine per cell

We begin with the simple case $x_{\max} = 1$, as in standard minesweeper games, in which case $P(X) = P(X = 1)$. we then consider the start of a new game. We suppose on the first click (which we assume to be away from the edges) that we neither hit a mine or an opening, revealing a number α, with $1 \leq \alpha \leq 8$, telling us that the surrounding eight cells contain a total of α mines. We may define this group of cells to be G_1, and we have $P(Y_1 = \alpha) = 1$, with the size of the group being $s_1 = 8$. We have $P(X^{(1)}) = \alpha/8$, giving the safety of all cells in G_1.

We now suppose our first click reveals a number '1' ($a = 1$), which is the most likely case (see the combinatorics approach). The probability of any of the 8 surrounding cells containing a mine is $1/8 = 0.125$, which is lower than ρ on any normal game, so the safest move is to choose an adjacent cell. If the first click were to reveal a 2 the probability of a surrounding cell containing a mine is $1/8 = 0.25$ is greater than ρ, so a random cell is safer. We suppose the second click is horizontally adjacent to the previously clicked cell, and let β represent the number revealed, an example of which is displayed in Figure 1, where we have numbered the cells so that we can take the groups to be $G_1 = \{1, 2, 3\}$, $G_2 = \{4, 5, 6, 7\}$, $G_3 = \{8, 9, 10\}$.

We calculate the probabilities $P(X_i)$ for $1 \leq i \leq 10$ (or equivalently $P(X^{(j)})$ for $j = 1, 2, 3$) by considering all possible configurations of mines in

1	4	5	8
2	1	2	9
3	6	7	10

Figure 1: Numbered cells around two clicked cells.

these three groups of cells. There are two configurations for the distribution of mines: $\mathbf{Y} \in \{(1,0,2),(0,1,1)\}$, which we refer to as configurations 1 and 2 respectively. For C_1 to occur we need exactly 1 mine in a group of 3 cells,

(a) Configuration 1. (b) Configuration 2.

Figure 2: Groups of equivalent cells represented by colours (numbered from left to right), mines represented by asterisks.

none in a group of 4, and 1 in a different group of 3.

Approximate solution

To illustrate the calculation we simplify the problem by assuming the grid is very large, so that the mine density, ρ, remains approximately constant. Considering the number of ways the required number of mines can occur in each group and the probability of there being a mine, we get

$$P(C_1) = \left[\binom{3}{1}\rho(1-\rho)^2\right]\left[(1-\rho)^4\right]\left[\binom{3}{2}\rho^2(1-\rho)\right] = 9\rho^3(1-\rho)^7,$$

which corresponds to 3 cells containing mines, 7 cells not containing mines, with the choice of 1 cell in a group of 3, and the choice of 2 cells in another group of 3. Thinking of it this way makes it much easier to just write down probabilities, and we see $P(C_2) = 12\rho^2(1-\rho)^8$. This gives the probabilities for the groups

$$
\begin{aligned}
P(\mathbf{Y}=(1,0,2)) &= \frac{P(C_1)}{P(C_1)+P(C_2)} \\
&= \frac{9\rho^2(1-\rho)^8}{9\rho^2(1-\rho)^8+12\rho(1-\rho)^9} \\
&= \frac{3\rho}{3\rho+4(1-\rho)}, \\
P(\mathbf{Y}=(0,1,1)) &= \frac{4(1-\rho)}{3\rho+4(1-\rho)}.
\end{aligned}
$$

From this we determine the probability distribution of mines in each of the groups. In this case there is no crossover of the number of mines between configurations, so we have

$$
\begin{aligned}
P(Y_1=1)=P(Y_2=0)=P(Y_3=2)=P(\mathbf{Y}=(1,0,2)) \\
P(Y_1=0)=P(Y_2=1)=P(Y_3=1)=P(\mathbf{Y}=(0,1,1)).
\end{aligned}
$$

