Multistage Bidding Model with Elements of Bargaining. Extension for a Countable State Space*

A.I. Pyanykh
Moscow State University, Moscow, Russia

We consider a simplified model of a financial market with two players bidding for one unit of a risky asset (a share) for $n \leq \infty$ consecutive stages. Player 1 (an insider) is informed about the liquidation price s^0 of the asset while Player 2 knows only its probability distribution p. At each stage players place integral bids. The higher bid wins and a share is transacted to the winning player. Each player aims to maximize the value of her final portfolio.

A model where the price s^0 has only two possible values $\{0, m\}$ is considered in [1]. It is reduced to a zero-sum game $G_n(p)$ with incomplete information on one side as in Aumann, Maschler [2]. In this model uninformed Player 2 uses the history of Player 1's moves to update the posterior probabilities over the liquidation price. Thus, Player 1 should find a strategy controlling posterior probabilities in such a way that allows her to use the private information without revealing too much of it to Player 2. The main results in [1] are explicit optimal strategies and the value of the infinitely long game $G_\infty(p)$. In [3] the model is extended so that the liquidation price can take any value $s \in S = \mathbb{Z}_+$ according to a probability distribution $p = (p_s, s \in S)$. It is shown that when a distribution p has a finite variation a game $G_\infty(p)$ is properly defined. For this game the value and optimal players strategies are found.

In both [1] and [3] the transaction price equals to the highest bid. Instead we could consider a transaction rule proposed in [4], and define a price at which the asset is transacted equal to a convex combination of proposed bids with some coefficient $\beta \in [0, 1]$. A model with such transaction rule and two possible values of the liquidation price is analyzed in [5]. Here these results are futher extended for the case of a countable state space.

The model is defined as follows. At stage 0 a chance move chooses a state of nature $s^0 \in S$ according to the distribution p. At each stage $t = \overline{1, n}$ players make bids $i_t \in I, j_t \in J$ where $I = J = \mathbb{Z}_+$. A stage

*The reported study was funded by RFBR according to the research project No.16-01-00353a.

payoff in state s equals to

$$a^s(i_t, j_t) = \begin{cases} (1-\beta)i_t + \beta j_t - s, & i_t < j_t, \\ 0, & i_t = j_t, \\ s - \beta i_t - (1-\beta)j_t, & i_t > j_t. \end{cases}$$

Player 1's strategy is a sequence of actions $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_t : S \times I^{t-1} \to \Delta(I)$ is a mapping to the set of probability distributions $\Delta(I)$ over I. That is, at each stage of the game Player 1 randomizes his bids depending on the history up to stage t and the state s. Player 2's strategy is similarly defined as a sequence of actions $\tau = (\tau_1, \ldots, \tau_n)$ where $\tau_t : J^{t-1} \to \Delta(J)$. The payoff in this zero-sum game $G_n(p)$ is defined as

$$K_n(p, \sigma, \tau) = \mathbb{E}_{(p,\sigma,\tau)} \sum_{t=1}^{n} a^s(i_t, j_t).$$

Let's denote $\Theta(x) = \{p : \mathbb{E}p = x\}$ and $\Lambda(x, y) = \{p : x < \mathbb{E}p \leq y\}$. Similar to [3], it can be shown that for $p \in \Lambda(k-1+\beta, k+\beta)$ a pure strategy τ^k defined as

$$\tau_1^k = k, \quad \tau_t^k(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1}, & i_{t-1} < j_{t-1}, \\ j_{t-1}, & i_{t-1} = j_{t-1}, \\ j_{t-1}, & i_{t-1} > j_{t-1}, \end{cases} \tag{1}$$

guarantees to Player 2 a payoff not more than $H_\infty(p)$ in game $G_n(p)$. Function $H_\infty(p)$ is piecewise linear with breakpoints at $\Theta(k+\beta)$ and domains of linearity $\Lambda(k-1+\beta, k+\beta)$. For distribution p such that $\mathbb{E}p = k - 1 + \beta + \xi,\ \xi \in [0, 1)$, it equals to

$$H_\infty(p) = \mathbb{D}p + \beta(1-\beta) - \xi(1-\xi). \tag{2}$$

Since $H_\infty(p)$ is finite for distributions p with finite variation, an infinitely long game $G_\infty(p)$ can be considered.

