UC Berkeley, Haas School of Business
Master in Financial Engineering, Industry Project, Spring 2022

Academic Project: The Recovery Theorem and Robust Estimation of the Probability Density of Returns

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1 Introduction

The project is focused on the article "The Recovery Theorem" by Steven Ross (2015) (3) and implement on the empirical estimation of the returns distribution from the options data.
I would like to apply the Recovery Theorem to divide the state prices into the risk aversion component and the estimation of the probability distribution and returns. This separation lets us get the pricing kernel, market risk premium, and the tail events probabilities in a model-independent fashion.
While some applications are used to check the Efficient Market Hypothesis, I would like to focus the work on the opportunities for more robust estimation of the probability densities, which show very transient behavior, using the techniques such as the Kalman Filter.
Moreover, the particular interest in the estimation lies in the application of the resulted methodology to the SOFR Options, which gained a lot of interest in recent years due to the transition from LIBOR-world. After this, we will be able to create an asset-independent framework applicable to any type of option.
The result of the work will be the application of the robust probability densities for returns forecast for the given asset class to measure the predictive power of the estimates.
The overall work will include working with the original model and extension of it for more robust estimations, data collection and cleaning for the SOFR options prices, using the trading terminals, estimation framework development, and forecastability hypothesis testing using Python.

2 Research methodology

In order to estimate the recovered probabilities and check the forecasting power of this feature, we are following the methodology outlined by Ross (2015) (3) and try to recover the pricing kernel from the option prices:

$$(x, y) = \frac{P(x, y)}{\mathcal{P}(x, y)}$$
$$P(x, y) \text{ is a Risk-Neutral Density Matrix}$$
$$\mathcal{P}(x, y) \text{ is a Transition Matrix} m(x, y) \text{ is a Pricing Kernel}$$

This pricing kernel allows us to capture the combined effect of the time-value-of-money and risk-aversion of the representative investor.
In order to do this, we are going to use the market data from Bloomberg, process it and interpolate. After this, we will estimate the Risk-Neutral density matrix using the Breeden-Litezenberger approach (4), and estimate the Transition Matrix using the Forward equation. Following this step, we will recover the pricing kernel and conver it to the recovered probabilities.
Our last step will be to check the predictive power of the recovered probabilities for the Delta-1 instrument, chosen to be the SOFR interest rate swaps, the options underlyings.

3 Data analysis

Using Bloomberg Terminal, I downloaded the data for the whole SOFR (S490) Interest Rate Swap (IRS) Curve and for the Cap Options Implied Volatilities contributed by interdealer broker Conticap over the available dates of November 9th 2021 to September 27th 2022.
For the interest rate curve we initially get the maturities from 1 day to 50 years with weekly and monthly data up to 1 year, 1 year data from 1 year to 10 years, and data with 2-10 years time step starting from 10 years.
For the options we initially get most dense data with the maturities from 1 to 30 years to expiry with step of 1 year in the short-end and the belly of the term structure, and 2-5 years in the long end, starting from 10 years. And I get strikes from -2% to 7% with the step of 0.5%.
As SOFR-options and IRS are the new instruments, not all the expiries and strikes are liquid and we can miss some data at given points. It is crucial for us to keep the dataset stable across the time, After the data preprocessing we keep only 1 to 10 years maturities with time step of 1 year across the options and swaps. For the strikes we keep the data from -0.75% to 6%.
Let's look at the example of the implied volatility surface we got for the date of September 27th 2022.

The Implied Volatilities are quoted in basis points for the interest rate swaps, so, for example, point -0.75% 1Y, which is equal to 114.3bps will be equivalent to 1.143%:

	-0.75%	-0.50%	-0.25%	0.00%	0.25%	0.50%	0.75%	1.00%	1.50%	2.00%	2.50%	3.00%	3.50%
1Y	133.9	134.2	114.3	112.3	109.9	107.4	104.8	102.7	101.5	103.3	103.8	103.8	112.1
2Y	163.3	162.2	160.7	158.8	156.7	154.5	152.2	150.1	146.2	143.2	140.4	137.9	139.3
3Y	166.4	166.6	166.3	165.6	164.4	163.0	161.4	159.7	156.4	153.5	151.1	149.5	150.9
4Y	158.2	159.0	159.4	159.4	158.9	158.2	157.3	156.3	154.3	152.6	151.4	150.9	152.6
5Y	151.4	152.4	153.0	153.2	153.0	152.6	152.0	151.3	149.9	148.7	148.1	148.2	150.2
6Y	145.0	146.0	146.7	146.9	146.9	146.6	146.2	145.7	144.6	143.8	143.6	144.1	146.2
7Y	137.6	138.7	139.4	139.9	140.0	139.9	139.6	139.3	138.7	138.3	138.5	139.4	141.6
8Y	130.7	131.8	132.6	133.1	133.3	133.3	133.2	133.0	132.7	132.7	133.2	134.4	136.9
9Y	125.2	126.3	127.1	127.6	127.9	128.0	127.9	127.8	127.7	127.8	128.6	130.1	132.8
10Y	121.0	122.0	122.8	123.3	123.5	123.6	123.5	123.4	123.3	123.5	124.4	126.0	128.6

In addition to the Implied Volatilities, I also downloaded price series from the ContiCap datasource in order to check the quality of our price-fitting process, which will be used further down in the analysis:

	-0.25%	-0.50%	0.00%	0.25%	0.50%	0.75%	1.00%	1.50%	2.00%	2.50%	3.00%	4.00%	4.50
1.0	440.0	465.0	416.0	391.0	366.0	341.0	317.0	267.0	218.0	169.0	122.0	51.6	3
2.0	866.0	915.0	818.0	770.0	722.0	674.0	627.0	532.0	440.0	350.0	267.0	139.0	9
3.0	1235.0	1304.0	1166.0	1097.0	1029.0	961.0	893.0	760.0	632.0	510.0	397.0	226.0	17
4.0	1581.0	1670.0	1493.0	1404.0	1317.0	1230.0	1145.0	977.0	816.0	664.0	526.0	315.0	24
5.0	1911.0	2019.0	1804.0	1697.0	1592.0	1488.0	1385.0	1184.0	992.0	813.0	650.0	400.0	31
6.0	2228.0	2354.0	2103.0	1979.0	1857.0	1735.0	1616.0	1383.0	1162.0	956.0	770.0	483.0	38
7.0	2519.0	2662.0	2377.0	2237.0	2098.0	1960.0	1825.0	1563.0	1315.0	1084.0	877.0	557.0	44
8.0	2799.0	2959.0	2641.0	2484.0	2329.0	2176.0	2026.0	1735.0	1460.0	1206.0	978.0	628.0	50
9.0	3071.0	3247.0	2897.0	2724.0	2554.0	2386.0	2221.0	1902.0	1602.0	1325.0	1078.0	699.0	56
10.0	3346.0	3538.0	3156.0	2968.0	2783.0	2600.0	2420.0	2074.0	1748.0	1448.0	1181.0	771.0	62

Now, with the trimmed data stable across all the timestamps, we are ready to start extracting our Risk-Neutral probability densities from the options prices.

4 Arrow-Debreu Densities

In order to do this, we are following the procedure outlined in the seminal paper by the Breeden and Litzenberger (1978) (4), and we want to get the densities from the options prices $p(K,T)$:

$$C(K,T) = \int_0^\infty [S-K]^+ p(S,T)dS = \int_K^\infty [S-K]p(S,T)dS$$

$$p(K,T) = \frac{\partial C(K,T)}{\partial K}$$

4.1 Forward Curve Construction

In order to price SOFR option, we have to follow several intermediary steps: SOFR rates are compounded overnight rates and the Cap options are the strip of Caplet options, which are fixed every day up to maturity with the strike equal to the forward rate. We can refer to the Bloomberg methodology (12) and see that:

$$\text{A daily overnight compounded rate is } 1 + \Delta(t_m, t_n)R_{[t_m,t_n]} = \prod_{k=m+1}^{n}(1+\tau_k R_k),\ \tau_k = \Delta(t_{k-1}, t_k)$$

$$\text{Swaplet is } Z(T) = R_{[T_s,T_e]} - K$$

$$\text{PV of the payment is } Z(t) = P(t,T)(\mathbb{E}^T[R_{[T_s,T_e]}|\mathcal{F}_t] - K)$$

$$\text{Forward Value of the compounded rate is } F(t,T_s,T_e,T) = \mathbb{E}^T[R_{T_s,T_e}|\mathcal{F}_t],\ t \le T$$

Therefore, we have to construct the forward curve with the values up to the longest maturity of the options, 10 years in our case, and we have to create the pricing mechanism for Caplets and Caps.
In order to do precise pricing, we take the raw yield curve with the tenors from 1 day up to 10 years

	1D	1W	2W	3W	1M	2M	3M	4M	5M	6M	7M	8M	9M	
Value	2.99	3.0281	3.0265	3.0285	3.032	3.319	3.5226	3.697	3.8375	3.9435	4.03	4.104	4.1602	4.2

And plug it into the QuantLib library, which will be used for the forwards pricing using the bootstrapping techniques and forward schedule generation.
We are using the following parameters for this:

- Calendar is the US Holidays
- Day Count Convention is ACT/360
- Settlement is T+0

After this, we are constructing the SOFR IRS Curve for each date in our sample (230 points).
For Forward Curve construction we are using Piecewise Logarithmic Cubic Discount Term Structure model with extrapolation. It works the following way:

- Term structure is bootstrapped on a number of interest rate instruments which are passed as a vector of RateHelper instances
- Their maturities mark the boundaries of the interpolated segments
- Each segment is determined sequentially starting from the earliest period to the latest and is chosen so that the instrument whose maturity marks the end of such segment is correctly repriced on the curve

After the interpolation we get SOFR IRS curve, Bootstrapped Implied Forwards curve and Discount Factors curve for each date in our sample:

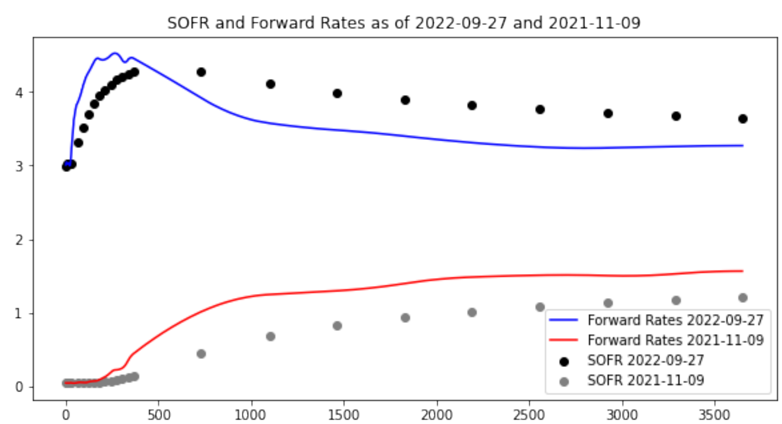

Figure 1: SOFR and Forward Rates as of 2022-09-27 and 2021-11-09

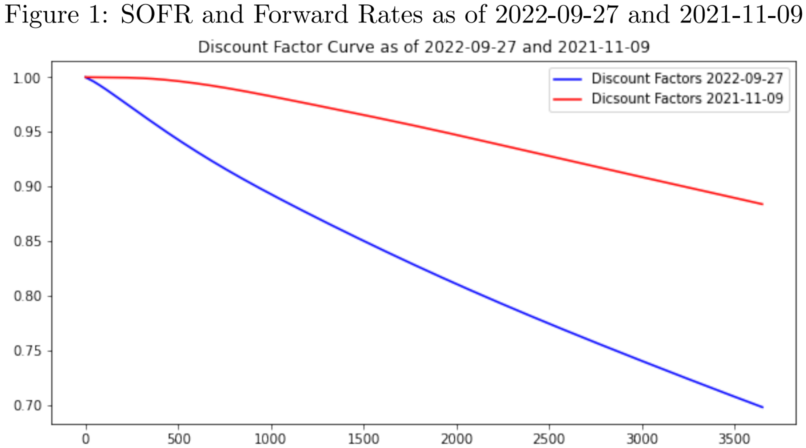

Figure 2: Discount Factor Curve as of 2022-09-27 and 2021-11-09

After getting this data, let's move to the Caplets and Caps pricing.

4.2 Implied Volatility Surface Construction

As the published volatilities correspond to the 3-month compounded overnight rates, we have to construct swaps for each of these values for each expiry from the closest one up to 10 years.
We get quarterly-quarterly swaps (QQ-Swaps) from the Market Overnight Forward Rates in order to imply the ATM Swap Rates:

- We have Compounded 3M Fwd OIS rates, which is used for pricing of Caps and have to annualize it to match with strike

- We use convention that we have 10 swaps, each has quarterly fixings, so we divide every swap rate by 4 and get fixing

Here is the example of the QQ-Swap rates for each Cap expiry:

Tenor	Q-Q Swaps
0	0.042722
1	0.042782
2	0.041187
3	0.039882
4	0.038918
5	0.038167
6	0.037480
7	0.036917
8	0.036491
9	0.036131

In order to get the Cap price as a function of daily Caplets, given the inputs, Cap Vol, Strike, Expiry, Forward Curve and Discount Factors.
As the current market practice for the interest rate options correspond to the negative-rate environment, instead of using the classic Black-Scholes model (1), we use Bachelier model (2):

$$Caplet(F,K,V) = \sqrt{V}\left[\phi\left(\frac{F-K}{\sqrt{V}}\right) + \frac{F-K}{\sqrt{V}}\mathcal{N}\left(\frac{F-K}{\sqrt{V}}\right)\right]$$

Where F, K, V are Forward, Strike, Variance

$$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$$\mathcal{N}(z) = \int_{-\infty}^{z}\phi(x)dx$$

$$Cap(F,K,V,T) = \sum_{t_i=t_1}^{T} Z(t,t_i) \times Caplet(F,K,V)$$

We get the following price surface in basis points:

	-0.25%	-0.50%	0.00%	0.25%	0.50%	0.75%	1.00%	1.50%
1Y	440.393542	464.763601	416.024099	391.654704	367.285403	342.916269	318.547578	269.81651
2Y	864.614434	912.054153	817.225229	769.906015	722.679566	675.568645	628.637817	535.51760
3Y	1235.250985	1303.695751	1167.057207	1099.099454	1031.492291	964.28427	897.578606	766.26357
4Y	1579.153331	1667.167676	1491.632936	1404.561629	1318.15675	1232.490805	1147.735639	981.78911
5Y	1906.976928	2013.520746	1801.137832	1696.017047	1591.861505	1488.759063	1386.947294	1188.3030
6Y	2220.381217	2344.624359	2096.896161	1974.574286	1853.342955	1733.612426	1615.53798	1385.5127
7Y	2514.91544	2656.394655	2374.827926	2235.837766	2098.362023	1962.509213	1828.899739	1569.4332
8Y	2796.608452	2954.622831	2640.025944	2484.911839	2331.571438	2180.407633	2031.618294	1743.4594
9Y	3070.610653	3244.680574	2898.143524	2727.599285	2559.090648	2392.741224	2229.427845	1913.6069
10Y	3336.212096	3525.626508	3148.561533	2962.69433	2779.383088	2598.441626	2420.879191	2077.7756

After comparing it to the ContiCap price surface we get the following differences in basis points:

	-0.25%	-0.50%	0.00%	0.25%	0.50%	0.75%	1.00%	1.50%	2.00%	2.5
1.0	0.393542	-0.236399	0.024099	0.654704	1.285403	1.916269	1.547578	2.816512	3.137087	3.7
2.0	-1.385566	-2.945847	-0.774771	-0.093985	0.679566	1.568645	1.637817	3.517607	4.101043	5.5
3.0	0.250985	-0.304249	1.057207	2.099454	2.492291	3.28427	4.578606	6.263579	6.930614	7.5
4.0	-1.846669	-2.832324	-1.367064	0.561629	1.15675	2.490805	2.735639	4.789114	6.259296	7.7
5.0	-4.023072	-5.479254	-2.862168	-0.982953	-0.138495	0.759063	1.947294	4.303006	6.199906	7.0
6.0	-7.618783	-9.375641	-6.103839	-4.425714	-3.657045	-1.387574	-0.46202	2.512787	4.391043	5.9
7.0	-4.08456	-5.605345	-2.172074	-1.162234	0.362023	2.509213	3.899739	6.433227	7.895021	9.7
8.0	-2.391548	-4.377169	-0.974056	0.911839	2.571438	4.407633	5.618294	8.45944	10.818552	11
9.0	-0.389347	-2.319426	1.143524	3.599285	5.090648	6.741224	8.427845	11.606979	13.019256	14
10.0	-9.787904	-12.373492	-7.438467	-5.30567	-3.616912	-1.558374	0.879191	3.775698	6.342841	8.8

We see that we get sufficiently precise pricing:

- Mean Error is 2.88 bps
- Relative Mean Error is 0.88%

Let's move to the interpolation of the implied volatility surface we got in order to compute the risk-neutral densities.

4.3 Volatility Interpolation

For the interpolation, we choose a step of 25bps for the strike, keeping the maturities the same.
After this, we take each volatility smile for each date and run a 5-degree Spline interpolation: we estimate uniformly divided polynomial $P(z)$ (14):

$$P_i(z) = a_i(z - z_i)^5 + b_i(z - z_i)^4 + c_i(z - z_i)^3 + d_i(z - z_i)^2 + e_i(z - z_i) + f_i,\ i = 0, 1, \ldots, n - 1$$

$$a_i = \frac{1}{120h}(F_{i+1} - F_i)$$

$$b_i = \frac{1}{24}F_i$$

$$c_i = 16h(M_{i+1} - M_i) - h36(F_{i+1} + 2F_i)$$

$$d_i = 12M_i$$

$$e_i = h(S_{i+1} - S_i) - h6(M_{i+1} + 2M_i) + h360(7F_{i+1} + 8F_i)$$

$$f_i = S_i$$

In essence, we do the following procedure:

- Create the polynomial of degree 5
- Construct piecewise version of it
- Join it at the knots
- Match the derivatives

In addition to this, to reduce the model complexity, we reduce the grid of strikes to -1% to 1% with the step equal to 25 basis points.
Let's look at the examples we got:

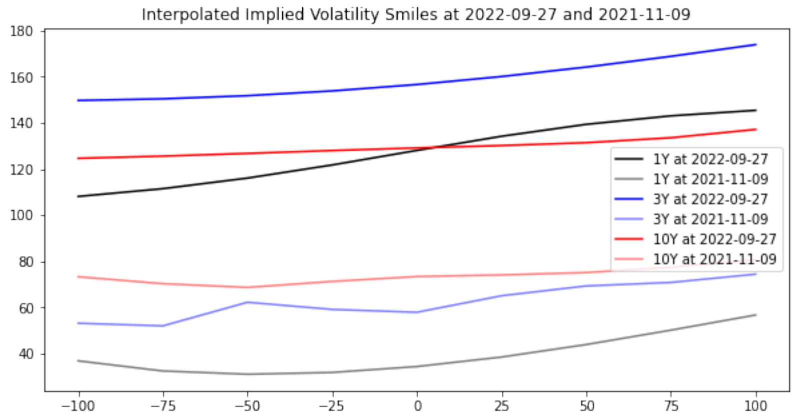

Figure 3: Interpolated Implied Volatility Smiles at 2022-09-27 and 2021-11-09

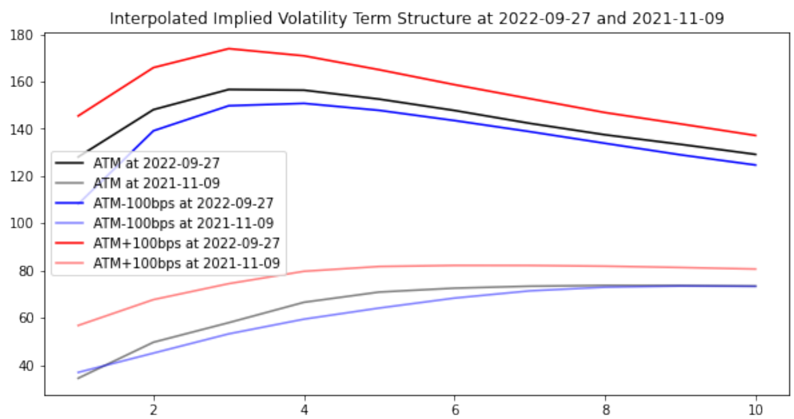

Figure 4: Interpolated Implied Volatility Term Structure at 2022-09-27 and 2021-11-09

In addition to this we constructed the price surface, which will be used for the Breeden-Litezenberger procedure later.
Let's also take a look at the surfaces of Implied Volatilities and Prices:

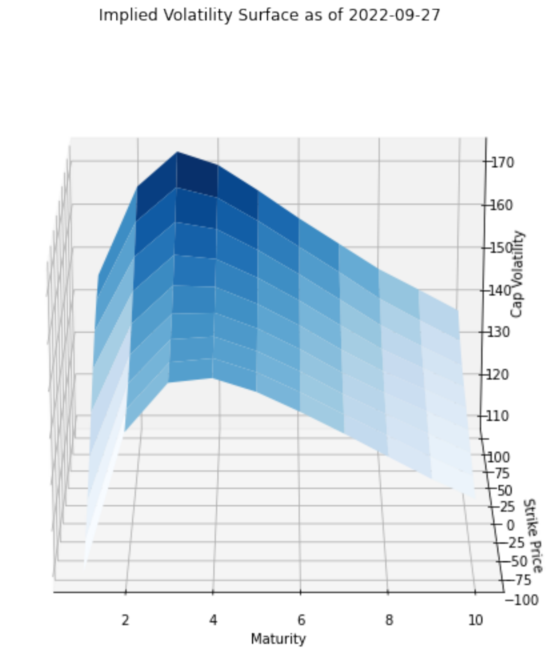

Figure 5: Interpolated Implied Volatility Surface at 2022-09-27

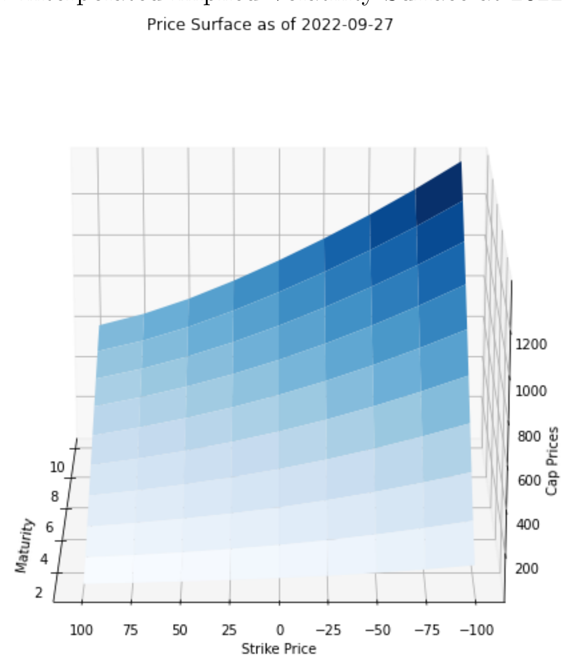

Figure 6: Interpolated Price Surface at 2022-09-27

Now we have everything to move to the risk-neutral probabilities.

