Contents

Introduction

0.1 Definitions

We will be considering various minesweeper boards and configurations within them. Before proceeding we first define the following parameters and variables:

- N, total number of cells;
- K, total number of mines;
- ρ, density of mines;
- $\rho_0 = K/N$, initial density of mines;
- $x_{\max}$, maximum number of mines in a cell.

The parameter values at the start of a game for the standard difficulties are given in Table 1.

Difficulty	Dimensions	N	K	ρ_0
Beginner (b)	8×8	64	10	0.156
Intermediate (i)	16×16	256	40	0.156
Expert (e)	16×30	480	99	0.206
Master (m)	30×16	900	200	0.222

Table 1: Characteristics of standard games.

1 Probabilities by combinatorics

1.1 One mine per cell

We initially consider the classic game, where each cell may contain either zero or one mine ($x_{\max} = 1$). We calculate the probabilities of different configurations occurring by considering the number of ways of placing the mines to create such a configuration. To demonstrate how we perform this count, consider a new game – a grid with N cells and K mines. Note that there is no need to distinguish between the K mines, so the number of ways that the mines could be arranged, which we call q, is given by the binomial coefficient

$$q = \binom{N}{K} = \frac{N!}{(N-K)!K!}. \tag{1.1}$$

Now suppose a cell somewhere in the middle is clicked, revealing a number α, where $0 \leq \alpha \leq 8$. We then know the number of mines in the surrounding 8 cells is exactly α, and there are $(K-\alpha)$ mines in the remaining $(N-9)$ cells. The number of ways to arrange α mines in 8 cells is $\binom{8}{\alpha}$ and the number of arrangements for the remaining mines is $\binom{N-9}{K-\alpha}$, giving a total number of arrangements

$$q = \binom{N-9}{K-\alpha}\binom{8}{\alpha}. \tag{1.2}$$

We can generalise this to the case where we have r equivalence groups (not including the group which isn't adjacent to any numbers), where group i has size s_i and contains m_i mines. Defining $S = \sum_{i=1}^{r} s_i$ and $M = \sum_{i=1}^{r} m_i$, and N' to be the number of remaining clickable cells, the number of arrangements is then

$$q = \binom{N'-S}{K-M}\prod_{i=1}^{r}\binom{s_i}{m_i}. \tag{1.3}$$

We may use this to calculate the probability distribution for the number revealed, α. We calculate the probabilities by comparing to the case where there are no restrictions, dividing the number of arrangements. The equation for the probability simplifies to

$$p_\alpha = \binom{8}{\alpha}\frac{(N-K)_{9-\alpha}(K)_\alpha}{(N)_9}, \tag{1.4}$$

where $(x)_n$ denotes the falling factorial $x!/(x-n)!$. For an expert grid with $N = 480$ and $K = 99$ the probabilities for the number being α are

$$p_0 = 12.3\%, \quad p_1 = 26.0\%, \quad p_2 = 23.9\%, \quad p_3 = 12.4\%, \quad p_4 = 4.9\%,$$

which may be compared to the probability of hitting a mine, $p_m = 20.6\%$.[1]

If we are given a board layout, we will not necessarily know the exact number of mines in each group. We may form a set of possible configurations, in the form of r-tuples containing the number of mines in each of the r groups. For example in the simple 1-2 situation, the set of possibilities would be $\{(0,1,1),(1,0,2)\}$. Denote the number of ways to arrange the mines in configuration i to be q_i, and the total number of configurations to be $Q = \sum_i q_i$. Then the probability of configuration i occurring is simply

$$P(C_i) = \frac{q_i}{Q} = \frac{1}{Q}\binom{N'-S}{K-M_j}\prod_{i=1}^{r}\binom{s_i}{m_{ij}}, \tag{1.5}$$

[1] As a comparison, on a beginner grid the probabilities are $p_0 = 19.3\%$, $p_1 = 33.6\%$, $p_2 = 22.5\%$, $p_3 = 7.5\%$, $p_4 = 1.3\%$ and $p_m = 15.6\%$.

