Solutions to selected problems in Brockwell and Davis

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This document contains solutions to selected problems in

Peter J. Brockwell and Richard A. Davis, *Introduction to Time Series and Forecasting,* 2nd Edition, Springer New York, 2002.

We provide solutions to most of the problems in the book that are not computer exercises. That is, you will not need a computer to solve these problems. We encourage students to come up with suggestions to improve the solutions and to report any misprints that may be found.

Contents

Notation: We will use the following notation.

- The indicator function

$$\mathbf{1}_A(h) = \begin{cases} 1 & \text{if } h \in A, \\ 0 & \text{if } h \notin A. \end{cases}$$

- Dirac's delta function

$$\delta(t) = \begin{cases} +\infty & \text{if} \quad t = 0, \\ 0 & \text{if} \quad t \neq 0, \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0).$$

Chapter 1

Problem 1.1. a) First note that

$$\begin{aligned}\mathbb{E}[(Y-c)^2] &= \mathbb{E}[Y^2 - 2Yc + c^2] = \mathbb{E}[Y^2] - 2c\mathbb{E}[Y] + c^2 \\ &= \mathbb{E}[Y^2] - 2c\mu + c^2.\end{aligned}$$

Find the extreme point by differentiating,

$$\frac{d}{dc}(\mathbb{E}[Y^2] - 2c\mu + c^2) = -2\mu + 2c = 0 \quad \Rightarrow c = \mu.$$

Since, $\frac{d^2}{dc^2}(\mathbb{E}[Y^2] - 2c\mu + c^2) = 2 > 0$ this is a min-point.
b) We have

$$\begin{aligned}\mathbb{E}[(Y-f(X))^2 \mid X] &= \mathbb{E}[Y^2 - 2Yf(X) + f^2(X) \mid X] \\ &= \mathbb{E}[Y^2 \mid X] - 2f(X)\mathbb{E}[Y \mid X] + f^2(X),\end{aligned}$$

which is minimized by $f(X) = \mathbb{E}[Y \mid X]$ (take $c = f(X)$ and $\mu = \mathbb{E}[Y \mid X]$ in a).
c) We have

$$\mathbb{E}[(Y-f(X))^2] = \mathbb{E}\big[\mathbb{E}[(Y-f(X))^2 \mid X]\big],$$

so the result follows from b).

Problem 1.4. a) For the mean we have

$$\mu_X(t) = \mathbb{E}[a + bZ_t + cZ_{t-2}] = a,$$

and for the autocovariance

$$\begin{aligned}\gamma_X(t+h,t) &= \operatorname{Cov}(X_{t+h}, X_t) = \operatorname{Cov}(a + bZ_{t+h} + cZ_{t+h-2}, a + bZ_t + cZ_{t-2}) \\ &= b^2 \operatorname{Cov}(Z_{t+h}, Z_t) + bc \operatorname{Cov}(Z_{t+h}, Z_{t-2}) \\ &\quad + cb \operatorname{Cov}(Z_{t+h-2}, Z_t) + c^2 \operatorname{Cov}(Z_{t+h-2}, Z_{t-2}) \\ &= \sigma^2 b^2 \mathbf{1}_{\{0\}}(h) + \sigma^2 bc \mathbf{1}_{\{-2\}}(h) + \sigma^2 cb \mathbf{1}_{\{2\}}(h) + \sigma^2 c^2 \mathbf{1}_{\{0\}}(h) \\ &= \begin{cases} (b^2 + c^2)\sigma^2 & \text{if } h = 0, \\ bc\sigma^2 & \text{if } |h| = 2, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Since $\mu_X(t)$ and $\gamma_X(t+h,t)$ do not depend on t, $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary.
b) For the mean we have

$$\mu_X(t) = \mathbb{E}[Z_1]\cos(ct) + \mathbb{E}[Z_2]\sin(ct) = 0,$$

and for the autocovariance

$$\begin{aligned}\gamma_X(t+h,t) &= \operatorname{Cov}(X_{t+h}, X_t) \\ &= \operatorname{Cov}(Z_1\cos(c(t+h)) + Z_2\sin(c(t+h)), Z_1\cos(ct) + Z_2\sin(ct)) \\ &= \cos(c(t+h))\cos(ct)\operatorname{Cov}(Z_1, Z_1) + \cos(c(t+h))\sin(ct)\operatorname{Cov}(Z_1, Z_2) \\ &\quad + \sin(c(t+h))\cos(ct)\operatorname{Cov}(Z_1, Z_2) + \sin(c(t+h))\sin(ct)\operatorname{Cov}(Z_2, Z_2) \\ &= \sigma^2(\cos(c(t+h))\cos(ct) + \sin(c(t+h))\sin(ct)) \\ &= \sigma^2\cos(ch)\end{aligned}$$

where the last equality follows since $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. Since $\mu_X(t)$ and $\gamma_X(t+h,t)$ do not depend on t, $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary.
c) For the mean we have

$$\mu_X(t) = \mathbb{E}[Z_t]\cos(ct) + \mathbb{E}[Z_{t-1}]\sin(ct) = 0,$$

and for the autocovariance

$$\begin{aligned}
\gamma_X(t+h,t) &= \operatorname{Cov}(X_{t+h}, X_t) \\
&= \operatorname{Cov}(Z_{t+h}\cos(c(t+h)) + Z_{t+h-1}\sin(c(t+h)), Z_t\cos(ct) + Z_{t-1}\sin(ct)) \\
&= \cos(c(t+h))\cos(ct)\operatorname{Cov}(Z_{t+h}, Z_t) + \cos(c(t+h))\sin(ct)\operatorname{Cov}(Z_{t+h}, Z_{t-1}) \\
&\quad + \sin(c(t+h))\cos(ct)\operatorname{Cov}(Z_{t+h-1}, Z_t) \\
&\quad + \sin(c(t+h))\sin(ct)\operatorname{Cov}(Z_{t+h-1}, Z_{t-1}) \\
&= \sigma^2\cos^2(ct)\mathbf{1}_{\{0\}}(h) + \sigma^2\cos(c(t-1))\sin(ct)\mathbf{1}_{\{-1\}}(h) \\
&\quad + \sigma^2\sin(c(t+1))\cos(ct)\mathbf{1}_{\{1\}}(h) + \sigma^2\sin^2(ct)\mathbf{1}_{\{0\}}(h) \\
&= \begin{cases} \sigma^2\cos^2(ct) + \sigma^2\sin^2(ct) = \sigma^2 & \text{if } h = 0, \\ \sigma^2\cos(c(t-1))\sin(ct) & \text{if } h = -1, \\ \sigma^2\cos(ct)\sin(c(t+1)) & \text{if } h = 1, \end{cases}
\end{aligned}$$

We have that $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary for $c = \pm k\pi$, $k \in \mathbb{Z}$, since then $\gamma_X(t+h,t) = \sigma^2\mathbf{1}_{\{0\}}(h)$. For $c \neq \pm k\pi$, $k \in \mathbb{Z}$, $\{X_t : t \in \mathbb{Z}\}$ is not (weakly) stationary since $\gamma_X(t+h,t)$ depends on t.

d) For the mean we have

$$\mu_X(t) = \mathbb{E}[a + bZ_0] = a,$$

and for the autocovariance

$$\gamma_X(t+h,t) = \operatorname{Cov}(X_{t+h}, X_t) = \operatorname{Cov}(a + bZ_0, a + bZ_0) = b^2\operatorname{Cov}(Z_0, Z_0) = \sigma^2 b^2.$$

Since $\mu_X(t)$ and $\gamma_X(t+h,t)$ do not depend on t, $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary.

e) If $c = k\pi$, $k \in \mathbb{Z}$ then $X_t = (-1)^{kt}Z_0$ which implies that X_t is weakly stationary when $c = k\pi$. For $c \neq k\pi$ we have

$$\mu_X(t) = \mathbb{E}[Z_0]\cos(ct) = 0,$$

and for the autocovariance

$$\begin{aligned}
\gamma_X(t+h,t) &= \operatorname{Cov}(X_{t+h}, X_t) = \operatorname{Cov}(Z_0\cos(c(t+h)), Z_0\cos(ct)) \\
&= \cos(c(t+h))\cos(ct)\operatorname{Cov}(Z_0, Z_0) = \cos(c(t+h))\cos(ct)\sigma^2.
\end{aligned}$$

The process $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary when $c = \pm k\pi$, $k \in \mathbb{Z}$ and not (weakly) stationary when $c \neq \pm k\pi$, $k \in \mathbb{Z}$, see 1.4. c).

f) For the mean we have

$$\mu_X(t) = \mathbb{E}[Z_t Z_{t-1}] = 0,$$

and

$$\begin{aligned}
\gamma_X(t+h,t) &= \operatorname{Cov}(X_{t+h}, X_t) = \operatorname{Cov}(Z_{t+h}Z_{t+h-1}, Z_t Z_{t-1}) \\
&= \mathbb{E}[Z_{t+h}Z_{t+h-1}Z_t Z_{t-1}] = \begin{cases} \sigma^4 & \text{if } h = 0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Since $\mu_X(t)$ and $\gamma_X(t+h,t)$ do not depend on t, $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary.

Problem 1.5. a) We have

$$\begin{aligned}
\gamma_X(t+h,t) &= \operatorname{Cov}(X_{t+h}, X_t) = \operatorname{Cov}(Z_{t+h} + \theta Z_{t+h-2}, Z_t + \theta Z_{t-2}) \\
&= \operatorname{Cov}(Z_{t+h}, Z_t) + \theta\operatorname{Cov}(Z_{t+h}, Z_{t-2}) + \theta\operatorname{Cov}(Z_{t+h-2}, Z_t) \\
&\quad + \theta^2\operatorname{Cov}(Z_{t+h-2}, Z_{t-2}) \\
&= \mathbf{1}_{\{0\}}(h) + \theta\mathbf{1}_{\{-2\}}(h) + \theta\mathbf{1}_{\{2\}}(h) + \theta^2\mathbf{1}_{\{0\}}(h) \\
&= \begin{cases} 1 + \theta^2 & \text{if } h = 0, \\ \theta & \text{if } |h| = 2. \end{cases} = \begin{cases} 1.64 & \text{if } h = 0, \\ 0.8 & \text{if } |h| = 2. \end{cases}
\end{aligned}$$

Hence the ACVF depends only on h and we write $\gamma_X(h) = \gamma_X(t+h, h)$. The ACF is then

$$\rho(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1 & \text{if } h = 0, \\ 0.8/1.64 \approx 0.49 & \text{if } |h| = 2. \end{cases}$$

b) We have

$$\begin{aligned} \operatorname{Var}\left(\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\right) &= \frac{1}{16}\operatorname{Var}(X_1 + X_2 + X_3 + X_4) \\ = \frac{1}{16}\Big(\operatorname{Var}(X_1) &+ \operatorname{Var}(X_2) + \operatorname{Var}(X_3) + \operatorname{Var}(X_4) + 2\operatorname{Cov}(X_1, X_3) \\ &+ 2\operatorname{Cov}(X_2, X_4)\Big) \\ = \frac{1}{16}\big(4\gamma_X(0) + 4\gamma_X(2)\big) &= \frac{1}{4}\big(\gamma_X(0) + \gamma_X(2)\big) = \frac{1.64 + 0.8}{4} = 0.61. \end{aligned}$$

c) $\theta = -0.8$ implies $\gamma_X(h) = -0.8$ for $|h| = 2$ so

$$\operatorname{Var}\left(\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\right) = \frac{1.64 - 0.8}{4} = 0.21.$$

Because of the negative covariance at lag 2 the variance in c) is considerably smaller.

Problem 1.8. a) First we show that $\{X_t : t \in \mathbb{Z}\}$ is WN $(0, 1)$. For t even we have $\mathbb{E}[X_t] = \mathbb{E}[Z_t] = 0$ and for t odd

$$\mathbb{E}[X_t] = \mathbb{E}\left[\frac{Z_{t-1}^2 - 1}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}\mathbb{E}[Z_{t-1}^2 - 1] = 0.$$

Next we compute the ACVF. If t is even we have $\gamma_X(t, t) = \mathbb{E}[Z_t^2] = 1$ and if t is odd

$$\gamma_X(t, t) = \mathbb{E}\left[\left(\frac{Z_{t-1}^2 - 1}{\sqrt{2}}\right)^2\right] = \frac{1}{2}\mathbb{E}[Z_{t-1}^4 - 2Z_{t-1}^2 + 1] = \frac{1}{2}(3 - 2 + 1) = 1.$$

If t is even we have

$$\gamma_X(t+1, t) = \mathbb{E}\left[\frac{Z_t^2 - 1}{\sqrt{2}} Z_t\right] = \frac{1}{\sqrt{2}}\mathbb{E}[Z_t^3 - Z_t] = 0,$$

and if t is odd

$$\gamma_X(t+1, t) = \mathbb{E}\left[Z_{t+1}\frac{Z_{t-1}^2 - 1}{\sqrt{2}}\right] = \mathbb{E}[Z_{t+1}]\mathbb{E}\left[\frac{Z_{t-1}^2 - 1}{\sqrt{2}}\right] = 0.$$

Clearly $\gamma_X(t+h, t) = 0$ for $|h| \geq 2$. Hence

$$\gamma_X(t+h, h) = \begin{cases} 1 & \text{if } h = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\{X_t : t \in \mathbb{Z}\}$ is WN $(0, 1)$. If t is odd X_t and X_{t-1} is obviously dependent so $\{X_t : t \in \mathbb{Z}\}$ is *not* IID $(0, 1)$.
b) If n is odd

$$\mathbb{E}[X_{n+1} \mid X_1, \dots, X_n] = \mathbb{E}[Z_{n+1} \mid Z_0, Z_2, Z_4 \dots, Z_{n-1}] = \mathbb{E}[Z_{n+1}] = 0.$$

If n is even

$$\mathbb{E}[X_{n+1} \mid X_1, \dots, X_n] = \mathbb{E}\left[\frac{Z_n^2 - 1}{\sqrt{2}} \mid Z_0, Z_2, Z_4, \dots, Z_n\right] = \frac{Z_n^2 - 1}{\sqrt{2}} = \frac{X_n^2 - 1}{\sqrt{2}}.$$

This again shows that $\{X_t : t \in \mathbb{Z}\}$ is not IID $(0, 1)$.