4.4 Risk-Neutral Probability Surfaces

We follow the procedure outlined in Breeden, Litzenberger (4) article and numerically solve the equation we got

$$\begin{aligned} p(K,T) &= e^{rT}\frac{\partial^2 C(K,T)}{\partial K^2} \\ &\approx e^{rT}\frac{C(K+\Delta_K,T)-2C(K,T)+C(K-\Delta_K,T)}{(\Delta_K)^2} \end{aligned}$$

The equation we get is the compounded butterfly option spread price. After solving the equation, we get the following probability surface:

	-75	-50	-25	0	25	50	75
1.0	0.194839	0.206928	0.188070	0.149538	0.109216	0.080973	0.070436
2.0	0.161696	0.170652	0.167983	0.155606	0.136807	0.114866	0.092391
3.0	0.148574	0.159660	0.162113	0.156000	0.142966	0.125335	0.105352
4.0	0.140657	0.152449	0.157525	0.155233	0.146329	0.132431	0.115376
5.0	0.136113	0.148253	0.154769	0.154653	0.148185	0.136584	0.121442
6.0	0.133095	0.145362	0.152782	0.154126	0.149364	0.139454	0.125817
7.0	0.124564	0.139451	0.150494	0.155586	0.153742	0.145190	0.130973
8.0	0.111065	0.119783	0.130049	0.142302	0.157039	0.171230	0.168533
9.0	0.118191	0.115549	0.109931	0.113960	0.141094	0.192281	0.208993
10.0	0.119734	0.115331	0.106061	0.106686	0.133748	0.193214	0.225227

We see that each maturity probability sums up to 1.0 and that our strike space was trimmed due to the NaN values for the wings.
If we look at the estimations over time, we see that the change a lot, what depends on the changed market regime, but also is hinting at the fact that we have to use some smoothing techniques in order to get the values correctly:

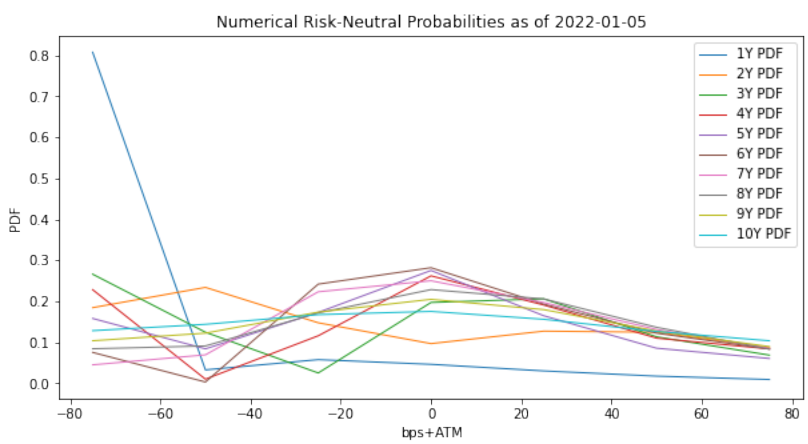

Figure 7: Numerical Risk-Neutral Probabilities as of 2022-01-05

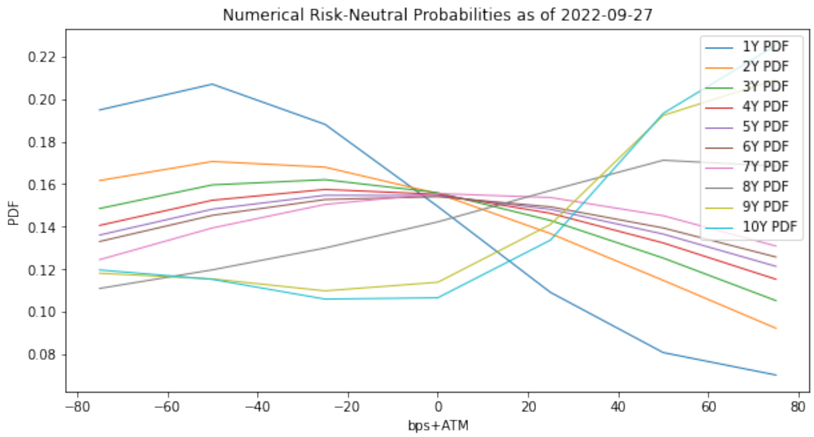

Figure 8: Numerical Risk-Neutral Probabilities as of 2022-09-27

After this step, it is time to estimate the transition matrix.

5 Transition Matrix

In order to estimate the transition matrix, we use are trying to find the matrix, which corresponds to the following equation with respect to our implied risk-neutral probability matrix:

$$P^{t+1} = P^t \mathcal{P}, \ t = 1, \ldots, m-1$$
$$\text{Where our transition matrix is } \mathcal{P}$$

And P^0 is an Arrow-Debreu state price vector with 0 everywhere apart from current, ATM, state.
In order to solve it, we are using not the dynamic, but static solution, using numerical optimization and Sequential Least Squares method for Quadratic Programming (15) with $m^2 - m$ restrictions, ensuring that every strike probability sums up to 1, as we have to get the unimodal rows for the transition matrix.
Let's look at the example of the transition probabilities we get over time:

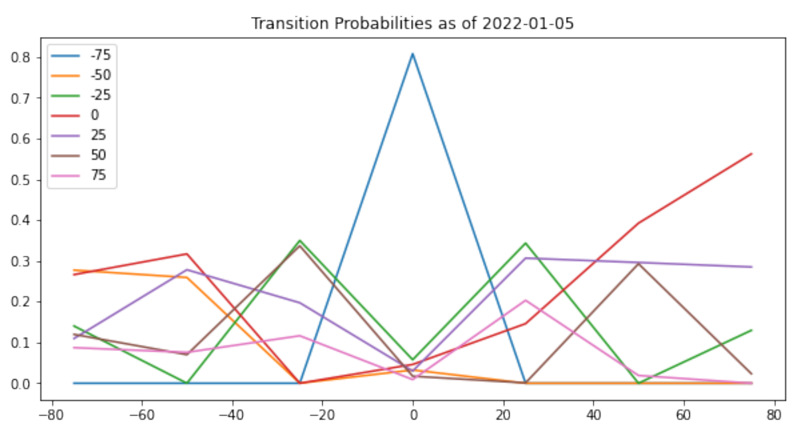

Figure 9: Transition Probabilities as of 2022-01-05

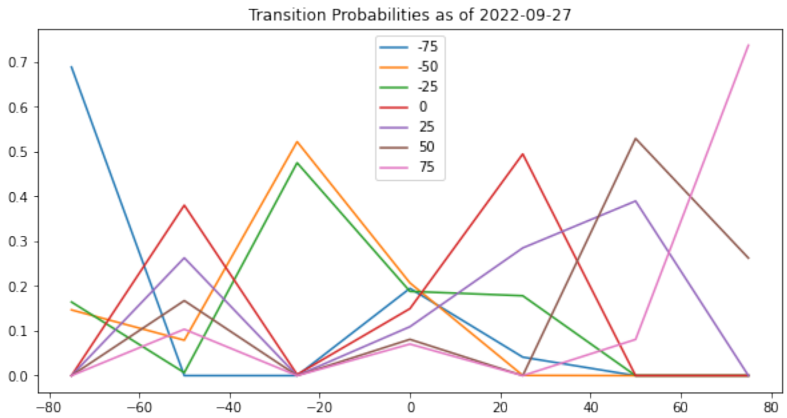

Figure 10: Transition Probabilities as of 2022-09-27

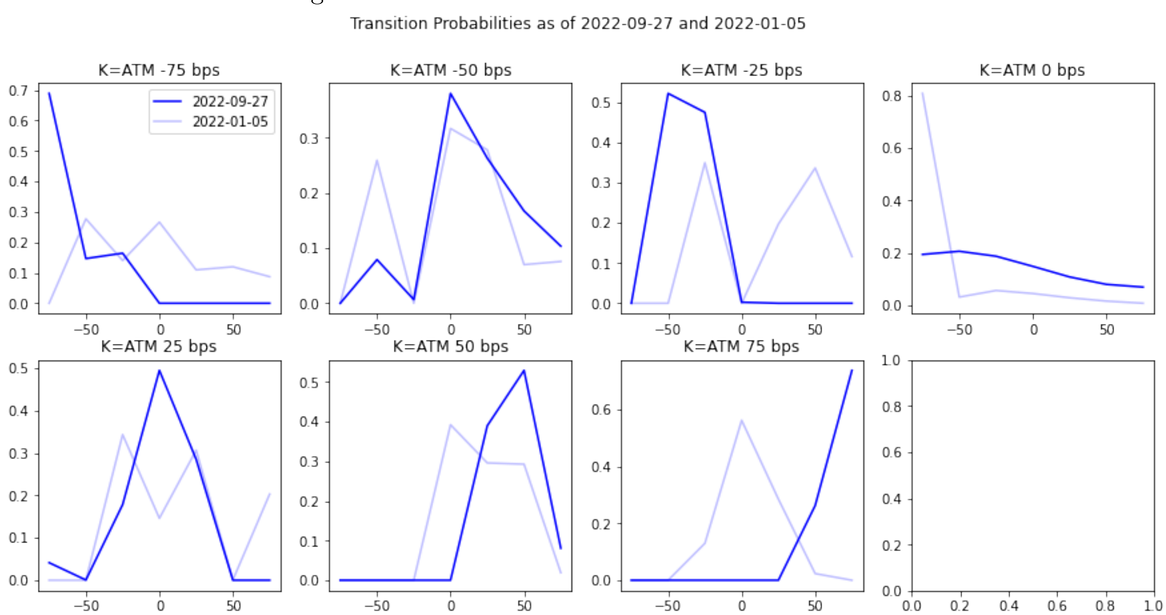

Figure 11: Transition Probabilities as of 2022-09-27 and 2022-01-05

We see that transition probabilities also reflect the change in the market regime as well as some inherent numerical instability, coming from the risk-neutral probability density function estimation and the transition probabilities itself.
Moreover, we see that:

- Probability to go further up or down from the wings, -75bps from ATM and +75bps to ATM, are significantly higher, as before the matrix indicated mean-reversion to the ATM. This behavior correspond to the current market expectations
- If rates fall 50bps from ATM, they will likely retrace back
- If we have a rise by 25bps, we likely stay at this level
- If there is jump by 50bps, we go further up Nevertheless, we see that some points are inconsistent, which is the result of the numerical estimation.

6 Recovered Probabilities

Following the previous step, we are ready to estimate the recovered probabilities. According to Ross (3), we can estimate the pricing kernel and the recovered probabilities from the transition probabilities:

$$p_{i,j} = \frac{\mathcal{P}_{i,j}}{m_{i,j}} = \frac{1}{\delta} \times \frac{\mathcal{P}_{ij} \times u^{'}_{i}}{u^{'}_{j}}$$

$$p_{i,j} \text{ is a Recovered Probability}$$

$$m_{i,j} \text{ is a Pricing Kernel}$$

$$\mathcal{P}_{i,j} \text{ is a Transition Matrix}$$

$$\delta \text{ is a Risk-Neutral Discount Factor}$$

$$u^{'}_{i} \text{ is a Marginal Utility}$$

We can apply the Perron-Frobenius theorem and can solve the problem as an N-dimensional eigenvalue problem

$$\Pi z = \delta z,\ z_i = \frac{1}{u^{'}_{i}}$$

$$\delta \text{is the largest eigenvalue}$$

$$z_i \text{ is a corresponding eigenvector}$$

Let's look at the values we got:

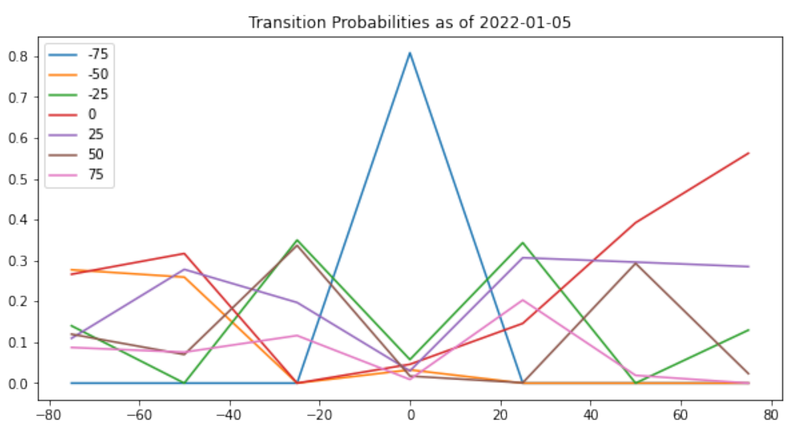

Figure 12: Transition Probabilities as of 2022-01-05

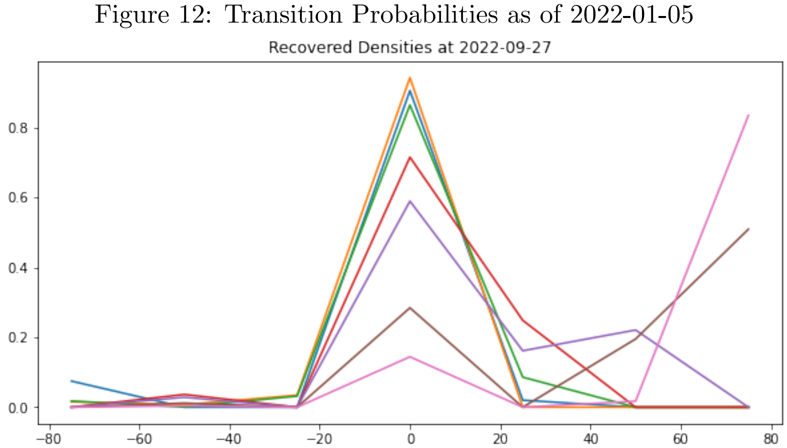

Figure 13: Recovered Probabilities as of 2022-01-05

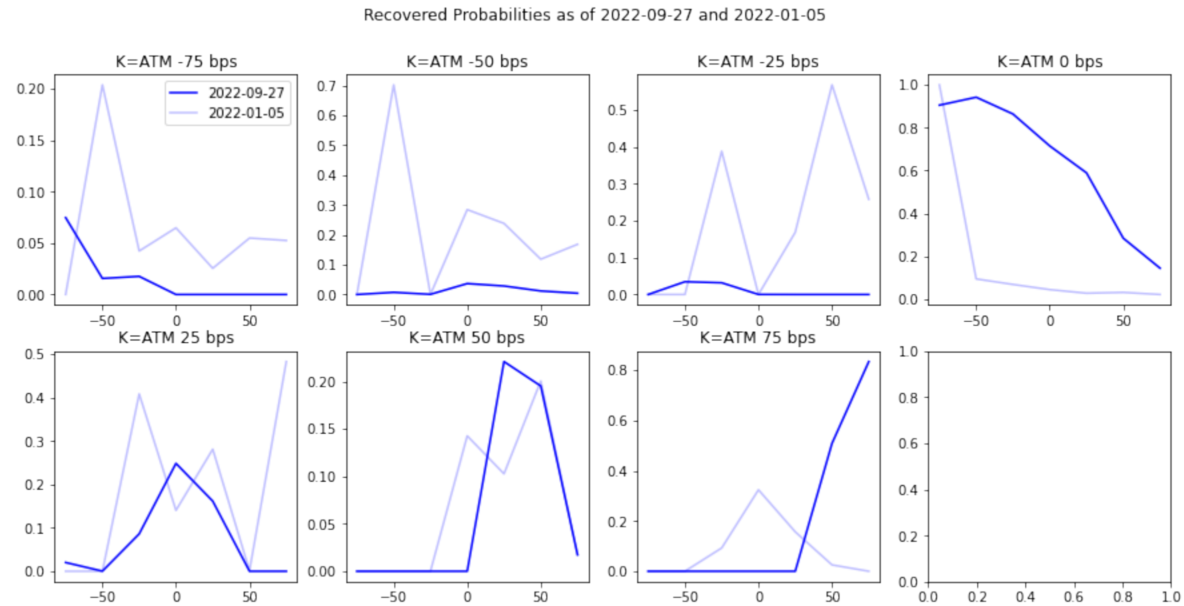

Figure 14: Recovered Probabilities as of 2022-09-27 and 2022-01-05

We see similar behavior to the transition probabilities:

- Probability to go further up or down from the wings, -75bps from ATM and +75bps to ATM, are significantly higher, as before the matrix indicated mean-reversion to the ATM. This behavior correspond to the current market expectations
- If rates fall 50bps or 25bps from ATM, they will likely remain there
- If rates rise by 25 or 50 basis points, there is room for further hikes

Again, we see that some points are inconsistent, which is the result of the numerical estimation. Let's check the forecastability of the data we got.

7 Signal Generation

Let's also create time-series and look at the data in order to make a signal:

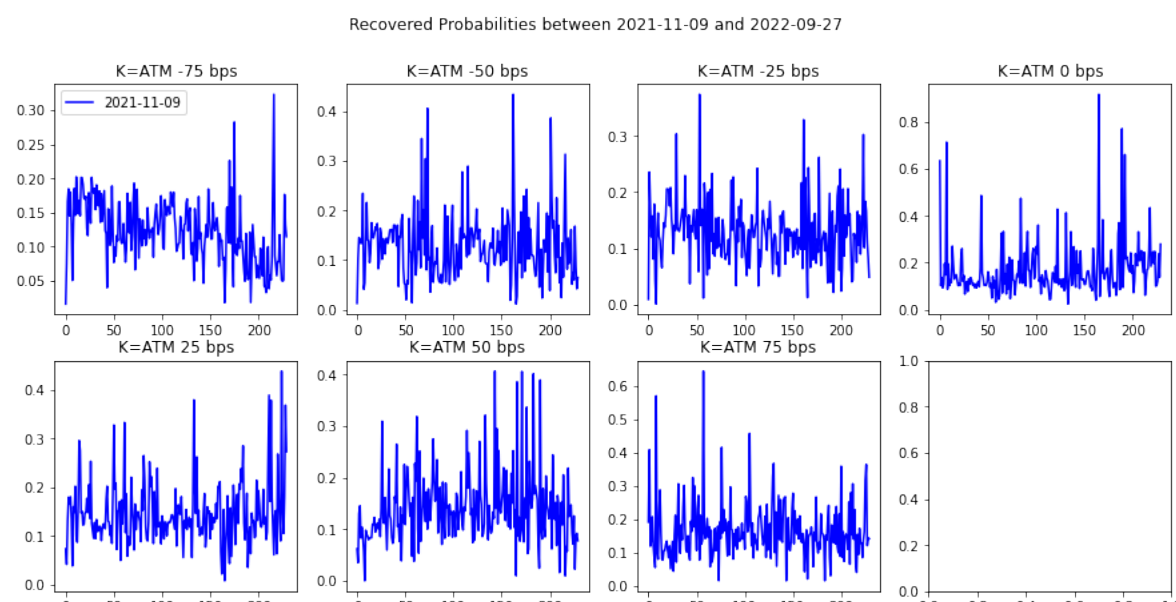

Figure 15: Recovered Probabilities over time

We see that we get very noisy data for the returns. Therefore, it makes sense for us to use some smoothing techniques.

8 Kalman Filter

In order to smooth our values, we are using Kalman Filter as a state-space estimator. The mechanics that is used is that we:

- Mark the unobserved space
- Model the inertion in our underlyings
- Define the time grid for the changes
- Find the values which contribute excessively to the estimated covariance matrix and mask them
- Construct the smoothed values using the masked data

In mathematical terms, we follow the approach of Welch and Bishop (16):

$$
\begin{aligned}
x_k &= F_k x_{k-1} + B_k u_k + w_k \\
F_k &\text{ is a State Transition Model} \\
B_k &\text{ is a Control-Input model} \\
w_k &\text{ is a Process Noise} \\
z_k &= H_k x_k + v_k \\
H_k &\text{ is the Observation model} \\
v_k &\text{ is an Observation Noise}
\end{aligned}
$$

We get the following estimations. Indeed, we see that the extreme values were removed by Kalman Filter:

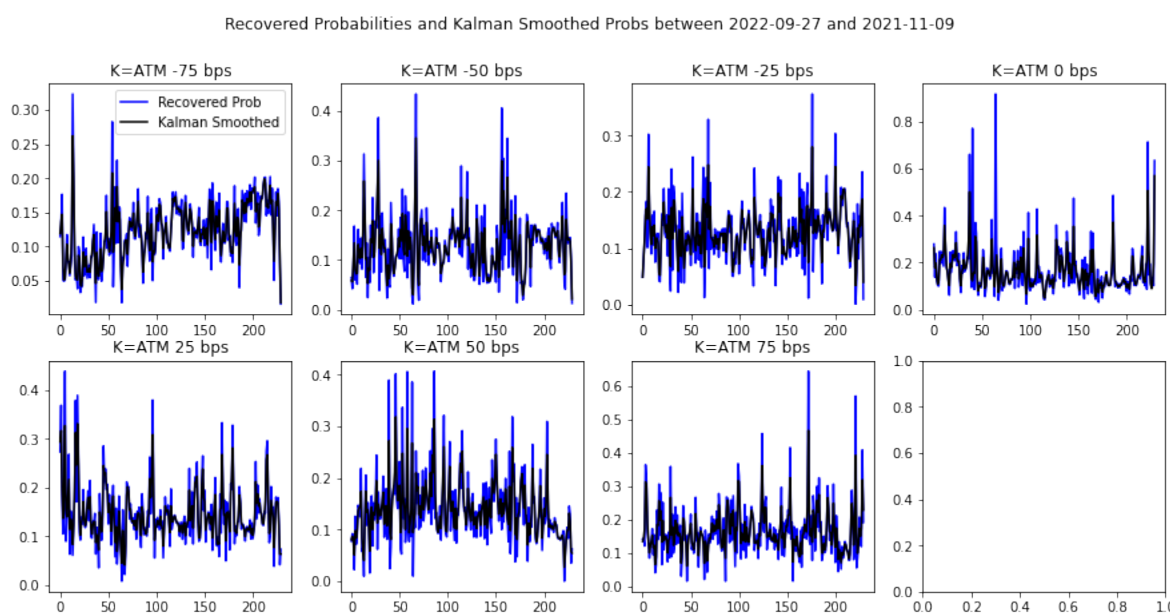

Figure 16: Recovered Probabilities and Kalman Smoother Probabilities over time

8.1 Test for Forward Returns

In order to estimate the forecasting power of our varaibles, let's check the simple OLS model based on the mean swap quotes shifted 30 business days forward. As the exogeneous variable we choose our smoothed recovered probabilities: We get that strikes ATM-50bps, ATM and ATM+25bps are insignificant, while the oterh values have very high t-statistics:

Dep. Variable:	mean_swap_fwd	**R-squared (uncentered):**	0.941
Model:	OLS	**Adj. R-squared (uncentered):**	0.940
Method:	Least Squares	**F-statistic:**	778.7
Date:	Mon, 17 Oct 2022	**Prob (F-statistic):**	4.55e-119
Time:	23:44:26	**Log-Likelihood:**	-183.12
No. Observations:	200	**AIC:**	374.2
Df Residuals:	196	**BIC:**	387.4
Df Model:	4		

	coef	**std err**	**t**	**P> |t|**	**[0.025**	**0.975]**
-75_K	10.0543	1.091	9.218	0.000	7.903	12.205
-25_K	2.0266	1.002	2.023	0.044	0.051	4.002
50_K	2.7243	0.747	3.647	0.000	1.251	4.197
75_K	3.2489	0.634	5.125	0.000	1.999	4.499

Omnibus:	4.701	**Durbin-Watson:**	0.307
Prob(Omnibus):	0.095	**Jarque-Bera (JB):**	4.470
Skew:	-0.364	**Prob(JB):**	0.107
Kurtosis:	3.085	**Cond. No.**	8.40

Notes:
[1] R^2 is computed without centering (uncentered) since the model does not contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly specified.
We run uncentered regression, as otherwise we would get multicollinearity issue, as the probabilities sum up to 1.
We see that there is no evidence of multicollinearity. Moreover, data shows close-to-normal Skew and Kurtosis. We get that all the variables have positive relationship with forward swap value. Let's see the plots:

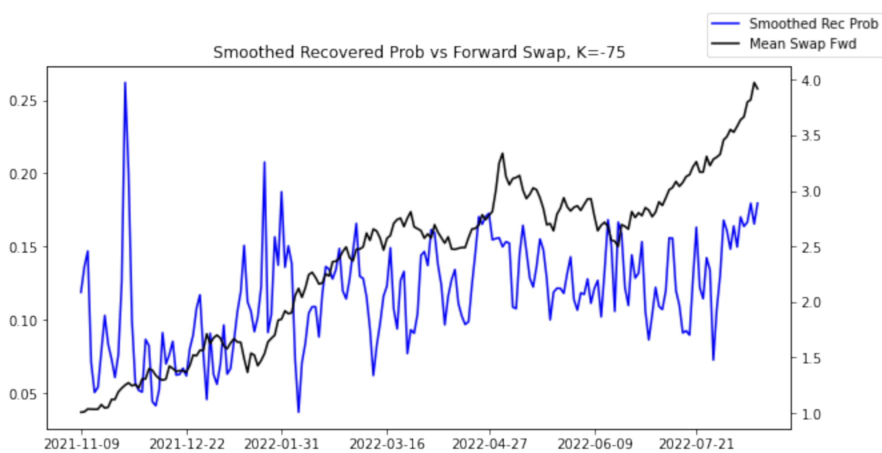

Figure 17: Smoothed Recovered Probability vs Mean Forward Swap, K=-75

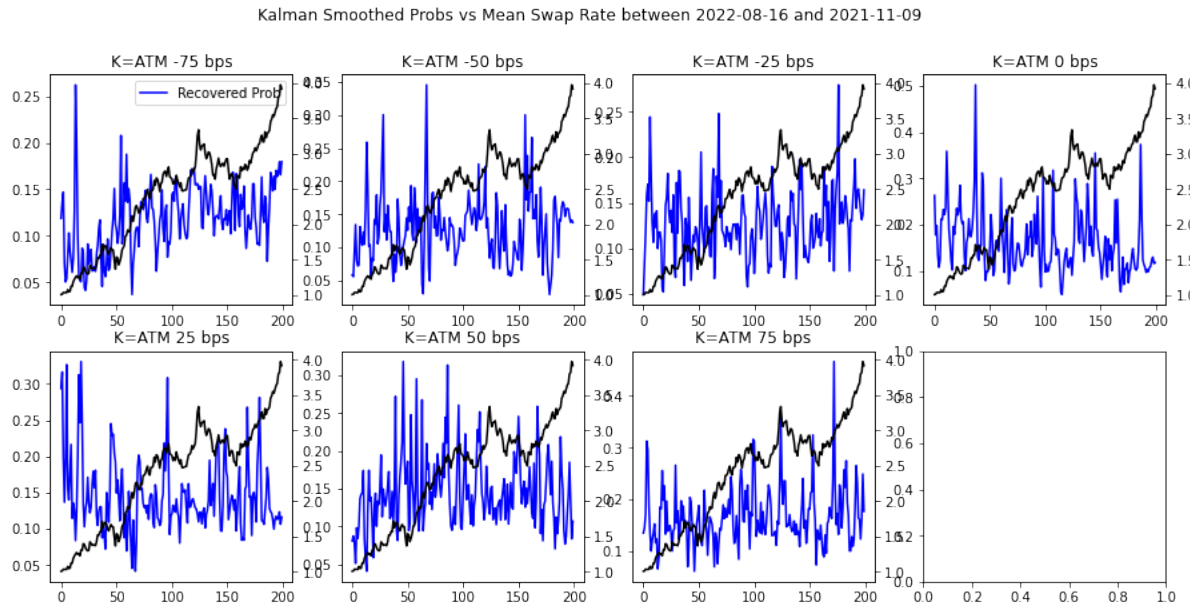

Figure 18: Smoothed Recovered Probability vs Mean Forward Swap

Let's also check the marginal regressions for all the strikes: we get that all the variables are significant, while together we see that some of them are redundant