where m_{ij} is the number of mines in group i, configuration j, and $M_j = \sum_{i=1}^{r} m_{ij}$ is the total number of mines in the main groups in configuration j. We find that there are factors that cancel due to the 'left-over' group which may be large and contain relatively few mines. We can therefore simplify by defining $\hat{M} = \max_j M_j$ and defining the relative probability of C_j to be

$$\xi(C_j) = (K)_{M_j}(N' - S - K + \hat{M})_{\hat{M}-M_j} \prod_{i=1}^{r} \binom{s_i}{m_{ij}}, \tag{1.6}$$

which restricts the falling factorials to only contain a total of $\hat{M}$ terms[2], and this still gives the form

$$P(C_i) = \frac{\xi(C_i)}{\sum_{j=1}^{r} \xi(C_j)}. \tag{1.7}$$

1.2 Multiple mines per cell

We now allow the maximum number of mines in a cell to be greater than one, $x_{\max} > 1$. Let us begin by returning to the situation where we click in the middle at the beginning of a game to reveal a number, α. It is now slightly more complicated to count the number of arrangements, since different arrangements have different probabilities of occurring – there are two ways for two mines to be placed with one in each of two cells but only one way for them to both be placed in the first cell. For this reason we now need to treat the mines as distinct.

We first check this gives us the same result for $x_{\max} = 1$. Consider again the case that one number in the centre is revealed to be α, then the number of ways to place the α distinct mines in the group of 8 is $(8)_\alpha$. We must also consider the number of ways to choose the α mines out of the K total mines, which there are $\binom{K}{\alpha}$ ways to do. There are then $(K - \alpha)$ mines left to place in the remaining $(N - 9)$ cells, which has $(N - 9)_{K-\alpha}$ arrangements. This gives the total number of arrangements

$$q = \frac{8!}{(8-\alpha)!} \frac{K!}{\alpha!(K-\alpha)!} \frac{(N-9)!}{(K-9-K+\alpha)!} = K! \binom{8}{\alpha} \binom{N-9}{K-\alpha}, \tag{1.8}$$

which has an additional $K!$ term arising compared to (1.2), which is due to the distinction of the K mines, and this term will cancel out when calculating

[2] Note this could be improved further by defining $M_0 = \max_j M_j$ and replacing M_j with $M_j - M_0$, meaning there will only be $\hat{M} - M_0$ terms.

the probabilities. For r groups this generalises to

$$\begin{aligned}
q &= (s_1)_{m_1}\binom{K}{m_1}(s_2)_{m_2}\binom{K-m_1}{m_2}\dots(s_r)_{m_r}\binom{K-\sum_{i=1}^{r-1}m_i}{m_r}(N'-S)_{K-M}\\
&= (N'-S)_{K-M}\binom{K}{m_1,m_2,\dots,m_r,K-M}\prod_{i=1}^{r}(s_i)_{m_i}\\
&= K!\frac{(N'-S)_{K-M}}{(K-M)!}\prod_{i=1}^{r}\frac{(s_i)_{m_i}}{m_i!}.
\end{aligned}$$

We could simplify with the binomial notation to arrive at an analogous expression to (1.3), however the distinction made here will allow us to extend to multiple mines per cell, since the falling factorial $(s_i)_{m_i}$ is the number of ways to arrange m_i distinct mines in s_i cells. Using the above formulation, we can write the number of arrangements as

$$q = K!\frac{\tilde{w}}{(K-M)!}\prod_{i=1}^{r}\frac{w_i}{m_i!}, \tag{1.9}$$

where w_i is the number of ways to arrange the mines in group i and $\tilde{w}$ is the number of ways to arrange the mines in the outer group. We are then able to redefine ξ for multiple mines per cell as

$$\xi(C_j) = \frac{\tilde{w}_j}{(K-M_j)!}\prod_{i=1}^{r}\frac{w_{ij}}{m_{ij}!}, \tag{1.10}$$

where (1.7) still holds.