Problem 1.11. a) Since $a_j = (2q+1)^{-1}$, $-q \le j \le q$, we have

$$\begin{aligned}
\sum_{j=-q}^{q} a_j m_{t-j} &= \frac{1}{2q+1}\sum_{j=-q}^{q}\left(c_0 + c_1\,(t-j)\right) \\
&= \frac{1}{2q+1}\left(c_0\,(2q+1) + c_1\sum_{j=-q}^{q}(t-j)\right) = c_0 + \frac{c_1}{2q+1}\left(t\,(2q+1) - \sum_{j=-q}^{q} j\right) \\
&= c_0 + c_1 t - \frac{c_1}{2q+1}\left(\sum_{j=1}^{q} j + \sum_{j=1}^{q} -j\right) \\
&= c_0 + c_1 t = m_t
\end{aligned}$$

b) We have

$$\mathbb{E}\left[A_t\right] = \mathbb{E}\left[\sum_{j=-q}^{q} a_j Z_{t-j}\right] = \sum_{j=-q}^{q} a_j \mathbb{E}\left[Z_{t-j}\right] = 0 \quad \text{and}$$

$$\operatorname{Var}\left(A_t\right) = \operatorname{Var}\left(\sum_{j=-q}^{q} a_j Z_{t-j}\right) = \sum_{j=-q}^{q} a_j^2 \operatorname{Var}\left(Z_{t-j}\right) = \frac{1}{\left(2q+1\right)^2}\sum_{j=-q}^{q}\sigma^2 = \frac{\sigma^2}{2q+1}$$

We see that the variance $\operatorname{Var}(A_t)$ is small for large q. Hence, the process A_t will be close to its mean (which is zero) for large q.

Problem 1.15. a) Put

$$\begin{aligned}
Z_t &= \nabla\nabla_{12} X_t = (1-B)(1-B^{12})X_t = (1-B)(X_t - X_{t-12}) \\
&= X_t - X_{t-12} - X_{t-1} + X_{t-13} \\
&= a + bt + s_t + Y_t - a - b(t-12) - s_{t-12} - Y_{t-12} - a - b(t-1) - s_{t-1} - Y_{t-1} \\
&\qquad + a + b(t-13) + s_{t-13} + Y_{t-13} \\
&= Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}.
\end{aligned}$$

We have $\mu_Z(t) = \mathbb{E}[Z_t] = 0$ and

$$\begin{aligned}
\gamma_Z(t+h,t) &= \operatorname{Cov}\left(Z_{t+h}, Z_t\right) \\
&= \operatorname{Cov}\left(Y_{t+h} - Y_{t+h-1} - Y_{t+h-12} + Y_{t+h-13}, Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}\right) \\
&= \gamma_Y(h) - \gamma_Y(h+1) - \gamma_Y(h+12) + \gamma_Y(h+13) - \gamma_Y(h-1) + \gamma_Y(h) \\
&\quad + \gamma_Y(h+11) - \gamma_Y(h+12) - \gamma_Y(h-12) + \gamma_Y(h-11) \\
&\quad + \gamma_Y(h) - \gamma_Y(h+1) + \gamma_Y(h-13) - \gamma_Y(h-12) - \gamma_Y(h-1) + \gamma_Y(h) \\
&= 4\gamma_Y(h) - 2\gamma_Y(h+1) - 2\gamma_Y(h-1) + \gamma_Y(h+11) + \gamma_Y(h-11) \\
&\quad - 2\gamma_Y(h+12) - 2\gamma_Y(h-12) + \gamma_Y(h+13) + \gamma_Y(h-13).
\end{aligned}$$

Since $\mu_Z(t)$ and $\gamma_Z(t+h,t)$ do not depend on t, $\{Z_t : t \in \mathbb{Z}\}$ is (weakly) stationary.
b) We have $X_t = (a+bt)s_t + Y_t$. Hence,

$$\begin{aligned}
Z_t &= \nabla_{12}^2 X_t = (1-B^{12})(1-B^{12})X_t = (1-B^{12})(X_t - X_{t-12}) \\
&= X_t - X_{t-12} - X_{t-12} + X_{t-24} = X_t - 2X_{t-12} + X_{t-24} \\
&= (a+bt)s_t + Y_t - 2(a + b(t-12)s_{t-12} + Y_{t-12}) + (a + b(t-24))s_{t-24} + Y_{t-24} \\
&= a(s_t - 2s_{t-12} + s_{t-24}) + b(ts_t - 2(t-12)s_{t-12} + (t-24)s_{t-24}) \\
&\quad + Y_t - 2Y_{t-12} + Y_{t-24} \\
&= Y_t - 2Y_{t-12} + Y_{t-24}.
\end{aligned}$$

Now we have $\mu_Z(t) = \mathbb{E}[Z_t] = 0$ and

$$\begin{aligned}
\gamma_Z(t+h,t) &= \operatorname{Cov}\left(Z_{t+h}, Z_t\right) \\
&= \operatorname{Cov}\left(Y_{t+h} - 2Y_{t+h-12} + Y_{t+h-24}, Y_t - 2Y_{t-12} + Y_{t-24}\right) \\
&= \gamma_Y(h) - 2\gamma_Y(h+12) + \gamma_Y(h+24) - 2\gamma_Y(h-12) + 4\gamma_Y(h) \\
&\quad - 2\gamma_Y(h+12) + \gamma_Y(h-24) - 2\gamma_Y(h-12) + \gamma_Y(h) \\
&= 6\gamma_Y(h) - 4\gamma_Y(h+12) - 4\gamma_Y(h-12) + \gamma_Y(h+24) + \gamma_Y(h-24).
\end{aligned}$$

Since $\mu_Z(t)$ and $\gamma_Z(t+h,t)$ do not depend on t, $\{Z_t : t \in \mathbb{Z}\}$ is (weakly) stationary.

Chapter 2

Problem 2.1. We find the best linear predictor $\hat{X}_{n+h} = aX_n + b$ of X_{n+h} by finding a and b such that $\mathbb{E}[X_{n+h} - \hat{X}_{n+h}] = 0$ and $\mathbb{E}[(X_{n+h} - \hat{X}_{n+h})X_n] = 0$. We have

$$\mathbb{E}[X_{n+h} - \hat{X}_{n+h}] = \mathbb{E}[X_{n+h} - aX_n - b] = \mathbb{E}[X_{n+h}] - a\mathbb{E}[X_n] - b = \mu\,(1-a) - b$$

and

$$\begin{aligned}
&\mathbb{E}[(X_{n+h} - \hat{X}_{n+h})X_n] = \mathbb{E}[(X_{n+h} - aX_n - b)X_n] \\
&\quad = \mathbb{E}[X_{n+h}X_n] - a\mathbb{E}[X_n^2] - b\mathbb{E}[X_n] \\
&\quad = \mathbb{E}[X_{n+h}X_n] - \mathbb{E}[X_{n+h}]\mathbb{E}[X_n] + \mathbb{E}[X_{n+h}]\mathbb{E}[X_n] \\
&\qquad - a\left(\mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2 + \mathbb{E}[X_n]^2\right) - b\mathbb{E}[X_n] \\
&\quad = \operatorname{Cov}(X_{n+h}, X_n) + \mu^2 - a\left(\operatorname{Cov}(X_n, X_n) + \mu^2\right) - b\mu \\
&\quad = \gamma(h) + \mu^2 - a\left(\gamma(0) + \mu^2\right) - b\mu,
\end{aligned}$$

which implies that

$$b = \mu\,(1-a)\,, \qquad a = \frac{\gamma(h) + \mu^2 - b\mu}{\gamma(0) + \mu^2}.$$

Solving this system of equations we get $a = \gamma(h)/\gamma(0) = \rho(h)$ and $b = \mu(1 - \rho(h))$ i.e. $\hat{X}_{n+h} = \rho(h)X_n + \mu(1 - \rho(h))$.

Problem 2.4. a) Put $X_t = (-1)^t Z$ where Z is random variable with $\mathbb{E}[Z] = 0$ and $\operatorname{Var}(Z) = 1$. Then

$$\gamma_X(t+h, t) = \operatorname{Cov}((-1)^{t+h}Z, (-1)^t Z) = (-1)^{2t+h}\operatorname{Cov}(Z, Z) = (-1)^h = \cos(\pi h).$$

b) Recall problem 1.4 b) where $X_t = Z_1\cos(ct) + Z_2\sin(ct)$ implies that $\gamma_X(h) = \cos(ch)$. If we let Z_1, Z_2, Z_3, Z_4, W be independent random variables with zero mean and unit variance and put

$$X_t = Z_1\cos\left(\frac{\pi}{2}t\right) + Z_2\sin\left(\frac{\pi}{2}t\right) + Z_3\cos\left(\frac{\pi}{4}t\right) + Z_4\sin\left(\frac{\pi}{4}t\right) + W.$$

Then we see that $\gamma_X(h) = \kappa(h)$.
c) Let $\{Z_t : t \in \mathbb{Z}\}$ be WN $\left(0, \sigma^2\right)$ and put $X_t = Z_t + \theta Z_{t-1}$. Then $\mathbb{E}[X_t] = 0$ and

$$\begin{aligned}
&\gamma_X(t+h, t) = \operatorname{Cov}(Z_{t+h} + \theta Z_{t+h-1}, Z_t + \theta Z_{t-1}) \\
&\quad = \operatorname{Cov}(Z_{t+h}, Z_t) + \theta\operatorname{Cov}(Z_{t+h}, Z_{t-1}) + \theta\operatorname{Cov}(Z_{t+h-1}, Z_t) \\
&\qquad + \theta^2\operatorname{Cov}(Z_{t+h-1}, Z_{t-1}) \\
&\quad = \begin{cases} \sigma^2(1+\theta^2) & \text{if } h = 0, \\ \sigma^2\theta & \text{if } |h| = 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

If we let $\sigma^2 = 1/(1+\theta^2)$ and choose θ such that $\sigma^2\theta = 0.4$, then we get $\gamma_X(h) = \kappa(h)$. Hence, we choose θ so that $\theta/(1+\theta^2) = 0.4$, which implies that $\theta = 1/2$ or $\theta = 2$.

Problem 2.8. Assume that there exists a stationary solution $\{X_t : t \in \mathbb{Z}\}$ to

$$X_t = \phi X_{t-1} + Z_t, \qquad t = 0, \pm 1, \ldots$$

where $\{Z_t : t \in \mathbb{Z}\} \sim \mathrm{WN}\left(0, \sigma^2\right)$ and $|\phi_1| = 1$. Use the recursions

$$X_t = \phi X_{t-1} + Z_t = \phi^2 X_{t-2} + \phi Z_{t-1} + Z_t = \ldots = \phi^{n+1} X_{t-(n+1)} + \sum_{i=0}^{n} \phi^i Z_{t-i},$$

which yields that

$$X_t - \phi^{n+1} X_{t-(n+1)} = \sum_{i=0}^{n} \phi^i Z_{t-i}.$$

We have that

$$\mathrm{Var}\left(\sum_{i=0}^{n} \phi^i Z_{t-i}\right) = \sum_{i=0}^{n} \phi^{2i}\, \mathrm{Var}\left(Z_{t-i}\right) = \sum_{i=0}^{n} \sigma^2 = (n+1)\,\sigma^2.$$

On the other side we have that

$$\mathrm{Var}\left(X_t - \phi^{n+1} X_{t-(n+1)}\right) = 2\gamma(0) - 2\phi^{n+1}\gamma(n+1) \le 2\gamma(0) + 2\gamma(n+1) \le 4\gamma(0).$$

This mean that $(n+1)\,\sigma^2 \le 4\gamma(0),\ \forall n$. Letting $n \to \infty$ implies that $\gamma(0) = \infty$, which is a contradiction, i.e. there exists no stationary solution.

Problem 2.11. We have that $\{X_t : t \in \mathbb{Z}\}$ is an AR(1) process with mean μ so $\{X_t : t \in \mathbb{Z}\}$ satisfies

$$X_t - \mu = \phi(X_{t-1} - \mu) + Z_t, \qquad \{Z_t : t \in \mathbb{Z}\} \sim \mathrm{WN}\left(0, \sigma^2\right),$$

with $\phi = 0.6$ and $\sigma^2 = 2$. Since $\{X_t : t \in \mathbb{Z}\}$ is AR(1) we have that $\gamma_X(h) = \frac{\phi^{|h|}\sigma^2}{1-\phi^2}$. We estimate μ by $\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$. For large values of n $\overline{X}_n$ is approximately normally distributed with mean μ and variance $\frac{1}{n}\sum_{|h|<\infty} \gamma(h)$ (see Section 2.4 in Brockwell and Davis). In our case the variance is

$$\begin{aligned}
\frac{1}{n}\left(1 + 2\sum_{h=1}^{\infty} \phi^h\right)\frac{\sigma^2}{1-\phi^2} &= \frac{1}{n}\left(1 + 2\left(\frac{1}{1-\phi} - 1\right)\right)\frac{\sigma^2}{1-\phi^2} \\
&= \frac{1}{n}\left(\frac{2}{1-\phi} - 1\right)\frac{\sigma^2}{1-\phi^2} = \frac{1}{n}\left(\frac{1+\phi}{1-\phi}\right)\frac{\sigma^2}{1-\phi^2} = \frac{\sigma^2}{n(1-\phi)^2}.
\end{aligned}$$

Hence, $\overline{X}_n$ is approximately $N(\mu, \frac{\sigma^2}{n(1-\phi)^2})$. A 95% confidence interval is given by $I = (\overline{x}_n - \lambda_{0.025}\frac{\sigma}{\sqrt{n}(1-\phi)}, \overline{x}_n + \lambda_{0.025}\frac{\sigma}{\sqrt{n}(1-\phi)})$. Putting in the numeric values gives $I = 0.271 \pm 0.69$. Since $0 \in I$ the hypothesis that $\mu = 0$ can not be rejected.