Dep. Variable:	mean_swap_fwd	**R-squared (uncentered):**	0.914
Model:	OLS	**Adj. R-squared (uncentered):**	0.913
Method:	Least Squares	**F-statistic:**	2110.
Date:	Mon, 17 Oct 2022	**Prob (F-statistic):**	6.90e-108
Time:	23:45:15	**Log-Likelihood:**	-220.65
No. Observations:	200	**AIC:**	443.3
Df Residuals:	199	**BIC:**	446.6
Df Model:	1		

	coef	**std err**	**t**	**P>|t|**	**[0.025**	**0.975]**
-75_K	19.3090	0.420	45.939	0.000	18.480	20.138

Omnibus:	71.679	**Durbin-Watson:**	0.596
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	267.593
Skew:	-1.401	**Prob(JB):**	7.82e-59
Kurtosis:	7.926	**Cond. No.**	1.00

Notes:
[1] R^2 is computed without centering (uncentered) since the model does not contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Dep. Variable:	mean_swap_fwd	**R-squared (uncentered):**	0.800
Model:	OLS	**Adj. R-squared (uncentered):**	0.799
Method:	Least Squares	**F-statistic:**	796.9
Date:	Mon, 17 Oct 2022	**Prob (F-statistic):**	1.63e-71
Time:	23:45:15	**Log-Likelihood:**	-304.76
No. Observations:	200	**AIC:**	611.5
Df Residuals:	199	**BIC:**	614.8
Df Model:	1		

	coef	**std err**	**t**	**P>|t|**	**[0.025**	**0.975]**
-50_K	16.1228	0.571	28.230	0.000	14.997	17.249

Omnibus:	13.734	**Durbin-Watson:**	0.485
Prob(Omnibus):	0.001	**Jarque-Bera (JB):**	15.089
Skew:	-0.565	**Prob(JB):**	0.000529
Kurtosis:	3.730	**Cond. No.**	1.00

Notes:
[1] R^2 is computed without centering (uncentered) since the model does not contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Dep. Variable:	mean_swap_fwd	**R-squared (uncentered):**	0.859
Model:	OLS	**Adj. R-squared (uncentered):**	0.858
Method:	Least Squares	**F-statistic:**	1213.
Date:	Mon, 17 Oct 2022	**Prob (F-statistic):**	1.34e-86
Time:	23:45:15	**Log-Likelihood:**	-269.88
No. Observations:	200	**AIC:**	541.8
Df Residuals:	199	**BIC:**	545.1
Df Model:	1		

	coef	**std err**	**t**	**P> |t|**	**[0.025**	**0.975]**
-25_K	17.3877	0.499	34.822	0.000	16.403	18.372

Omnibus:	12.693	**Durbin-Watson:**	0.555
Prob(Omnibus):	0.002	**Jarque-Bera (JB):**	13.332
Skew:	-0.619	**Prob(JB):**	0.00127
Kurtosis:	3.256	**Cond. No.**	1.00

Notes:
[1] R^2 is computed without centering (uncentered) since the model does not contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Dep. Variable:	mean_swap_fwd	**R-squared (uncentered):**	0.729
Model:	OLS	**Adj. R-squared (uncentered):**	0.727
Method:	Least Squares	**F-statistic:**	534.0
Date:	Mon, 17 Oct 2022	**Prob (F-statistic):**	3.02e-58
Time:	23:45:15	**Log-Likelihood:**	-335.42
No. Observations:	200	**AIC:**	672.8
Df Residuals:	199	**BIC:**	676.1
Df Model:	1		

	coef	**std err**	**t**	**P> |t|**	**[0.025**	**0.975]**
0_K	12.1335	0.525	23.108	0.000	11.098	13.169

Omnibus:	13.730	**Durbin-Watson:**	0.349
Prob(Omnibus):	0.001	**Jarque-Bera (JB):**	14.574
Skew:	-0.644	**Prob(JB):**	0.000684
Kurtosis:	3.298	**Cond. No.**	1.00

Notes:
[1] R^2 is computed without centering (uncentered) since the model does not contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Dep. Variable:	mean_swap_fwd	**R-squared (uncentered):**	0.778
Model:	OLS	**Adj. R-squared (uncentered):**	0.777
Method:	Least Squares	**F-statistic:**	699.3
Date:	Mon, 17 Oct 2022	**Prob (F-statistic):**	4.74e-67
Time:	23:45:15	**Log-Likelihood:**	-315.08
No. Observations:	200	**AIC:**	632.2
Df Residuals:	199	**BIC:**	635.5
Df Model:	1		

	coef	std err	t	P> abs(t)	[0.025	0.975]
25_K	14.2200	0.538	26.444	0.000	13.160	15.280

Omnibus:	32.906	**Durbin-Watson:**	0.293
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	45.773
Skew:	-0.985	**Prob(JB):**	1.15e-10
Kurtosis:	4.271	**Cond. No.**	1.00

Notes:
[1] R^2 is computed without centering (uncentered) since the model does not contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Dep. Variable:	mean_swap_fwd	**R-squared (uncentered):**	0.838
Model:	OLS	**Adj. R-squared (uncentered):**	0.837
Method:	Least Squares	**F-statistic:**	1031.
Date:	Mon, 17 Oct 2022	**Prob (F-statistic):**	1.20e-80
Time:	23:45:15	**Log-Likelihood:**	-283.65
No. Observations:	200	**AIC:**	569.3
Df Residuals:	199	**BIC:**	572.6
Df Model:	1		

	coef	std err	t	P> abs(t)	[0.025	0.975]
50_K	14.8860	0.464	32.109	0.000	13.972	15.800

Omnibus:	3.888	**Durbin-Watson:**	0.571
Prob(Omnibus):	0.143	**Jarque-Bera (JB):**	3.474
Skew:	-0.278	**Prob(JB):**	0.176
Kurtosis:	3.330	**Cond. No.**	1.00

Notes:
[1] R^2 is computed without centering (uncentered) since the model does not contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Dep. Variable:	mean_swap_fwd	**R-squared (uncentered):**	0.852
Model:	OLS	**Adj. R-squared (uncentered):**	0.852
Method:	Least Squares	**F-statistic:**	1149.
Date:	Mon, 17 Oct 2022	**Prob (F-statistic):**	1.36e-84
Time:	23:45:15	**Log-Likelihood:**	-274.53
No. Observations:	200	**AIC:**	551.1
Df Residuals:	199	**BIC:**	554.3
Df Model:	1		

	coef	std err	t	P> abs(t)	[0.025	0.975]
75_K	13.0966	0.386	33.890	0.000	12.335	13.859

Omnibus:	18.923	**Durbin-Watson:**	0.601
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	22.703
Skew:	-0.675	**Prob(JB):**	1.17e-05
Kurtosis:	3.951	**Cond. No.**	1.00

Notes:
[1] R^2 is computed without centering (uncentered) since the model does not contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly specified.
Given the reasonable theoretical justification let's construct the trading strategy, which works the following way:

- If the expected value of the rate is higher than 0, we long the agregate swap curve - Pay Fixed (bet on swap rate rise)
- If less than 0, we short - Receive Fixed (bet on swap rate fall)

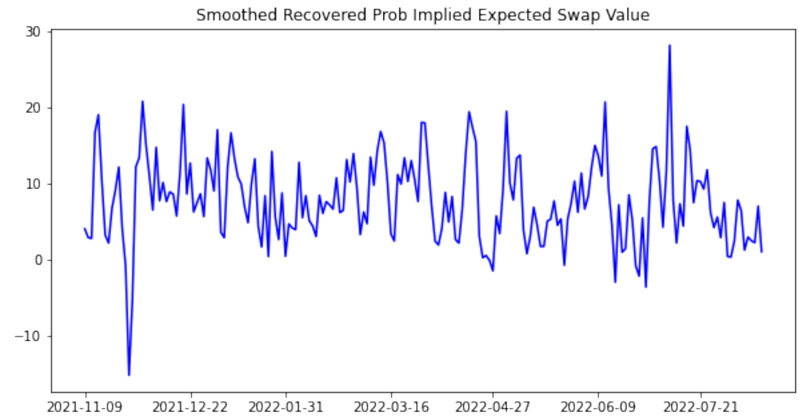

Figure 19: Smoothed Recovered Probability Implied Expected Swap Value

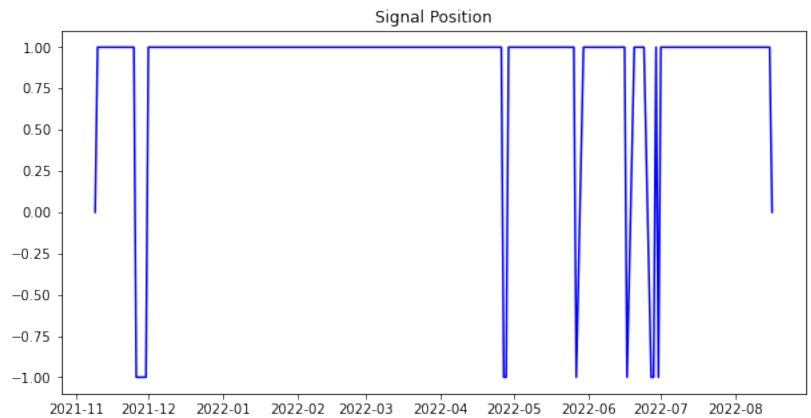

Figure 20: Signal Posiiton

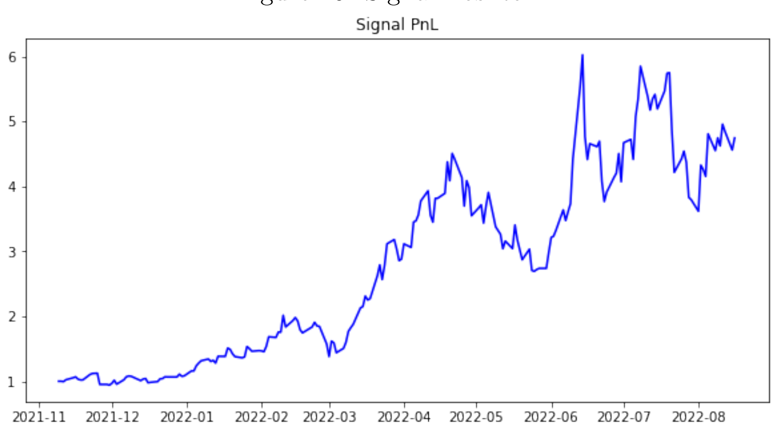

Figure 21: Signal PnL

We get the signal that in the current market environment, during the last year, most of the time we had to keep the long position in the aggregated swap curve. We see that it produced significant PnL, while the volatility of it was very high.
The interesting thing to look into is the prediction for the negative signals:

	pnl	mean_swap	position	exp_val
2021-11-19	-0.00011	0.99572	1	6.753158
2021-11-22	0.08289	1.07861	1	9.105183
2021-11-23	0.01356	1.09217	1	12.140045
2021-11-24	0.00212	1.09429	1	4.121661
2021-11-25	0.00173	1.09602	1	-0.858330
2021-11-26	-0.15358	0.94244	-1	-15.183108
2021-11-29	-0.00012	0.94256	-1	-5.341180
2021-11-30	-0.01156	0.95412	-1	12.172705
2021-12-01	0.02790	0.92622	1	13.365422
2021-12-02	0.05038	0.97660	1	20.784968
2021-12-03	-0.05937	0.91723	1	15.033549

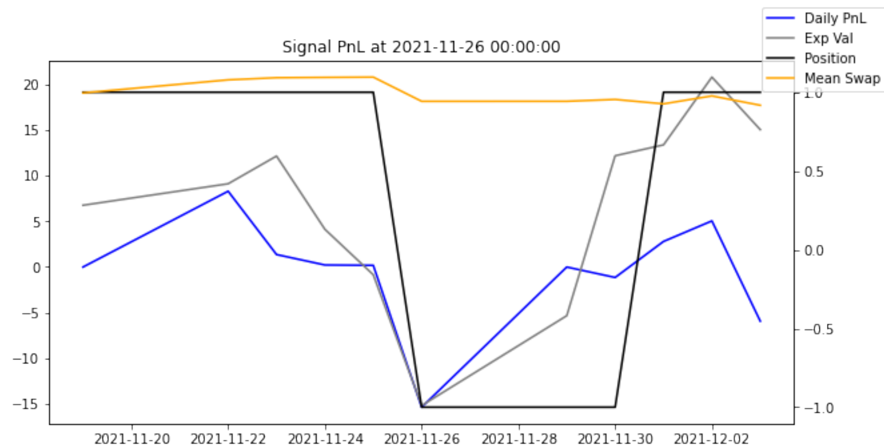

Figure 22: PnL at 2021-11-26

	pnl	mean_swap	position	exp_val
2021-11-22	0.08289	1.07861	1	9.105183
2021-11-23	0.01356	1.09217	1	12.140045
2021-11-24	0.00212	1.09429	1	4.121661
2021-11-25	0.00173	1.09602	1	-0.858330
2021-11-26	-0.15358	0.94244	-1	-15.183108
2021-11-29	-0.00012	0.94256	-1	-5.341180
2021-11-30	-0.01156	0.95412	-1	12.172705
2021-12-01	0.02790	0.92622	1	13.365422
2021-12-02	0.05038	0.97660	1	20.784968
2021-12-03	-0.05937	0.91723	1	15.033549
2021-12-06	0.07060	0.98783	1	10.932793

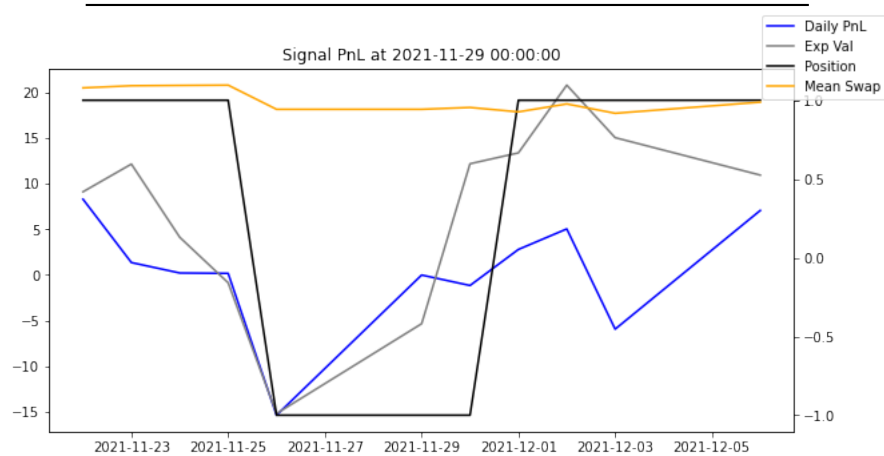

Figure 23: PnL at 2021-11-29

	pnl	mean_swap	position	exp_val
2021-11-23	0.01356	1.09217	1	12.140045
2021-11-24	0.00212	1.09429	1	4.121661
2021-11-25	0.00173	1.09602	1	-0.858330
2021-11-26	-0.15358	0.94244	-1	-15.183108
2021-11-29	-0.00012	0.94256	-1	-5.341180
2021-11-30	-0.01156	0.95412	-1	12.172705
2021-12-01	0.02790	0.92622	1	13.365422
2021-12-02	0.05038	0.97660	1	20.784968
2021-12-03	-0.05937	0.91723	1	15.033549
2021-12-06	0.07060	0.98783	1	10.932793
2021-12-07	0.04485	1.03268	1	6.496641

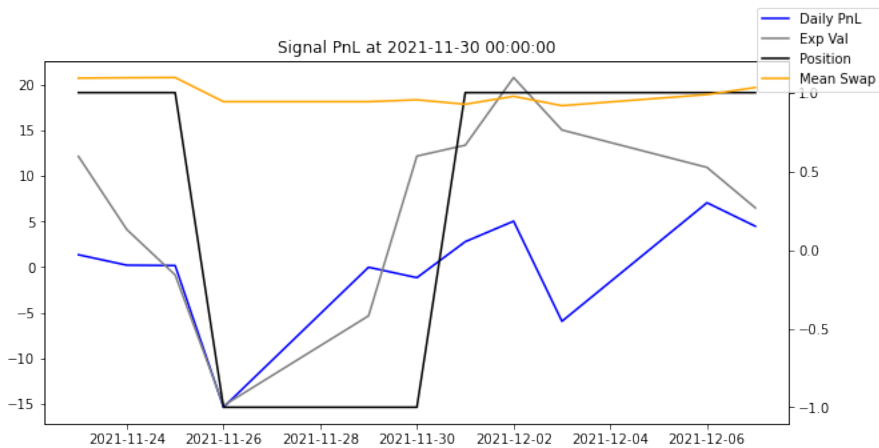

Figure 24: PnL at 2021-11-30

	pnl	mean_swap	position	exp_val
2022-04-20	-0.06622	2.55440	1	15.504003
2022-04-21	0.10267	2.65707	1	3.076724
2022-04-22	-0.01885	2.63822	1	0.256691
2022-04-25	-0.06453	2.57369	1	0.528700
2022-04-26	-0.10630	2.46739	1	-0.090780
2022-04-27	0.10570	2.57309	-1	-1.477216
2022-04-28	-0.02567	2.59876	-1	5.738454
2022-04-29	-0.10910	2.70786	1	3.408595
2022-05-02	0.03445	2.74231	1	9.293731
2022-05-03	0.01253	2.75484	1	19.462155
2022-05-04	-0.07567	2.67917	1	10.057492

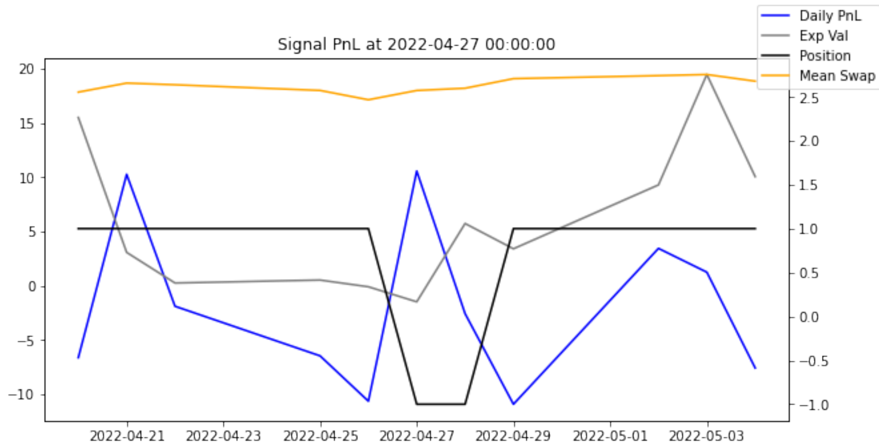

Figure 25: PnL at 2022-04-27

	pnl	mean_swap	position	exp_val
2022-04-21	0.10267	2.65707	1	3.076724
2022-04-22	-0.01885	2.63822	1	0.256691
2022-04-25	-0.06453	2.57369	1	0.528700
2022-04-26	-0.10630	2.46739	1	-0.090780
2022-04-27	0.10570	2.57309	-1	-1.477216
2022-04-28	-0.02567	2.59876	-1	5.738454
2022-04-29	-0.10910	2.70786	1	3.408595
2022-05-02	0.03445	2.74231	1	9.293731
2022-05-03	0.01253	2.75484	1	19.462155
2022-05-04	-0.07567	2.67917	1	10.057492
2022-05-05	0.07516	2.75433	1	7.832490

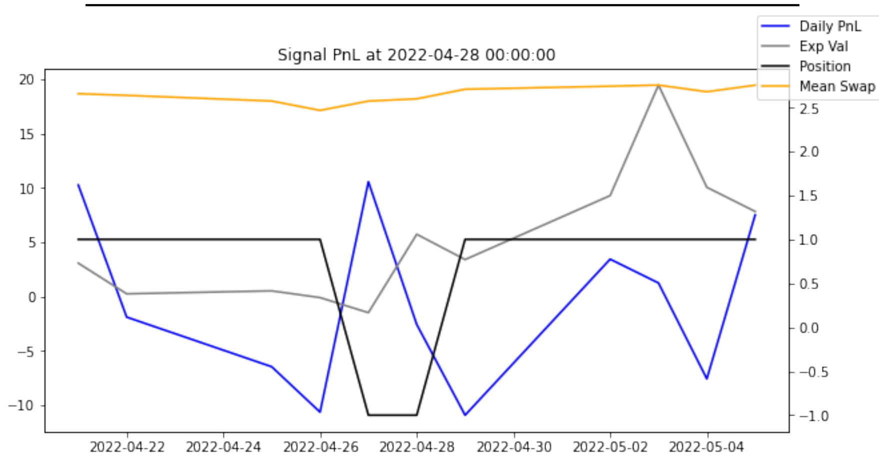

Figure 26: PnL at 2022-04-28

	pnl	mean_swap	position	exp_val
2022-05-20	-0.05009	2.52983	1	5.307967
2022-05-23	0.05648	2.58631	1	7.695054
2022-05-24	-0.10800	2.47831	1	4.499782
2022-05-25	-0.00485	2.47346	1	5.337023
2022-05-26	0.01000	2.48346	1	-0.746280
2022-05-27	0.00697	2.49043	-1	5.347120
2022-05-30	-0.00002	2.49045	1	7.398284
2022-05-31	0.08958	2.58003	1	10.295527
2022-06-01	0.07678	2.65681	1	6.198352
2022-06-02	0.00658	2.66339	1	11.352957
2022-06-03	0.02848	2.69187	1	6.640405

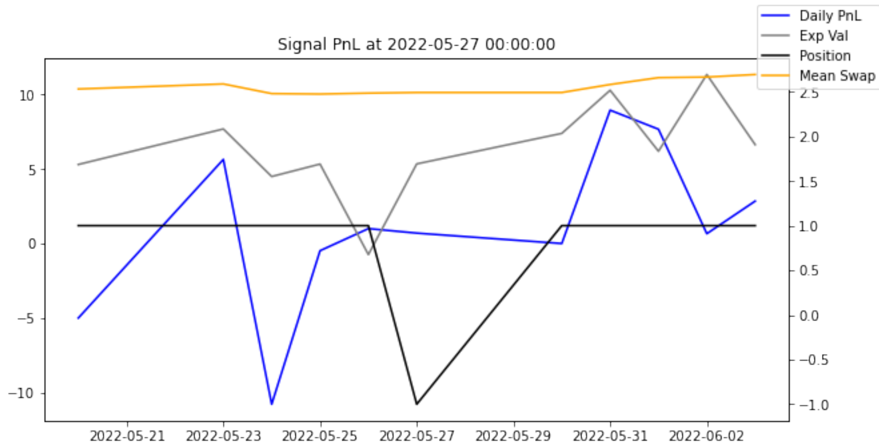

Figure 27: PnL at 2022-11-26

	pnl	mean_swap	position	exp_val
2022-06-10	0.18549	2.99937	1	10.974319
2022-06-13	0.25048	3.24985	1	20.671028
2022-06-14	0.08884	3.33869	1	9.349077
2022-06-15	-0.20910	3.12959	1	4.530647
2022-06-16	-0.07394	3.05565	1	-2.984454
2022-06-17	0.05515	3.11080	-1	7.185208
2022-06-20	-0.00974	3.12054	1	0.960761
2022-06-21	0.01802	3.13856	1	1.442416
2022-06-22	-0.13145	3.00711	1	8.491379
2022-06-23	-0.07606	2.93105	1	5.192505
2022-06-24	0.03810	2.96915	1	-0.822617