The task we are left with is calculating the values of w_{ij} and $\tilde{w}_j$. We once again revisit the case of α mines in a group of 8 cells after one central cell is clicked. If $\alpha \leq x_{\max}$ each cell can contain any number of the α mines, so each mine has 8 places it can be, giving 8^α arrangements. This provides an upper bound for any $x_{\max}$. There is no simple formula for the case $\alpha > x_{\max}$, however we are able to form a fairly straightforward algorithm to count the arrangements. To illustrate the method, consider the example of $\alpha = 6$ and $x_{\max} = 3$. The arrangements are then:

- $\binom{8}{2} \times \binom{6}{3}$ ways to arrange $(3,3,0,0,0,0,0,0)$,
- $\binom{8}{1,1,1,5} \times \binom{6}{3,2,1}$ ways to arrange $(3,2,1,0,0,0,0,0)$,
- $\binom{8}{1,3,4} \times \binom{6}{3,1,1,1}$ ways to arrange $(3,1,1,1,0,0,0,0)$,
- $\binom{8}{3} \times \binom{6}{2,2,2}$ ways to arrange $(2,2,2,0,0,0,0,0)$,

- $\binom{8}{2,2,4} \times \binom{6}{2,2,1,1}$ ways to arrange $(2,2,1,1,0,0,0,0)$,
- $\binom{8}{1,4,3} \times \binom{6}{2,1,1,1,1}$ ways to arrange $(2,1,1,1,1,0,0,0)$,
- the usual $\binom{8}{6} \times 6!$ ways to arrange $(1,1,1,1,1,1,0,0)$.

These can be summed to give the total number of arrangements of the group, which in this case gives $w_1 = 255,920$. The largest possible inner group will be of size 8, so it is a simple matter to calculate all the required arrangements. In fact in this case there will be a fixed number of mines in the group anyway and for variable number of mines the largest possible groups size is 7, where only a corner cell is shared between two numbers. It is not as simple for the outer group, since the group size could easily be as large as 500. A standard computer is capable of computing the number of arrangements in a few seconds for large groups, however in practice we would like the performance to be much quicker.

Outer group approximation

We now focus on approximating the number of arrangements in a general outer group.

2 Direct probability approach

Setup

Numbering the cells from 1 to n, we define the random variable X_i to be the number of mines in cell i and we use a slight abuse of notation such that $P(X_i) := P(X_i \geq 1)$. Suppose there are r sets of equivalent cells (not including the outer group), then we denote a set of equivalent cells as G_j, for $1 \leq j \leq r$, for example a group of four neighbouring cells might be $G_1 = \{11, 12, 13, 14\}$. We also define the random variable representing the total number of mines in the jth group to be Y_j, which is given by $Y_j = \sum_{i \in G_j} X_i$, and we represent the distribution of mines across the groups with the vector $\mathbf{Y} = (Y_j)_{j=1}^{r}$. We also use the notation $X^{(j)}$ to denote the number of mines in a single cell within group j.

Given the situation of the board, there are a number of valid configurations for the mines in the unclicked cells, corresponding to vectors $\mathbf{y}^{(i)}$ with $P(\mathbf{Y} = \mathbf{y}^{(i)}) > 0$. We use C_i to denote the event that configuration i represents the actual hidden distribution of mines across these groups, which does not take into account the full board situation, so in general $\sum_i P(C_i) < 1$. We then have $P(\mathbf{Y} = \mathbf{y}^{(i)}) = P(C_i) / \sum_j P(C_j)$.

2.1 One mine per cell

We begin with the simple case $x_{\max} = 1$, as in standard minesweeper games, in which case $P(X) = P(X = 1)$. we then consider the start of a new game. We suppose on the first click (which we assume to be away from the edges) that we neither hit a mine or an opening, revealing a number α, with $1 \leq \alpha \leq 8$, telling us that the surrounding eight cells contain a total of α mines. We may define this group of cells to be G_1, and we have $P(Y_1 = \alpha) = 1$, with the size of the group being $s_1 = 8$. We have $P(X^{(1)}) = \alpha/8$, giving the safety of all cells in G_1.

