0.1 One mine per cell

We initially consider the classic game, where each cell may contain either zero or one mine ($x_{\max} = 1$). We calculate the probabilities of different configurations occurring by considering the number of ways of placing the mines to create such a configuration. To demonstrate how we perform this count, consider a new game – a grid with N cells and K mines. Note that there is no need to distinguish between the K mines, so the number of ways that the mines could be arranged, which we call q, is given by the binomial coefficient

$$q = \binom{N}{K} = \frac{N!}{(N-K)!K!}. \tag{0.1}$$

Now suppose a cell somewhere in the middle is clicked, revealing a number α, where $0 \leq \alpha \leq 8$. We then know the number of mines in the surrounding 8 cells is exactly α, and there are $(K - \alpha)$ mines in the remaining $(N - 9)$ cells. The number of ways to arrange α mines in 8 cells is $\binom{8}{\alpha}$ and the number of arrangements for the remaining mines is $\binom{N-9}{K-\alpha}$, giving a total number of arrangements

$$q = \binom{N-9}{K-\alpha}\binom{8}{\alpha}. \tag{0.2}$$

We can generalise this to the case where we have r equivalence groups (not including the group which isn't adjacent to any numbers), where group i has size s_i and contains m_i mines. Defining $S = \sum_{i=1}^{r} s_i$ and $M = \sum_{i=1}^{r} m_i$, and N' to be the number of remaining clickable cells, the number of arrangements is then

$$q = \binom{N'-S}{K-M}\prod_{i=1}^{r}\binom{s_i}{m_i}. \tag{0.3}$$

We may use this to calculate the probability distribution for the number revealed, α. We calculate the probabilities by comparing to the case where there are no restrictions, dividing the number of arrangements. The equation for the probability simplifies to

$$p_\alpha = \binom{8}{\alpha}\frac{(N-K)_{9-\alpha}(K)_\alpha}{(N)_9}, \tag{0.4}$$

where $(x)_n$ denotes the falling factorial $x!/(x-n)!$. For an expert grid with $N = 480$ and $K = 99$ the probabilities for the number being α are

$$p_0 = 12.3\%, \quad p_1 = 26.0\%, \quad p_2 = 23.9\%, \quad p_3 = 12.4\%, \quad p_4 = 4.9\%,$$

which may be compared to the probability of hitting a mine, $p_m = 20.6\%$.[1]

If we are given a board layout, we will not necessarily know the exact number of mines in each group. We may form a set of possible configurations, in the form of r-tuples containing the number of mines in each of the r groups. For example in the simple 1-2 situation, the set of possibilities would be $\{(0,1,1),(1,0,2)\}$. Denote the number of ways to arrange the mines in configuration i to be q_i, and the total number of configurations to be $Q = \sum_i q_i$. Then the probability of configuration i occurring is simply

$$P(C_i) = \frac{q_i}{Q} = \frac{1}{Q}\binom{N'-S}{K-M_j}\prod_{i=1}^{r}\binom{s_i}{m_{ij}}, \tag{0.5}$$

where m_{ij} is the number of mines in group i, configuration j, and $M_j = \sum_{i=1}^{r} m_{ij}$ is the total number of mines in the main groups in configuration j. We find that there are factors that cancel due to the 'left-over' group which may be large and contain relatively few mines. We can therefore simplify by defining $\hat{M} = \max_j M_j$ and defining the relative probability of C_j to be

$$\xi(C_j) = (K)_{M_j}(N'-S-K+\hat{M})_{\hat{M}-M_j}\prod_{i=1}^{r}\binom{s_i}{m_{ij}}, \tag{0.6}$$

which restricts the falling factorials to only contain a total of $\hat{M}$ terms[2], and this still gives the form

$$P(C_i) = \frac{\xi(C_i)}{\sum_{j=1}^{r}\xi(C_j)}. \tag{0.7}$$

0.2 Multiple mines per cell

We now allow the maximum number of mines in a cell to be greater than one, $x_{\max} > 1$. Let us begin by returning to the situation where we click in the middle at the beginning of a game to reveal a number, α. It is now slightly more complicated to count the number of arrangements, since different arrangements have different probabilities of occurring – there are two ways for two mines to be placed with one in each of two cells but only one way for them to both be placed in the first cell. For this reason we now need to treat the mines as distinct.

[1] As a comparison, on a beginner grid the probabilities are $p_0 = 19.3\%$, $p_1 = 33.6\%$, $p_2 = 22.5\%$, $p_3 = 7.5\%$, $p_4 = 1.3\%$ and $p_m = 15.6\%$.

[2] Note this could be improved further by defining $M_0 = \max_j M_j$ and replacing M_j with $M_j - M_0$, meaning there will only be $\hat{M} - M_0$ terms.