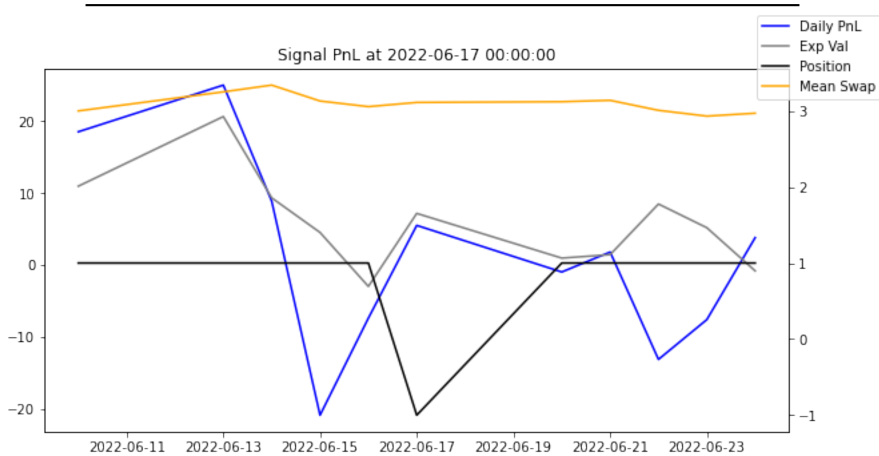

Figure 22: PnL at 2022-05-27

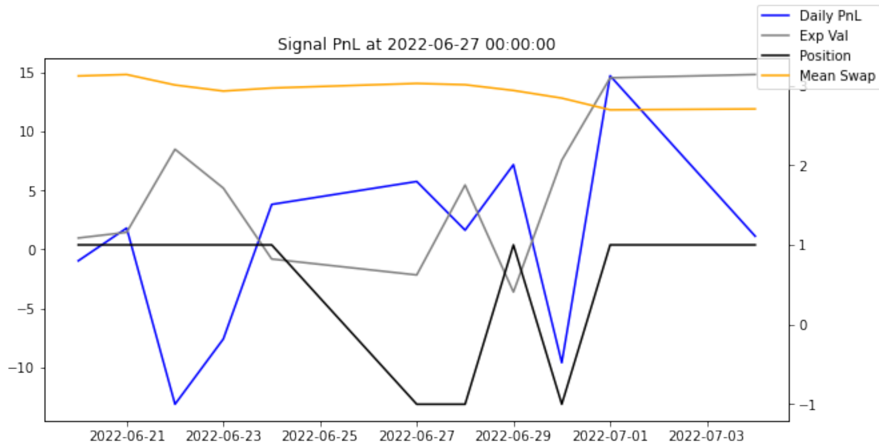

Figure 28: PnL at 2022-06-17

	pnl	mean_swap	position	exp_val
2022-06-20	-0.00974	3.12054	1	0.960761
2022-06-21	0.01802	3.13856	1	1.442416
2022-06-22	-0.13145	3.00711	1	8.491379
2022-06-23	-0.07606	2.93105	1	5.192505
2022-06-24	0.03810	2.96915	1	-0.822617
2022-06-27	0.05757	3.02672	-1	-2.175269
2022-06-28	0.01626	3.01046	-1	5.455327
2022-06-29	0.07183	2.93863	1	-3.621882
2022-06-30	-0.09621	2.84242	-1	7.560011
2022-07-01	0.14724	2.69518	1	14.550637
2022-07-04	0.01115	2.70633	1	14.832692

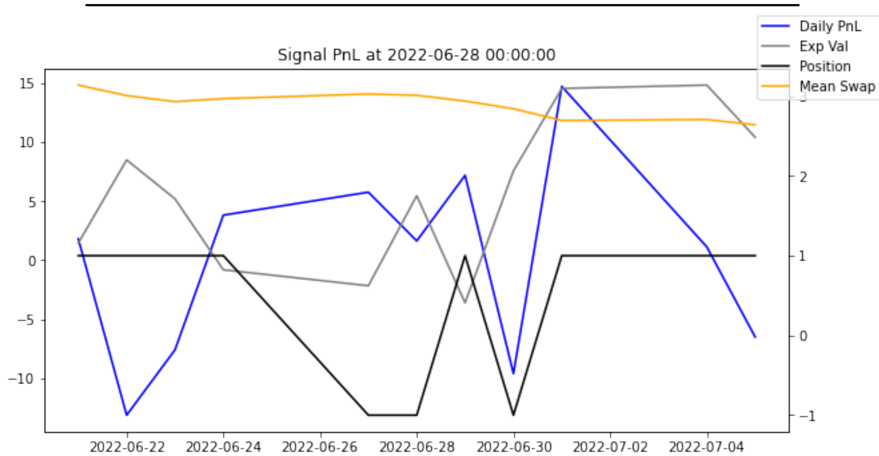

Figure 22: PnL at 2022-06-27

	pnl	mean_swap	position	exp_val
2022-06-21	0.01802	3.13856	1	1.442416
2022-06-22	-0.13145	3.00711	1	8.491379
2022-06-23	-0.07606	2.93105	1	5.192505
2022-06-24	0.03810	2.96915	1	-0.822617
2022-06-27	0.05757	3.02672	-1	-2.175269
2022-06-28	0.01626	3.01046	-1	5.455327
2022-06-29	0.07183	2.93863	1	-3.621882
2022-06-30	-0.09621	2.84242	-1	7.560011
2022-07-01	0.14724	2.69518	1	14.550637
2022-07-04	0.01115	2.70633	1	14.832692
2022-07-05	-0.06518	2.64115	1	10.412682

	pnl	mean_swap	position	exp_val
2022-06-23	-0.07606	2.93105	1	5.192505
2022-06-24	0.03810	2.96915	1	-0.822617
2022-06-27	0.05757	3.02672	-1	-2.175269
2022-06-28	0.01626	3.01046	-1	5.455327
2022-06-29	0.07183	2.93863	1	-3.621882
2022-06-30	-0.09621	2.84242	-1	7.560011
2022-07-01	0.14724	2.69518	1	14.550637
2022-07-04	0.01115	2.70633	1	14.832692
2022-07-05	-0.06518	2.64115	1	10.412682
2022-07-06	0.14873	2.78988	1	4.242454
2022-07-07	0.05380	2.84368	1	11.535816

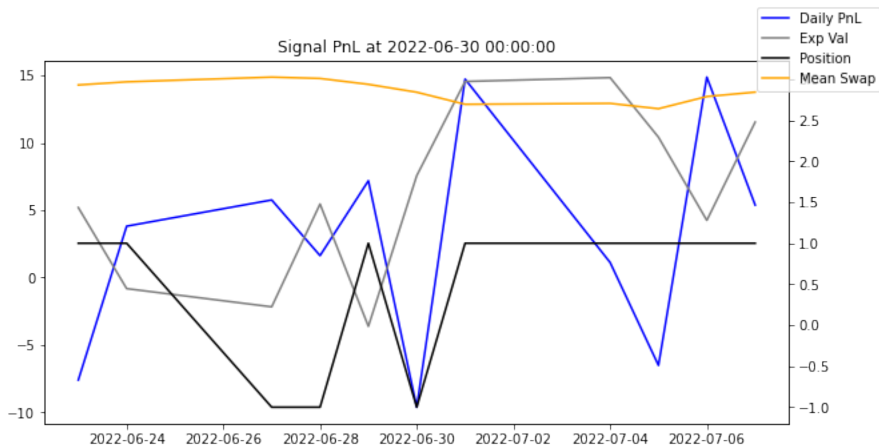

Figure 27: PnL at 2022-06-30

Despite the decent predictive power, our estimated PnL during the short positions is -0.45%. We see that the signal has a significant delay, what is caused by the trading strategy we use: to avoid data leakage, we put the trades the next day we saw the value. In reality, we can do this faster.

9 Results

Let's estimate monthly daily mean, standard deviation and sharpe of the PnL we get:

Mean PnL is

	pnl
2021-11-30	-0.002740
2021-12-31	0.006704
2022-01-31	0.015692
2022-02-28	0.005692
2022-03-31	0.029600
2022-04-30	0.012730
2022-05-31	-0.005812
2022-06-30	0.019050
2022-07-31	-0.000034
2022-08-31	0.021780

Standard Deviation is

	pnl
2021-11-30	0.047472
2021-12-31	0.033878
2022-01-31	0.045870
2022-02-28	0.066987
2022-03-31	0.079522
2022-04-30	0.076585
2022-05-31	0.064908
2022-06-30	0.100584
2022-07-31	0.082886
2022-08-31	0.083621

Share Ratio is

	pnl
2021-11-30	-0.057718
2021-12-31	0.197897
2022-01-31	0.342095
2022-02-28	0.084965
2022-03-31	0.372228
2022-04-30	0.166215
2022-05-31	-0.089547
2022-06-30	0.189390
2022-07-31	-0.000408
2022-08-31	0.260462

And here are the final in-sample daily and annualized results:

- Mean Monthly PnL is 0.0102
- Mean Annual PnL is 2.5805
- Std Daily PnL is 0.0701
- Std Annual PnL is 1.1126
- Sharpe Ratio is 2.32

We see that our signal parsed from the implied volatilities for SOFR options and transformed into transition probabilities for predictions, showed forecasting power for the forward values of the SOFR interest rate swaps.

10 Suggestions for Further Work

We received decent is-sample results for the given method of estimation of Recovered probability densities and using them for the forecasting.
As we have not enough data, due to recent launch of SOFR products, we have to obtain the larger sample later and check the values.
Moreover, we could parse the data from different brokers and check the stability of results.
After this, we had numerical issues for estimating the risk-neutral probabilities, probably we need more dense strikes grid, transition matrix and recovery probabilities, which are subject to ill-conditioned matrices we get.
Finally, we could work more on our signal, as we see significant drawdowns of the strategy and very high volatility, what is undesirable.

11 Bibliography

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RecoveryTheoremProject

October 18, 2022

[1200]:
```python
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.dates as dates
import statsmodels.api as sm
import datetime
import re
import QuantLib as ql
import copy
import pickle

from datetime import datetime

from filterpy.kalman import KalmanFilter

from matplotlib.ticker import LinearLocator
from matplotlib import cm
from mpl_toolkits.mplot3d import Axes3D

from scipy import optimize
from scipy.optimize import root
from scipy.optimize import fsolve
from scipy import interpolate
from scipy.stats import norm

from functools import reduce

from pykalman import KalmanFilter
```

[1049]:
```python
def pickle_dict(filename, dic=None, serialize='wb'):
    if serialize == 'wb':
        with open(f'{filename}.pickle', serialize) as handle:
            pickle.dump(dic, handle, protocol=pickle.HIGHEST_PROTOCOL)
    else:
        with open(f'{filename}.pickle', serialize) as handle:
            return pickle.load(handle)
```

1 EDA

1.1 Import Data

[116]:
```
vol_tickers = pd.read_excel('SOFR_data.xlsx', sheet_name=1).iloc[:-5, :]
curve_tickers = pd.read_excel('SOFR_data.xlsx', sheet_name=2).iloc[:-7, :]
curve = pd.read_excel('SOFR_data.xlsx', sheet_name='S490')
prices = pd.read_excel('SOFR_data.xlsx', sheet_name=3, skiprows=18, index_col=0).
 ↪iloc[:10, 1:28]

vol_tickers.index = [val[:-1] for val in vol_tickers['Expiry']]
vol_tickers.drop(['Expiry'], axis=1, inplace=True)
```

1.2 Analyze Vol

[117]:
```
def parse_vol(vol_surface):
    max_len = 0
    max_col = 0

    for col in vol_surface.iloc[:, 2::2].columns:
        cur_len = len(vol_surface.iloc[:, 2::2][col].dropna())
        if cur_len > max_len:
            max_len = cur_len
            max_col = col
        else:
            continue

    vol_df = pd.DataFrame(vol_surface[max_col].dropna().values,
                         columns=['Dates'])

    for j in range(len(vol_surface.columns)//2-2):
        try:
            vol_df = vol_df.merge(vol_surface.iloc[1:, 2+j*2:2+(j+1)*2].
 ↪set_index('Unnamed: {}'.format(2+j*2)),
                                 left_on='Dates', right_index=True, how='left').
 ↪fillna(method='bfill')
        except ValueError:
            pass

    vol_df.set_index('Dates', inplace=True)

    return vol_df
```

Create stable panel of data for volatility surfaces

[118]:
```
tenors = vol_tickers.index
vol_dict = {tenor : pd.DataFrame for tenor in tenors}
```

```
for tenor in vol_dict.keys():
    vol_surface = pd.read_excel('SOFR_data.xlsx', sheet_name=tenor)
    vol_dict[tenor] = parse_vol(vol_surface)
```

Let's see the example

[119]:
```
vol_dict['1Y'].head()
```

[119]:
```
            USCLQD1  USCLQC1  USCLQB1  USCNSQ1  USCNQA1  USCNQB1  USCNQC1  \
Dates
2022-09-27    133.9    134.2    114.3    127.0    112.3    109.9    107.4
2022-09-26    133.9    118.3    117.9    120.8    115.9    113.6    111.1
2022-09-23    133.9    118.3    119.2    113.1    117.2    114.9    112.3
2022-09-22    133.9    118.3    119.3     91.2    117.2    114.8    112.1
2022-09-21    133.9    118.3    118.0     91.0    115.8    113.4    110.7

            USCNQD1  USCNQE1  USCNQG1  USCNQH1  USCNQI1  USCNQJ1  USCNQK1  \
Dates
2022-09-27    104.8    102.7    101.5    103.3    103.8    103.8    112.1
2022-09-26    108.5    105.8    101.3     99.7     99.0     98.3    106.1
2022-09-23    109.6    106.8    101.3     97.1     94.4     93.4    101.0
2022-09-22    109.4    106.5    100.7     95.0     89.1     85.6     89.6
2022-09-21    107.8    104.8     98.9     93.1     87.5     84.5     88.6

            USCNQL1  USCNQM1  USCNQO1  USCNQQ1
Dates
2022-09-27    122.8    133.1    143.1    162.4
2022-09-26    115.3    124.0    133.2    151.6
2022-09-23    108.6    116.1    124.6    142.7
2022-09-22     90.1     93.3    100.3    120.7
2022-09-21     89.7     93.8    101.4    122.8
```

1.3 Analyze Curve

Create stable panel of data for interest rate swaps

[120]:
```
def parse_curve(curve):
    max_len = 0
    max_col = 0

    for col in curve.iloc[:, 0::2].columns:
        cur_len = len(curve.iloc[:, 0::2][col].dropna())
        if cur_len > max_len:
            max_len = cur_len
            max_col = col
        else:
            continue
```

```
    curve_df = pd.DataFrame(curve[max_col].values,
                            columns=['Dates'])

    for j in range(len(curve.columns)//2):
        curve_df = curve_df.merge(curve.iloc[:, j*2:(j+1)*2].set_index('Unnamed:␣
↪{}'.format(j*2)),
                             left_on='Dates', right_index=True, how='left').
↪fillna(method='bfill')

    curve_df.set_index('Dates', inplace=True)

    return curve_df
```

[123]:
```
s_490 = parse_curve(curve).iloc[:, :-7]
```

[124]:
```
s_490.head()
```

[124]:
```
            SOFRRATE Index  USOSFR1Z BGN Curncy  USOSFR2Z BGN Curncy  \
Dates
2022-09-27            2.99               3.0281               3.0265
2022-09-26            2.99               3.0407               3.0415
2022-09-23            2.99               3.0304               3.0250
2022-09-22            2.99               3.0301               3.0310
2022-09-21            2.25               3.0351               3.0370

            USOSFR3Z BGN Curncy  USOSFRA BGN Curncy  USOSFRB BGN Curncy  \
Dates
2022-09-27               3.0285              3.0320              3.3190
2022-09-26               3.0425              3.0438              3.3331
2022-09-23               3.0322              3.0367              3.3235
2022-09-22               3.0315              3.0320              3.3152
2022-09-21               3.0369              3.0387              3.2689

            USOSFRC BGN Curncy  USOSFRD BGN Curncy  USOSFRE BGN Curncy  \
Dates
2022-09-27              3.5226              3.6970              3.8375
2022-09-26              3.5466              3.7400              3.8900
2022-09-23              3.5270              3.7125              3.8735
2022-09-22              3.5175              3.6963              3.8575
2022-09-21              3.4711              3.6697              3.8229

            USOSFRF BGN Curncy  ...  USOSFR1 BGN Curncy  USOSFR2 BGN Curncy  \
Dates                           ...
2022-09-27              3.9435  ...              4.2762              4.2800
2022-09-26              3.9990  ...              4.3870              4.3895
2022-09-23              3.9850  ...              4.3807              4.3292
2022-09-22              3.9690  ...              4.3482              4.2571
```

```
2022-09-21              3.9326  ...             4.2960             4.1874

              USOSFR3 BGN Curncy  USOSFR4 BGN Curncy  USOSFR5 BGN Curncy  \
Dates
2022-09-27              4.1200              3.9945              3.9039
2022-09-26              4.2065              4.0640              3.9571
2022-09-23              4.0955              3.9093              3.7735
2022-09-22              4.0185              3.8514              3.7408
2022-09-21              3.9302              3.7256              3.5849

              USOSFR6 BGN Curncy  USOSFR7 BGN Curncy  USOSFR8 BGN Curncy  \
Dates
2022-09-27              3.8288              3.7652              3.7131
2022-09-26              3.8692              3.7959              3.7368
2022-09-23              3.6701              3.5893              3.5278
2022-09-22              3.6570              3.5923              3.5425
2022-09-21              3.4928              3.4245              3.3726

              USOSFR9 BGN Curncy  USOSFR10 BGN Curncy
Dates
2022-09-27              3.6734               3.6434
2022-09-26              3.6917               3.6579
2022-09-23              3.4818               3.4507
2022-09-22              3.5059               3.4827
2022-09-21              3.3379               3.3164

[5 rows x 25 columns]
```

1.4 Merge the dates between Vols and Rates

[125]:
```python
for tenor in vol_dict.keys():
    index_intersection = set(s_490.index).intersection(vol_dict[tenor].index)
    vol_dict[tenor] = vol_dict[tenor].loc[index_intersection, :].
 ↪sort_index(ascending=False)
```

[1076]:
```python
# Save data to reuse it
# pickle_dict('vol_dict', vol_dict)
vol_dict = pickle_dict('vol_dict', serialize='rb')
```

1.5 Create Auxillary Data

[126]:
```python
for i in range(len(curve_tickers)):

    tenor = curve_tickers.Tenor[i]

    if len(tenor) == 2:
        tenor_numeric = int(tenor[0])
```

```
    elif len(tenor) == 3:
        tenor_numeric = int(tenor[:2])

    if 'D' in tenor:
        curve_tickers.loc[i, 'year_frac'] = tenor_numeric / 360
    elif 'W' in tenor:
        curve_tickers.loc[i, 'year_frac'] = tenor_numeric * 7 / 360
    elif 'M' in tenor:
        curve_tickers.loc[i, 'year_frac'] = tenor_numeric * 30 / 360
    else:
        if tenor_numeric <= 10:
            curve_tickers.loc[i, 'year_frac'] = tenor_numeric
```

Create mapping for ticker to year fraction:

[127]:
```
ticker_yearfrac_dict = curve_tickers[['Cusip', 'year_frac']].set_index('Cusip').
 ↪to_dict()['year_frac']
```

[1075]:
```
# Save data to reuse it
# pickle_dict('ticker_yearfrac_dict', ticker_yearfrac_dict)
ticker_yearfrac_dict = pickle_dict('ticker_yearfrac_dict', serialize='rb')
```

Create mapping for tenor and strike to ticker:

[128]:
```
ticker_strike_dict = {}
ticker_strike_dict_unique = {}
strikes_unique = []
pattern = r'[0-9]'

for tenor in vol_tickers.index:
    for strike in vol_tickers.columns:
        strike_generic = re.sub(pattern, '', vol_tickers.loc[tenor, strike]).
 ↪replace(' ', '')
        ticker_strike_dict[(tenor, strike)] = vol_tickers.loc[tenor, strike]
        strikes_unique.append(strike_generic)
        ticker_strike_dict_unique[strike_generic] = strike

strikes_unique = set(strikes_unique)
```

[129]:
```
strikes_unique_keep = []

for strike in strikes_unique:
    count = 0
    for key in vol_dict:
        if strike+key[:-1] in vol_dict[key].columns:
            count += 1
        if count == len(vol_dict.keys()):
            strikes_unique_keep.append(strike)
```

```python
strikes_unique_keep = set(strikes_unique_keep)
```

[130]:
```python
vol_tickers
```

[130]:
```
       -2.00%    -1.50%    -1.25%    -1.00%    -0.75%    -0.50%    -0.25%  \
1Y    USCLQH1   USCLQG1   USCLQF1   USCLQE1   USCLQD1   USCLQC1   USCLQB1
2Y    USCLQH2   USCLQG2   USCLQF2   USCLQE2   USCLQD2   USCLQC2   USCLQB2
3Y    USCLQH3   USCLQG3   USCLQF3   USCLQE3   USCLQD3   USCLQC3   USCLQB3
4Y    USCLQH4   USCLQG4   USCLQF4   USCLQE4   USCLQD4   USCLQC4   USCLQB4
5Y    USCLQH5   USCLQG5   USCLQF5   USCLQE5   USCLQD5   USCLQC5   USCLQB5
6Y    USCLQH6   USCLQG6   USCLQF6   USCLQE6   USCLQD6   USCLQC6   USCLQB6
7Y    USCLQH7   USCLQG7   USCLQF7   USCLQE7   USCLQD7   USCLQC7   USCLQB7
8Y    USCLQH8   USCLQG8   USCLQF8   USCLQE8   USCLQD8   USCLQC8   USCLQB8
9Y    USCLQH9   USCLQG9   USCLQF9   USCLQE9   USCLQD9   USCLQC9   USCLQB9
10Y  USCLQH10  USCLQG10  USCLQF10  USCLQE10  USCLQD10  USCLQC10  USCLQB10

       -0.10%       ATM     0.00%  ...     2.50%     3.00%     3.50%  \
1Y    USCLQA1   USCNSQ1   USCNQA1  ...   USCNQI1   USCNQJ1   USCNQK1
2Y    USCLQA2   USCNSQ2   USCNQA2  ...   USCNQI2   USCNQJ2   USCNQK2
3Y    USCLQA3   USCNSQ3   USCNQA3  ...   USCNQI3   USCNQJ3   USCNQK3
4Y    USCLQA4   USCNSQ4   USCNQA4  ...   USCNQI4   USCNQJ4   USCNQK4
5Y    USCLQA5   USCNSQ5   USCNQA5  ...   USCNQI5   USCNQJ5   USCNQK5
6Y    USCLQA6   USCNSQ6   USCNQA6  ...   USCNQI6   USCNQJ6   USCNQK6
7Y    USCLQA7   USCNSQ7   USCNQA7  ...   USCNQI7   USCNQJ7   USCNQK7
8Y    USCLQA8   USCNSQ8   USCNQA8  ...   USCNQI8   USCNQJ8   USCNQK8
9Y    USCLQA9   USCNSQ9   USCNQA9  ...   USCNQI9   USCNQJ9   USCNQK9
10Y  USCLQA10  USCNSQ10  USCNQA10  ...  USCNQI10  USCNQJ10  USCNQK10

        4.00%     4.50%     5.00%     5.50%     6.00%     6.50%     7.00%
1Y    USCNQL1   USCNQM1   USCNQO1   USCNQP1   USCNQQ1   USCNQR1   USCNQS1
2Y    USCNQL2   USCNQM2   USCNQO2   USCNQP2   USCNQQ2   USCNQR2   USCNQS2
3Y    USCNQL3   USCNQM3   USCNQO3   USCNQP3   USCNQQ3   USCNQR3   USCNQS3
4Y    USCNQL4   USCNQM4   USCNQO4   USCNQP4   USCNQQ4   USCNQR4   USCNQS4
5Y    USCNQL5   USCNQM5   USCNQO5   USCNQP5   USCNQQ5   USCNQR5   USCNQS5
6Y    USCNQL6   USCNQM6   USCNQO6   USCNQP6   USCNQQ6   USCNQR6   USCNQS6
7Y    USCNQL7   USCNQM7   USCNQO7   USCNQP7   USCNQQ7   USCNQR7   USCNQS7
8Y    USCNQL8   USCNQM8   USCNQO8   USCNQP8   USCNQQ8   USCNQR8   USCNQS8
9Y    USCNQL9   USCNQM9   USCNQO9   USCNQP9   USCNQQ9   USCNQR9   USCNQS9
10Y  USCNQL10  USCNQM10  USCNQO10  USCNQP10  USCNQQ10  USCNQR10  USCNQS10

[10 rows x 27 columns]
```

[1074]:
```python
# Save data to reuse it
# pickle_dict('ticker_strike_dict', ticker_strike_dict)
# pickle_dict('ticker_strike_dict_unique', ticker_strike_dict_unique)

ticker_strike_dict = pickle_dict('ticker_strike_dict', serialize='rb')
```

```
ticker_strike_dict_unique = pickle_dict('ticker_strike_dict_unique',␣
↪serialize='rb')
```

1.6 Trim the Strikes

We see that now all the data has the same strikes and tenors.