Problem 2.15. Let $\hat{X}_{n+1} = P_n X_{n+1} = a_0 + a_1 X_n + \cdots + a_n X_1$. We may assume that $\mu_X(t) = 0$. Otherwise we can consider $Y_t = X_t - \mu$. Let $S(a_0, a_1, \ldots, a_n) = \mathbb{E}[(X_{n+1} - \hat{X}_{n+1})^2]$ and minimize this w.r.t. $a_0, a_1, \ldots, a_n$.

$$\begin{aligned}
S(a_0, a_1, \ldots, a_n) &= \mathbb{E}[(X_{n+1} - \hat{X}_{n+1})^2] \\
&= \mathbb{E}[(X_{n+1} - a_0 - a_1 X_n - \cdots - a_n X_1)^2] \\
&= a_0^2 - 2a_0\mathbb{E}[X_{n+1} - a_1 X_n - \cdots - a_n X_1] \\
&\quad + \mathbb{E}[(X_{n+1} - a_1 X_n - \cdots - a_n X_1)^2] \\
&= a_0^2 + \mathbb{E}[(X_{n+1} - a_1 X_n - \cdots - a_n X_1)^2].
\end{aligned}$$

Differentiation with respect to a_i gives

$$\begin{aligned}
\frac{\partial S}{\partial a_0} &= 2a_0, \\
\frac{\partial S}{\partial a_i} &= -2\mathbb{E}[((X_{n+1} - a_1 X_n - \cdots - a_n X_1) X_{n+1-i}], \qquad i = 1, \ldots, n.
\end{aligned}$$

Putting the partial derivatives equal to zero we get that $S(a_0, a_1, \dots, a_n)$ is minimized if

$$\begin{aligned} a_0 &= 0 \\ \mathbb{E}[(X_{n+1} - \hat{X}_{n+1})X_k] &= 0, \qquad \text{for each } k = 1, \dots, n. \end{aligned}$$

Plugging in the expression for X_{n+1} we get that for $k = 1, \dots, n$.

$$\begin{aligned} 0 &= \mathbb{E}[(X_{n+1} - \hat{X}_{n+1})X_k] \\ &= \mathbb{E}[(\phi_1 X_n + \cdots + \phi_p X_{n-p+1} + Z_{n+1} - a_1 X_n - \cdots - a_n X_1)X_k]. \end{aligned}$$

This is clearly satisfied if we let

$$\begin{cases} a_i = \phi_i, & \text{if } 1 \le i \le p \\ a_i = 0, & \text{if } i > p \end{cases}$$

Since there is best linear predictor is unique this is the one. The mean square error is

$$\mathbb{E}[(X_{n+1} - \hat{X}_{n+1})^2] = \mathbb{E}[Z_{n+1}^2] = \sigma^2.$$

Chapter 3

Problem 3.1. We write the ARMA processes as $\phi(B)X_t = \theta(B)Z_t$. The process $\{X_t : t \in \mathbb{Z}\}$ is causal if and only if $\phi(z) \neq 0$ for each $|z| \leq 1$ and invertible if and only if $\theta(z) \neq 0$ for each $|z| \leq 1$.

a) $\phi(z) = 1 + 0.2z - 0.48z^2 = 0$ is solved by $z_1 = 5/3$ and $z_2 = -5/4$.
Hence $\{X_t : t \in \mathbb{Z}\}$ is causal.
$\theta(z) = 1$. Hence $\{X_t : t \in \mathbb{Z}\}$ is invertible.

b) $\phi(z) = 1 + 1.9z + 0.88z^2 = 0$ is solved by $z_1 = -10/11$ and $z_2 = -5/4$.
Hence $\{X_t : t \in \mathbb{Z}\}$ is not causal.
$\theta(z) = 1 + 0.2z + 0.7z^2 = 0$ is solved by $z_1 = -(1 - i\sqrt{69})/7$
and $z_2 = -(1 + i\sqrt{69})/7$. Since $|z_1| = |z_2| = \sqrt{70}/7 > 1$, $\{X_t : t \in \mathbb{Z}\}$
is invertible.

c) $\phi(z) = 1 + 0.6z = 0$ is solved by $z = -5/3$. Hence $\{X_t : t \in \mathbb{Z}\}$ is causal.
$\theta(z) = 1 + 1.2z = 0$ is solved by $z = -5/6$. Hence $\{X_t : t \in \mathbb{Z}\}$ is not invertible.

d) $\phi(z) = 1 + 1.8z + 0.81z^2 = 0$ is solved by $z_1 = z_2 = -10/9$.
Hence $\{X_t : t \in \mathbb{Z}\}$ is causal.
$\theta(z) = 1$. Hence $\{X_t : t \in \mathbb{Z}\}$ is invertible.

e) $\phi(z) = 1 + 1.6z = 0$ is solved by $z = -5/8$. Hence $\{X_t : t \in \mathbb{Z}\}$ is not causal.
$\theta(z) = 1 - 0.4z + 0.04z^2 = 0$ is solved by $z_1 = z_2 = 5$.
Hence $\{X_t : t \in \mathbb{Z}\}$ is invertible.

Problem 3.4. We have $X_t = 0.8X_{t-2} + Z_t$, where $\{Z_t : t \in \mathbb{Z}\} \sim \mathrm{WN}\left(0, \sigma^2\right)$. To obtain the Yule-Walker equations we multiply each side by X_{t-k} and take expected value. Then we get

$$\mathbb{E}[X_t X_{t-k}] = 0.8\mathbb{E}[X_{t-2}X_{t-k}] + \mathbb{E}[Z_t X_{t-k}],$$

which gives us

$$\begin{aligned} \gamma(0) &= 0.8\gamma(2) + \sigma^2 \\ \gamma(k) &= 0.8\gamma(k-2), \qquad k \geq 1. \end{aligned}$$

We use that $\gamma(k) = \gamma(-k)$. Thus, we need to solve

$$\begin{aligned} \gamma(0) - 0.8\gamma(2) &= \sigma^2 \\ \gamma(1) - 0.8\gamma(1) &= 0 \\ \gamma(2) - 0.8\gamma(0) &= 0 \end{aligned}$$

First we see that $\gamma(1) = 0$ and therefore $\gamma(h) = 0$ if h is odd. Next we solve for $\gamma(0)$ and we get $\gamma(0) = \sigma^2(1 - 0.8^2)^{-1}$. It follows that $\gamma(2k) = \gamma(0)0.8^k$ and hence the ACF is

$$\rho(h) = \begin{cases} 1 & h = 0, \\ 0.8^h, & h = 2k,\ k = \pm 1, \pm 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

The PACF can be computed as $\alpha(0) = 1$, $\alpha(h) = \phi_{hh}$ where ϕ_{hh} comes from that the best linear predictor of X_{h+1} has the form

$$\hat{X}_{h+1} = \sum_{i=1}^{h} \phi_{hi} X_{h+1-i}.$$

For an AR(2) process we have $\hat{X}_{h+1} = \phi_1 X_h + \phi_2 X_{h-1}$ where we can identify $\alpha(0) = 1$, $\alpha(1) = 0$, $\alpha(2) = 0.8$ and $\alpha(h) = 0$ for $h \geq 3$.

Problem 3.6. The ACVF for $\{X_t : t \in \mathbb{Z}\}$ is

$$\begin{aligned}\gamma_X(t+h,t) &= \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(Z_{t+h} + \theta Z_{t+h-1}, Z_t + \theta Z_{t-1}) \\ &= \gamma_Z(h) + \theta\gamma_Z(h+1) + \theta\gamma_Z(h-1) + \theta^2\gamma_Z(h) \\ &= \begin{cases} \sigma^2(1+\theta^2), & h = 0 \\ \sigma^2\theta, & |h| = 1. \end{cases}\end{aligned}$$

On the other hand, the ACVF for $\{Y_t : t \in \mathbb{Z}\}$ is

$$\begin{aligned}\gamma_Y(t+h,t) &= \mathrm{Cov}(Y_{t+h}, Y_t) = \mathrm{Cov}(\tilde{Z}_{t+h} + \theta^{-1}\tilde{Z}_{t+h-1}, \tilde{Z}_t + \theta^{-1}\tilde{Z}_{t-1}) \\ &= \gamma_{\tilde{Z}}(h) + \theta^{-1}\gamma_{\tilde{Z}}(h+1) + \theta^{-1}\gamma_{\tilde{Z}}(h-1) + \theta^{-2}\gamma_{\tilde{Z}}(h) \\ &= \begin{cases} \sigma^2\theta^2(1+\theta^{-2}) = \sigma^2(1+\theta^2), & h = 0 \\ \sigma^2\theta^2\theta^{-1} = \sigma^2\theta, & |h| = 1. \end{cases}\end{aligned}$$

Hence they are equal.

Problem 3.7. First we show that $\{W_t : t \in \mathbb{Z}\}$ is WN$\left(0, \sigma_w^2\right)$.

$$\mathbb{E}[W_t] = \mathbb{E}\left[\sum_{j=0}^{\infty}(-\theta)^{-j}X_{t-j}\right] = \sum_{j=0}^{\infty}(-\theta)^{-j}\mathbb{E}[X_{t-j}] = 0,$$

since $\mathbb{E}[X_{t-j}] = 0$ for each j. Next we compute the ACVF of $\{W_t : t \in \mathbb{Z}\}$ for $h \geq 0$.

$$\begin{aligned}\gamma_W(t+h,t) &= \mathbb{E}[W_{t+h}W_t] = \mathbb{E}\left[\sum_{j=0}^{\infty}(-\theta)^{-j}X_{t+h-j}\sum_{k=0}^{\infty}(-\theta)^{-k}X_{t-k}\right] \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-\theta)^{-j}(-\theta)^{-k}\mathbb{E}[X_{t+h-j}X_{t-k}] = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-\theta)^{-j}(-\theta)^{-k}\gamma_X(h-j+k) \\ &= \left\{\gamma_X(r) = \sigma^2(1+\theta^2)\mathbf{1}_{\{0\}}(r) + \sigma^2\theta\mathbf{1}_{\{1\}}(|r|)\right\} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-\theta)^{-(j+k)}\left(\sigma^2(1+\theta^2)\mathbf{1}_{\{j-k\}}(h) + \sigma^2\theta\mathbf{1}_{\{j-k+1\}}(h) + \sigma^2\theta\mathbf{1}_{\{j-k-1\}}(h)\right) \\ &= \sum_{j=h}^{\infty}(-\theta)^{-(j+j-h)}\sigma^2(1+\theta^2) + \sum_{j=h-1,j\geq 0}^{\infty}(-\theta)^{-(j+j-h+1)}\sigma^2\theta \\ &\quad + \sum_{j=h+1}^{\infty}(-\theta)^{-(j+j-h-1)}\sigma^2\theta \\ &= \sigma^2(1+\theta^2)(-\theta)^{-h}\sum_{j=h}^{\infty}(-\theta)^{-2(j-h)} + \sigma^2\theta(-\theta)^{-(h-1)}\sum_{j=h-1,j\geq 0}^{\infty}(-\theta)^{-2(j-(h-1))} \\ &\quad + \sigma^2\theta(-\theta)^{-(h+1)}\sum_{j=h+1}^{\infty}(-\theta)^{-2(j-(h+1))} \\ &= \sigma^2(1+\theta^2)(-\theta)^{-h}\frac{\theta^2}{\theta^2-1} + \sigma^2\theta(-\theta)^{-(h-1)}\frac{\theta^2}{\theta^2-1} + \sigma^2\theta^2\mathbf{1}_{\{0\}}(h) \\ &\quad + \sigma^2\theta(-\theta)^{-(h+1)}\frac{\theta^2}{\theta^2-1} \\ &= \sigma^2(-\theta)^{-h}\frac{\theta^2}{\theta^2-1}\left(1+\theta^2-\theta^2-1\right) + \sigma^2\theta^2\mathbf{1}_{\{0\}}(h) \\ &= \sigma^2\theta^2\mathbf{1}_{\{0\}}(h)\end{aligned}$$

Hence, $\{W_t : t \in \mathbb{Z}\}$ is WN $\left(0, \sigma_w^2\right)$ with $\sigma_w^2 = \sigma^2\theta^2$. To continue we have that

$$W_t = \sum_{j=0}^{\infty} (-\theta)^{-j} X_{t-j} = \sum_{j=0}^{\infty} \pi_j X_{t-j},$$

with $\pi_j = (-\theta)^{-j}$ and $\sum_{j=0}^{\infty} |\pi_j| = \sum_{j=0}^{\infty} \theta^{-j} < \infty$ so $\{X_t : t \in \mathbb{Z}\}$ is invertible and solves $\phi(B)X_t = \theta(B)W_t$ with $\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \phi(z)/\theta(z)$. This implies that we must have

$$\sum_{j=0}^{\infty} \pi_j z^j = \sum_{j=0}^{\infty} \left(-\frac{z}{\theta}\right)^j = \frac{1}{1 + z/\theta} = \frac{\phi(z)}{\theta(z)}.$$

Hence, $\phi(z) = 1$ and $\theta(z) = 1 + z/\theta$, i.e. $\{X_t : t \in \mathbb{Z}\}$ satisfies $X_t = W_t + \theta^{-1}W_{t-1}$.

