Multistage Bidding Model with Elements of Bargaining. Extension for a Countable State Space*

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We consider a simplified model of a financial market with two players bidding for one unit of a risky asset (a share) for $n \leq \infty$ consecutive stages. Player 1 (an insider) is informed about the liquidation price s of the asset while Player 2 knows only its probability distribution p. At each stage players place integral bids. The higher bid wins and a share is transacted to the winning player. Each player aims to maximize the value of her final portfolio.

A model where the price s has only two possible values $\{0, m\}$ is considered in [1]. It is reduced to a zero-sum game $G_n(p)$ with incomplete information on one side as in Aumann, Maschler [2]. In this model uninformed Player 2 uses the history of Player 1's moves to update the posterior probabilities over the liquidation price. Thus, Player 1 should find a strategy controlling posterior probabilities in such a way that allows her to use the private information without revealing too much of it to Player 2. The main results in [1] are explicit optimal strategies and the value of the game $G_\infty(p)$. In [3] the model is extended so that the liquidation price can take any value $s \in S = \mathbb{Z}_+$ according to a probability distribution $p = (p_0, p_1, \ldots)$. It is shown that when $\mathbb{D}p < \infty$ a game $G_\infty(p)$ is properly defined. For this game the value and optimal players strategies are found.

In both [1] and [3] the transaction price equals to the highest bid. Instead we could consider a transaction rule proposed in [4], and define a price at which the asset is transacted equal to a convex combination of proposed bids with some coefficient $\beta \in [0, 1]$. A model with such transaction rule and two possible values of the liquidation price is analyzed in [5]. Here these results are futher extended for the case of a countable state space.

Formally the model is defined as follows. At stage 0 a chance move chooses a state of nature $s \in S$ according to the distribution p. At each stage $t = \overline{1, n}$ players make bids $i_t \in I, j_t \in J$ where $I = J = \mathbb{Z}_+$.

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Denoting $\overline{\beta} = 1 - \beta$ a stage payoff in state s equals to

$$a^s(i,j) = \begin{cases} \overline{\beta} i + \beta j - s, & i < j, \\ 0, & i = j, \\ s - \beta i - \overline{\beta} j, & i > j. \end{cases}$$

Player 1's strategy is a sequence of actions $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_t : S \times I^{t-1} \to \Delta(I)$ is a mapping to the set of probability distributions $\Delta(I)$ over I. That is, at each stage Player 1 randomizes his bids depending on the history up to stage t and the liquidation price s. Similarly, Player 2's strategy $\tau = (\tau_1, \ldots, \tau_n)$ where $\tau_t : J^{t-1} \to \Delta(J)$. Player 1's payoff then defined as $K_n(p, \sigma, \tau) = \mathbb{E}_{(p,\sigma,\tau)} \sum_{t=1}^{n} a^s(i_t, j_t)$. Player 2's payoff equals to $-K_n(p, \sigma, \tau)$.

Following [3], Player 1's strategy σ in n–stage game can be represented as a pair $(\sigma_1, \sigma(i))$ where σ_1 is a one stage action and $\sigma(i)$ is a strategy in $(n-1)$–stage game dependent on the actual first bid. Similarly Player 2's strategy τ can be represented as a pair $(\tau_1, \tau(i))$. Denoting $q = (q_0, q_1, \ldots)$ a marginal distribution of the first bid, and $p(i) = \{p(0|i), p(1, i), \ldots\}$ – a posterior distribution of the liquidation price given a bid i, the following recursive formula holds

$$K_n(p, \sigma, \tau) = K_1(p, \sigma_1, \tau_1) + \sum_{i \in I} q_i K_{n-1}(p(i), \sigma(i), \tau(i)).$$

Thus to define a strategy in $G_n(p)$ it is suffice to define a stage action for any posterior probability p. Let's define a pure strategy τ^k as

$$\tau_1^k = k, \quad \tau_t^k(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1}, & i_{t-1} < j_{t-1}, \\ j_{t-1}, & i_{t-1} = j_{t-1}, \\ j_{t-1}, & i_{t-1} > j_{t-1}. \end{cases}$$

It can be shown that for p such that $\mathbb{E}p = k - 1 + \beta + \xi$, $\xi \in [0, 1)$ Player 2 can guarantee a payoff not more than $H^\infty(p) = \mathbb{D}p + \beta\overline{\beta} - \xi(1 - \xi)$. Function $H^\infty(p)$ is piecewise linear with breakpoints at $p \in \Theta(k + \beta)$ where $\Theta(x) = \{p : \mathbb{E}p = x\}$ and domains of linearity $\Lambda(k - \overline{\beta}, k + \beta)$ where $\Lambda(x, y) = \{p : x \le \mathbb{E}p \le y\}$.

Given $\sigma_1 = \{q_1, p_1(i)\}$ and $\sigma_2 = \{q_2, p_2(i)\}$ are two first-stage actions and $p = \lambda p_1 + (1 - \lambda) p_2$, it can be shown that a first-stage action σ with $q_i = \lambda q_{1,i} + (1 - \lambda) q_{2,i}$ and $p(s|i) = (\lambda q_{1,i} p_1(s|i) + (1 - \lambda) q_{2,i} p_2(s|i))/q_i$ guarantees Player 1 a payoff of at least $L_n(p, \sigma) = \lambda L_{n-1}(p_1, \sigma_1) + (1 -$

$\lambda)L_{n-1}(p_2,\sigma_2)$. Since every p can be represented as a convex combination of distributions taking only 2 different values it is suffice to show that $H^\infty(p^{k+\beta}(r,l)) = B_\infty(p^{k+\beta}(r,l))$ to prove that $H^\infty(p) = B_\infty(p)$ for all p.

An optimal strategy for $p^x(0,m)$ is described in [5]. Adjusting it for $p^{k+\beta}(l,r)$ gives us $L_\infty(p^{k+\beta}(l,r)) = ((r-k-\beta)(k-l+\beta)+\beta(1-\beta))/2 = H_\infty(p^{k+\beta}(l,r))$. Thus the game $G_\infty(p)$ has a value $V_\infty(p) = H_\infty(p)$.

It is important to note that Player 1's strategy described above is properly defined only for $\beta \in (0,1)$. In case of $\beta \in \{0,1\}$ one should use strategies described in [3].

References

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