[131]:
```
for key in vol_dict.keys():
    print(key, vol_dict[key].shape)
```

```
1Y (230, 18)
2Y (230, 18)
3Y (230, 18)
4Y (230, 18)
5Y (230, 18)
6Y (230, 18)
7Y (230, 18)
8Y (230, 18)
9Y (230, 18)
10Y (230, 18)
```

Therefore, let's create historical surafces for all timestamps

[506]:
```
strikes_numeric = sorted([ticker_strike_dict_unique[key] for key in␣
↪strikes_unique_keep])

vol_surface_dict = {key.date().strftime('%Y-%m-%d') :
                    pd.DataFrame(index=vol_dict.keys(), columns=strikes_numeric)
                    for key in vol_dict['1Y'].index}
```

[507]:
```
for key in vol_surface_dict:
    for tenor in vol_surface_dict[key].index:
        for strike in vol_surface_dict[key].columns:
            ticker = ticker_strike_dict[(tenor, strike)].replace(' ', '')
            vol_value = vol_dict[tenor][key][ticker].values[0]

            vol_surface_dict[key].loc[tenor, strike] = vol_value
```

```
/var/folders/zt/2mmqxh5j3ms4m8_dlf7n891c0000gn/T/ipykernel_9363/2435240141.py:5:
FutureWarning: Indexing a DataFrame with a datetimelike index using a single
string to slice the rows, like `frame[string]`, is deprecated and will be
removed in a future version. Use `frame.loc[string]` instead.
  vol_value = vol_dict[tenor][key][ticker].values[0]
```

This is the example of the surface

[508]:
```
vol_surface_dict['2022-09-27']
```

```
[508]:      -0.25%  -0.50%  -0.75%  0.00%  0.25%  0.50%  0.75%  1.00%  1.50%  2.00%  \
       1Y    114.3   134.2   133.9  112.3  109.9  107.4  104.8  102.7  101.5  103.3
       2Y    160.7   162.2   163.3  158.8  156.7  154.5  152.2  150.1  146.2  143.2
       3Y    166.3   166.6   166.4  165.6  164.4  163.0  161.4  159.7  156.4  153.5
       4Y    159.4   159.0   158.2  159.4  158.9  158.2  157.3  156.3  154.3  152.6
       5Y    153.0   152.4   151.4  153.2  153.0  152.6  152.0  151.3  149.9  148.7
       6Y    146.7   146.0   145.0  146.9  146.9  146.6  146.2  145.7  144.6  143.8
       7Y    139.4   138.7   137.6  139.9  140.0  139.9  139.6  139.3  138.7  138.3
       8Y    132.6   131.8   130.7  133.1  133.3  133.3  133.2  133.0  132.7  132.7
       9Y    127.1   126.3   125.2  127.6  127.9  128.0  127.9  127.8  127.7  127.8
       10Y   122.8   122.0   121.0  123.3  123.5  123.6  123.5  123.4  123.3  123.5

             2.50%   3.00%   3.50%  4.00%  4.50%  5.00%  6.00%    ATM
       1Y    103.8   103.8   112.1  122.8  133.1  143.1  162.4  127.0
       2Y    140.4   137.9   139.3  144.8  152.2  160.3  178.0  147.8
       3Y    151.1   149.5   150.9  155.9  163.1  171.9  192.1  156.6
       4Y    151.4   150.9   152.6  157.3  163.9  171.8  190.2  156.6
       5Y    148.1   148.2   150.2  154.4  160.4  167.6  183.9  153.0
       6Y    143.6   144.1   146.2  150.2  155.7  162.2  177.0  148.3
       7Y    138.5   139.4   141.6  145.6  150.8  156.9  170.6  143.2
       8Y    133.2   134.4   136.9  140.9  146.0  151.8  164.8  138.1
       9Y    128.6   130.1   132.8  136.7  141.7  147.3  159.7  133.6
       10Y   124.4   126.0   128.6  132.6  137.3  142.7  154.6  129.3
```

Let's trim the price surface in order to check the convergence of the method further

```
[135]: prices = prices[vol_surface_dict['2022-09-27'].columns].dropna(axis=1)
```

```
[136]: prices
```

```
[136]:        -0.25%  -0.50%   0.00%   0.25%   0.50%   0.75%   1.00%   1.50%   2.00%  \
       1.0    440.0   465.0   416.0   391.0   366.0   341.0   317.0   267.0   218.0
       2.0    866.0   915.0   818.0   770.0   722.0   674.0   627.0   532.0   440.0
       3.0   1235.0  1304.0  1166.0  1097.0  1029.0   961.0   893.0   760.0   632.0
       4.0   1581.0  1670.0  1493.0  1404.0  1317.0  1230.0  1145.0   977.0   816.0
       5.0   1911.0  2019.0  1804.0  1697.0  1592.0  1488.0  1385.0  1184.0   992.0
       6.0   2228.0  2354.0  2103.0  1979.0  1857.0  1735.0  1616.0  1383.0  1162.0
       7.0   2519.0  2662.0  2377.0  2237.0  2098.0  1960.0  1825.0  1563.0  1315.0
       8.0   2799.0  2959.0  2641.0  2484.0  2329.0  2176.0  2026.0  1735.0  1460.0
       9.0   3071.0  3247.0  2897.0  2724.0  2554.0  2386.0  2221.0  1902.0  1602.0
       10.0  3346.0  3538.0  3156.0  2968.0  2783.0  2600.0  2420.0  2074.0  1748.0

               2.50%   3.00%  4.00%  4.50%  5.00%  6.00%    ATM
       1.0    169.0   122.0   51.6   31.9   19.4    7.2   42.5
       2.0    350.0   267.0  139.0   97.9   69.2   35.8  120.0
       3.0    510.0   397.0  226.0  171.0  130.0   81.1  219.0
       4.0    664.0   526.0  315.0  244.0  192.0  127.0  324.0
       5.0    813.0   650.0  400.0  316.0  252.0  170.0  430.0
```

```
6.0     956.0    770.0  483.0  385.0  311.0  213.0  533.0
7.0    1084.0    877.0  557.0  447.0  363.0  251.0  633.0
8.0    1206.0    978.0  628.0  507.0  414.0  290.0  730.0
9.0    1325.0   1078.0  699.0  567.0  466.0  329.0  824.0
10.0   1448.0   1181.0  771.0  628.0  518.0  369.0  915.0
```

[1073]:
```
# Save data to reuse it
# pickle_dict('vol_surface_dict', vol_surface_dict)

vol_surface_dict= pickle_dict('vol_surface_dict', serialize='rb')
```

2 Arrow-Debreau Densities

2.1 Forward Curve Construction

Create tenors

[138]:
```
tenors = [curve_tickers[curve_tickers.Cusip == ticker].Tenor.values[0] for␣
 ↪ticker in s_490.columns]
```

[139]:
```
tenors
```

[139]:
```
['1D',
 '1W',
 '2W',
 '3W',
 '1M',
 '2M',
 '3M',
 '4M',
 '5M',
 '6M',
 '7M',
 '8M',
 '9M',
 '10M',
 '11M',
 '12M',
 '2Y',
 '3Y',
 '4Y',
 '5Y',
 '6Y',
 '7Y',
 '8Y',
 '9Y',
 '10Y']
```

Create yield curve for each date

```
[140]: yield_curve_dict = {date.strftime('%Y-%m-%d') : pd.DataFrame(s_490.loc[date, :].
       ↪values,
                              index=tenors, columns=['Value']).T for date in s_490.index}
```

```
[ ]: # Save data to reuse it
     # pickle_dict('yield_curve_dict', yield_curve_dict)

     yield_curve_dict = pickle_dict('yield_curve_dict', serialize='rb')
```

```
[141]: yield_curve_dict['2022-09-27']
```

```
[141]:         1D      1W      2W      3W     1M     2M      3M     4M      5M  \
       Value  2.99  3.0281  3.0265  3.0285  3.032  3.319  3.5226  3.697  3.8375

                 6M  ...     12M    2Y    3Y      4Y      5Y      6Y      7Y  \
       Value  3.9435  ...  4.2762  4.28  4.12  3.9945  3.9039  3.8288  3.7652

                 8Y      9Y     10Y
       Value  3.7131  3.6734  3.6434

       [1 rows x 25 columns]
```

And make the version for pricing

```
[142]: yield_curve_dict_ql = {date.strftime('%Y-%m-%d') : 0 for date in s_490.index}
```

Set up the schedule parameters: - US Holidays - ACT/360 day count - T+0 settlement convention

```
[143]: calendar = ql.UnitedStates()
       day_count = ql.Actual360()
       settlement_days_SOFR = 0

       SOFR = ql.OvernightIndex("SOFR", settlement_days_SOFR,
                                ql.USDCurrency(), calendar, day_count)
```

Create SOFR Swap curve for each date

Convert each yield curve into the QuantLib format:

```
[144]: for date in yield_curve_dict.keys():
           ois_helper_i = []
           for tenor in yield_curve_dict[date].columns:
               rate = yield_curve_dict[date].loc['Value', tenor]
               if 'D' in tenor:
                   tenor_ql = ql.Period(int(tenor[:-1]), ql.Days)
                   ois_helper_i.append(ql.OISRateHelper(settlement_days_SOFR, tenor_ql,
                                                        ql.QuoteHandle(ql.SimpleQuote(rate/
       ↪100)), SOFR))
```

```
        if 'W' in tenor:
            tenor_ql = ql.Period(int(tenor[:-1]), ql.Weeks)
            ois_helper_i.append(ql.OISRateHelper(settlement_days_SOFR, tenor_ql,
                                                 ql.QuoteHandle(ql.
↪SimpleQuote(rate/100)), SOFR))
        if 'M' in tenor:
            tenor_ql = ql.Period(int(tenor[:-1]), ql.Months)
            ois_helper_i.append(ql.OISRateHelper(settlement_days_SOFR, tenor_ql,
                                                 ql.QuoteHandle(ql.
↪SimpleQuote(rate/100)), SOFR))
        if 'Y' in tenor:
            tenor_ql = ql.Period(int(tenor[:-1]), ql.Years)
            ois_helper_i.append(ql.OISRateHelper(settlement_days_SOFR, tenor_ql,
                                                 ql.QuoteHandle(ql.
↪SimpleQuote(rate/100)), SOFR))
    yield_curve_dict_ql[date] = ois_helper_i
```

Here is the example of the object

[145]:
```
yield_curve_dict_ql['2022-09-27']
```

[145]:
```
[<QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f85618f1630> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f85618f1690> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f85618f1390> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f85618f1420> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f852826c990> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f85618f1360> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f852826c7b0> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f85618f1bd0> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f85618f1150> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f854805a0f0> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f854805a3c0> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f854805a120> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f854805a6c0> >,
```

```
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f8528264db0> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f853823a6c0> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f854805a570> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f854805a420> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f854805a840> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f854805ab10> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f854805a030> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f854805a960> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f8568cfb4e0> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f8568cfbe10> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f8568cfba80> >,
 <QuantLib.QuantLib.OISRateHelper; proxy of <Swig Object of type
'ext::shared_ptr< OISRateHelper > *' at 0x7f8568cfb780> >]
```

Let's create the dictionaries for SOFR, Forward Curves and Discount Factors:

[146]:
```
sofr_curves_dict = {date.strftime('%Y-%m-%d') : 0 for date in s_490.index}

fwd_curves_dict = {date.strftime('%Y-%m-%d') : 0 for date in s_490.index}

dfs_curves_dict = {date.strftime('%Y-%m-%d') : 0 for date in s_490.index}
```

And interpolate:

[160]:
```
# for date in list(sofr_curves_dict.keys()):
#     print(date)

#     # Set the pricing date
#     date_datetime = pd.to_datetime(date)
#     year = date_datetime.year
#     month = date_datetime.month
#     day = date_datetime.day

#     ql.Settings.instance().evaluationDate = ql.Date(day, month, year)

# #     # Create the curves
```

```
#     sofr_curve_i = ql.PiecewiseLogCubicDiscount(settlement_days_SOFR,␣
↪calendar,
#                                                   yield_curve_dict_ql[date],␣
↪day_count)
#     sofr_curve_i.enableExtrapolation()

#     all_days = ql.MakeSchedule(calendar.advance(ql.Date(day, month, year),
#                                                  1, ql.Days),
#                                 ql.Date(day, month, year+10),
#                                 ql.Period('1D'))

# #     all_days = all_days[all_days.index(ql.Date(day,month,year)):all_days.
↪index(ql.Date(day,month,year + 10))]

#     rates_fwd = [sofr_curve_i.forwardRate(d, calendar.advance(d, 1, ql.Days),
#                                            day_count, ql.Simple).rate()*100 for␣
↪d in all_days]

#     dfs = [sofr_curve_i.discount(d) for d in all_days]

#     sofr_curves_dict[date] = sofr_curve_i
#     fwd_curves_dict[date] = rates_fwd
#     dfs_curves_dict[date] = dfs
```

2022-09-27
2022-09-26
2022-09-23
2022-09-22
2022-09-21
2022-09-20
2022-09-19
2022-09-16
2022-09-15
2022-09-14
2022-09-13
2022-09-12
2022-09-09
2022-09-08
2022-09-07
2022-09-06
2022-09-05
2022-09-02
2022-09-01
2022-08-31
2022-08-30
2022-08-29
2022-08-26

2022-08-25
2022-08-24
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2022-08-16
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2022-08-12
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2022-08-05
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2022-08-02
2022-08-01
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2022-02-28
2022-02-25
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2022-02-17
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2021-12-10
2021-12-09
2021-12-08
2021-12-07
2021-12-06
2021-12-03
2021-12-02
2021-12-01
2021-11-30

```
2021-11-29
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2021-11-22
2021-11-19
2021-11-18
2021-11-17
2021-11-16
2021-11-15
2021-11-12
2021-11-11
2021-11-10
2021-11-09
```

[1071]:
```
# Save data to reuse it
# pickle_dict('fwd_curves_dict', fwd_curves_dict)
# pickle_dict('dfs_curves_dict', dfs_curves_dict)

fwd_curves_dict = pickle_dict('fwd_curves_dict', serialize='rb')
dfs_curves_dict = pickle_dict('dfs_curves_dict', serialize='rb')
```

[346]:
```
key = list(fwd_curves_dict.keys())[0]
key_init = list(fwd_curves_dict.keys())[-1]

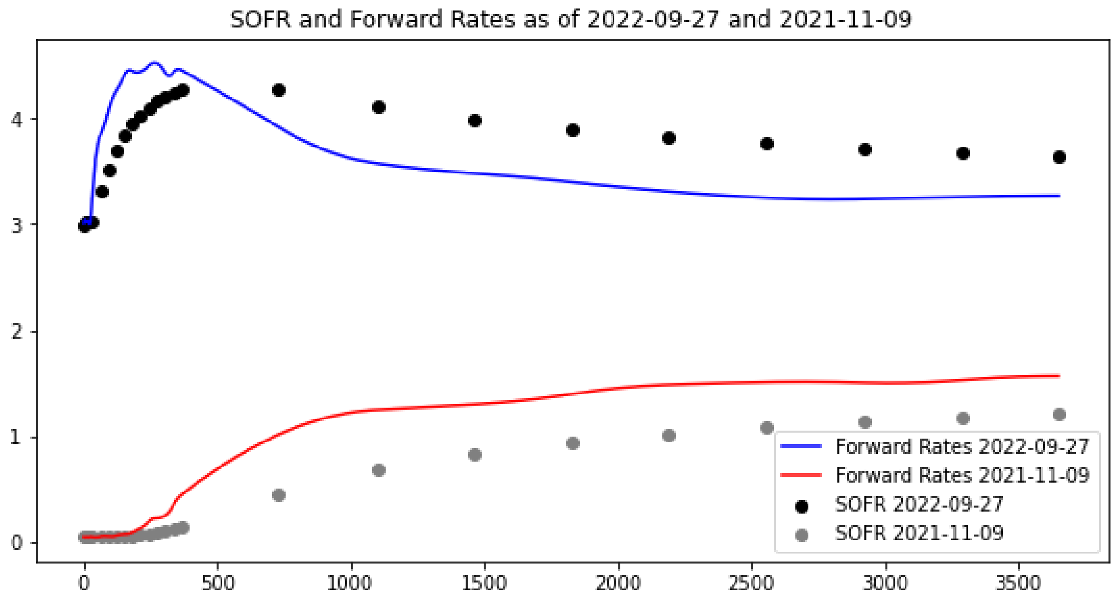
plt.figure(figsize=(10, 5))

plt.plot([d for d in range(len([d for d in all_days])+1)],
         fwd_curves_dict[key], color='blue')
plt.plot([d for d in range(len([d for d in all_days]))],
         fwd_curves_dict[key_init], color='red')

plt.scatter([time*360 for time in sofr_curve_i.times()[1:] if time < 11],
            yield_curve_dict[key], color='black')
plt.scatter([time*360 for time in sofr_curve_i.times()[1:] if time < 11],
            yield_curve_dict[key_init], color='grey')

plt.legend([f'Forward Rates {key}', f'Forward Rates {key_init}',
           f'SOFR {key}', f'SOFR {key_init}'])
plt.title(f'SOFR and Forward Rates as of {key} and {key_init}')

plt.show()
```



[351]:

```
plt.figure(figsize=(10, 5))

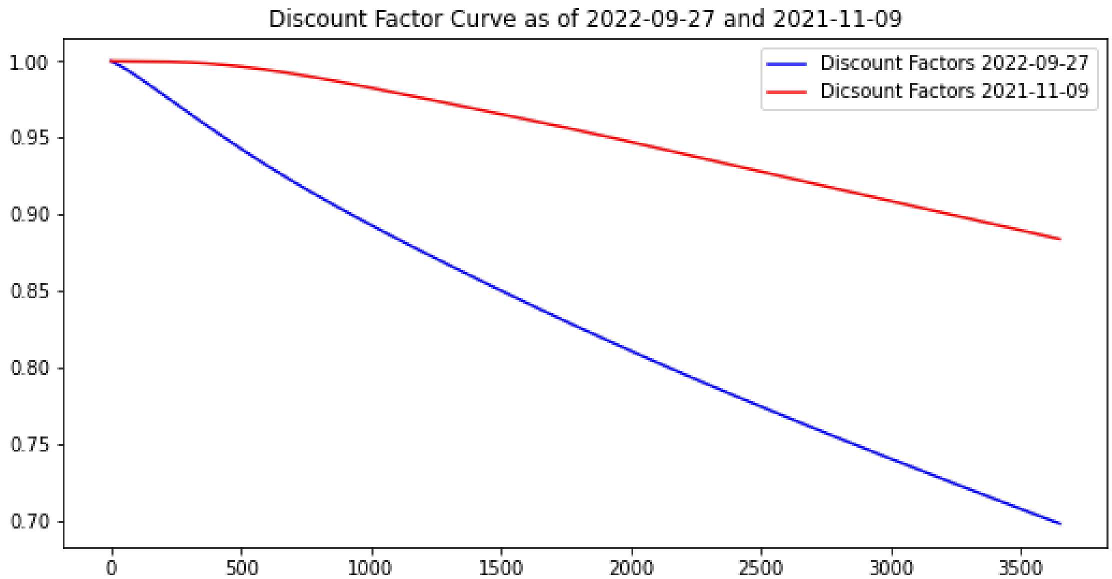
plt.plot([d for d in range(len([d for d in all_days])+1)],
         dfs_curves_dict[key], color='blue')

plt.plot([d for d in range(len([d for d in all_days]))],
         dfs_curves_dict[key_init], color='red')

plt.legend([f'Discount Factors {key}', f'Dicsount Factors {key_init}'])

plt.title(f'Discount Factor Curve as of {key} and {key_init}')

plt.show()
```



2.2 Implied Volatility Surface Construction

We get quarterly-quarterly swaps (QQ-Swaps) from the Market Overnight Forward Rates in order to imply the ATM Swap Rates: - We have Compounded 3M Fwd OIS rates, which is used for pricing of Caps and have to annualize it to match with strike - We use convention that we have 10 swaps, each has quarterly fixings, so we divide every swap rate by 4 and get fixing

[223]:
```
qq_swaps_dict = {key : {'qq_swaps': [],
                        'dfs': [],
                        'quarter_comps': []} for key in fwd_curves_dict.keys()}
```

[239]:
```
# for key in fwd_curves_dict.keys():
#     date_datetime = pd.to_datetime(key)
#     year = date_datetime.year
#     month = date_datetime.month
#     day = date_datetime.day
#     today = ql.Date(day, month, year)

#     quarters = []

#     i = today
#     while i <= calendar.advance(today, 10, ql.Years):
#         quarters.append(calendar.advance(i, 3, ql.Months))
#         i = quarters[-1]

#     days_in_quarters = []
#     for i in quarters:
```

```
#         days_in_quarters.append(i - today)
#     days_in_quarters = [0]+days_in_quarters
#     days_in_quarters[-1] = 3653

#     tenors_by_quarters = [4*i for i in range(1, 11)]

#     quarter_comps = []

#     for j in range(len(days_in_quarters)-1):
#         q = 1
#         for i in fwd_curves_dict[key][days_in_quarters[j]:
↪days_in_quarters[j+1]]:
#             x = 1 + i/360/100
#             q *= x
#         quarter_comps.append(q)

#     q_swaps = []
#     for i in tenors_by_quarters:
#         k = 1
#         for j in quarter_comps[:i]:
#             k *= j
#         res = (k**(1/i)-1)*4
#         q_swaps.append(res)

#     qq_swaps_dict[key]['qq_swaps'] = q_swaps

#     #here we have dfs for every quarter(end)
#     DFs = []
#     for i in days_in_quarters[1:]:
#         try:
#             DFs.append(dfs_curves_dict[key][i])
#         except IndexError:
#             DFs.append(dfs_curves_dict[key][-1])
#     qq_swaps_dict[key]['dfs'] = DFs

#     qq_swaps_dict[key]['quarter_comps'] = quarter_comps
```

Here is the Example

[1819]:
```
pd.DataFrame(qq_swaps_dict['2022-09-27']['qq_swaps']
```

[1819]:
```
[0.04272202160856426,
 0.04278212660699943,
 0.041186561116353104,
 0.039881962173724084,
 0.0389182705838067,
 0.038167197626648,
```

```
 0.037480081462804904,
 0.03691693658608486,
 0.03649072835189582,
 0.036130568582940725]
```

```
[1069]: # Save data to reuse it
        # pickle_dict('qq_swaps_dict', qq_swaps_dict)

        qq_swaps_dict = pickle_dict('qq_swaps_dict', serialize='rb')
```

We get the Cap price as a function of daily Caplets, given the inputs, Cap Vol, Strike, Expiry, Forward Curve and Discount Factors

```
[251]: def caplet_fv(tau, rate, strike, vol, expiry):
           z = ( rate - strike ) / (vol * expiry ** 0.5)
           return tau * (( rate - strike ) * norm.cdf(z) + vol * (expiry ** 0.5) * norm.
        ↪pdf(z))

       def cap_pv(tau, fwd_vector, dfs_vector, strike, vol, expiry):
           caplets = []
           for i in range(1, 4 * expiry + 1):
               df = dfs_vector[i-1]
               rate = (fwd_vector[i-1] - 1) * 4
               expiry = i * tau
               caplets.append(df * caplet_fv(tau, rate, strike, vol, expiry))
           return sum(caplets)
```

```
[303]: price_surface_dict = {key : pd.DataFrame(index=vol_surface_dict[key].index,
                                                columns=[col for col in
                                                         vol_surface_dict[key].columns
                                                         if col not in ['-0.75%', '3.
        ↪50%']])
                             for key in vol_surface_dict.keys()}
```

```
[306]: # for key in price_surface_dict.keys():
       #     for strike in price_surface_dict[key].columns:
       #         for expiry in price_surface_dict[key].index:
       #             expiry_num = int(expiry.replace('Y', ''))

       #             if strike == 'ATM':
       #                 strike_num = qq_swaps_dict[key]['qq_swaps'][expiry_num-1]
       #             else:
       #                 strike_num = float(strike.replace('%', '')) / 100

       #             vol = vol_surface_dict[key].loc[expiry, strike]/10000
       #             tau = 0.25

       #             price_surface_dict[key].loc[expiry, strike] = cap_pv(tau,
```

```
#                      qq_swaps_dict[key]['quarter_comps'],
#                      qq_swaps_dict[key]['dfs'],
#                      strike_num, vol, expiry_num) * 10000
```

```
/var/folders/zt/2mmqxh5j3ms4m8_dlf7n891c0000gn/T/ipykernel_9363/1990442262.py:2:
RuntimeWarning: divide by zero encountered in double_scalars
  z = ( rate - strike ) / (vol * expiry ** 0.5)
```

Here is the result

[1826]:
```
price_surface_dict['2022-09-27']
```

[1826]:

	-0.25%	-0.50%	0.00%	0.25%	0.50%
1Y	440.393542	464.763601	416.024099	391.654704	367.285403
2Y	864.614434	912.054153	817.225229	769.906015	722.679566
3Y	1235.250985	1303.695751	1167.057207	1099.099454	1031.492291
4Y	1579.153331	1667.167676	1491.632936	1404.561629	1318.15675
5Y	1906.976928	2013.520746	1801.137832	1696.017047	1591.861505
6Y	2220.381217	2344.624359	2096.896161	1974.574286	1853.342955
7Y	2514.91544	2656.394655	2374.827926	2235.837766	2098.362023
8Y	2796.608452	2954.622831	2640.025944	2484.911839	2331.571438
9Y	3070.610653	3244.680574	2898.143524	2727.599285	2559.090648
10Y	3336.212096	3525.626508	3148.561533	2962.69433	2779.383088

	0.75%	1.00%	1.50%	2.00%	2.50%
1Y	342.916269	318.547578	269.816512	221.137087	172.735966
2Y	675.568645	628.637817	535.517607	444.101043	355.538381
3Y	964.28427	897.578606	766.263579	638.930614	517.546564
4Y	1232.490805	1147.735639	981.789114	822.259296	671.721533
5Y	1488.759063	1386.947294	1188.303006	998.199906	820.026081
6Y	1733.612426	1615.53798	1385.512787	1166.391043	961.920052
7Y	1962.509213	1828.899739	1569.433227	1322.895021	1093.772747
8Y	2180.407633	2031.618294	1743.45944	1470.818552	1217.979057
9Y	2392.741224	2229.427845	1913.606979	1615.019256	1339.773472
10Y	2598.441626	2420.879191	2077.775698	1754.342841	1456.892907