Problem 3.11. The PACF can be computed as $\alpha(0) = 1$, $\alpha(h) = \phi_{hh}$ where ϕ_{hh} comes from that the best linear predictor of X_{h+1} has the form

$$\hat{X}_{h+1} = \sum_{i=1}^{h} \phi_{hi} X_{h+1-i}.$$

In particular $\alpha(2) = \phi_{22}$ in the expression

$$\hat{X}_3 = \phi_{21}X_2 + \phi_{22}X_1.$$

The best linear predictor satisfies

$$\operatorname{Cov}(X_3 - \hat{X}_3, X_i) = 0, \qquad i = 1, 2.$$

This gives us

$$\begin{aligned}
&\operatorname{Cov}(X_3 - \hat{X}_3, X_1) = \operatorname{Cov}(X_3 - \phi_{21}X_2 - \phi_{22}X_1, X_1) \\
&\quad = \operatorname{Cov}(X_3, X_1) - \phi_{21}\operatorname{Cov}(X_2, X_1) - \phi_{22}\operatorname{Cov}(X_1, X_1) \\
&\quad = \gamma(2) - \phi_{21}\gamma(1) - \phi_{22}\gamma(0) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Cov}(X_3 - \hat{X}_3, X_2) = \operatorname{Cov}(X_3 - \phi_{21}X_2 - \phi_{22}X_1, X_2) \\
&\quad = \gamma(1) - \phi_{21}\gamma(0) - \phi_{22}\gamma(1) = 0.
\end{aligned}$$

Since we have an MA(1) process it has ACVF

$$\gamma(h) = \begin{cases} \sigma^2(1 + \theta^2), & h = 0, \\ \sigma^2\theta, & |h| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we have to solve the equations

$$\begin{aligned}
\phi_{21}\gamma(1) + \phi_{22}\gamma(0) &= 0 \\
(1 - \phi_{22})\gamma(1) - \phi_{21}\gamma(0) &= 0.
\end{aligned}$$

Solving this system of equations we find

$$\phi_{22} = -\frac{\theta^2}{\theta^4 + \theta^2 + 1}.$$

Chapter 4

Problem 4.4. By Corollary 4.1.1 we know that a function $\gamma(h)$ with $\sum_{|h|<\infty} |\gamma(h)|$ is ACVF for some stationary process if and only if it is an even function and

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h) \geq 0, \qquad \text{for } \lambda \in (-\pi, \pi].$$

We have that $\gamma(h)$ is even, $\gamma(h) = \gamma(-h)$ and

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi} \sum_{h=-3}^{3} e^{-ih\lambda}\gamma(h) \\ &= \frac{1}{2\pi}\left(-0.25e^{i3\lambda} - 0.5e^{i2\lambda} + 1 - 0.5e^{-i2\lambda} - 0.25e^{-i3\lambda}\right) \\ &= \frac{1}{2\pi}\left(1 - 0.25(e^{i3\lambda} + e^{-i3\lambda}) - 0.5(e^{i2\lambda} + e^{-i2\lambda})\right) \\ &= \frac{1}{2\pi}\left(1 - 0.5\cos(3\lambda) - \cos(2\lambda)\right). \end{aligned}$$

Do we have $f(\lambda) \geq 0$ on $\lambda \in (-\pi, \pi]$? The answer is NO, for instance $f(0) = -1/(4\pi)$. Hence, $\gamma(h)$ is NOT an ACVF for a stationary time series.

Problem 4.5. Let $Z_t = X_t + Y_t$. First we show that $\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$.

$$\begin{aligned} \gamma_Z(t+h, t) &= \mathrm{Cov}(Z_{t+h}, Z_t) = \mathrm{Cov}(X_{t+h} + Y_{t+h}, X_t + Y_t) \\ &= \mathrm{Cov}(X_{t+h}, X_t) + \mathrm{Cov}(X_{t+h}, Y_t) + \mathrm{Cov}(Y_{t+h}, X_t) + \mathrm{Cov}(Y_{t+h}, Y_t) \\ &= \mathrm{Cov}(X_{t+h}, X_t) + \mathrm{Cov}(Y_{t+h}, Y_t) \\ &= \gamma_X(t+h, t) + \gamma_Y(t+h, t). \end{aligned}$$

We have that

$$\gamma_Z(h) = \int_{(-\pi,\pi]} e^{ih\lambda} dF_Z(\lambda)$$

but we also know that

$$\begin{aligned} \gamma_Z(h) &= \gamma_X(h) + \gamma_Y(h) = \int_{(-\pi,\pi]} e^{ih\lambda} dF_X(\lambda) + \int_{(-\pi,\pi]} e^{ih\lambda} dF_Y(\lambda) \\ &= \int_{(-\pi,\pi]} e^{ih\lambda} \left(dF_X(\lambda) + dF_Y(\lambda)\right) \end{aligned}$$

Hence we have that $dF_Z(\lambda) = dF_X(\lambda) + dF_Y(\lambda)$, which implies that

$$F_Z(\lambda) = \int_{(-\pi,\lambda]} dF_Z(\nu) = \int_{(-\pi,\lambda]} (dF_X(\nu) + dF_Y(\nu)) = F_X(\lambda) + F_Y(\lambda).$$

Problem 4.6. Since $\{Y_t : t \in \mathbb{Z}\}$ is MA(1)-process we have

$$\gamma_Y(h) = \begin{cases} \sigma^2(1+\theta^2), & h = 0, \\ \sigma^2\theta, & |h| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

By Problem 2.2 the process $S_t = A\cos(\pi t/3) + B\sin(\pi t/3)$ has ACVF $\gamma_S(h) = \nu^2\cos(\pi h/3)$. Since the processes are uncorrelated, Problem 4.5 gives that $\gamma_X(h) = \gamma_S(h) + \gamma_Y(h)$. Moreover,

$$\nu^2\cos(\pi h/3) = \frac{\nu^2}{2}(e^{i\pi h/3} + e^{-i\pi h/3}) = \int_{-\pi}^{\pi} e^{i\lambda h} dF_S(\lambda),$$

where

$$dF_S(\lambda) = \frac{\nu^2}{2}\delta\left(\lambda - \pi/3\right) d\lambda + \frac{\nu^2}{2}\delta\left(\lambda + \pi/3\right) d\lambda$$

This implies

$$F_S(\lambda) = \begin{cases} 0, & \lambda < -\pi/3, \\ \nu^2/2, & -\pi/3 \leq \lambda < \pi/3, \\ \nu^2, & \lambda \geq \pi/3. \end{cases}$$

Furthermore we have that

$$\begin{aligned} f_Y(\lambda) &= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_Y(h) = \frac{1}{2\pi}\left(e^{i\lambda}\gamma_Y(-1) + \gamma_Y(0) + e^{-i\lambda}\gamma_Y(1)\right) \\ &= \frac{1}{2\pi}\left(\sigma^2\left(1 + 2.5^2\right) + 2.5\sigma^2\left(e^{i\lambda} + e^{-i\lambda}\right)\right) = \frac{\sigma^2}{2\pi}(7.25 + 5\cos(\lambda)). \end{aligned}$$

This implies that

$$\begin{aligned} F_Y(\lambda) &= \int_{-\pi}^{\lambda} f_Y(\xi)d\xi = \int_{-\pi}^{\lambda}\frac{\sigma^2}{2\pi}(7.25 + 5\cos(\xi))d\xi = \frac{\sigma^2}{2\pi}\left[7.25\xi + 5\sin(\xi)\right]_{-\pi}^{\lambda} \\ &= \frac{\sigma^2}{2\pi}(7.25(\lambda + \pi) + 5\sin(\lambda)). \end{aligned}$$

Finally we have $F_X(\lambda) = F_S(\lambda) + F_Y(\lambda)$.

Problem 4.9. a) We start with $\gamma_X(0)$,

$$\gamma_X(0) = \int_{-\pi}^{\pi} e^{i0\lambda} f_X(\lambda)d\lambda = 100\int_{-\frac{\pi}{6}-0.01}^{-\frac{\pi}{6}+0.01} d\lambda + 100\int_{\frac{\pi}{6}-0.01}^{\frac{\pi}{6}+0.01} d\lambda = 100 \cdot 0.04 = 4.$$

For $\gamma_X(1)$ we have,

$$\begin{aligned} \gamma_X(1) &= \int_{-\pi}^{\pi} e^{i\lambda} f_X(\lambda)d\lambda \\ &= 100\int_{-\frac{\pi}{6}-0.01}^{-\frac{\pi}{6}+0.01} e^{i\lambda}d\lambda + 100\int_{\frac{\pi}{6}-0.01}^{\frac{\pi}{6}+0.01} e^{i\lambda}d\lambda \\ &= 100\left[\frac{e^{i\lambda}}{i}\right]_{-\frac{\pi}{6}-0.01}^{-\frac{\pi}{6}+0.01} + 100\left[\frac{e^{i\lambda}}{i}\right]_{\frac{\pi}{6}-0.01}^{\frac{\pi}{6}+0.01} \\ &= \frac{100}{i}\left(e^{i(-\frac{\pi}{6}+0.01)} - e^{-i(\frac{\pi}{6}+0.01)} + e^{i(\frac{\pi}{6}+0.01)} - e^{-i(-\frac{\pi}{6}+0.01)}\right) \\ &= 200\left(\sin\left(-\frac{\pi}{6} + 0.01\right) + \sin\left(\frac{\pi}{6} + 0.01\right)\right) \\ &= 200\sqrt{3}\sin(0.01) \approx 3.46. \end{aligned}$$

The spectral density $f_X(\lambda)$ is plotted in Figure 4.9(a).

b) Let

$$Y_t = \nabla_{12}X_t = X_t - X_{t-12} = \sum_{k=-\infty}^{\infty}\psi_k X_{t-k},$$

with $\psi_0 = 1$, $\psi_{12} = -1$ and $\psi_j = 0$ otherwise. Then we have the spectral density $f_Y(\lambda) = |\psi(e^{-i\lambda})|^2 f_X(\lambda)$ where

$$\psi(e^{-i\lambda}) = \sum_{k=-\infty}^{\infty} \psi_k e^{-ik\lambda} = 1 - e^{-i12\lambda}.$$

Hence,

$$\begin{aligned} f_Y(\lambda) &= \left|1 - e^{-12i\lambda}\right|^2 f_X(\lambda) = (1 - e^{-12i\lambda})(1 - e^{12i\lambda}) f_X(\lambda) \\ &= 2(1 - \cos(12\lambda)) f_X(\lambda). \end{aligned}$$

The power transfer function $|\psi(e^{-i\lambda})|^2$ is plotted in Figure 4.9(b) and the resulting spectral density $f_Y(\lambda)$ is plotted in Figure 4.9(c).

c) The variance of Y_t is $\gamma_Y(0)$ which is computed by

$$\begin{aligned} \gamma_Y(0) &= \int_{-\pi}^{\pi} f_Y(\lambda) d\lambda \\ &= 200 \int_{-\frac{\pi}{6}-0.01}^{-\frac{\pi}{6}+0.01} (1 - \cos(12\lambda)) d\lambda + 200 \int_{\frac{\pi}{6}-0.01}^{\frac{\pi}{6}+0.01} (1 - \cos(12\lambda)) d\lambda \\ &= 200 \left(\left[\lambda - \frac{\sin(12\lambda)}{12}\right]_{-\frac{\pi}{6}-0.01}^{-\frac{\pi}{6}+0.01} + \left[\lambda - \frac{\sin(12\lambda)}{12}\right]_{\frac{\pi}{6}-0.01}^{\frac{\pi}{6}+0.01} \right) \\ &= 200 \left(0.02 - \frac{\sin(12(-\pi/6 + 0.01)) - \sin(12(-\pi/6 - 0.01))}{12} \right. \\ &\quad \left. +0.02 - \frac{\sin(12(\pi/6 + 0.01)) - \sin(12(\pi/6 - 0.01))}{12} \right) \\ &= 200 \left(0.04 + \frac{\sin(2\pi - 0.12) - \sin(2\pi + 0.12)}{6} \right) \\ &= 200 \left(0.04 - \frac{1}{3} \sin(0.12) \right) = 0.0192. \end{aligned}$$

Problem 4.10. a) Let $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 - \theta z$. Then $X_t = \frac{\theta(B)}{\phi(B)} Z_t$ and

$$f_X(\lambda) = \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2 f_Z(\lambda) = \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2 \frac{\sigma^2}{2\pi}.$$

For $\{W_t : t \in \mathbb{Z}\}$ we get

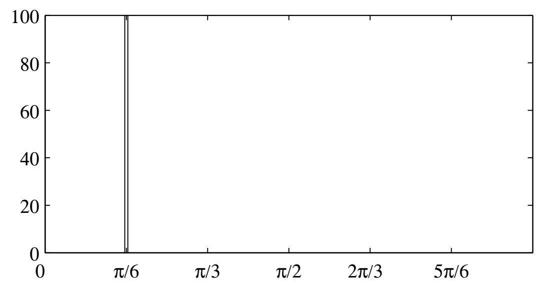
$$f_W(\lambda) = \left| \frac{\tilde{\phi}(e^{-i\lambda})}{\tilde{\theta}(e^{-i\lambda})} \right|^2 \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2 \frac{\sigma^2}{2\pi} = \frac{\left|1 - \frac{1}{\phi} e^{-i\lambda}\right|^2 \left|1 - \theta e^{-i\lambda}\right|^2}{\left|1 - \frac{1}{\theta} e^{-i\lambda}\right|^2 \left|1 - \phi e^{-i\lambda}\right|^2} \frac{\sigma^2}{2\pi}.$$