Finally, we find $P(X_i)$ for each cell i by considering the expected value of $\mathbf{Y}$ and dividing by the size of the group. Taking $\rho = 0.207$ (expert board with two cells clicked), we calculate the probabilities for the situation in Figure 1:

$$
\begin{aligned}
P(X^{(1)}) &= \frac{1}{3}\left(\frac{3\rho}{3\rho+4(1-\rho)}\right) \approx 0.055, \\
P(X^{(2)}) &= \frac{1}{4}\left(\frac{4(1-\rho)}{3\rho+4(1-\rho)}\right) \approx 0.209, \\
P(X^{(3)}) &= \frac{1}{3}\left(\frac{6\rho+4(1-\rho)}{3\rho+4(1-\rho)}\right) \approx 0.388.
\end{aligned}
$$

0.055	0.209	0.209	0.388
0.055	**1**	**2**	0.388
0.055	0.209	0.209	0.388

So we conclude that the best next move would be to click on one of the side cells on the left where there is a low chance of hitting a mine.

Refinements

To obtain an exact answer, we drop the assumption that ρ is constant, so that choosing one cell to be safe makes the others slightly more likely to contain a mine. Going back to the first configuration as in Figure 2a, we calculate the probability to be

$$\begin{aligned} P(C_1) &= \binom{3}{1}\binom{3}{2}\prod_{j=1}^{3}\left(\frac{K-j+1}{N'-j+1}\right)\prod_{j=1}^{7}\left(1-\frac{k-3}{n-j-2}\right) \\ &= \binom{3}{1}\binom{3}{2}\frac{K_{(3)}(N'-K)_{(7)}}{N_{(10)}}, \end{aligned}$$

where $x_{(a)} = x!/(x-a)!$ is the falling factorial, and N' is the remaining number of clickable cells (in this case $N' = N-2$). For the start of an expert game this new equation gives the adjusted values a shown below.

0.044	0.217	0.217	0.378
0.044	**1**	**2**	0.378
0.044	0.217	0.217	0.378

In general we have a set of groups $\{G_i : 1 \leq i \leq r\}$ with size $|G_i| = s_i$, which span $S = \sum_{i=1}^{r} s_i$ cells and have various mine configurations C_j with a total of M_j. Suppose the number of mines in group G_i, configuration C_j, is m_{ij}, where $M_j = \sum_{i=1}^{r} m_{ij}$, then the relative probability of a configuration occurring is given by

$$P(C_j) = \frac{K_{(M_j)}(N'-K)_{(S-M_j)}}{N'_{(S)}}\prod_{i=1}^{r}\binom{m_{ij}}{s_i}. \tag{0.1}$$

Since this is normalised by dividing by the sum of probabilities for all configurations to get the distribution of $\mathbf{Y}$, it is sufficient to consider the relative probability

$$\xi(C_j) = K_{(M_j)}(N'-K)_{(S-M_j)}\prod_{i=1}^{r}\binom{m_{ij}}{s_i}, \tag{0.2}$$

as seen in **??**. From this we compute the individual probabilities:

$$P(X^{(i)}) = \frac{1}{s_i}\sum_j x_j P(Y_i = x_j).. \tag{0.3}$$

0.2 Multiple mines per cell

In the case $x_{\max} \geq a$ we think of a mines being allocated randomly to the 8 cells, which gives $X^{(1)} \sim \text{Bin}(a, 1/8)$ and $P\big(\widehat{X}^{(1)}\big) = 1 - (7/8)^a$. These two equations provide upper and lower bounds for the probabilities when $x_{\max}$ is small, and the second provides a very good approximation for $x_{\max} \gg 1$ (and reasonable for $x_{\max} \geq 3$). To calculate the probability of a cell containing a mine for $x_{\max} \leq a$ we...

We note that the only change we would see for $x_{\max} > 1$ is for the groups which can contain more than one mine, in this case G_3, and we would obtain a lower value for $P\big(\widehat{X}^{(3)}\big)$. We find the probability of a cell in group 3 containing at least one mine given there are two in total to be $P\big(\widehat{X}^{(3)}\big) = 1 - (2/3)^2 = 5/9$ (this will be more complicated if a group contains more than two mines), and we obtain

$$P\big(\widehat{X}^{(3)}\big) = \frac{\frac{5}{9}3\rho + \frac{1}{3}4(1-\rho)}{3\rho + 4(1-\rho)} \approx 0.370.$$