	3.00%	4.00%	4.50%	5.00%	6.00%	ATM
1Y	125.720368	54.512788	34.01337	20.922871	7.877918	41.88187
2Y	272.146243	143.132869	101.582156	72.04408	37.390676	117.594767
3Y	405.326447	232.860088	175.953673	134.820322	84.035787	216.621484
4Y	533.804936	321.419243	249.971233	196.99961	129.653345	322.25529
5Y	657.699346	406.845344	321.290218	257.169417	173.172865	427.828618
6Y	776.388832	489.058306	389.934276	314.780652	215.254425	531.690376
7Y	886.703768	565.795967	454.302296	369.346014	255.50836	632.839741
8Y	990.273533	638.51639	515.98579	421.861066	295.175867	730.136377
9Y	1092.728107	710.825313	577.647656	474.913834	335.627669	825.346053
10Y	1190.527892	779.374174	634.750892	523.806347	372.956258	916.704045

[1068]:
```
# Save data to reuse it
# pickle_dict('price_surface_dict', price_surface_dict)

price_surface_dict = pickle_dict('price_surface_dict', serialize='rb')
```

We can check the prices on one slice to see that they are close to the market ones

[315]:
```
prices_diff_example = pd.DataFrame(price_surface_dict['2022-09-27'].values -␣
 ↪prices.values,
                                   index=prices.index, columns=prices.columns)
```

[1827]:
```
prices_diff_example
```

[1827]:
```
          -0.25%     -0.50%     0.00%     0.25%     0.50%     0.75%     1.00%  \
1.0    0.393542  -0.236399  0.024099  0.654704  1.285403  1.916269  1.547578
2.0   -1.385566  -2.945847 -0.774771 -0.093985  0.679566  1.568645  1.637817
3.0    0.250985  -0.304249  1.057207  2.099454  2.492291   3.28427  4.578606
4.0   -1.846669  -2.832324 -1.367064  0.561629   1.15675  2.490805  2.735639
5.0   -4.023072  -5.479254 -2.862168 -0.982953 -0.138495  0.759063  1.947294
6.0   -7.618783  -9.375641 -6.103839 -4.425714 -3.657045 -1.387574  -0.46202
7.0    -4.08456  -5.605345 -2.172074 -1.162234  0.362023  2.509213  3.899739
8.0   -2.391548  -4.377169 -0.974056  0.911839  2.571438  4.407633  5.618294
9.0   -0.389347  -2.319426  1.143524  3.599285  5.090648  6.741224  8.427845
10.0  -9.787904 -12.373492 -7.438467  -5.30567 -3.616912 -1.558374  0.879191

           1.50%      2.00%      2.50%      3.00%      4.00%      4.50%  \
1.0     2.816512   3.137087   3.735966   3.720368   2.912788    2.11337
2.0     3.517607   4.101043   5.538381   5.146243   4.132869   3.682156
3.0     6.263579   6.930614   7.546564   8.326447   6.860088   4.953673
4.0     4.789114   6.259296   7.721533   7.804936   6.419243   5.971233
5.0     4.303006   6.199906   7.026081   7.699346   6.845344   5.290218
6.0     2.512787   4.391043   5.920052   6.388832   6.058306   4.934276
7.0     6.433227   7.895021   9.772747   9.703768   8.795967   7.302296
8.0      8.45944  10.818552  11.979057  12.273533   10.51639    8.98579
9.0    11.606979  13.019256  14.773472  14.728107  11.825313  10.647656
10.0    3.775698   6.342841   8.892907   9.527892   8.374174   6.750892

          5.00%     6.00%       ATM
1.0    1.522871  0.677918  -0.61813
2.0     2.84408  1.590676 -2.405233
3.0    4.820322  2.935787 -2.378516
4.0     4.99961  2.653345  -1.74471
5.0    5.169417  3.172865 -2.171382
6.0    3.780652  2.254425 -1.309624
7.0    6.346014   4.50836 -0.160259
8.0    7.861066  5.175867  0.136377
9.0    8.913834  6.627669  1.346053
10.0   5.806347  3.956258  1.704045
```

Average error is:

[337]:
```
print(f'Mean Error is {round(np.mean(prices_diff_example).mean(), 2)} bps')
```

```
Mean Error is 2.88 bps
```

And check the relative difference

[327]:
```
prices_rel_diff_example = pd.DataFrame(price_surface_dict['2022-09-27'].values /␣
↪prices.values - 1,
                                  index=prices.index, columns=prices.columns)
```

[328]:
```
prices_rel_diff_example
```

[328]:

	-0.25%	-0.50%	0.00%	0.25%	0.50%	0.75%	1.00%	\
1.0	0.000894	-0.000508	0.000058	0.001674	0.003512	0.00562	0.004882	
2.0	-0.0016	-0.00322	-0.000947	-0.000122	0.000941	0.002327	0.002612	
3.0	0.000203	-0.000233	0.000907	0.001914	0.002422	0.003418	0.005127	
4.0	-0.001168	-0.001696	-0.000916	0.0004	0.000878	0.002025	0.002389	
5.0	-0.002105	-0.002714	-0.001587	-0.000579	-0.000087	0.00051	0.001406	
6.0	-0.00342	-0.003983	-0.002902	-0.002236	-0.001969	-0.0008	-0.000286	
7.0	-0.001622	-0.002106	-0.000914	-0.00052	0.000173	0.00128	0.002137	
8.0	-0.000854	-0.001479	-0.000369	0.000367	0.001104	0.002026	0.002773	
9.0	-0.000127	-0.000714	0.000395	0.001321	0.001993	0.002825	0.003795	
10.0	-0.002925	-0.003497	-0.002357	-0.001788	-0.0013	-0.000599	0.000363	

	1.50%	2.00%	2.50%	3.00%	4.00%	4.50%	5.00%	\
1.0	0.010549	0.01439	0.022106	0.030495	0.056449	0.06625	0.078499	
2.0	0.006612	0.009321	0.015824	0.019274	0.029733	0.037611	0.041099	
3.0	0.008242	0.010966	0.014797	0.020973	0.030354	0.028969	0.037079	
4.0	0.004902	0.007671	0.011629	0.014838	0.020379	0.024472	0.02604	
5.0	0.003634	0.00625	0.008642	0.011845	0.017113	0.016741	0.020514	
6.0	0.001817	0.003779	0.006193	0.008297	0.012543	0.012816	0.012156	
7.0	0.004116	0.006004	0.009015	0.011065	0.015792	0.016336	0.017482	
8.0	0.004876	0.00741	0.009933	0.01255	0.016746	0.017723	0.018988	
9.0	0.006103	0.008127	0.01115	0.013662	0.016917	0.018779	0.019128	
10.0	0.00182	0.003629	0.006142	0.008068	0.010861	0.01075	0.011209	

	6.00%	ATM
1.0	0.094155	-0.014544
2.0	0.044432	-0.020044
3.0	0.0362	-0.010861
4.0	0.020892	-0.005385
5.0	0.018664	-0.00505
6.0	0.010584	-0.002457
7.0	0.017962	-0.000253
8.0	0.017848	0.000187
9.0	0.020145	0.001634

```
10.0  0.010722  0.001862
```

[334]:
```
print(f'Relative Mean Error is {round(np.mean(prices_rel_diff_example).mean() *␣
↪100, 2)}%')
```

```
Relative Mean Error is 0.88%
```

[362]:
```
vol_surface_dict[key_init]
```

[362]:
```
     -0.25% -0.50% -0.75% 0.00% 0.25% 0.50% 0.75% 1.00% 1.50%  2.00%  2.50%  \
1Y     30.6   31.8   34.6  33.2  38.7  46.4  55.1  64.3  82.8  101.1  119.0
2Y     39.9   43.8   49.0  41.0  46.8  53.9  61.1  68.0  81.7   95.6  109.5
3Y     52.9   57.0   61.4  51.9  55.5  60.6  65.9  71.2  82.8   95.4  108.5
4Y     61.3   64.4   67.4  60.0  62.1  65.7  69.7  74.0  83.7   94.6  106.2
5Y     64.8   66.8   68.5  63.9  65.5  68.4  71.7  75.3  83.5   92.9  102.9
6Y     66.0   67.2   68.0  65.5  66.9  69.4  72.2  75.3  82.4   90.5   99.2
7Y     66.5   67.2   67.5  66.3  67.6  69.8  72.3  75.0  81.3   88.5   96.3
8Y     66.8   67.1   67.1  66.8  68.0  69.9  72.2  74.6  80.3   86.8   93.8
9Y     66.9   66.9   66.7  67.0  68.1  69.9  71.9  74.1  79.3   85.3   91.9
10Y    66.8   66.7   66.2  67.1  68.1  69.7  71.5  73.6  78.3   83.9   90.1

      3.00%  3.50%  4.00%  4.50%  5.00%  6.00%   ATM
1Y    136.6  153.8  170.8  187.5  204.0  236.6  36.8
2Y    123.3  137.1  150.8  164.3  177.6  203.9  53.9
3Y    121.7  134.9  147.9  160.8  173.5  198.4  65.6
4Y    118.1  129.9  141.8  153.4  165.0  187.6  71.8
5Y    113.3  123.7  134.1  144.5  154.7  174.9  74.7
6Y    108.3  117.4  126.6  135.7  144.8  162.8  76.0
7Y    104.4  112.6  121.0  129.3  137.6  154.1  76.3
8Y    101.3  108.9  116.6  124.4  132.2  147.6  76.3
9Y     98.8  106.0  113.3  120.7  128.1  142.8  76.1
10Y    96.6  103.5  110.4  117.5  124.5  138.7  75.7
```

2.3 Volatility Interpolation

[722]:
```
vol_surface_int_dict = {key : {'vol_surface':pd.
↪DataFrame(index=vol_surface_dict[key].index,
                                          columns=vol_surface_dict[key].columns),
                               'price_surface': pd.
↪DataFrame(index=vol_surface_dict[key].index,
                                          columns=vol_surface_dict[key].columns)}
                        for key in vol_surface_dict.keys()}
```

[723]:
```
vol_surface_dict['2022-09-13'].shape
```

[723]:
```
(10, 18)
```

Create the 5-degree Spline to get interpolated volatilities per each smile:

[733]:

```
# for key in vol_surface_int_dict.keys():
#     print(key)
#     atm_replace = f'{round(qq_swaps_dict[key]["qq_swaps"][0]*100,2):.1f}'

#     vol_surface_dict[key].rename(columns={'ATM':f'{atm_replace}%'},␣
↪inplace=True)
#     vol_surface_dict[key].columns = [f'{float(val.replace("%", "")):.2f}%' for␣
↪val in vol_surface_dict[key].columns]

#     # Sort Strikes
#     tenors = [float(tenor.replace('Y', '')) for tenor in␣
↪list(vol_surface_dict[key].index)]
#     strikes = [float(strike.replace('%', '')) for strike in␣
↪list(set(list(vol_surface_dict[key].columns)))]

#     vol_surface_dict[key] = vol_surface_dict[key][[f'{strike:.2f}'+'%' for␣
↪strike in sorted(strikes)]]

#     # Convert to Numbers
#     strikes = [float(strike.replace('%', ''))/ 100 for strike in list(vol_temp.
↪columns)]
#     strikes_new = [np.arange(atm_i - 0.01, atm_i + 0.0125, 0.0025)
#                    for atm_i in qq_swaps_dict[key]["qq_swaps"]]

#     vol_temp_new = pd.DataFrame(index=range(len(tenors)),
#                                 columns=range(len(strikes_new[0])))

#     # Interpolate
#     for i in range(vol_surface_dict[key].shape[0]):
#         x = strikes
#         y = vol_surface_dict[key].iloc[i, :].values
#         f = interpolate.UnivariateSpline(x, y, k=5)

#         strike_i = strikes_new[i]
#         vol_new = f(strike_i)

#         for j in range(len(vol_new)):
#             try:
#                 vol_temp_new.iloc[i, j] = vol_new[j]
#             except IndexError:
#                 pass

#     # Address the NaNs
#     vol_temp_new = vol_temp_new.T.fillna(method='ffill').T
#     new_cols = [f'ATM-{25*i}' for i in \
#                 range(4, 0, -1)]+['ATM']+[f'ATM+{25*i}' for i in range(1, 5)]
```

```
#     # Format data
#     vol_temp_new.columns = new_cols
#     vol_temp_new.index = tenors

#     K = np.arange(-100, 125, 25)
#     vol_temp_new.columns = K

#     vol_surface_int_dict[key]['vol_surface'] = vol_temp_new

#     # Calculate the Price matrix
#     pv_temp_new = vol_temp_new.copy()
#     for i in range(len(strikes_new)):
#         for j in range(len(strikes_new[i])):
#             expiry = int(pv_temp_new.index[i])
#             strike = strikes_new[i][j]
#             vol = vol_temp_new.iloc[i, min(vol_temp_new.shape[1]-1, j)] / 10000
#             tau = 0.25
#             pv_temp_new.iloc[i, min(vol_temp_new.shape[1]-1, j)] = cap_pv(tau,
#                   qq_swaps_dict[key]['quarter_comps'],
#                   qq_swaps_dict[key]['dfs'],
#                   strike, vol, expiry) * 10000

#     pv_temp_new.columns = K
#     vol_surface_int_dict[key]['price_surface'] = pv_temp_new
```

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```

[1077]:
```python
# Save data to reuse it
# pickle_dict('vol_surface_int_dict', vol_surface_int_dict)

vol_surface_int_dict = pickle_dict('vol_surface_int_dict', serialize='rb')
```

Here is our interpolated volatility surface

[735]:
```python
vol_surface_int_dict['2021-11-09']['vol_surface']
```

[735]:

	-100	-75	-50	-25	0	25	\
1.0	36.879176	32.493270	31.078558	31.849497	34.406792	38.524366	
2.0	45.038807	39.570780	41.217824	50.636831	49.645523	51.153025	
3.0	53.188185	52.035331	62.270712	59.171975	57.925982	65.091370	
4.0	59.418613	65.471634	68.446312	62.967415	66.558435	71.730657	
5.0	64.120089	72.090413	68.590269	66.823569	70.891596	73.574952	
6.0	68.337491	73.728787	68.116796	69.050376	72.534178	74.609571	
7.0	71.396720	72.941407	68.191194	70.331133	73.367423	74.949181	
8.0	72.952254	71.925845	68.474500	70.990142	73.706698	74.882119	
9.0	73.454780	71.035634	68.677307	71.278418	73.625817	74.531800	
10.0	73.317750	70.322737	68.704529	71.304175	73.417072	74.114435	

	50	75	100
1.0	43.936706	50.179959	56.748803
2.0	59.987746	65.593634	67.686038
3.0	69.340089	70.856092	74.438572
4.0	73.394633	75.026378	79.674102
5.0	75.551575	78.036432	81.710332
6.0	76.187520	78.547846	82.157621
7.0	76.248337	78.579376	82.141944
8.0	76.019956	78.363969	81.842542
9.0	75.622761	77.942720	81.294113
10.0	75.144135	77.413305	80.628251

[737]:

```python
plt.figure(figsize=(10, 5))

key = list(fwd_curves_dict.keys())[0]
key_init = list(fwd_curves_dict.keys())[-1]

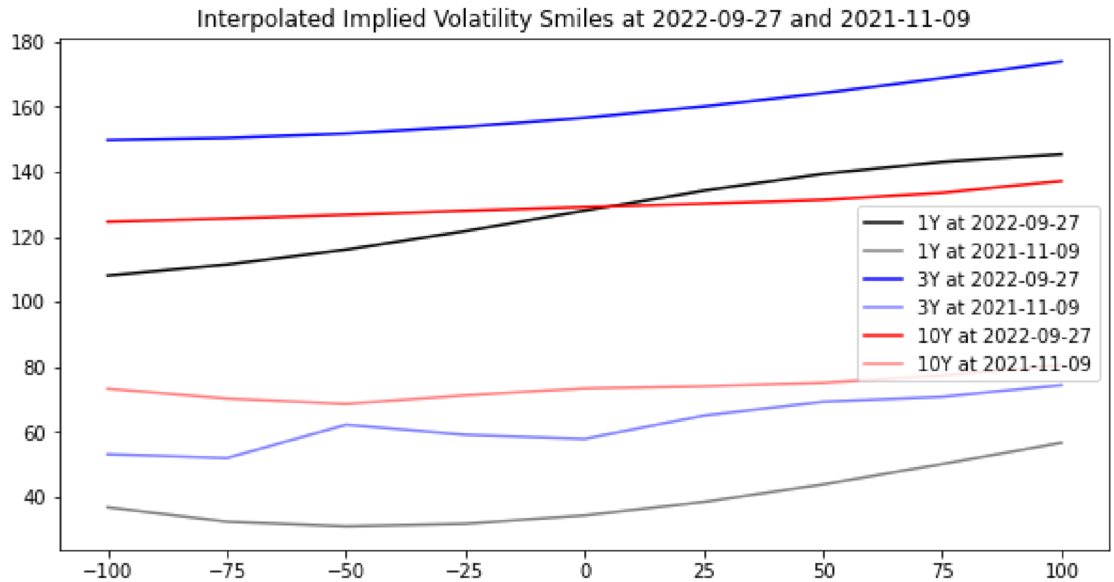
strikes = vol_surface_int_dict[key]['vol_surface'].columns

plt.plot(strikes, vol_surface_int_dict[key]['vol_surface'].iloc[0, :],␣
 ↪color='black')
plt.plot(strikes, vol_surface_int_dict[key_init]['vol_surface'].iloc[0, :],␣
 ↪color='black',
        alpha=0.5)
plt.plot(strikes, vol_surface_int_dict[key]['vol_surface'].iloc[2, :],␣
 ↪color='blue')
plt.plot(strikes, vol_surface_int_dict[key_init]['vol_surface'].iloc[2, :],␣
 ↪color='blue',
        alpha=0.5)
plt.plot(strikes, vol_surface_int_dict[key]['vol_surface'].iloc[9, :],␣
 ↪color='red')
plt.plot(strikes, vol_surface_int_dict[key_init]['vol_surface'].iloc[9, :],␣
 ↪color='red',
        alpha=0.5)


plt.legend([f'1Y at {key}', f'1Y at {key_init}',
           f'3Y at {key}', f'3Y at {key_init}',
           f'10Y at {key}', f'10Y at {key_init}'])

plt.title(f'Interpolated Implied Volatility Smiles at {key} and {key_init}')

plt.show()
```



[739]:

```
plt.figure(figsize=(10, 5))
cut_strikes = 0

key = list(fwd_curves_dict.keys())[0]
key_init = list(fwd_curves_dict.keys())[-1]

strikes = vol_surface_int_dict[key]['vol_surface'].index

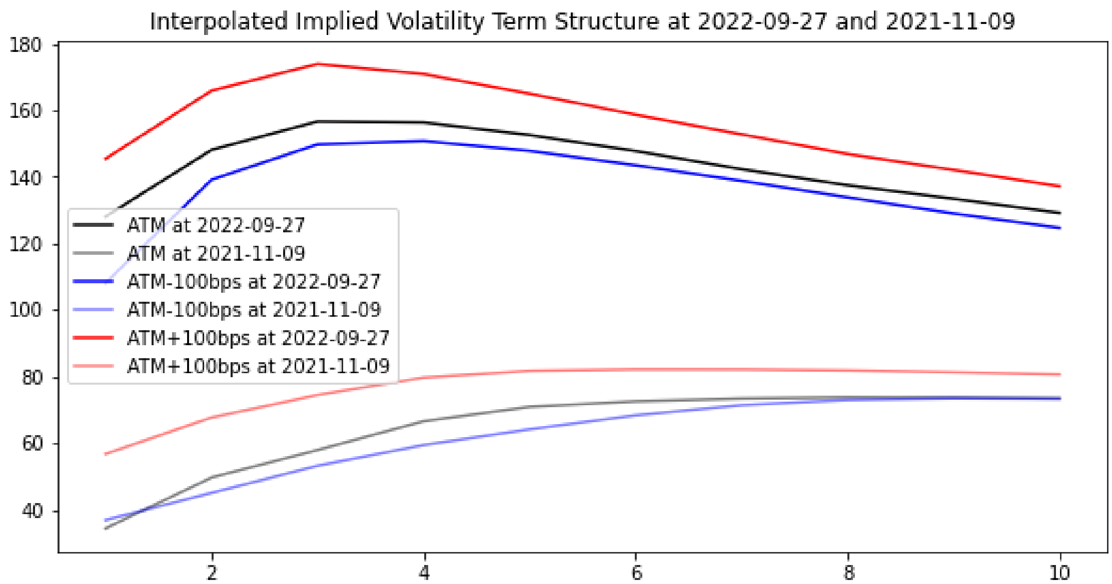
plt.plot(strikes, vol_surface_int_dict[key]['vol_surface'].loc[:, 0],␣
 ↪color='black')
plt.plot(strikes, vol_surface_int_dict[key_init]['vol_surface'].loc[:, 0],␣
 ↪color='black',
        alpha=0.5)
plt.plot(strikes, vol_surface_int_dict[key]['vol_surface'].loc[:, -100],␣
 ↪color='blue')
plt.plot(strikes, vol_surface_int_dict[key_init]['vol_surface'].loc[:, -100],␣
 ↪color='blue',
        alpha=0.5)
plt.plot(strikes, vol_surface_int_dict[key]['vol_surface'].loc[:, 100],␣
 ↪color='red')
plt.plot(strikes, vol_surface_int_dict[key_init]['vol_surface'].loc[:, 100],␣
 ↪color='red',
        alpha=0.5)


plt.legend([f'ATM at {key}', f'ATM at {key_init}',
           f'ATM-100bps at {key}', f'ATM-100bps at {key_init}',
```

```
            f'ATM+100bps at {key}', f'ATM+100bps at {key_init}'])

plt.title(f'Interpolated Implied Volatility Term Structure at {key} and␣
 ↪{key_init}')

plt.show()
```



And we also have price surface

[740]:
```
vol_surface_int_dict['2022-09-27']['price_surface']
```

[740]:

	-100	-75	-50	-25	0	\
1.0	102.385396	83.076719	66.563238	53.018391	42.171639	
2.0	230.730345	196.680662	166.418339	140.153147	117.822566	
3.0	381.033871	332.652587	288.944216	250.257436	216.669393	
4.0	536.625009	474.478154	417.692636	366.717911	321.747452	
5.0	690.488985	615.010828	545.559665	482.673041	426.639485	
6.0	840.829866	752.529218	670.889876	596.525770	529.808275	
7.0	987.752204	886.605542	792.580746	706.528986	629.081644	
8.0	1128.682612	1017.097670	912.562162	815.629461	726.951135	
9.0	1263.005971	1141.077410	1027.475677	922.014602	824.298413	
10.0	1393.514829	1260.860140	1137.315841	1022.546933	915.848070	

	25	50	75	100
1.0	33.470194	26.335585	20.362638	15.400175
2.0	99.136709	83.655232	70.864219	60.237245
3.0	187.987822	163.802797	143.559792	126.630281
4.0	282.693914	249.217912	220.789702	196.759216

5.0	377.453854	334.829731	298.253465	267.054576
6.0	470.804646	419.276552	374.728006	336.476509
7.0	560.529855	500.768158	449.307616	405.335392
8.0	647.304926	577.626207	518.815680	470.702217
9.0	734.610929	654.863829	588.663330	537.186852
10.0	817.266796	728.862257	655.159187	598.593369

2.4 3D Volatility Surface Plot

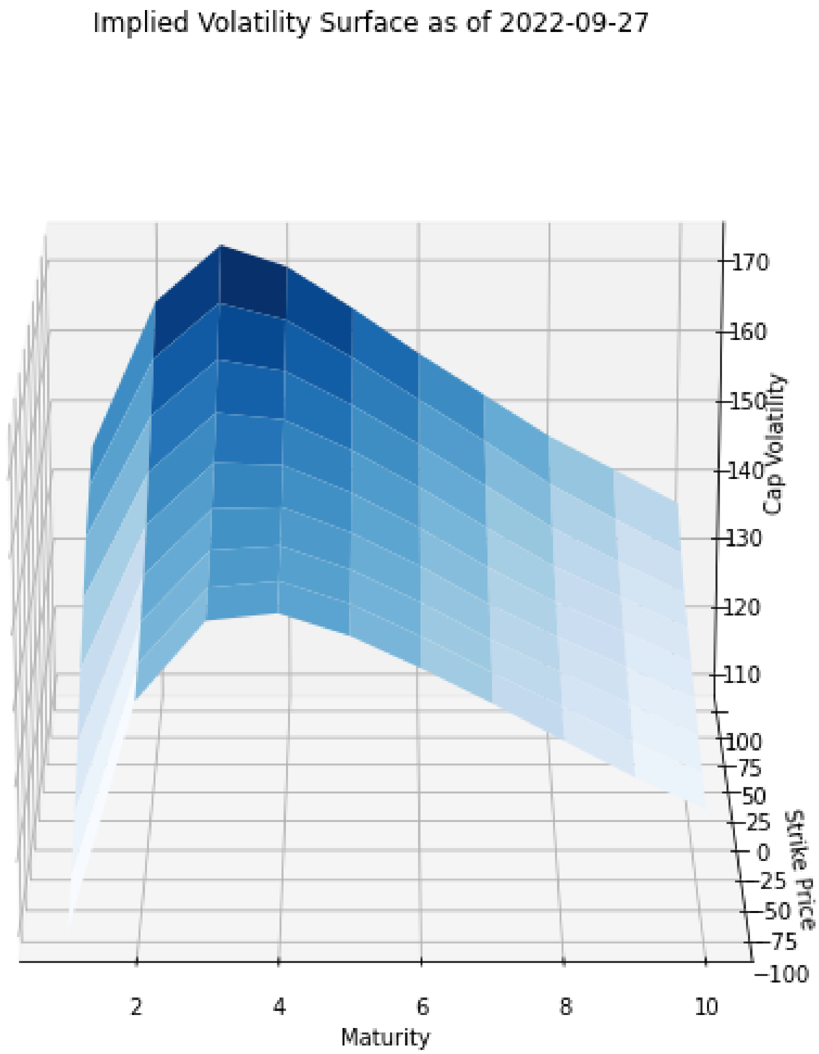
[741]:
```
Ts, Ks = np.meshgrid(range(1,11), K)
vol_matrix = np.zeros((len(range(1,11)), len(K)))

def vol_plot(tenor, strike):
    return vol_surface_int_dict[key]['vol_surface'].loc[tenor, strike]

vol_3d=np.vectorize(vol_plot)
Vol = vol_3d(Ts, Ks)

fig = plt.figure(figsize=(10,7))
ax = Axes3D(fig)
ax.plot_surface(Ts, Ks, Vol, cmap = "Blues")
ax.view_init(20,270)
plt.xlabel('Maturity')
plt.ylabel('Strike Price')
ax.set_zlabel('Cap Volatility')
plt.title(f'Implied Volatility Surface as of {key}')

plt.show()
```