We first check this gives us the same result for $x_{\max} = 1$. Consider again the case that one number in the centre is revealed to be α, then the number of ways to place the α distinct mines in the group of 8 is $(8)_\alpha$. We must also consider the number of ways to choose the α mines out of the K total mines, which there are $\binom{K}{\alpha}$ ways to do. There are then $(K-\alpha)$ mines left to place in the remaining $(N-9)$ cells, which has $(N-9)_{K-\alpha}$ arrangements. This gives the total number of arrangements

$$q = \frac{8!}{(8-\alpha)!}\frac{K!}{\alpha!(K-\alpha)!}\frac{(N-9)!}{(K-9-K+\alpha)!} = K!\binom{8}{\alpha}\binom{N-9}{K-\alpha}, \tag{0.8}$$

which has an additional $K!$ term arising compared to (0.2), which is due to the distinction of the K mines, and this term will cancel out when calculating the probabilities. For r groups this generalises to

$$\begin{aligned} q &= (s_1)_{m_1}\binom{K}{m_1}(s_2)_{m_2}\binom{K-m_1}{m_2}\dots(s_r)_{m_r}\binom{K-\sum_{i=1}^{r-1}m_i}{m_r}(N'-S)_{K-M} \\ &= (N'-S)_{K-M}\binom{K}{m_1, m_2, \dots, m_r, K-M}\prod_{i=1}^{r}(s_i)_{m_i} \\ &= K!\frac{(N'-S)_{K-M}}{(K-M)!}\prod_{i=1}^{r}\frac{(s_i)_{m_i}}{m_i!}. \end{aligned}$$

We could simplify with the binomial notation to arrive at a an analogous expression to (0.3), however the distinction made here will allow us to extend to multiple mines per cell, since the falling factorial $(s_i)_{m_i}$ is the number of ways to arrange m_i distinct mines in s_i cells. Using the above formulation, we can write the number of arrangements as

$$q = K!\frac{\tilde{w}}{(K-M)!}\prod_{i=1}^{r}\frac{w_i}{m_i!}, \tag{0.9}$$

where w_i is the number of ways to arrange the mines in group i and $\tilde{w}$ is the number of ways to arrange the mines in the outer group. We are then able to redefine ξ for multiple mines per cell as

$$\xi(C_j) = \frac{\tilde{w}_j}{(K-M_j)!}\prod_{i=1}^{r}\frac{w_{ij}}{m_{ij}!}, \tag{0.10}$$

where (0.7) still holds.

The task we are left with is calculating the values of w_{ij} and $\tilde{w}_j$. We once again revisit the case of α mines in a group of 8 cells after one central cell is clicked. If $\alpha \leq x_{\max}$ each cell can contain any number of the α mines, so each mine has 8 places it can be, giving 8^α arrangements. This provides an upper bound for any $x_{\max}$. There is no simple formula for the case $\alpha > x_{\max}$, however we are able to form a fairly straightforward algorithm to count the arrangements. To illustrate the method, consider the example of $\alpha = 6$ and $x_{\max} = 3$. The arrangements are then:

- $\binom{8}{2} \times \binom{6}{3}$ ways to arrange $(3,3,0,0,0,0,0,0)$,
- $\binom{8}{1,1,1,5} \times \binom{6}{3,2,1}$ ways to arrange $(3,2,1,0,0,0,0,0)$,
- $\binom{8}{1,3,4} \times \binom{6}{3,1,1,1}$ ways to arrange $(3,1,1,1,0,0,0,0)$,
- $\binom{8}{3} \times \binom{6}{2,2,2}$ ways to arrange $(2,2,2,0,0,0,0,0)$,
- $\binom{8}{2,2,4} \times \binom{6}{2,2,1,1}$ ways to arrange $(2,2,1,1,0,0,0,0)$,
- $\binom{8}{1,4,3} \times \binom{6}{2,1,1,1,1}$ ways to arrange $(2,1,1,1,1,0,0,0)$,
- the usual $\binom{8}{6} \times 6!$ ways to arrange $(1,1,1,1,1,1,0,0)$.

These can be summed to give the total number of arrangements of the group, which in this case gives $w_1 = 255,920$. The largest possible inner group will be of size 8, so it is a simple matter to calculate all the required arrangements. In fact in this case there will be a fixed number of mines in the group anyway and for variable number of mines the largest possible groups size is 7, where only a corner cell is shared between two numbers. It is not as simple for the outer group, since the group size could easily be as large as 500. A standard computer is capable of computing the number of arrangements in a few seconds for large groups, however in practice we would like the performance to be much quicker.

We now focus on approximating the calculation for the number of arrangements in the outer group.