Now note that we can write

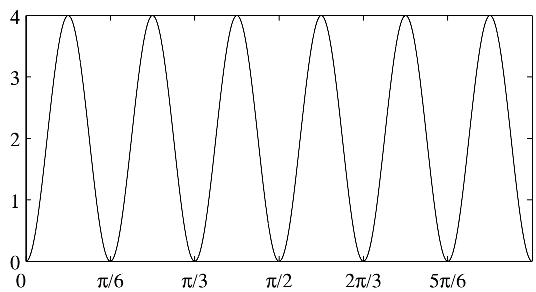
$$\begin{aligned} \left|1 - \frac{1}{\phi} e^{-i\lambda}\right|^2 &= \frac{1}{\phi^2} \left|\phi - e^{-i\lambda}\right|^2 = \frac{\left|e^{i\lambda}\right|^2}{\phi^2} \left|\phi - e^{-i\lambda}\right|^2 = \frac{1}{\phi^2} \left|\phi e^{i\lambda} - 1\right|^2 \\ &= \frac{1}{\phi^2} \left|1 - \phi e^{i\lambda}\right|^2 = \frac{1}{\phi^2} \left|1 - \phi e^{-i\lambda}\right|^2. \end{aligned}$$

Inserting this and the corresponding expression with ϕ substituted by θ in the computation above we get

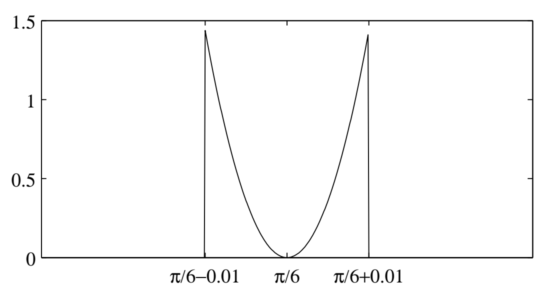
$$f_W(\lambda) = \frac{\frac{1}{\phi^2} \left|1 - \phi e^{-i\lambda}\right|^2 \left|1 - \theta e^{-i\lambda}\right|^2}{\frac{1}{\theta^2} \left|1 - \theta e^{-i\lambda}\right|^2 \left|1 - \phi e^{-i\lambda}\right|^2} \frac{\sigma^2}{2\pi} = \frac{\theta^2}{\phi^2} \frac{\sigma^2}{2\pi}$$



(a) $f_X(\lambda)$



(b) $\left|\psi\left(e^{-i\lambda}\right)\right|^2$



(c) $f_Y(\lambda)$

Figure 1: Exercise 4.9

which is constant.
b) Since $\{W_t : t \in \mathbb{Z}\}$ has constant spectral density it is white noise and

$$\sigma_w^2 = \gamma_W(0) = \int_{-\pi}^{\pi} f_W(\lambda) d\lambda = \frac{\theta^2}{\phi^2} \frac{\sigma^2}{2\pi} 2\pi = \frac{\theta^2}{\phi^2} \sigma^2.$$

c) From definition of $\{W_t : t \in \mathbb{Z}\}$ we get that $\tilde{\phi}(B) X_t = \tilde{\theta}(B) W_t$ which is a causal and invertible representation.

Chapter 5

Problem 5.1. We begin by writing the Yule-Walker equations. $\{Y_t : t \in \mathbb{Z}\}$ satisfies

$$Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} = Z_t, \quad \{Z_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2).$$

Multiplying this equation with Y_{t-k} and take expectation gives

$$\gamma(k) - \phi_1\gamma(k-1) - \phi_2\gamma(k-2) = \begin{cases} \sigma^2 & k = 0, \\ 0 & k \geq 1. \end{cases}$$

We rewrite the first three equations as

$$\phi_1\gamma(k-1) + \phi_2\gamma(k-2) = \begin{cases} \gamma(k) & k = 1, 2, \\ \gamma(0) - \sigma^2 & k = 0. \end{cases}$$

Introducing the notation

$$\mathbf{\Gamma}_2 = \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}, \ \boldsymbol{\gamma}_2 = \begin{pmatrix} \gamma(1) \\ \gamma(2) \end{pmatrix}, \ \boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

we have $\mathbf{\Gamma}_2\boldsymbol{\phi} = \boldsymbol{\gamma}_2$ and $\sigma^2 - \gamma(0) - \boldsymbol{\phi}^T\boldsymbol{\gamma}_2$. We replace $\mathbf{\Gamma}_2$ by $\hat{\mathbf{\Gamma}}_2$ and $\boldsymbol{\gamma}_2$ by $\hat{\boldsymbol{\gamma}}_2$ and solve to get an estimate $\hat{\boldsymbol{\phi}}$ for $\boldsymbol{\phi}$. That is, we solve

$$\hat{\mathbf{\Gamma}}_2\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\gamma}}_2 \qquad \hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\boldsymbol{\phi}}^T\hat{\boldsymbol{\gamma}}_2.$$

Hence

$$\begin{aligned} \hat{\boldsymbol{\phi}} = \hat{\mathbf{\Gamma}}_2^{-1}\hat{\boldsymbol{\gamma}}_2 &= \frac{1}{\hat{\gamma}(0)^2 - \hat{\gamma}(1)^2} \begin{pmatrix} \hat{\gamma}(0) & -\hat{\gamma}(1) \\ -\hat{\gamma}(1) & \hat{\gamma}(0) \end{pmatrix} \begin{pmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \end{pmatrix} \\ &= \frac{1}{\hat{\gamma}(0)^2 - \hat{\gamma}(1)^2} \begin{pmatrix} \hat{\gamma}(0)\hat{\gamma}(1) & -\hat{\gamma}(1)\hat{\gamma}(2) \\ -\hat{\gamma}(1)^2 & \hat{\gamma}(0)\hat{\gamma}(2) \end{pmatrix}. \end{aligned}$$

We get that

$$\begin{aligned} \hat{\phi}_1 &= \frac{(\hat{\gamma}(0) - \hat{\gamma}(2))\hat{\gamma}(1)}{\hat{\gamma}(0)^2 - \hat{\gamma}(1)^2} = 1.32 \\ \hat{\phi}_2 &= \frac{\hat{\gamma}(0)\hat{\gamma}(2) - \hat{\gamma}(1)^2}{\hat{\gamma}(0)^2 - \hat{\gamma}(1)^2} = -0.634 \\ \hat{\sigma}^2 &= \hat{\gamma}(0) - \hat{\phi}_1\hat{\gamma}(1) - \hat{\phi}_2\hat{\gamma}(2) = 289.18. \end{aligned}$$

We also have that $\hat{\boldsymbol{\phi}} \sim \mathrm{AN}(\boldsymbol{\phi}, \sigma^2\mathbf{\Gamma}_2^{-1}/n)$ and approximately $\hat{\boldsymbol{\phi}} \sim \mathrm{AN}(\boldsymbol{\phi}, \hat{\sigma}^2\hat{\mathbf{\Gamma}}_2^{-1}/n)$. Here

$$\hat{\sigma}^2\hat{\mathbf{\Gamma}}_2^{-1}/n = \frac{289.18}{100} \begin{pmatrix} 0.0021 & -0.0017 \\ -0.0017 & 0.0021 \end{pmatrix} = \begin{pmatrix} 0.0060 & -0.0048 \\ -0.0048 & 0.0060 \end{pmatrix}$$

So we have approximately $\hat{\phi}_1 \sim N(\phi_1, 0.0060)$ and $\hat{\phi}_2 \sim N(\phi_2, 0.0060)$ and the confidence intervals are

$$\begin{aligned} I_{\phi_1} &= \hat{\phi}_1 \pm \lambda_{0.025}\sqrt{0.006} = 1.32 \pm 0.15 \\ I_{\phi_2} &= \hat{\phi}_2 \pm \lambda_{0.025}\sqrt{0.006} = -0.634 \pm 0.15. \end{aligned}$$

Problem 5.3. a) $\{X_t : t \in \mathbb{Z}\}$ is causal if $\phi(z) \neq 0$ for $|z| \leq 1$ so let us check for which values of ϕ this can happen. $\phi(z) = 1 - \phi z - \phi^2 z^2$ so putting this equal to zero implies

$$z^2 + \frac{z}{\phi} - \frac{1}{\phi^2} = 0 \Rightarrow z_1 = -\frac{1-\sqrt{5}}{2\phi} \text{ and } z_2 = -\frac{1+\sqrt{5}}{2\phi}$$

Furthermore $|z_1| > 1$ if $|\phi| < (\sqrt{5}-1)/2 = 0.61$ and $|z_2| > 1$ if $|\phi| < (1+\sqrt{5})/2 = 1.61$. Hence, the process is causal if $|\phi| < 0.61$.
b) The Yule-Walker equations are

$$\gamma(k) - \phi\gamma(k-1) - \phi^2\gamma(k-2) = \begin{cases} \sigma^2 & k = 0, \\ 0 & k \geq 1. \end{cases}$$

Rewriting the first 3 equations and using $\gamma(k) = \gamma(-k)$ gives

$$\begin{aligned} \gamma(0) - \phi\gamma(1) - \phi^2\gamma(2) &= \sigma^2 \\ \gamma(1) - \phi\gamma(0) - \phi^2\gamma(1) &= 0 \\ \gamma(2) - \phi\gamma(1) - \phi^2\gamma(0) &= 0. \end{aligned}$$

Multiplying the third equation by ϕ^2 and adding the first gives

$$\begin{aligned} -\phi^3\gamma(1) - \phi\gamma(1) - \phi^4\gamma(0) + \gamma(0) &= \sigma^2 \\ \gamma(1) - \phi\gamma(0) - \phi^2\gamma(1) &= 0. \end{aligned}$$

We solve the second equation to obtain

$$\phi = -\frac{1}{2\rho(1)} \pm \sqrt{\frac{1}{4\rho(1)^2} + 1}.$$

Inserting the estimated values of $\hat{\gamma}(0)$ and $\hat{\gamma}(1) = \hat{\gamma}(0)\hat{\rho}(1)$ gives the solutions $\hat{\phi} = \{0.509, -1.965\}$ and we choose the causal solution $\hat{\phi} = 0.509$. Inserting this value in the expression for σ^2 we get

$$\hat{\sigma}^2 = -\hat{\phi}^3\hat{\gamma}(1) - \hat{\phi}\hat{\gamma}(1) - \hat{\phi}^4\hat{\gamma}(0) + \hat{\gamma}(0) = 2.985.$$

Problem 5.4. a) Let us construct a test to see if the assumption that $\{X_t - \mu : t \in \mathbb{Z}\}$ is WN $(0, \sigma^2)$ is reasonable. To this end suppose that $\{X_t - \mu : t \in \mathbb{Z}\}$ is WN $(0, \sigma^2)$. Then, since $\rho(k) = 0$ for $k \geq 1$ we have that $\hat{\rho}(k) \sim \text{AN}(0, 1/n)$. A 95% confidence interval for $\rho(k)$ is then $I_{\rho(k)} = \hat{\rho}(k) \pm \lambda_{0.025}/\sqrt{200}$. This gives us

$$\begin{aligned} I_{\rho(1)} &= 0.427 \pm 0.139 \\ I_{\rho(2)} &= 0.475 \pm 0.139 \\ I_{\rho(3)} &= 0.169 \pm 0.139. \end{aligned}$$

Clearly $0 \notin I_{\rho(k)}$ for any of the observed $k = 1, 2, 3$ and we conclude that it is not reasonable to assume that $\{X_t - \mu : t \in \mathbb{Z}\}$ is white noise.
b) We estimate the mean by $\hat{\mu} = \overline{x}_{200} = 3.82$. The Yule-Walker estimates is given by

$$\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}_2^{-1}\hat{\boldsymbol{\rho}}_2, \quad \hat{\sigma}^2 = \hat{\gamma}(0)(1 - \hat{\boldsymbol{\rho}}_2{}^T\hat{\mathbf{R}}_2^{-1}\hat{\boldsymbol{\rho}}_2),$$

where

$$\hat{\boldsymbol{\phi}} = \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix}, \ \hat{\mathbf{R}}_2 = \begin{pmatrix} \hat{\rho}(0) & \hat{\rho}(1) \\ \hat{\rho}(1) & \hat{\rho}(0) \end{pmatrix}, \ \hat{\boldsymbol{\rho}}_2 = \begin{pmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \end{pmatrix}.$$

Solving this system gives the estimates $\hat{\phi}_1 = 0.2742$, $\hat{\phi}_2 = 0.3579$ and $\hat{\sigma}^2 = 0.8199$.
c) We construct a 95% confidence interval for μ to test if we can reject the hypothesis that $\mu = 0$. We have that $\overline{X}_{200} \sim \mathrm{AN}(\mu, \nu/n)$ with

$$\nu = \sum_{h=-\infty}^{\infty} \gamma(h) \approx \hat{\gamma}(-3) + \hat{\gamma}(-2) + \hat{\gamma}(-1) + \hat{\gamma}(0) + \hat{\gamma}(1) + \hat{\gamma}(2) + \hat{\gamma}(3) = 3.61.$$

An approximate 95% confidence interval for μ is then

$$I = \overline{x}_n \pm \lambda_{0.025}\sqrt{\nu/n} = 3.82 \pm 1.96\sqrt{3.61/200} = 3.82 \pm 0.263.$$

Since $0 \notin I$ we reject the hypothesis that $\mu = 0$.
d) We have that approximately $\hat{\boldsymbol{\phi}} \sim \mathrm{AN}(\boldsymbol{\phi}, \hat{\sigma}^2\hat{\boldsymbol{\Gamma}}_2^{-1}/n)$. Inserting the observed values we get

$$\frac{\hat{\sigma}^2\hat{\boldsymbol{\Gamma}}_2^{-1}}{n} = \begin{pmatrix} 0.0050 & -0.0021 \\ -0.0021 & 0.0050 \end{pmatrix},$$

and hence $\hat{\phi}_1 \sim \mathrm{AN}(\phi_1, 0.0050)$ and $\hat{\phi}_2 \sim \mathrm{AN}(\phi_2, 0.0050)$. We get the 95% confidence intervals

$$\begin{aligned} I_{\phi_1} &= \hat{\phi}_1 \pm \lambda_{0.025}\sqrt{0.005} = 0.274 \pm 0.139 \\ I_{\phi_2} &= \hat{\phi}_2 \pm \lambda_{0.025}\sqrt{0.005} = 0.358 \pm 0.139. \end{aligned}$$

e) If the data were generated from an AR(2) process, then the PACF would be $\alpha(0) = 1$, $\hat{\alpha}(1) = \hat{\rho}(1) = 0.427$, $\hat{\alpha}(2) = \hat{\phi}_2 = 0.358$ and $\hat{\alpha}(h) = 0$ for $h \geq 3$.