[742]:
```
Ts, Ks = np.meshgrid(range(1,11), K)
vol_matrix = np.zeros((len(range(1,11)), len(K)))

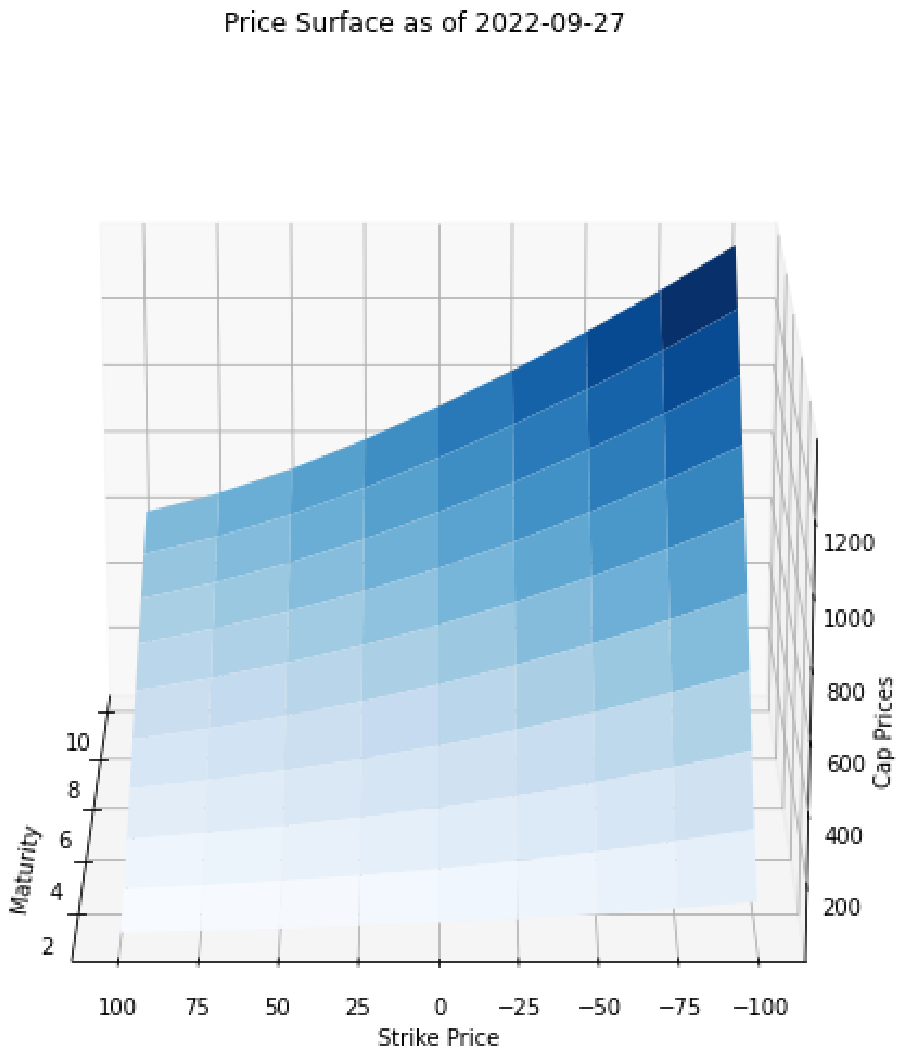
def vol_plot(tenor, strike):
    return vol_surface_int_dict[key]['price_surface'].loc[tenor, strike]

vol_3d=np.vectorize(vol_plot)
Vol = vol_3d(Ts, Ks)

fig = plt.figure(figsize=(10,7))
ax = Axes3D(fig)
ax.plot_surface(Ts, Ks, Vol, cmap = "Blues")
```

```
ax.view_init(20,180)
plt.xlabel('Maturity')
plt.ylabel('Strike Price')
ax.set_zlabel('Cap Prices')
plt.title(f'Price Surface as of {key}')

plt.show()
```



2.5 Risk-Neutral Probabilty Matrix

We solve Breeden-Litzenberger Numerically

$$f(K) = e^{rT}\frac{\partial^2 C(K,T)}{\partial K^2} \tag{1}$$

$$\approx e^{rT}\frac{C(K+\Delta_K,T) - 2C(K,T) + C(K-\Delta_K,T)}{(\Delta_K)^2} \tag{2}$$

[743]:
```
rn_pdf_dict = {key : pd.DataFrame() for key in vol_surface_int_dict.keys()}
```

Here we get annual DFs and after that convert them to ZC Bonds, because probablities are measured in future terms

[751]:
```
def breedenlitzenberger(pv_matrix, FVs):
    pdf_implied = pv_matrix.copy()
    tenors = range(1, len(pv_matrix.loc[:,0])+1)
    strikes = pv_matrix.columns
    for t in tenors:
        for k in strikes:
            if k == -100 or k == 100:
                pdf_implied.loc[t,k] = "NaN"
            else:
                pdf_implied.loc[t,k] = FVs[t-1] * (pv_matrix.loc[t, k + 25] - 2␣
 ↪* pv_matrix.loc[t, k] + pv_matrix.loc[t, k - 25] ) / (25**2)
    return pdf_implied
```

[752]:
```
def prob(bl_pdf):
    prob_values = bl_pdf.copy()
    for t in range(1,11):
        prob_values.loc[t,:] = abs(bl_pdf.loc[t,:])/sum(abs(bl_pdf.loc[t,:]))
    return prob_values
```

[753]:
```
# for key in rn_pdf_dict.keys():
#     print(key)
#     DFs_yearly = [qq_swaps_dict[key]['dfs'][3 + 4*i] for i in range(10)]
#     FVs = [1/DFs_yearly[i] for i in range(10)]

#     rn_pdf_dict[key] =␣
 ↪prob(breedenlitzenberger(vol_surface_int_dict[key]['price_surface'],
#                                           FVs).iloc[:,1:-1])
```

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```

[1078]:

```python
# Save data to reuse it
# pickle_dict('rn_pdf_dict', rn_pdf_dict)

rn_pdf_dict = pickle_dict('rn_pdf_dict', serialize='rb')
```

Let's see the example:

[1829]:

```python
rn_pdf_dict['2022-09-27']
```

[1829]:

	-75	-50	-25	0	25	50	75
1.0	0.194839	0.206928	0.188070	0.149538	0.109216	0.080973	0.070436
2.0	0.161696	0.170652	0.167983	0.155606	0.136807	0.114866	0.092391
3.0	0.148574	0.159660	0.162113	0.156000	0.142966	0.125335	0.105352
4.0	0.140657	0.152449	0.157525	0.155233	0.146329	0.132431	0.115376
5.0	0.136113	0.148253	0.154769	0.154653	0.148185	0.136584	0.121442
6.0	0.133095	0.145362	0.152782	0.154126	0.149364	0.139454	0.125817
7.0	0.124564	0.139451	0.150494	0.155586	0.153742	0.145190	0.130973
8.0	0.111065	0.119783	0.130049	0.142302	0.157039	0.171230	0.168533

```
9.0    0.118191  0.115549  0.109931  0.113960  0.141094  0.192281  0.208993
10.0   0.119734  0.115331  0.106061  0.106686  0.133748  0.193214  0.225227
```

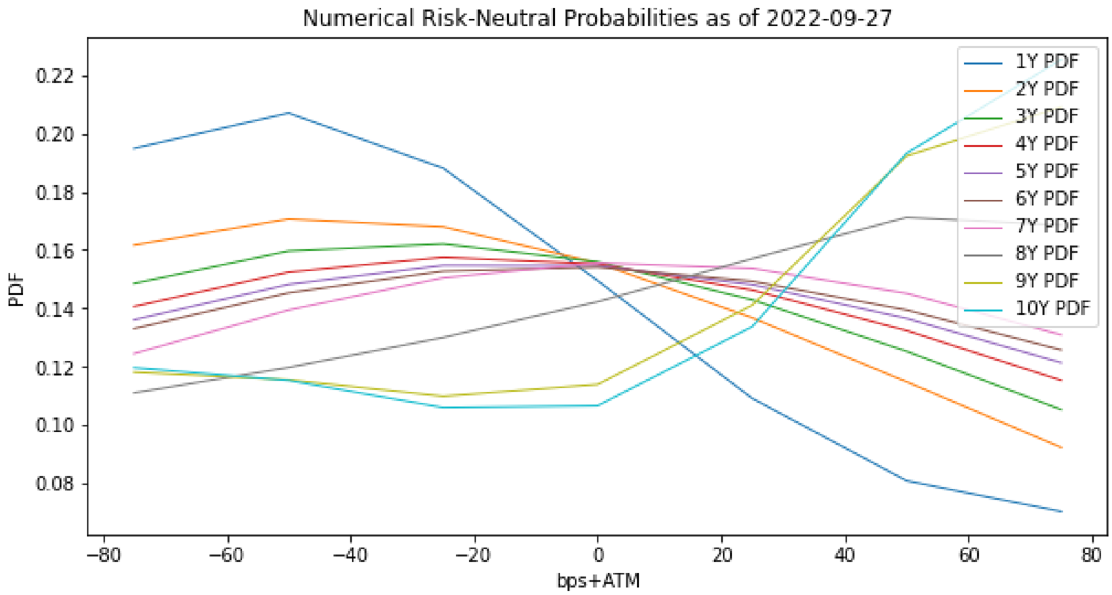
And we see that they sum up to 1

[755]:
```
rn_pdf_dict['2022-09-27'].sum(axis=1)
```

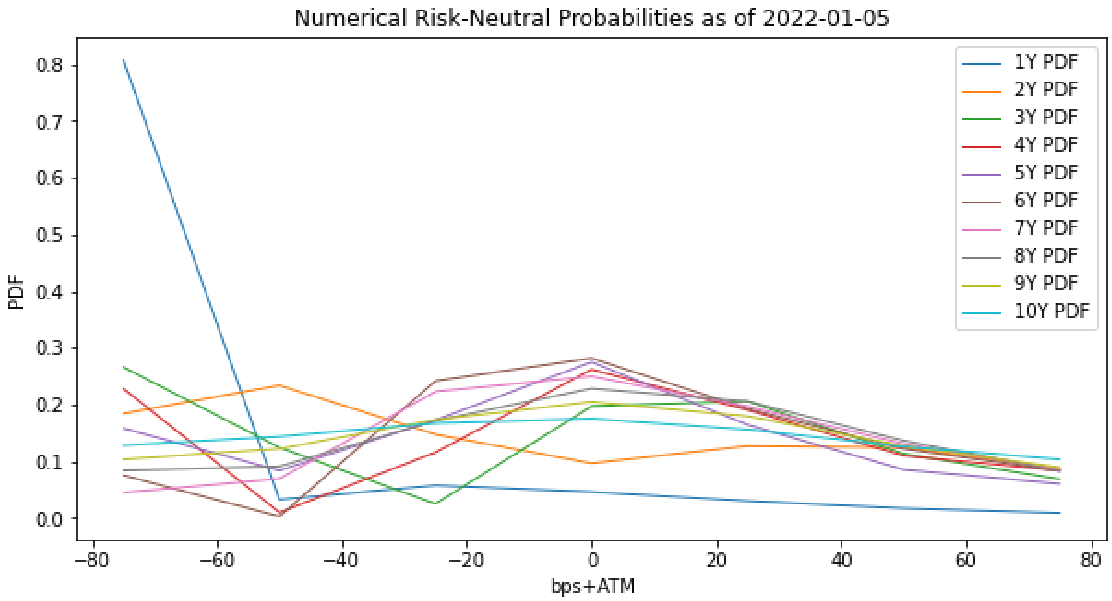
[755]:
```
1.0     1.0
2.0     1.0
3.0     1.0
4.0     1.0
5.0     1.0
6.0     1.0
7.0     1.0
8.0     1.0
9.0     1.0
10.0    1.0
dtype: float64
```

2.5.1 PDFs Charts

[756]:
```
key = list(fwd_curves_dict.keys())[0]
key_init = list(fwd_curves_dict.keys())[-1]

new_K = K[1:-1]
fig = plt.figure(figsize=(10,5))
for i in range(1,11):
    plt.plot(new_K, rn_pdf_dict[key].loc[i,:], linewidth = 1, linestyle='-',␣
 ↪label=str(i)+'Y PDF')
plt.gca().set_prop_cycle(None)

plt.xlabel("bps+ATM")
plt.ylabel("PDF")
plt.title(f'Numerical Risk-Neutral Probabilities as of {key}')
plt.legend()
plt.show()
```



[757]:
```
key = list(fwd_curves_dict.keys())[0]
key_init = list(fwd_curves_dict.keys())[-42]

new_K = K[1:-1]
fig = plt.figure(figsize=(10,5))
for i in range(1,11):
    plt.plot(new_K, rn_pdf_dict[key_init].loc[i,:], linewidth = 1,␣
 ↪linestyle='-', label=str(i)+'Y PDF')
plt.gca().set_prop_cycle(None)

plt.xlabel("bps+ATM")
plt.ylabel("PDF")
plt.title(f'Numerical Risk-Neutral Probabilities as of {key_init}')
plt.legend()
plt.show()
```



3 Transition Matrix

Now let's estimate the transition matrix

[758]:
```
transition_matrix_dict = {key : pd.DataFrame() for key in rn_pdf_dict.keys()}
```

[805]:
```
# for key in transition_matrix_dict.keys():
#     print(key)
#     strks = rn_pdf_dict[key].T.shape[0]

#     P = pd.DataFrame(np.zeros((strks,strks)),
#                      index = rn_pdf_dict[key].T.index,
#                      columns = rn_pdf_dict[key].T.index)

#     P.loc[0,:] = rn_pdf_dict[key].T.iloc[:,0].values
#     P = P.values
#     pi_0 = [0 if strike != 0 else 1 for strike in rn_pdf_dict[key].T.index]
#     pi_1 = rn_pdf_dict[key].T.iloc[:, 0].values

#     def objective(P):
#         x1, x2, x3, x4, x5, x6, x7 = P.reshape(strks, strks)

#         P_tot = np.vstack([x1, x2, x3, x4, x5, x6, x7])

#         res1 = 0
```

```
#         for i in range(strks):
#             res1 = res1 + np.sum(np.abs(rn_pdf_dict[key].T.iloc[:,i + 1].
↪values - \
#                                         np.dot(pi_0, np.linalg.
↪matrix_power(P_tot, i+ 2)))**1)

#         cross_results = np.zeros(strks)

#         for i in range(strks):
#             for j in range(strks - i):
#                 cross_results[i] = cross_results[i] + \
#                                     np.sum(np.abs(rn_pdf_dict[key].T.iloc[:,␣
↪strks - j].values - \
#                                               np.dot(rn_pdf_dict[key].T.iloc[:
↪,strks - 1 - i - j].values,
#                                                 np.linalg.matrix_power(P_tot,␣
↪i + 1)))**1)
#         res = res1 + sum(cross_results)
#         return res


#     constr = []
#     middle_strk = strks // 2

#     res = optimize.minimize(objective, P, method='SLSQP',
#                         tol=1e-5,bounds=[(0.0, 1.0)]*strks*strks,
#         constraints=({'type':'eq', 'fun': lambda X: 1.0 - sum([X[strks * 0 +␣
↪i] for i in range(strks)])},
#                      {'type':'eq', 'fun': lambda X: 1.0 - sum([X[strks * 1 +␣
↪i] for i in range(strks)])},
#                      {'type':'eq', 'fun': lambda X: 1.0 - sum([X[strks * 2 +␣
↪i] for i in range(strks)])},

#                      {'type':'eq', 'fun': lambda X: pi_1[0]-X[strks *␣
↪middle_strk + 0]},
#                      {'type':'eq', 'fun': lambda X: pi_1[1]-X[strks *␣
↪middle_strk + 1]},
#                      {'type':'eq', 'fun': lambda X: pi_1[2]-X[strks *␣
↪middle_strk + 2]},
#                      {'type':'eq', 'fun': lambda X: pi_1[3]-X[strks *␣
↪middle_strk + 3]},
#                      {'type':'eq', 'fun': lambda X: pi_1[4]-X[strks *␣
↪middle_strk + 4]},
#                      {'type':'eq', 'fun': lambda X: pi_1[5]-X[strks *␣
↪middle_strk + 5]},
```

```
#                        {'type':'eq', 'fun': lambda X: pi_1[6]-X[strks *␣
↪middle_strk + 6]},

#                        {'type':'eq', 'fun': lambda X: 1.0 - sum([X[strks * 4 +␣
↪i] for i in range(strks)])},
#                        {'type':'eq', 'fun': lambda X: 1.0 - sum([X[strks * 5 +␣
↪i] for i in range(strks)])},
#                        {'type':'eq', 'fun': lambda X: 1.0 - sum([X[strks * 6 +␣
↪i] for i in range(strks)])}
#                        ))

#     rec_df = pd.DataFrame(res.x.reshape(strks, strks), index=rn_pdf_dict[key].
↪T.index,
#                        columns=rn_pdf_dict[key].T.index)

#     transition_matrix_dict[key] = rec_df
```

```
2022-09-27
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2021-11-10
2021-11-09
```
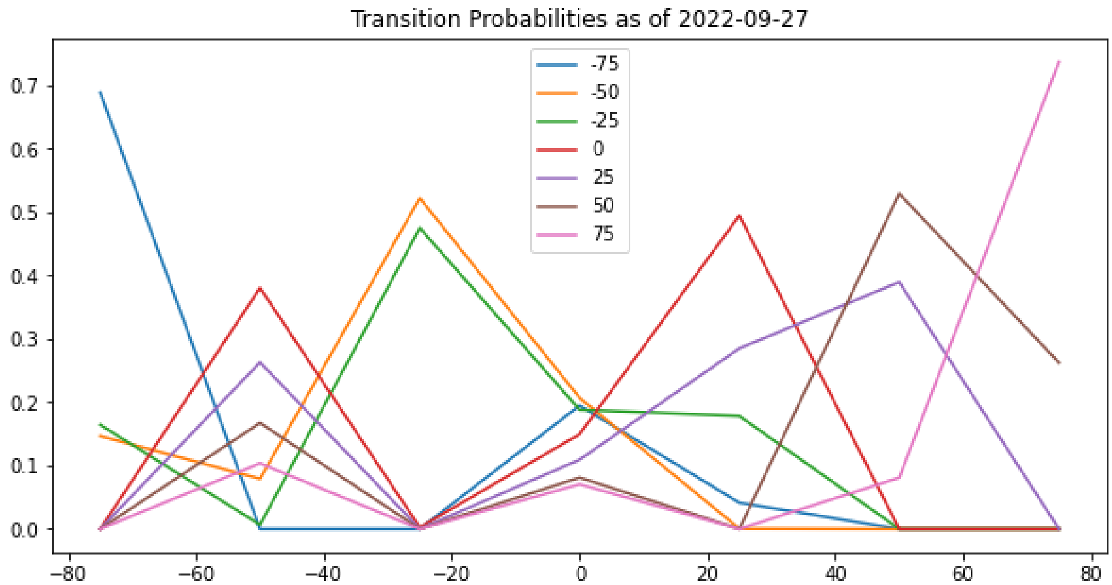

[1079]:
```
# Save data to reuse it
# pickle_dict('transition_matrix_dict', transition_matrix_dict)

transition_matrix_dict = pickle_dict('transition_matrix_dict', serialize='rb')
```

Let's look at the example of the Transition Matrix

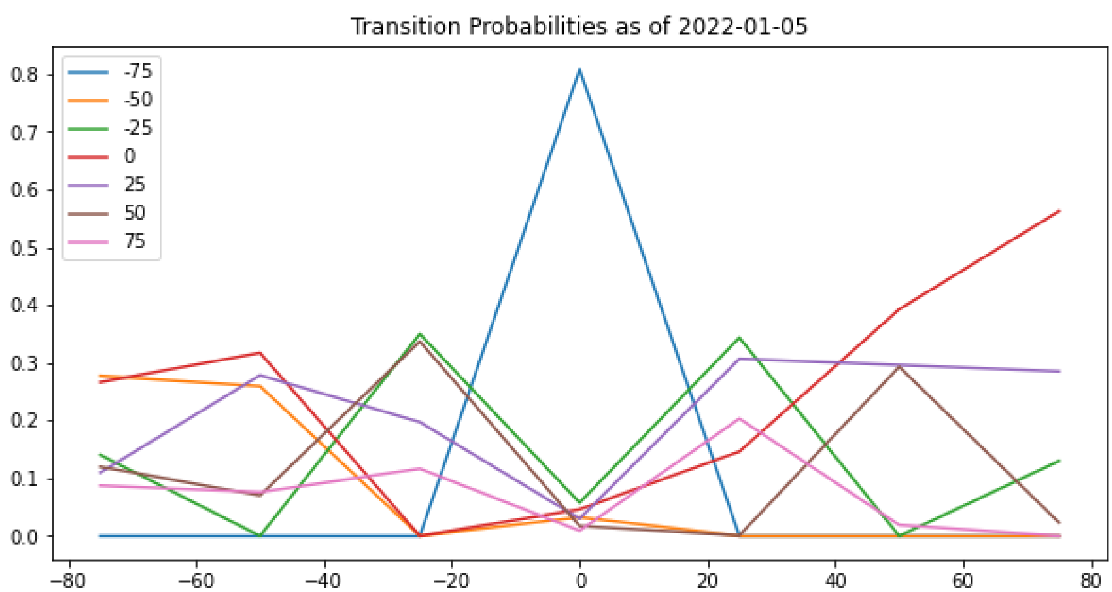
[806]:
```
transition_matrix_dict['2022-09-27']
```

[806]:
```
              -75           -50           -25             0            25  \
-75  6.889143e-01  1.467364e-01  1.643493e-01  1.980308e-17  1.107881e-17
-50  6.511673e-18  7.909911e-02  6.630280e-03  3.804476e-01  2.629488e-01
-25  3.821182e-17  5.223131e-01  4.751570e-01  2.529892e-03  4.680678e-18
 0   1.948386e-01  2.069281e-01  1.880702e-01  1.495382e-01  1.092160e-01
 25  4.131939e-02  4.997899e-04  1.783062e-01  4.948350e-01  2.850396e-01
 50  2.927340e-17  8.705765e-18  3.937417e-17  4.404462e-18  3.897847e-01
 75  1.659542e-17  5.438438e-18  7.004841e-18  2.819991e-17  3.079491e-17

               50            75
-75  8.772537e-18  5.080904e-18
-50  1.673437e-01  1.035306e-01
-25  5.634808e-18  6.990003e-18
 0   8.097337e-02  7.043557e-02
 25  2.713681e-17  2.346598e-18
 50  5.294402e-01  8.077507e-02
 75  2.624750e-01  7.375250e-01
```

[807]:
```
key = list(fwd_curves_dict.keys())[0]
key_init = list(fwd_curves_dict.keys())[-42]

plt.figure(figsize=(10, 5))
plt.plot(transition_matrix_dict[key])
plt.title(f'Transition Probabilities as of {key}')
plt.legend(rec_df.columns)
plt.show();
```