We now suppose our first click reveals a number '1' ($a = 1$), which is the most likely case (see the combinatorics approach). The probability of any of the 8 surrounding cells containing a mine is $1/8 = 0.125$, which is lower than ρ on any normal game, so the safest move is to choose an adjacent cell. If the first click were to reveal a 2 the probability of a surrounding cell containing a mine is $1/8 = 0.25$ is greater than ρ, so a random cell is safer. We suppose the second click is horizontally adjacent to the previously clicked cell, and let β represent the number revealed, an example of which is displayed in Figure 1, where we have numbered the cells so that we can take the groups to be $G_1 = \{1, 2, 3\}$, $G_2 = \{4, 5, 6, 7\}$, $G_3 = \{8, 9, 10\}$.

1	4	5	8
2	1	2	9
3	6	7	10

Figure 1: Numbered cells around two clicked cells.

We calculate the probabilities $P(X_i)$ for $1 \leq i \leq 10$ (or equivalently $P(X^{(j)})$ for $j = 1, 2, 3$) by considering all possible configurations of mines in these three groups of cells. There are two configurations for the distribution of mines: $\mathbf{Y} \in \{(1, 0, 2), (0, 1, 1)\}$, which we refer to as configurations 1 and 2 respectively. For C_1 to occur we need exactly 1 mine in a group of 3 cells,

(a) Configuration 1. (b) Configuration 2.

Figure 2: Groups of equivalent cells represented by colours (numbered from left to right), mines represented by asterisks.

none in a group of 4, and 1 in a different group of 3.

Approximate solution

To illustrate the calculation we simplify the problem by assuming the grid is very large, so that the mine density, ρ, remains approximately constant. Considering the number of ways the required number of mines can occur in each group and the probability of there being a mine, we get

$$P(C_1) = \left[\binom{3}{1}\rho(1-\rho)^2\right]\left[(1-\rho)^4\right]\left[\binom{3}{2}\rho^2(1-\rho)\right] = 9\rho^3(1-\rho)^7,$$

which corresponds to 3 cells containing mines, 7 cells not containing mines, with the choice of 1 cell in a group of 3, and the choice of 2 cells in another group of 3. Thinking of it this way makes it much easier to just write down probabilities, and we see $P(C_2) = 12\rho^2(1-\rho)^8$. This gives the probabilities for the groups

$$\begin{aligned} P(\mathbf{Y} = (1,0,2)) &= \frac{P(C_1)}{P(C_1) + P(C_2)} \\ &= \frac{9\rho^2(1-\rho)^8}{9\rho^2(1-\rho)^8 + 12\rho(1-\rho)^9} \\ &= \frac{3\rho}{3\rho + 4(1-\rho)}, \\ P(\mathbf{Y} = (0,1,1)) &= \frac{4(1-\rho)}{3\rho + 4(1-\rho)}. \end{aligned}$$

From this we determine the probability distribution of mines in each of the groups. In this case there is no crossover of the number of mines between configurations, so we have

$$\begin{aligned} P(Y_1 = 1) = P(Y_2 = 0) = P(Y_3 = 2) = P(\mathbf{Y} = (1,0,2)) \\ P(Y_1 = 0) = P(Y_2 = 1) = P(Y_3 = 1) = P(\mathbf{Y} = (0,1,1)). \end{aligned}$$

Finally, we find $P(X_i)$ for each cell i by considering the expected value of $\mathbf{Y}$ and dividing by the size of the group. Taking $\rho = 0.207$ (expert board with two cells clicked), we calculate the probabilities for the situation in Figure 1:

$$\begin{aligned} P(X^{(1)}) &= \frac{1}{3}\left(\frac{3\rho}{3\rho + 4(1-\rho)}\right) \approx 0.055, \\ P(X^{(2)}) &= \frac{1}{4}\left(\frac{4(1-\rho)}{3\rho + 4(1-\rho)}\right) \approx 0.209, \\ P(X^{(3)}) &= \frac{1}{3}\left(\frac{6\rho + 4(1-\rho)}{3\rho + 4(1-\rho)}\right) \approx 0.388. \end{aligned}$$

0.055	0.209	0.209	0.388
0.055	**1**	**2**	0.388
0.055	0.209	0.209	0.388

So we conclude that the best next move would be to click on one of the side cells on the left where there is a low chance of hitting a mine.