Problem 5.11. To obtain the maximum likelihood estimator we compute as if the process were Gaussian. Then the innovations

$$\begin{aligned} X_1 - \hat{X}_1 &= X_1 \sim N(0, \nu_0), \\ X_2 - \hat{X}_2 &= X_2 - \phi X_1 \sim N(0, \nu_1), \end{aligned}$$

where $\nu_0 = \sigma^2 r_0 = \mathbb{E}[(X_1 - \hat{X}_1)^2]$, $\nu_1 = \sigma^2 r_1 = \mathbb{E}[(X_2 - \hat{X}_2)^2]$. This implies $\nu_0 = \mathbb{E}[X_1^2] = \gamma(0)$, $r_0 = 1/(1-\phi^2)$ and $\nu_1 = \mathbb{E}[(X_2 - \hat{X}_2)^2] = \gamma(0) - 2\phi\gamma(1) + \phi^2\gamma(0)$ and hence

$$r_1 = \frac{\gamma(0)(1+\phi^2) - 2\phi\gamma(1)}{\sigma^2} = \frac{1 + \phi^2 - 2\phi^2}{1 - \phi^2} = 1.$$

Here we have used that $\gamma(1) = \sigma^2\phi/(1-\phi^2)$. Since the distribution of the innovations is normal the density for $X_j - \hat{X}_j$ is

$$f_{X_j - \hat{X}_j} = \frac{1}{\sqrt{2\pi\sigma^2 r_{j-1}}} \exp\left(-\frac{x^2}{2\sigma^2 r_{j-1}}\right)$$

and the likelihood function is

$$\begin{aligned} L(\phi, \sigma^2) = \prod_{j=1}^{2} f_{X_j - \hat{X}_j} &= \frac{1}{\sqrt{(2\pi\sigma^2)^2 r_0 r_1}} \exp\left\{-\frac{1}{2\sigma^2}\left(\frac{(x_1 - \hat{x}_1)^2}{r_0} + \frac{(x_2 - \hat{x}_2)^2}{r_1}\right)\right\} \\ &= \frac{1}{\sqrt{(2\pi\sigma^2)^2 r_0 r_1}} \exp\left\{-\frac{1}{2\sigma^2}\left(\frac{x_1^2}{r_0} + \frac{(x_2 - \phi x_1)^2}{r_1}\right)\right\}. \end{aligned}$$

We maximize this by taking logarithm and then differentiate:

$$\begin{aligned}
\log L(\phi,\sigma^2) &= -\frac{1}{2}\log(4\pi^2\sigma^4 r_0 r_1) - \frac{1}{2\sigma^2}\Big(\frac{x_1^2}{r_0} + \frac{(x_2-\phi x_1)^2}{r_1}\Big) \\
&= -\frac{1}{2}\log(4\pi^2\sigma^4/(1-\phi^2)) - \frac{1}{2\sigma^2}\big(x_1^2(1-\phi^2) + (x_2-\phi x_1)^2\big) \\
&= -\log(2\pi) - \log(\sigma^2) + \frac{1}{2}\log(1-\phi^2) - \frac{1}{2\sigma^2}\big(x_1^2(1-\phi^2) + (x_2-\phi x_1)^2\big).
\end{aligned}$$

Differentiating yields

$$\begin{aligned}
\frac{\partial l(\phi,\sigma^2)}{\partial\sigma^2} &= -\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\big(x_1^2(1-\phi^2) + (x_2-\phi x_1)^2\big), \\
\frac{\partial l(\phi,\sigma^2)}{\partial\phi} &= \frac{1}{2}\cdot\frac{-2\phi}{1-\phi^2} + \frac{x_1 x_2}{\sigma^2}.
\end{aligned}$$

Putting these expressions equal to zero gives $\sigma^2 = \frac{1}{2}\big(x_1^2(1-\phi^2) + (x_2-\phi x_1)^2\big)$ and then after some computations $\phi = 2x_1x_2/(x_1^2+x_2^2)$. Inserting the expression for ϕ is the equation for σ gives the maximum likelihood estimators

$$\hat{\sigma}^2 = \frac{(x_1^2-x_2^2)^2}{2(x_1^2+x_2^2)} \text{ and } \hat{\phi} = \frac{2x_1x_2}{x_1^2+x_2^2}$$

Chapter 6

Problem 6.5. The best linear predictor of Y_{n+1} in terms of $1, X_0, Y_1, \ldots, Y_n$ i.e.

$$\hat{Y}_{n+1} = a_0 + cX_0 + a_1 Y_1 + \cdots + a_n Y_n,$$

must satisfy the orthogonality relations

$$\begin{aligned}
\mathrm{Cov}(Y_{n+1} - \hat{Y}_{n+1}, 1) &= 0 \\
\mathrm{Cov}(Y_{n+1} - \hat{Y}_{n+1}, X_0) &= 0 \\
\mathrm{Cov}(Y_{n+1} - \hat{Y}_{n+1}, Y_j) &= 0, \quad j = 1, \ldots, n.
\end{aligned}$$

The second equation can be written as

$$\mathrm{Cov}(Y_{n+1} - \hat{Y}_{n+1}, X_0) = \mathbb{E}[(Y_{n+1} - a_0 + cX_0 + a_1 Y_1 + \cdots + a_n Y_n)X_0] = c\mathbb{E}[X_0^2] = 0$$

so we must have $c = 0$. This does not effect the other equations since $\mathbb{E}[Y_j X_0] = 0$ for each j.

Problem 6.6. Put $Y_t = \nabla X_t$. Then $\{Y_t : t \in \mathbb{Z}\}$ is an AR(2) process. We can rewrite this as $X_{t+1} = Y_t + X_{t-1}$. Putting $t = n + h$ and using the linearity of the projection operator P_n gives $P_n X_{n+h} = P_n Y_{n+h} + P_n X_{n+h-1}$. Since $\{Y_t : t \in \mathbb{Z}\}$ is AR(2) process we have $P_n Y_{n+1} = \phi_1 Y_n + \phi_2 Y_{n-1}$, $P_n Y_{n+2} = \phi_1 P_n Y_{n+1} + \phi_2 Y_n$ and iterating we find $P_n Y_{n+h} = \phi_1 P_n Y_{n+h-1} + \phi_2 P_n Y_{n+h-2}$. Let $\phi^*(z) = (1-z)\phi(z) = 1 - \phi_1^* z - \phi_2^* z^2 - \phi_3^* z^3$. Then

$$(1-z)\phi(z) = 1 - \phi_1 z - \phi_2 z - z + \phi_1 z^2 + \phi_2 z^3,$$

i.e. $\phi_1^* = \phi_1 + 1$, $\phi_2^* = \phi_2 - \phi_1$ and $\phi_3^* = -\phi_2$. Then

$$P_n X_{n+h} = \sum_{j=1}^{3} \phi_j^* X_{n+h-j}.$$

This can be verified by first noting that

$$\begin{aligned}
P_n Y_{n+h} &= \phi_1 P_n Y_{n+h-1} + \phi_2 P_n Y_{n+h-2} \\
&= \phi_1 (P_n X_{n+h-1} - P_n X_{n+h-2}) + \phi_2 (P_n X_{n+h-2} - P_n X_{n+h-3}) \\
&= \phi_1 P_n X_{n+h-1} + (\phi_2 - \phi_1) P_n X_{n+h-2} - \phi_2 P_n X_{n+h-3}.
\end{aligned}$$

and then

$$\begin{aligned}
P_n X_{n+h} &= P_n Y_{n+h} + P_n X_{n+h-1} \\
&= (\phi_1 + 1) P_n X_{n+h-1} + (\phi_2 - \phi_1) P_n X_{n+h-2} - \phi_2 P_n X_{n+h-3} \\
&= \phi_1^* P_n X_{n+h-1} + \phi_2^* P_n X_{n+h-2} + \phi_3^* P_n X_{n+h-3}.
\end{aligned}$$

Hence, we have

$$g(h) = \begin{cases} \phi_1^* g(h-1) + \phi_2^* g(h-2) + \phi_3^* g(h-3), & h \geq 1, \\ X_{n+h}, & h \leq 0. \end{cases}$$

We may suggest a solution of the form $g(h) = a + b\xi_1^{-h} + c\xi_2^{-h}$, $h > -3$ where ξ_1 and ξ_2 are the solutions to $\phi(z) = 0$ and $g(-2) = X_{n-2}$, $g(-1) = X_{n-1}$ and $g(0) = X_n$. Let us first find the roots ξ_1 and ξ_2.

$$\phi(z) = 1 - 0.8z + 0.25z^2 = 1 - \frac{4}{5}z + \frac{1}{4}z^2 = 0 \Rightarrow z^2 - \frac{16}{5}z + 4 = 0.$$

We get that $z = 8/5 \pm \sqrt{(8/5)^2 - 4} = (8 \pm 6i)/5$. Then $\xi_1^{-1} = 5/(8+6i) = \cdots = 0.4 - 0.3i$ and $\xi_2^{-1} = 0.4 + 0.3i$. Next we find the constants a, b and c by solving

$$\begin{aligned} X_{n-2} &= g(-2) = a + b\xi_1^{-2} + c\xi_2^{-2}, \\ X_{n-1} &= g(-1) = a + b\xi_1^{-1} + c\xi_2^{-1}, \\ X_n &= g(0) = a + b + c. \end{aligned}$$

Note that $(0.4 - 0.3i)^2 = 0.07 - 0.24i$ and $(0.4 + 0.3i)^2 = 0.07 + 0.24i$ so we get the equations

$$\begin{aligned} X_{n-2} &= a + b(0.07 - 0.24i) + c(0.07 + 0.24i), \\ X_{n-1} &= a + b(0.4 - 0.3i) + c(0.4 + 0.3i), \\ X_n &= a + b + c. \end{aligned}$$

Let $a = a_1 + a_2 i$, $b = b_1 + b_2 i$ and $c = c_1 + c_2 i$. Then we split the equations into a real part and an imaginary part and get

$$\begin{aligned} X_{n-2} &= a_1 + 0.07b_1 + 0.24b_2 + 0.07c_1 - 0.24c_2, \\ X_{n-1} &= a_1 + 0.4b_1 + 0.3b_2 + 0.4c_1 - 0.4c_2, \\ X_n &= a_1 + b_1 + c_1, \\ 0 &= a_2 + 0.07b_2 - 0.24b_1 + 0.07c_2 + 0.24c_1, \\ 0 &= a_2 + 0.4b_2 - 0.3b_1 + 4c_2 + 0.3c_1, \\ 0 &= a_2 + b_2 + c_2. \end{aligned}$$

We can write this as a matrix equation by

$$\begin{pmatrix} 1 & 0 & 0.07 & 0.24 & 0.07 & -0.24 \\ 1 & 0 & 0.4 & 0.3 & 0.4 & -0.3 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -0.24 & 0.07 & 0.24 & 0.07 \\ 0 & 1 & -0.3 & 0.4 & 0.3 & 0.4 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} X_{n-2} \\ X_{n-1} \\ X_n \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which has the solution $a = 2.22X_n - 1.77X_{n-1} + 0.55X_{n-2}$, $b = \overline{c} = -1.1X_{n-2} + 0.88X_{n-1} + 0.22X_n + (-2.22X_{n-2} + 3.44X_{n-1} - 1.22X_n)i$.

Chapter 7

Problem 7.1. The problem is not very well formulated; we replace the condition $\rho_Y(h) \to 0$ as $h \to \infty$ by the condition that $\rho_Y(h)$ is strictly decreasing.

The process is stationary if $\bar{\mu}_t = \mathbb{E}[(X_{1,t}, X_{2,t})^T] = (\mu_1, \mu_2)^T$ and $\Gamma(t+h, t)$ does not depend on t. We may assume that $\{Y_t\}$ has mean zero so that

$$\mathbb{E}[X_{1,t}] = \mathbb{E}[Y_t] = 0$$
$$\mathbb{E}[X_{2,t}] = \mathbb{E}[Y_{t-d}] = 0,$$

and the covariance function is

$$\Gamma(t+h, t) = \mathbb{E}[(X_{1,t+h}, X_{2,t+h})^T (X_{1,t}, X_{2,t})] = \begin{pmatrix} \mathbb{E}[Y_{t+h}Y_t] & \mathbb{E}[Y_{t+h}Y_{t-d}] \\ \mathbb{E}[Y_{t+h-d}Y_t] & \mathbb{E}[Y_{t+h-d}Y_{t-d}] \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_Y(h) & \gamma_Y(h+d) \\ \gamma_Y(h-d) & \gamma_Y(h) \end{pmatrix}.$$

Since neither $\bar{\mu}_t$ or $\Gamma(t+h, t)$ depend on t, the process is stationary. We assume that $\rho_Y(h) \to 0$ as $h \to \infty$. Then we have that the cross-correlation

$$\rho_{12}(h) = \frac{\gamma_{12}(h)}{\sqrt{\gamma_{11}(0)\gamma_{22}(0)}} = \frac{\gamma_Y(h+d)}{\gamma_Y(0)} = \rho_Y(h+d).$$

In particular, $\rho_{12}(0) = \rho_Y(d) < 1$ whereas $\rho_{12}(-d) = \rho_Y(0) = 1$.

