# Multistage bidding model with elements of bargaining: extension for a countable state space*

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We consider a simplified model of a financial market with two players bidding for one unit of a risky asset for $n \leq \infty$ consecutive stages. Player 1 (an insider) is informed about the liquidation price $s^0$ of the asset while Player 2 knows only its probability distribution $p$. At each stage players place integral bids. The higher bid wins, and an asset is transacted to the winning player. Each player aims to maximize the value of her final portfolio.

A model where the price $s^0$ has only two possible values $\{0, m\}$ is considered in [1]. It is reduced to a zero-sum game $G_n(p)$ with incomplete information on one side as in [2]. In this model uninformed Player 2 uses the history of Player 1's moves to update posterior probabilities over the liquidation price. Thus, Player 1 should find a strategy controlling posterior probabilities in such a way that allows her to use the private information without revealing too much of it to Player 2. In [3] the model is extended so that the liquidation price can take any value $s \in S = \mathbb{Z}_+$ according to a probability distribution $p = (p_s, s \in S)$. It is shown that when $\mathbb{D}p$ is finite a game $G_\infty(p)$ is properly defined. For this game the value and optimal players strategies are found.

In both [1] and [3] the transaction price equals to the highest bid. Instead we can consider a transaction rule proposed in [4], and define a transaction price equal to a convex combination of proposed bids with a coefficient $\beta \in [0, 1]$. A model with such transaction rule and two possible values of the liquidation price is studied in [5]. Here those results are further extended for the case of a countable state space.

The model is defined as follows. At stage 0 a chance move chooses a state of nature $s^0 \in S$ according to the distribution $p$. At each stage $t = \overline{1, n}$ players make bids $i_t \in I, j_t \in J$ where $I = J = \mathbb{Z}_+$. A stage payoff in state $s$ equals to

$$a^s(i_t, j_t) = \begin{cases} (1-\beta)i_t + \beta j_t - s, & i_t < j_t, \\ 0, & i_t = j_t, \\ s - \beta i_t - (1-\beta) j_t, & i_t > j_t. \end{cases}$$

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Player 1's strategy is a sequence of actions $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_t : S \times I^{t-1} \to \Delta(I)$ is a mapping to the set of probability distributions $\Delta(I)$ over $I$. So, at each stage of the game Player 1 randomizes his bids depending on the history before stage $t$ and the state $s$. Player 2's strategy is defined as a sequence of actions $\tau = (\tau_1, \ldots, \tau_n)$ where $\tau_t : J^{t-1} \to \Delta(J)$. The payoff in this zero-sum game $G_n(p)$ is defined as

$$K_n(p, \sigma, \tau) = \mathbb{E}_{(p,\sigma,\tau)} \sum_{t=1}^{n} a^s(i_t, j_t).$$

Let's denote distribution sets $\Theta(x) = \{p' \in \Delta(S) : \mathbb{E}p' = x\}$ and $\Lambda(x, y) = \{p' \in \Delta(S) : x < \mathbb{E}p' \leq y\}$. Similar to [3], it can be shown that for $p \in \Lambda(k - 1 + \beta, k + \beta)$ a pure strategy $\tau^k$ defined as

$$\tau_1^k = k, \quad \tau_t^k(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1}, & i_{t-1} < j_{t-1}, \\ j_{t-1}, & i_{t-1} = j_{t-1}, \\ j_{t-1}, & i_{t-1} > j_{t-1}, \end{cases}$$

guarantees to Player 2 a payoff not greater than $H_\infty(p)$ in game $G_n(p)$. Function $H_\infty(p)$ is piecewise linear with breakpoints at $\Theta(k + \beta)$ and domains of linearity $\Lambda(k - 1 + \beta, k + \beta)$. For distribution $p$ such that $\mathbb{E}p = k - 1 + \beta + \xi$, $\xi \in [0, 1)$, it equals to

$$H_\infty(p) = \mathbb{D}p + \beta(1 - \beta) - \xi(1 - \xi).$$

Since $\mathbb{D}p$ is assumed finite, the value $H_\infty(p)$ is finite as well. Hence an infinitely long game $G_\infty(p)$ can be considered.

Let's denote $L_\infty(p)$ a guaranteed payoff to Player 1 in game $G_\infty(p)$, and $p^x(l, r) \in \Theta(x)$ a probability distribution taking only values $l$ and $r$. It can be shown that Player 1 can guarantee herself for $p = \lambda p_1 + (1-\lambda)p_2$ a payoff of at least $\lambda L_\infty(p_1) + (1 - \lambda)L_\infty(p_2)$. Since every distribution $p$ can be represented as a convex combination of some $p^x(l, r)$, proving that $H_\infty(p) = L_\infty(p)$ requires an explicit proof only for $p = p^{k+\beta}(l, r)$.

Let's denote $q = (q_i, i \in I)$ a marginal distribution of Player 1's first bid and $p^i = (p^{s|i}, s \in S)$ a posterior distribution over the liquidation price given a bid $i$ was made. Let's also denote $\sigma_i^s$ a component of Player 1's stage action, i.e. a probability of making a bid $i$ in state $s$. Then from the Bayes rule $\sigma_i^s = p^{s|i} q_i / p_s$. Thus in order to define a stage action, it is suffice to specify $q$ and $(p^i, i \in I)$.

An optimal strategy for $p^x(0, m)$ as described in [5] can be adjusted to $p^{k+\beta}(l, r)$ in the following way. For $p = p^l(l, r)$ and $p = p^r(l, r)$ Player

1 uses bids $l$ and $r$ respectively with probability 1 at the first stage of the game. For $p \in \{p^k(l,r), p^{k+\beta}(l,r)\}$ she uses a stage action with parameters

$$p^k(l,r) : q_k = \beta, q_{k+1} = 1-\beta, p^k = p^{k-1+\beta}(l,r), p^{k+1} = p^{k+\beta}(l,r),$$
$$p^{k+\beta}(l,r) : q_k = 1-\beta, q_{k+1} = \beta, p^k = p^k(l,r), p^{k+1} = p^{k+1}(l,r).$$

Applied recursively for respective posterior probabilities at subsequent stages this strategy guarantees to Player 1 a payoff at least

$$L_\infty\left(p^{k+\beta}(l,r)\right) = \big((r-k-\beta)(k-l+\beta) + \beta(1-\beta)\big)/2.$$

This coincides with the value of $H_\infty\left(p^{k+\beta}(l,r)\right)$. Thus the game $G_\infty(p)$ has a value $V_\infty(p) = H_\infty(p)$, and strategies described above are optimal.

It must be noted that Player 2's strategy is surprisingly robust in regard to changes in the payoff function. At the same time Player 1's strategy becomes more complex. For initial $p \in \Theta(k)$ posterior probabilities in [3] form a symmetric random walk, i.e. posterior $p'$ will be either in $\Theta(k-1)$ or $\Theta(k+1)$ with equal to $1/2$ probabilities. This is no longer true when $\beta \in (0,1)$. The strategy described above essentially differs from that in [3], e.g. it doesn't collapse to that of [3] when $\beta \to 1$.

## References


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