Refinements

To obtain an exact answer, we drop the assumption that ρ is constant, so that choosing one cell to be safe makes the others slightly more likely to contain a mine. Going back to the first configuration as in Figure 2a, we calculate the probability to be

$$
\begin{aligned}
P(C_1) &= \binom{3}{1}\binom{3}{2}\prod_{j=1}^{3}\left(\frac{K-j+1}{N'-j+1}\right)\prod_{j=1}^{7}\left(1-\frac{k-3}{n-j-2}\right) \\
&= \binom{3}{1}\binom{3}{2}\frac{K_{(3)}(N'-K)_{(7)}}{N_{(10)}},
\end{aligned}
$$

where $x_{(a)} = x!/(x-a)!$ is the falling factorial, and N' is the remaining number of clickable cells (in this case $N' = N-2$). For the start of an expert game this new equation gives the adjusted values a shown below.

0.044	0.217	0.217	0.378
0.044	**1**	**2**	0.378
0.044	0.217	0.217	0.378

In general we have a set of groups $\{G_i : 1 \leq i \leq r\}$ with size $|G_i| = s_i$, which span $S = \sum_{i=1}^{r} s_i$ cells and have various mine configurations C_j with a total of M_j. Suppose the number of mines in group G_i, configuration C_j, is m_{ij}, where $M_j = \sum_{i=1}^{r} m_{ij}$, then the relative probability of a configuration occurring is given by

$$
P(C_j) = \frac{K_{(M_j)}(N'-K)_{(S-M_j)}}{N'_{(S)}}\prod_{i=1}^{r}\binom{m_{ij}}{s_i}. \tag{2.1}
$$

Since this is normalised by dividing by the sum of probabilities for all configurations to get the distribution of $\mathbf{Y}$, it is sufficient to consider the relative probability

$$
\xi(C_j) = K_{(M_j)}(N'-K)_{(S-M_j)}\prod_{i=1}^{r}\binom{m_{ij}}{s_i}, \tag{2.2}
$$

as seen in Section 1. From this we compute the individual probabilities:

$$P(X^{(i)}) = \frac{1}{s_i} \sum_j x_j P(Y_i = x_j). \tag{2.3}$$

2.2 Multiple mines per cell

In the case $x_{\max} \geq a$ we think of a mines being allocated randomly to the 8 cells, which gives $X^{(1)} \sim \text{Bin}(a, 1/8)$ and $P\big(\widehat{X}^{(1)}\big) = 1 - (7/8)^a$. These two equations provide upper and lower bounds for the probabilities when $x_{\max}$ is small, and the second provides a very good approximation for $x_{\max} \gg 1$ (and reasonable for $x_{\max} \geq 3$). To calculate the probability of a cell containing a mine for $x_{\max} \leq a$ we...

We note that the only change we would see for $x_{\max} > 1$ is for the groups which can contain more than one mine, in this case G_3, and we would obtain a lower value for $P\big(\widehat{X}^{(3)}\big)$. We find the probability of a cell in group 3 containing at least one mine given there are two in total to be $P\big(\widehat{X}^{(3)}\big) = 1 - (2/3)^2 = 5/9$ (this will be more complicated if a group contains more than two mines), and we obtain

$$P\big(\widehat{X}^{(3)}\big) = \frac{\frac{5}{9}3\rho + \frac{1}{3}4(1-\rho)}{3\rho + 4(1-\rho)} \approx 0.370.$$

Conclusion

2.3 Optimal strategy

A good opening strategy is as follows: 1. Click randomly in the middle of the grid (20% chance of losing). 2. If a number 1 is revealed, click an adjacent cell, say the one to the right (13% chance of losing). 3. - If we now have a 1 1 configuration, click one of the six cells adjacent to only one of the revealed 1s, say the one directly to the right (12% chance of losing). - If we have a 1 2 configuration, click one of the 3 cells only next to the 1, say the one directly to the left (5% chance of losing) 4. - If we now have a 1 1 1 configuration, click one of the 6 cells on the ends, next to only one of the 1s (10% chance of losing) - If we have a 1 1 2 configuration, click one of the two cells shared by only the two 1s (6% chance of losing), or maybe click one of the 3 cells on the far left for a higher chance of getting an opening?? (13% chance of losing)