Problem 7.3. We want to estimate the cross-correlation

$$\rho_{12}(h) = \gamma_{12}(h)/\sqrt{\gamma_{11}(0)\gamma_{22}(0)}.$$

We estimate

$$\Gamma(h) = \begin{pmatrix} \gamma_{11}(h) & \gamma_{12}(h) \\ \gamma_{21}(h) & \gamma_{22}(h) \end{pmatrix}$$

by

$$\hat{\Gamma}(h) = \begin{cases} \frac{1}{n}\sum_{t=1}^{n-h}(\mathbf{X}_{t+h} - \bar{\mathbf{X}}_n)(\mathbf{X}_t - \bar{\mathbf{X}}_n)^T & 0 \le h \le n-1 \\ \Gamma^T(-h) & -n+1 \le h < 0. \end{cases}$$

Then we get $\hat{\rho}_{12}(h) = \hat{\gamma}_{12}(h)/\sqrt{\hat{\gamma}_{11}(0)\hat{\gamma}_{22}(0)}$. According to Theorem 7.3.1 in Brockwell and Davis we have, for $h \neq k$, that

$$\begin{pmatrix} \sqrt{n}\hat{\rho}_{12}(h) \\ \sqrt{n}\hat{\rho}_{21}(h) \end{pmatrix} \sim \text{approx. N}(\mathbf{0}, \Lambda)$$

where

$$\Lambda_{11} = \Lambda_{22} = \sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j)$$
$$\Lambda_{12} = \Lambda_{21} = \sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j+k-h).$$

Since $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are MA(1) processes we know that their ACF's are

$$\rho_{X_1}(h) = \begin{cases} 1 & h = 0 \\ 0.8/(1+0.8^2) & h = \pm 1 \end{cases}$$
$$\rho_{X_2}(h) = \begin{cases} 1 & h = 0 \\ -0.6/(1+0.6^2) & h = \pm 1 \end{cases}$$

Hence

$$\sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j) = \rho_{11}(-1)\rho_{22}(-1) + \rho_{11}(0)\rho_{22}(0) + \rho_{11}(1)\rho_{22}(1)$$
$$= \frac{0.8}{1+0.8^2} \cdot \frac{-0.6}{1+0.6^2} + 1 + \frac{0.8}{1+0.8^2} \cdot \frac{-0.6}{1+0.6^2} \approx 0.57.$$

For the covariance we see that $\rho_{11}(j) \neq 0$ if $j = -1, 0, 1$ and $\rho_{22}(j+k-h) \neq 0$ if $j+k-h = -1, 0, 1$. Hence, the covariance is

$$\sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j+k-h) = \rho_{11}(-1)\rho_{22}(0) + \rho_{11}(0)\rho_{22}(1) \approx 0.0466, \quad \text{if } k-h = 1$$
$$\sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j+k-h) = \rho_{11}(0)\rho_{22}(-1) + \rho_{11}(1)\rho_{22}(0) \approx 0.0466, \quad \text{if } k-h = -1$$
$$\sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j+k-h) = \rho_{11}(-1)\rho_{22}(1) \approx -0.2152, \quad \text{if } k-h = 2$$
$$\sum_{j=-\infty}^{\infty} \rho_{11}(j)\rho_{22}(j+k-h) = \rho_{11}(1)\rho_{22}(-1) \approx -0.2152, \quad \text{if } k-h = -2.$$

Problem 7.5. We have $\{X_t : t \in \mathbb{Z}\}$ is a causal process if $\det(\Phi(z)) \neq 0$ for all $|z| \leq 1$, due to Brockwell-Davis page 242. Further more we have that if $\{X_t : t \in \mathbb{Z}\}$ is a causal process, then

$$\mathbf{X}_t = \sum_{j=0}^{\infty} \mathbf{\Psi}_j \mathbf{Z}_{t-j},$$

where

$$\begin{aligned}
\mathbf{\Psi}_j &= \mathbf{\Theta}_j + \sum_{k=1}^{\infty} \mathbf{\Phi}_k \mathbf{\Psi}_{j-k} \\
\mathbf{\Theta}_0 &= \mathbf{I} \\
\mathbf{\Theta}_j &= \mathbf{0} \quad \text{for} \quad j > q \\
\mathbf{\Phi}_j &= \mathbf{0} \quad \text{for} \quad j > p \\
\mathbf{\Psi}_j &= \mathbf{0} \quad \text{for} \quad j < 0
\end{aligned}$$

and

$$\mathbf{\Gamma}(h) = \sum_{j=0}^{\infty} \mathbf{\Psi}_{h+j} \mathbf{\Sigma} \mathbf{\Psi}_j^T, \quad h = 0, \pm 1, \pm 2, \ldots$$

(where in this case $\mathbf{\Sigma} = \mathbf{I}_2$). We have to establish that $\{X_t : t \in \mathbb{Z}\}$ is a causal process and then derive $\mathbf{\Gamma}(h)$.

$$\det(\mathbf{\Phi}(z)) = \det(\mathbf{I} - z\mathbf{\Phi}_1) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{z}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$$
$$= \det\left(\begin{bmatrix} 1-\frac{z}{2} & \frac{z}{2} \\ 0 & 1-\frac{z}{2} \end{bmatrix}\right) = \frac{1}{4}(2-z)^2$$

Which implies that $|z_1| = |z_2| = 2 > 1$ and hence $\{X_t : t \in \mathbb{Z}\}$ is a causal process. We have that $\mathbf{\Psi}_j = \mathbf{\Theta}_j + \mathbf{\Phi}_1 \mathbf{\Psi}_{j-1}$ and

$$\begin{aligned}
\mathbf{\Psi}_0 &= \mathbf{\Theta}_0 + \mathbf{\Phi}_1 \mathbf{\Psi}_{-1} = \mathbf{\Theta}_0 = \mathbf{I} \\
\mathbf{\Psi}_1 &= \mathbf{\Theta}_1 + \mathbf{\Phi}_1 \mathbf{\Psi}_0 = \mathbf{\Phi}_1^T + \mathbf{\Phi}_1 \\
\mathbf{\Psi}_{n+1} &= \mathbf{\Phi}_1 \mathbf{\Psi}_n \quad \text{for} \quad n \geq 1.
\end{aligned}$$

From the last equation we get that $\boldsymbol{\Psi}_{n+1} = \boldsymbol{\Phi}_1^n \boldsymbol{\Psi}_1 = \boldsymbol{\Phi}_1^n (\boldsymbol{\Phi}_1^T + \boldsymbol{\Phi}_1)$ and from the definition of $\boldsymbol{\Phi}_1$

$$\boldsymbol{\Phi}_1^n = \frac{1}{2^n} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \quad \left(\boldsymbol{\Phi}_1^T + \boldsymbol{\Phi}_1\right)^2 = \frac{1}{4} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

Assume that $h \geq 0$, then

$$\begin{aligned}
\boldsymbol{\Gamma}(h) &= \sum_{j=0}^{\infty} \boldsymbol{\Psi}_{h+j} \boldsymbol{\Psi}_j^T = \boldsymbol{\Psi}_h + \sum_{j=1}^{\infty} \boldsymbol{\Psi}_{h+j} \boldsymbol{\Psi}_j^T \\
&= \boldsymbol{\Psi}_h + \sum_{j=1}^{\infty} \boldsymbol{\Phi}_1^{h+j-1} \left(\boldsymbol{\Phi}_1^T + \boldsymbol{\Phi}_1\right) \left(\boldsymbol{\Phi}_1^{j-1} \left(\boldsymbol{\Phi}_1^T + \boldsymbol{\Phi}_1\right)\right)^T \\
&= \boldsymbol{\Psi}_h + \boldsymbol{\Phi}_1^h \sum_{j=0}^{\infty} \boldsymbol{\Phi}_1^j \left(\boldsymbol{\Phi}_1^T + \boldsymbol{\Phi}_1\right)^2 \left(\boldsymbol{\Phi}_1^j\right)^T \\
&= \boldsymbol{\Psi}_h + \boldsymbol{\Phi}_1^h \sum_{j=0}^{\infty} \frac{1}{2^j} \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \frac{1}{2^j} \begin{bmatrix} 1 & 0 \\ j & 1 \end{bmatrix} \\
&= \boldsymbol{\Psi}_h + \boldsymbol{\Phi}_1^h \frac{1}{4} \sum_{j=0}^{\infty} \frac{1}{2^{2j}} \begin{bmatrix} 5 + 8j + 5j^2 & 4 + 5j \\ 4 + 5j & 5 \end{bmatrix} \\
&= \boldsymbol{\Psi}_h + \boldsymbol{\Phi}_1^h \begin{bmatrix} 94/27 & 17/9 \\ 17/9 & 5/3 \end{bmatrix}.
\end{aligned}$$

We have that

$$\boldsymbol{\Psi}_h = \begin{cases} \mathbf{I}, & h = 0 \\ \boldsymbol{\Phi}_1^{h-1} \left(\boldsymbol{\Phi}_1^T + \boldsymbol{\Phi}_1\right), & h > 0 \end{cases}$$

which gives that

$$\boldsymbol{\Gamma}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 94/27 & 17/9 \\ 17/9 & 5/3 \end{bmatrix} = \begin{bmatrix} 121/27 & 17/9 \\ 17/9 & 8/3 \end{bmatrix}$$

and for $h > 0$

$$\begin{aligned}
\boldsymbol{\Gamma}(h) &= \boldsymbol{\Phi}_1^{h-1} \left(\boldsymbol{\Phi}_1^T + \boldsymbol{\Phi}_1\right) + \boldsymbol{\Phi}_1^h \begin{bmatrix} 94/27 & 17/9 \\ 17/9 & 5/3 \end{bmatrix} \\
&= \boldsymbol{\Phi}_1^{h-1} \left(\frac{1}{2} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 94/27 & 17/9 \\ 17/9 & 5/3 \end{bmatrix}\right) \\
&= \frac{1}{2^h} \begin{bmatrix} 1 & h-1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 199/27 & 41/9 \\ 26/9 & 11/3 \end{bmatrix}.
\end{aligned}$$

Chapter 8

Problem 8.7. First we would like to show that

$$\mathbf{X}_{t+1} = \begin{bmatrix} 1 & \theta \\ \theta & 0 \end{bmatrix} \begin{bmatrix} Z_{t+1} \\ Z_t \end{bmatrix} \tag{8.1}$$

is a solution to

$$\mathbf{X}_{t+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{X}_t + \begin{bmatrix} 1 \\ \theta \end{bmatrix} Z_{t+1}. \tag{8.2}$$

Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ \theta \end{bmatrix},$$

and note that

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then equation (8.2) can be written as

$$\begin{aligned} \mathbf{X}_{t+1} &= A\mathbf{X}_t + BZ_{t+1} = A\left(A\mathbf{X}_{t-1} + BZ_t\right) + BZ_{t+1} = A^2\mathbf{X}_{t-1} + ABZ_t + BZ_{t+1} \\ &= \begin{bmatrix} \theta \\ 0 \end{bmatrix} Z_t + \begin{bmatrix} 1 \\ \theta \end{bmatrix} Z_{t+1} = \begin{bmatrix} \theta Z_t + Z_{t+1} \\ \theta Z_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \theta \\ \theta & 0 \end{bmatrix} \begin{bmatrix} Z_{t+1} \\ Z_t \end{bmatrix}, \end{aligned}$$

and hence (8.1) is a solution to equation (8.2). Next we prove that (8.1) is a unique solution to (8.2). Let $\mathbf{X}'_{t+1}$ be another solution to equation (8.2) and consider the difference

$$\begin{aligned} \mathbf{X}_{t+1} - \mathbf{X}'_{t+1} &= A\mathbf{X}_t + BZ_{t+1} - A\mathbf{X}'_t - BZ_{t+1} = A\left(\mathbf{X}_t - \mathbf{X}'_t\right) \\ &= A\left(A\mathbf{X}_{t-1} + BZ_t - A\mathbf{X}'_{t-1} - BZ_t\right) = A^2\left(\mathbf{X}_{t-1} - \mathbf{X}'_{t-1}\right) = \mathbf{0}, \end{aligned}$$

since $A^2 = \mathbf{0}$. This implies that $\mathbf{X}_{t+1} = \mathbf{X}'_{t+1}$, i.e. (8.1) is a unique solution to (8.2). Moreover, $\mathbf{X}_t$ is stationary since

$$\mu_{\mathbf{X}}(t) = \begin{bmatrix} 1 & \theta \\ \theta & 0 \end{bmatrix} \begin{bmatrix} \mathbb{E}[Z_t] \\ \mathbb{E}[Z_{t-1}] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{\Gamma}_{\mathbf{X}}(t+h,t) &= \begin{bmatrix} \gamma_{11}(t+h,t) & \gamma_{12}(t+h,t) \\ \gamma_{21}(t+h,t) & \gamma_{22}(t+h,t) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Cov}(Z_{t+h} + \theta Z_{t+h-1}, Z_t + \theta Z_{t-1}) & \mathrm{Cov}(Z_{t+h} + \theta Z_{t+h-1}, \theta Z_t) \\ \mathrm{Cov}(\theta Z_{t+h}, Z_t + \theta Z_{t-1}) & \mathrm{Cov}(\theta Z_{t+h}, \theta Z_t) \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} \left(1+\theta^2\right)\mathbf{1}_{\{0\}}(h) + \theta\mathbf{1}_{\{-1,1\}}(h) & \theta\mathbf{1}_{\{0\}}(h) + \theta^2\mathbf{1}_{\{1\}}(h) \\ \theta\mathbf{1}_{\{0\}}(h) + \theta^2\mathbf{1}_{\{-1\}}(h) & \theta^2\mathbf{1}_{\{0\}}(h) \end{bmatrix}, \end{aligned}$$

i.e. neither of them depend on t. Now we see that

$$Y_t = [1 \quad 0]\mathbf{X}_t = [1 \quad 0] \begin{bmatrix} 1 & \theta \\ \theta & 0 \end{bmatrix} \begin{bmatrix} Z_t \\ Z_{t-1} \end{bmatrix} = [1 \quad \theta] \begin{bmatrix} Z_t \\ Z_{t-1} \end{bmatrix} = Z_t + \theta Z_{t-1},$$

which is the MA(1) process.