Let's denote $L_\infty(p)$ a guaranteed payoff to Player 1 in game $G_\infty(p)$ and $p^x(l, r) \in \Theta(x)$ a probability distribution taking only values l and r. It can be shown that Player 1 can guarantee herself for $p = \lambda p_1 + (1-\lambda)p_2$ a payoff of at least $\lambda L_\infty(p_1) + (1-\lambda)L_\infty(p_2)$. Since every distribution p can be represented as a convex combination of some $p^x(l, r)$, proving that $H_\infty(p) = L_\infty(p)$ requires an explicit proof only for $p = p^{k+\beta}(l, r)$.

Let's denote $q = (q_i, i \in I)$ a marginal distribution of Player 1's first bid, and $p^i = (p^{s|i}, s \in S)$ a posterior distribution of the liquidation price given a bid i is made. Let's also denote σ_i^s a component of Player 1's stage action, that is a probability of making a bid i in state s. Then from the Bayes rule $\sigma_i^s = p^{s|i} q_i / p_s$. So, in order to define a stage action it is suffice to specify q and $(p^i, i \in I)$.

An optimal strategy for $p^x(0, m)$ as described in [5] can be adjusted to $p^{k+\beta}(l, r)$ in the following way. When $p = p^l(l, r)$ and $p = p^r(l, r)$ Player 1 uses bids l and r with probability 1 at the first stage of the game. For $p \in \left\{p^k(l, r), p^{k+\beta}(l, r)\right\}$ she uses a stage action with parameters

$$
\begin{aligned}
&p^k(l, r) : q_k = \beta, q_{k+1} = 1 - \beta, p^k = p^{k-1+\beta}(l, r), p^{k+1} = p^{k+\beta}(l, r),\\
&p^{k+\beta}(l, r) : q_k = 1 - \beta, q_{k+1} = \beta, p^k = p^k(l, r), p^{k+1} = p^{k+1}(l, r).
\end{aligned}
$$

Applied recursively for respective posterior probabilities at further stages this strategy guarantees Player 1 a payoff at least

$$L_\infty(p^{k+\beta}(l, r)) = \left((r - k - \beta)(k - l + \beta) + \beta(1 - \beta)\right)/2.$$

This coincides with the value of $H_\infty(p^{k+\beta}(l, r))$. Thus the game $G_\infty(p)$ has a value $V_\infty(p) = H_\infty(p)$ and strategies described above are optimal.

It is important to note that Player 1's strategy described above is properly defined only for $\beta \in (0, 1)$. In case of $\beta \in \{0, 1\}$ one should use strategies described in [3].

References

1. Domansky V. Repeated games with asymmetric information and random price fluctuations at finance markets // International Journal of Game Theory. 2007. V. 36, No. 2. P. 241–257.
2. Aumann R.J., Maschler M.B. Repeated Games with Incomplete Information. Cambridge, Massachusetts: The MIT Press, 1995.
3. Domansky V.C., Kreps V.L. Game Theoretic Bidding Model: Strategic Aspects of Price Formation at Stock Markets // The Journal of the New Economic Association. 2011. V. 11. P. 39–62.
4. Chatterjee K., Samuelson W. Bargaining under incomplete information // Operations Research. 1983. V. 31, No.5. P. 835–851.
5. P'yanykh A.I. A Multistage Exchange Trading Model with Asymmetric Information and Elements of Bargaining // Moscow University Computational Mathematics and Cybernetics. 2016. V. 40, No. 1. P. 35–40.