Problem 8.9. Let $\mathbf{Y}_t$ consist of $\mathbf{Y}_{t,1}$ and $\mathbf{Y}_{t,2}$, then we can write

$$\begin{aligned}\mathbf{Y}_t &= \begin{bmatrix} \mathbf{Y}_{t,1} \\ \mathbf{Y}_{t,1} \end{bmatrix} = \begin{bmatrix} G_1\mathbf{X}_{t,1} + \mathbf{W}_{t,1} \\ G_2\mathbf{X}_{t,2} + \mathbf{W}_{t,2} \end{bmatrix} = \begin{bmatrix} G_1\mathbf{X}_{t,1} \\ G_2\mathbf{X}_{t,2} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_{t,1} \\ \mathbf{W}_{t,2} \end{bmatrix} \\ &= \begin{bmatrix} G_1 & \mathbf{0} \\ \mathbf{0} & G_2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{t,1} \\ \mathbf{X}_{t,2} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_{t,1} \\ \mathbf{W}_{t,2} \end{bmatrix}.\end{aligned}$$

Set

$$G = \begin{bmatrix} G_1 & \mathbf{0} \\ \mathbf{0} & G_2 \end{bmatrix}, \quad \mathbf{X}_t = \begin{bmatrix} \mathbf{X}_{t,1} \\ \mathbf{X}_{t,1} \end{bmatrix} \quad \text{and} \quad \mathbf{W}_t = \begin{bmatrix} \mathbf{W}_{t,1} \\ \mathbf{W}_{t,2} \end{bmatrix}$$

then we have $\mathbf{Y}_t = G\mathbf{X}_t + \mathbf{W}_t$. Similarly we have that

$$\begin{aligned}\mathbf{X}_{t+1} &= \begin{bmatrix} \mathbf{X}_{t+1,1} \\ \mathbf{X}_{t+1,1} \end{bmatrix} = \begin{bmatrix} F_1\mathbf{X}_{t,1} + \mathbf{V}_{t,1} \\ F_2\mathbf{X}_{t,2} + \mathbf{V}_{t,2} \end{bmatrix} = \begin{bmatrix} F_1\mathbf{X}_{t,1} \\ F_2\mathbf{X}_{t,2} \end{bmatrix} + \begin{bmatrix} \mathbf{V}_{t,1} \\ \mathbf{V}_{t,2} \end{bmatrix} \\ &= \begin{bmatrix} F_1 & \mathbf{0} \\ \mathbf{0} & F_2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{t,1} \\ \mathbf{X}_{t,2} \end{bmatrix} + \begin{bmatrix} \mathbf{V}_{t,1} \\ \mathbf{V}_{t,2} \end{bmatrix}\end{aligned}$$

and set

$$F = \begin{bmatrix} F_1 & \mathbf{0} \\ \mathbf{0} & F_2 \end{bmatrix} \quad \text{and} \quad \mathbf{V}_t = \begin{bmatrix} \mathbf{V}_{t,1} \\ \mathbf{V}_{t,2} \end{bmatrix}.$$

Finally we have the state-space representation

$$\begin{aligned}\mathbf{Y}_t &= G\mathbf{X}_t + \mathbf{W}_t \\ \mathbf{X}_{t+1} &= F\mathbf{X}_t + \mathbf{V}_t.\end{aligned}$$

Problem 8.13. We have to solve

$$\Omega + \sigma_v^2 - \frac{\Omega^2}{\Omega + \sigma_w^2} = \Omega$$

which is equivalent to

$$\frac{\Omega^2}{\Omega + \sigma_w^2} - \sigma_v^2 = 0.$$

Multiplying with $\Omega + \sigma_w^2$ we get

$$\Omega^2 - \Omega\sigma_v^2 - \sigma_w^2\sigma_v^2 = 0,$$

which has the solutions

$$\Omega = \frac{1}{2}\sigma_v^2 \pm \sqrt{\frac{\sigma_v^4}{4} + \sigma_w^2\sigma_v^2} = \frac{\sigma_v^2 \pm \sqrt{\sigma_v^4 + 4\sigma_w^2\sigma_v^2}}{2}.$$

Since $\Omega \geq 0$ we have the positive root which is the solution we wanted.

Problem 8.14. We have that

$$\Omega_{t+1} = \Omega_t + \sigma_v^2 - \frac{\Omega_t^2}{\Omega_t + \sigma_w^2}$$

and since $\sigma_v^2 = \Omega^2/(\Omega + \sigma_w^2)$ substracting Ω yields

$$\begin{aligned}\Omega_{t+1} - \Omega &= \Omega_t + \frac{\Omega^2}{\Omega + \sigma_w^2} - \frac{\Omega_t^2}{\Omega_t + \sigma_w^2} - \Omega \\ &= \frac{\Omega_t\left(\Omega_t + \sigma_w^2\right) - \Omega_t^2}{\Omega_t + \sigma_w^2} - \frac{\Omega\left(\Omega + \sigma_w^2\right) - \Omega^2}{\Omega + \sigma_w^2} \\ &= \frac{\Omega_t\sigma_w^2}{\Omega_t + \sigma_w^2} - \frac{\Omega\sigma_w^2}{\Omega + \sigma_w^2} \\ &= \sigma_w^2\left(\frac{\Omega_t}{\Omega_t + \sigma_w^2} - \frac{\Omega}{\Omega + \sigma_w^2}\right).\end{aligned}$$

This implies that

$$(\Omega_{t+1}-\Omega)(\Omega_t-\Omega)=\sigma_w^2\left(\frac{\Omega_t}{\Omega_t+\sigma_w^2}-\frac{\Omega}{\Omega+\sigma_w^2}\right)(\Omega_t-\Omega).$$

Now, note that the function $f(x)=x/(x+\sigma_w^2)$ is increasing in x. Indeed, $f'(x)=\sigma_w^2/(x+\sigma_w^2)^2>0$. Thus we get that for $\Omega_t>\Omega$ both terms are >0 and for $\Omega_t<\Omega$ both terms are <0. Hence, $(\Omega_{t+1}-\Omega)(\Omega_t-\Omega)\geq 0$.

Problem 8.15. We have the equations for θ:

$$\begin{aligned}\theta\sigma^2&=-\sigma_w^2\\ \sigma^2(1+\theta^2)&=2\sigma_w^2+\sigma_v^2.\end{aligned}$$

From the first equation we get that $\sigma^2=-\sigma_w^2/\theta$ and inserting this in the second equation gives

$$2\sigma_w^2+\sigma_v^2=-\frac{\sigma_w^2}{\theta}(1+\theta^2),$$

and multiplying by θ gives the equation

$$(2\sigma_w^2+\sigma_v^2)\theta+\sigma_w^2+\sigma_w^2\theta^2=0.$$

This can be rewritten as

$$\theta^2+\theta\frac{2\sigma_w^2+\sigma_v^2}{\sigma_w^2}+1=0$$

which has the solution

$$\theta=-\frac{2\sigma_w^2+\sigma_v^2}{2\sigma_w^2}\pm\sqrt{\frac{(2\sigma_w^2+\sigma_v^2)^2}{4\sigma_w^4}-1}=-\frac{2\sigma_w^2+\sigma_v^2\pm\sqrt{\sigma_v^4+4\sigma_v^2\sigma_w^2}}{2\sigma_w^2}.$$

To get an invertible representation we choose the solution

$$\theta=-\frac{2\sigma_w^2+\sigma_v^2-\sqrt{\sigma_v^4+4\sigma_v^2\sigma_w^2}}{2\sigma_w^2}.$$

To show that $\theta=-\frac{\sigma_w^2}{\sigma_w^2+\Omega}$, recall the steady-state solution

$$\Omega=\frac{\sigma_v^2+\sqrt{\sigma_v^4+4\sigma_v^2\sigma_w^2}}{2},$$

which gives

$$\begin{aligned}\theta&=-\frac{2\sigma_w^2+\sigma_v^2-\sqrt{\sigma_v^4+4\sigma_v^2\sigma_w^2}}{2\sigma_w^2}\\ &=-\frac{\left(2\sigma_w^2+\sigma_v^2-\sqrt{\sigma_v^4+4\sigma_v^2\sigma_w^2}\right)\left(2\sigma_w^2+\sigma_v^2+\sqrt{\sigma_v^4+4\sigma_v^2\sigma_w^2}\right)}{2\sigma_w^2\left(2\sigma_w^2+\sigma_v^2+\sqrt{\sigma_v^4+4\sigma_v^2\sigma_w^2}\right)}\\ &=-\frac{4\sigma_w^4+4\sigma_v^2\sigma_w^2+\sigma_v^4-\sigma_v^4-4\sigma_v^2\sigma_w^2}{2\sigma_w^2\left(2\sigma_w^2+2\Omega\right)}=-\frac{4\sigma_w^4}{4\sigma_w^2\left(\sigma_w^2+\Omega\right)}=-\frac{\sigma_w^2}{\sigma_w^2+\Omega}.\end{aligned}$$

Chapter 10

Problem 10.5. First a remark on existence of such a process: We assume for simplicity that $p = 1$. A necessary and sufficient condition for the existence of a causal, stationary solution to the ARCH(1) equations with $\mathbb{E}[Z_t^4] < \infty$ is that $\alpha_1^2 < 1/3$. If $p > 1$ existence of a causal, stationary solution is much more complicated. Let us now proceed with the solution to the problem.

We have

$$e_t^2\left(1+\sum_{i=1}^{p}\alpha_i Y_{t-i}\right) = e_t^2\left(1+\sum_{i=1}^{p}\alpha_i \frac{Z_{t-i}^2}{\alpha_0}\right) = \frac{e_t^2}{\alpha_0}\left(\alpha_0+\sum_{i=1}^{p}\alpha_i Z_{t-i}^2\right) = \frac{e_t^2 h_t}{\alpha_0} = \frac{Z_t^2}{\alpha_0} = Y_t,$$

hence $Y_t = Z_t^2/\alpha_0$ satisfies the given equation. Let us now compute its ACVF. We assume $h \geq 1$, then

$$\begin{aligned}
\mathbb{E}[Y_tY_{t-h}] &= \mathbb{E}\left[e_t^2\left(1+\sum_{i=1}^{p}\alpha_i Y_{t-i}\right)Y_{t-h}\right] \\
&= \mathbb{E}[e_t^2]\mathbb{E}\left[Y_{t-h}+\sum_{i=1}^{p}\alpha_i Y_{t-i}Y_{t-h}\right] \\
&= \mathbb{E}[Y_{t-h}]+\sum_{i=1}^{p}\alpha_i\mathbb{E}[Y_{t-i}Y_{t-h}].
\end{aligned}$$

Since $\gamma_Y(h) = \mathrm{Cov}(Y_t, Y_{t-h}) = \mathbb{E}[Y_tY_{t-h}] - \mu_Y^2$ we get

$$\gamma_Y(h) + \mu_Y^2 = \mu_Y + \sum_{i=1}^{p}\alpha_i\left(\gamma_Y(h-i)+\mu_Y^2\right)$$

and then

$$\gamma_Y(h) - \sum_{i=1}^{p}\alpha_i\gamma_Y(h-i) = \mu_Y + \mu_Y^2\left(\sum_{i=1}^{p}\alpha_i - 1\right).$$

We can compute μ_Y as

$$\mu_Y = \mathbb{E}[Y_t] = \mathbb{E}\left[e_t^2\left(1+\sum_{i=1}^{p}\alpha_i Y_{t-i}\right)\right] = 1+\sum_{i=1}^{p}\alpha_i\mathbb{E}[Y_t] = 1+\mu_Y\sum_{i=1}^{p}\alpha_i.$$

From this expression we see that $\mu_Y = 1/(1-\sum_{i=1}^{p}\alpha_i)$. This means that we have

$$\gamma_Y(h) - \sum_{i=1}^{p}\alpha_i\gamma_Y(h-i) = \frac{1}{1-\sum_{i=1}^{p}\alpha_i} + \frac{\sum_{i=1}^{p}\alpha_i - 1}{(1-\sum_{i=1}^{p}\alpha_i)^2} = 0.$$

Dividing by $\gamma_Y(0)$ we find that the ACF $\rho_Y(h)$ satisfies

$$\begin{aligned}
&\rho_Y(0) = 1, \\
&\rho_Y(h) - \sum_{i=1}^{p}\alpha_i\rho_Y(h-i) = 0, \qquad h \geq 1,
\end{aligned}$$

which corresponds to the Yule-Walker equations for the ACF for an AR(p) process

$$W_t = \alpha_1 W_{t-1} + \cdots + \alpha_p W_{t-p} + Z_t.$$