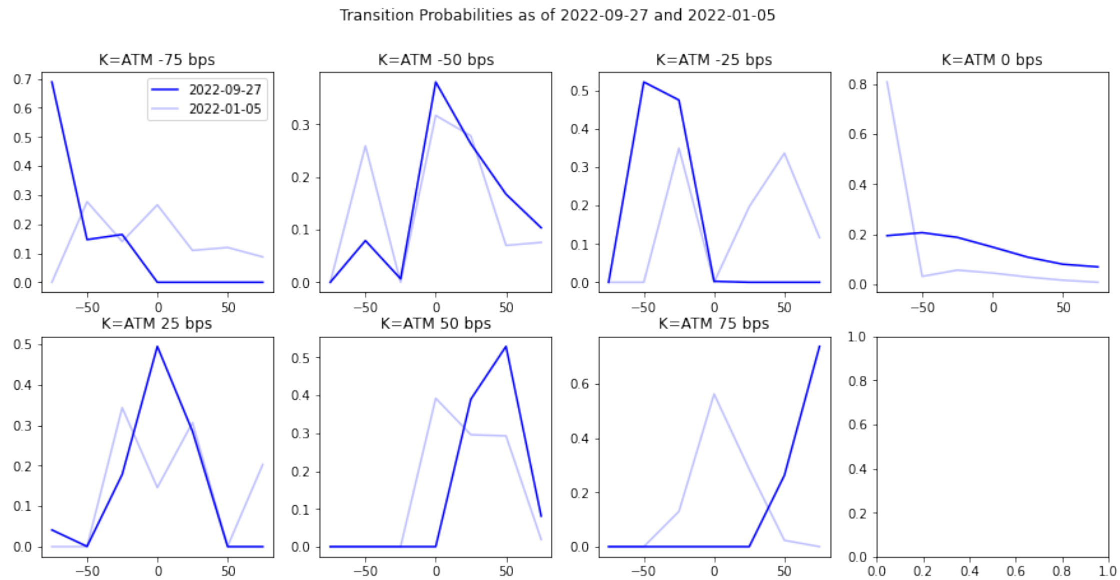


[808]:
```
key = list(fwd_curves_dict.keys())[0]
key_init = list(fwd_curves_dict.keys())[-42]

plt.figure(figsize=(10, 5))
plt.plot(transition_matrix_dict[key_init])
plt.title(f'Transition Probabilities as of {key_init}')
plt.legend(rec_df.columns)
plt.show();
```



Let's look at the slices

[896]:
```
fig, axes = plt.subplots(nrows=2, ncols=4, figsize=(15, 7))

fig.suptitle(f'Transition Probabilities as of {key} and {key_init}')
axes = axes.flatten()

for strike in transition_matrix_dict[key].index:
    idx = list(transition_matrix_dict[key].index).index(strike)
    axes[idx].plot(transition_matrix_dict[key].loc[strike, :], color='blue')
    axes[idx].plot(transition_matrix_dict[key_init].loc[strike, :],␣
↪color='blue', alpha=0.25)
    if idx == 0:
        axes[idx].legend([key, key_init])
    axes[idx].set_title(f'K=ATM {strike} bps')

plt.show();
```



4 Recovered Probabilities

[921]:
```
rec_prob_dict = {key : {'rec_df' : pd.
↪DataFrame(index=transition_matrix_dict[key].index,
                                    columns=transition_matrix_dict[key].columns),
                        'eigenval' : 0,
                        'eigenvec' : 0,
```

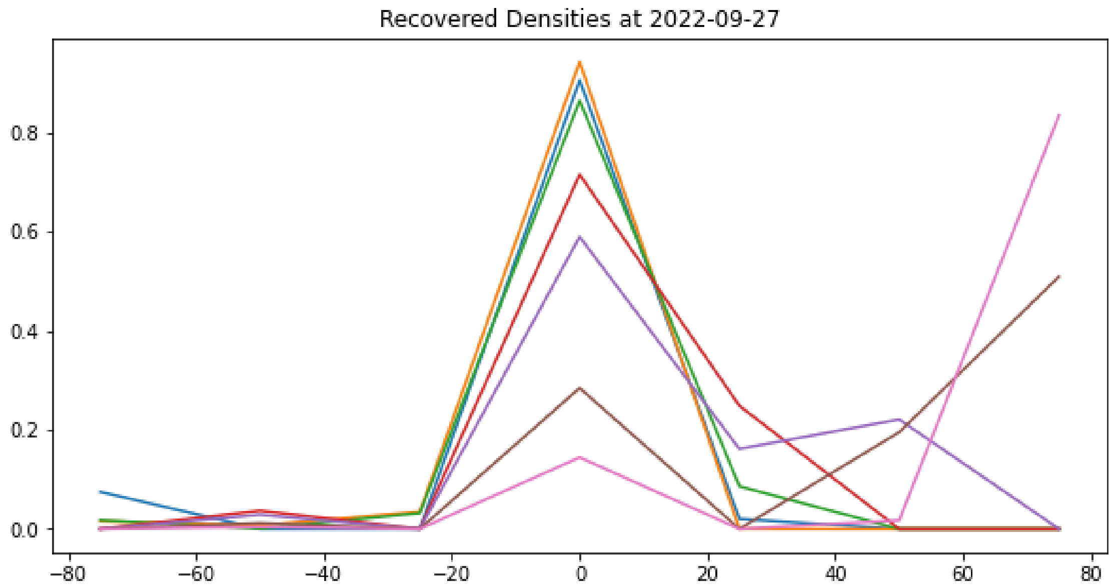
```
                        'expected_value': 0}
                  for key in transition_matrix_dict.keys()}
```

[937]:
```
for key in rec_prob_dict.keys():
    eigenvals = np.linalg.eig(transition_matrix_dict[key])[0]
    eigenvecs = np.linalg.eig(transition_matrix_dict[key])[1]
    delta = eigenvals[0]
    z = 1 / eigenvecs[0]
    rec_prob_dict[key]['eigenval'] = eigenvals
    rec_prob_dict[key]['eigenvec'] = eigenvecs

    for i in range(len(transition_matrix_dict[key])):
        for j in range(len(transition_matrix_dict[key])):
            rec_prob_dict[key]['rec_df'].iloc[i, j] = np.abs(1/delta *␣
 ↪transition_matrix_dict[key].iloc[i, j] * z[i] / z[j])
    # Normalize data
    rec_prob_dict[key]['rec_df'] = rec_prob_dict[key]['rec_df'] /␣
 ↪rec_prob_dict[key]['rec_df'].sum(axis=0)

    rec_prob_dict[key]['expected_value'] = np.sum(rec_prob_dict[key]['rec_df'].
 ↪index * \
                                                  rec_prob_dict[key]['rec_df'].
 ↪sum(axis=1))
```

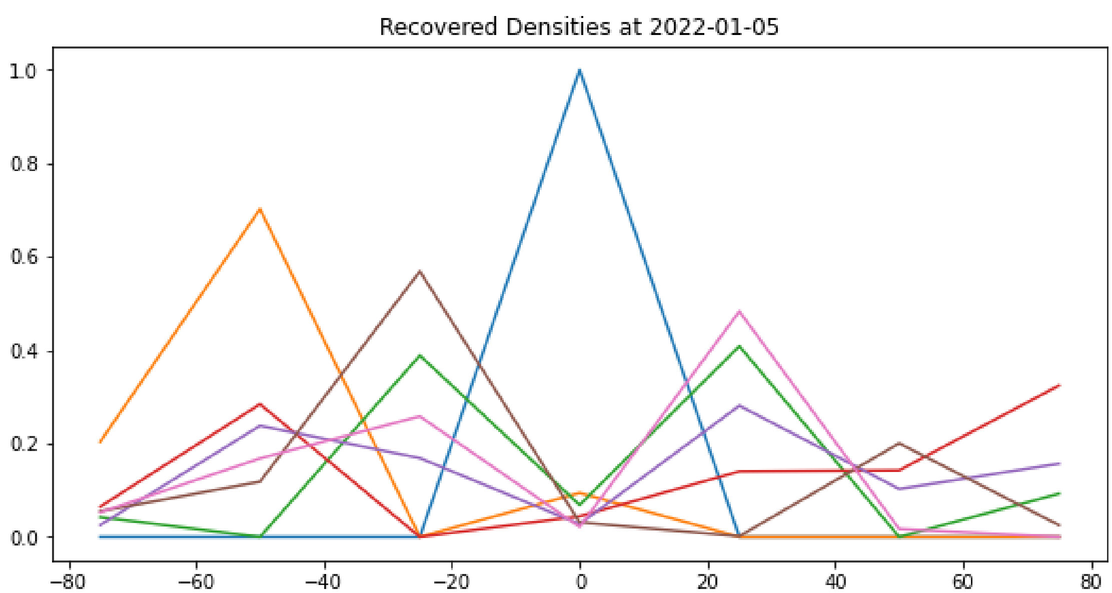
[1080]:
```
# Save data to reuse it
# pickle_dict('rec_prob_dict', rec_prob_dict)

rec_prob_dict = pickle_dict('rec_prob_dict', serialize='rb')
```

Let's look at the examples:

[924]:
```
key = list(fwd_curves_dict.keys())[0]
key_init = list(fwd_curves_dict.keys())[-42]

plt.figure(figsize=(10, 5))
plt.plot(rec_prob_dict[key]['rec_df'])
plt.title(f'Recovered Densities at {key}')
plt.show()
```

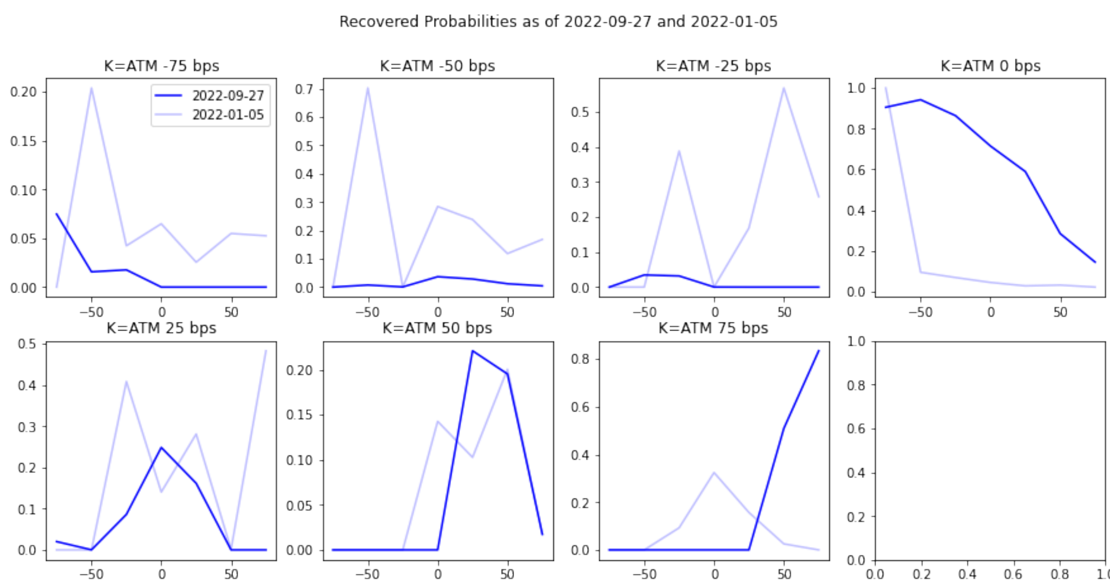


[925]:
```
key = list(fwd_curves_dict.keys())[0]
key_init = list(fwd_curves_dict.keys())[-42]

plt.figure(figsize=(10, 5))
plt.plot(rec_prob_dict[key_init]['rec_df'])
plt.title(f'Recovered Densities at {key_init}')
plt.show()
```



And see how our expected value moves through the time:

[926]:
```
fig, axes = plt.subplots(nrows=2, ncols=4, figsize=(15, 7))

fig.suptitle(f'Recovered Probabilities as of {key} and {key_init}')
axes = axes.flatten()

for strike in rec_prob_dict[key]['rec_df'].index:
    idx = list(rec_prob_dict[key]['rec_df'].index).index(strike)
    axes[idx].plot(rec_prob_dict[key]['rec_df'].loc[strike, :], color='blue')
    axes[idx].plot(rec_prob_dict[key_init]['rec_df'].loc[strike, :],␣
↪color='blue', alpha=0.25)
    if idx == 0:
        axes[idx].legend([key, key_init])
    axes[idx].set_title(f'K=ATM {strike} bps')

plt.show();
```



5 Signal Generation

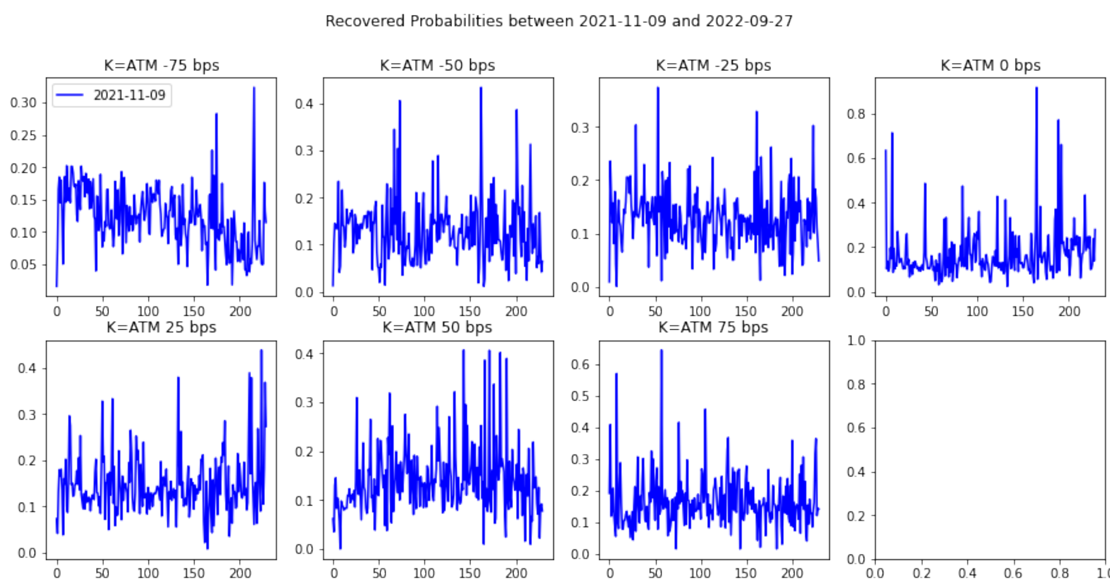
Let's also create time-series and look at the data in order to make a signal

[1832]:
```
rec_prob_df = pd.DataFrame(index=list(rec_prob_dict.keys()),
                           columns=list(rec_prob_dict[key]['rec_df'].columns))
```

[1833]:

```
for key in rec_prob_dict.keys():
    rec_prob_df.loc[key, :] = rec_prob_dict[key]['rec_df'].sum(axis=1) /␣
↪rec_prob_dict[key]['rec_df'].sum(axis=1).sum()
    rec_prob_df.loc[key, 'exp_val'] = rec_prob_dict[key]['expected_value']
```

[1834]:
```
fig, axes = plt.subplots(nrows=2, ncols=4, figsize=(15, 7))

fig.suptitle(f'Recovered Probabilities between {rec_prob_df.index[-1]} and␣
↪{rec_prob_df.index[0]}')
axes = axes.flatten()

for strike in rec_prob_df.iloc[:, :-1].columns:
    idx = list(rec_prob_df.iloc[:, :-1].columns).index(strike)
    axes[idx].plot(rec_prob_df.loc[:, strike].values, color='blue')
    if idx == 0:
        axes[idx].legend([key, key_init])
    axes[idx].set_title(f'K=ATM {strike} bps')

plt.show();
```



6 Kalman Filter

We see that we get very noisy data for the returns. Therefore, it makes sense for us to use some smoothing techniques

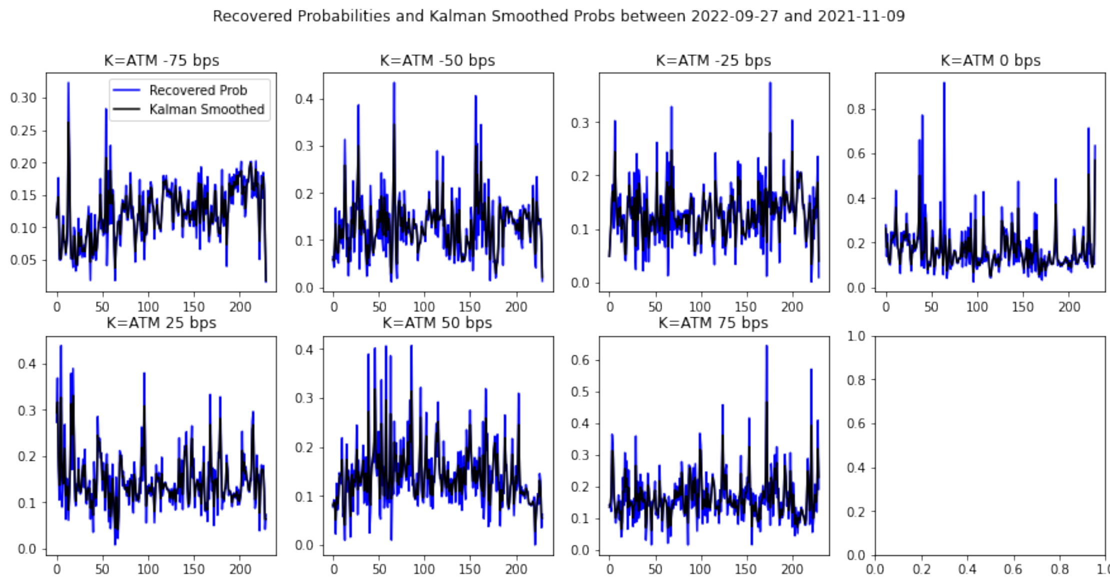
[1151]:
```
# for column in rec_prob_df.columns[:-1]:
```

```
#     outlier_thresh = 0.95

#     # Treat y as position, and that y-dot is
#     # an unobserved state - the velocity,
#     # which is modelled as changing slowly (inertia)

#     # state vector [y,
#     #               y_dot]

#     # transition_matrix =  [[1, dt],
#     #                        [0, 1]]

#     observation_matrix = np.asarray([[1, 0]])

#     # observations:
#     t = list(range(len(rec_prob_df.index)))

#     # dt betweeen observations:
#     dt = [np.mean(np.diff(t))] + list(np.diff(t))
#     transition_matrices = np.asarray([[[1, each_dt],[0, 1]]
#                                       for each_dt in dt])

#     # observations
#     y = np.transpose(np.asarray([list(rec_prob_df.loc[:, column].values)]))

#     y = np.ma.array(y)

#     leave_1_out_cov = []

#     for i in range(len(y)):
#         y_masked = np.ma.array(copy.deepcopy(y))
#         y_masked[i] = np.ma.masked

#         kf1 = KalmanFilter(transition_matrices = transition_matrices,
#                            observation_matrices = observation_matrix)

#         kf1 = kf1.em(y_masked)

#         leave_1_out_cov.append(kf1.observation_covariance[0,0])

#     # Find indexes that contributed excessively to observation covariance
#     outliers = (leave_1_out_cov / np.mean(leave_1_out_cov)) < outlier_thresh

#     for i in range(len(outliers)):
#         if outliers[i]:
#             y[i] = np.ma.masked
```

```
#     kf1 = KalmanFilter(transition_matrices = transition_matrices,
#                        observation_matrices = observation_matrix)

#     kf1 = kf1.em(y)

#     (smoothed_state_means, smoothed_state_covariances) = kf1.smooth(y)

#     rec_prob_df.loc[:, f'{column}_K'] = smoothed_state_means[:,0]
```

[1849]:
```
# rec_prob_df.to_pickle('rec_prob_df.pickle')

rec_prob_df = pd.read_pickle('rec_prob_df.pickle')
rec_prob_df = rec_prob_df.sort_index()
```

[1850]:
```
fig, axes = plt.subplots(nrows=2, ncols=4, figsize=(15, 7))

fig.suptitle(f'Recovered Probabilities and Kalman Smoothed Probs between␣
 ↪{rec_prob_df.index[-1]} and {rec_prob_df.index[0]}')
axes = axes.flatten()

for strike in rec_prob_df.iloc[:, :7].columns:
    idx = list(rec_prob_df.iloc[:, :-2].columns).index(strike)
    axes[idx].plot(rec_prob_df.loc[:, strike].values, color='blue')
    axes[idx].plot(rec_prob_df.loc[:, str(strike)+'_K'].values, color='black')

    if idx == 0:
        axes[idx].legend(['Recovered Prob', 'Kalman Smoothed'])
    axes[idx].set_title(f'K=ATM {strike} bps')

plt.show();
```



7 Test for the Forward Returns

Let's calculate the mean swap rate

[1851]:
```
for date in rec_prob_df.index:
    rec_prob_df.loc[date, 'mean_swap'] = yield_curve_dict[date].iloc[:, 15:25].
↪mean(axis=1).values[0]
```

[1852]:
```
rec_prob_df['mean_swap_fwd'] = rec_prob_df['mean_swap'].shift(-30)
rec_prob_df = rec_prob_df.dropna()
```

[1853]:
```
rec_prob_df.head()
```

[1853]:

	-75	-50	-25	0	25	50	\
2021-11-09	0.1145	0.064408	0.048971	0.279243	0.273139	0.077766	
2021-11-10	0.131841	0.041973	0.083144	0.138381	0.368266	0.09199	
2021-11-11	0.176407	0.086784	0.110993	0.237541	0.18058	0.086456	
2021-11-12	0.050257	0.168266	0.166233	0.123779	0.105405	0.022419	
2021-11-15	0.04894	0.059682	0.18327	0.102861	0.163048	0.125743	

	75	exp_val	-75_K	-50_K	-25_K	0_K	\
2021-11-09	0.141974	58.328462	0.118902	0.058253	0.049113	0.263159	
2021-11-10	0.144406	73.999101	0.136265	0.055833	0.083103	0.178280	
2021-11-11	0.121239	-16.900236	0.147011	0.092682	0.116625	0.197905	
2021-11-12	0.36364	102.834768	0.071179	0.134441	0.158274	0.134449	
2021-11-15	0.316456	160.028576	0.050596	0.080003	0.170425	0.108461	

```
                 25_K      50_K      75_K  mean_swap  mean_swap_fwd
2021-11-09  0.293772  0.080696  0.135161    0.86625        1.00809
2021-11-10  0.316499  0.086916  0.144551    0.97460        1.01170
2021-11-11  0.197829  0.077217  0.171531    0.96882        1.03716
2021-11-12  0.136782  0.050831  0.312539    0.99760        1.03578
2021-11-15  0.195873  0.097886  0.295769    1.02727        1.03430
```

[1854]:

```python
X = rec_prob_df[[col for col in rec_prob_df.columns[8:-2] if col not in␣
 ↪['-50_K', '0_K', '25_K']]]
# X = sm.add_constant(X)
y = rec_prob_df['mean_swap_fwd']

model = sm.OLS(y, X)

res = model.fit()

print(res.summary())
```

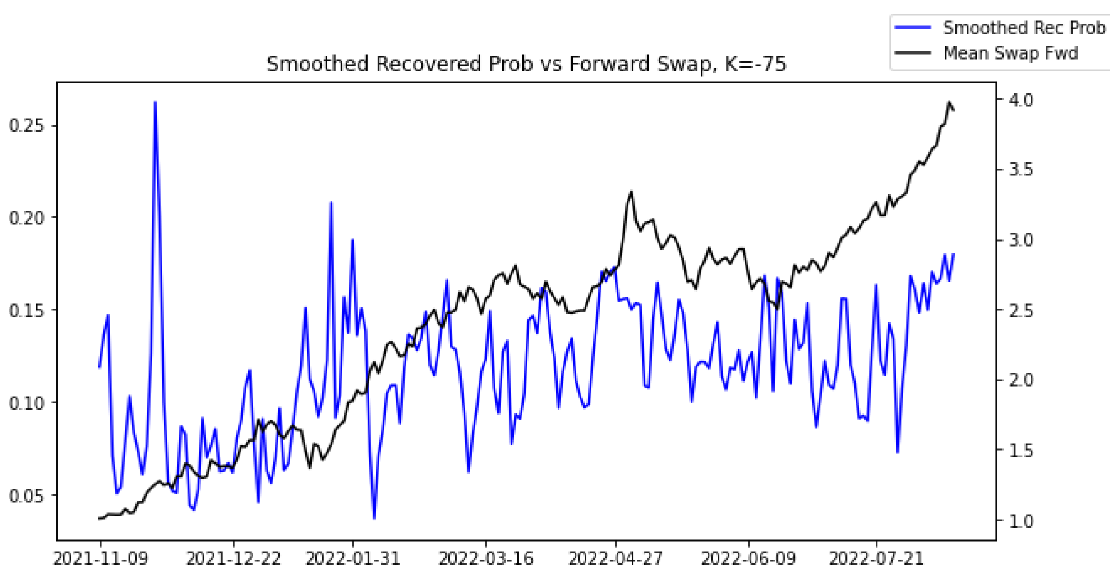
```
                                 OLS Regression Results
================================================================================
=======
Dep. Variable:          mean_swap_fwd   R-squared (uncentered):
0.941
Model:                            OLS   Adj. R-squared (uncentered):
0.940
Method:                 Least Squares   F-statistic:
778.7
Date:                Tue, 18 Oct 2022   Prob (F-statistic):
4.55e-119
Time:                        00:12:10   Log-Likelihood:
-183.12
No. Observations:                 200   AIC:
374.2
Df Residuals:                     196   BIC:
387.4
Df Model:                           4
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
-75_K         10.0543      1.091      9.218      0.000       7.903      12.205
-25_K          2.0266      1.002      2.023      0.044       0.051       4.002
50_K           2.7243      0.747      3.647      0.000       1.251       4.197
75_K           3.2489      0.634      5.125      0.000       1.999       4.499
==============================================================================
Omnibus:                        4.701   Durbin-Watson:                   0.307
Prob(Omnibus):                  0.095   Jarque-Bera (JB):                4.470
Skew:                          -0.364   Prob(JB):                        0.107
```

```
Kurtosis:                       3.085   Cond. No.                    8.40
==============================================================================

Notes:
[1] R² is computed without centering (uncentered) since the model does not
contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly
specified.
```

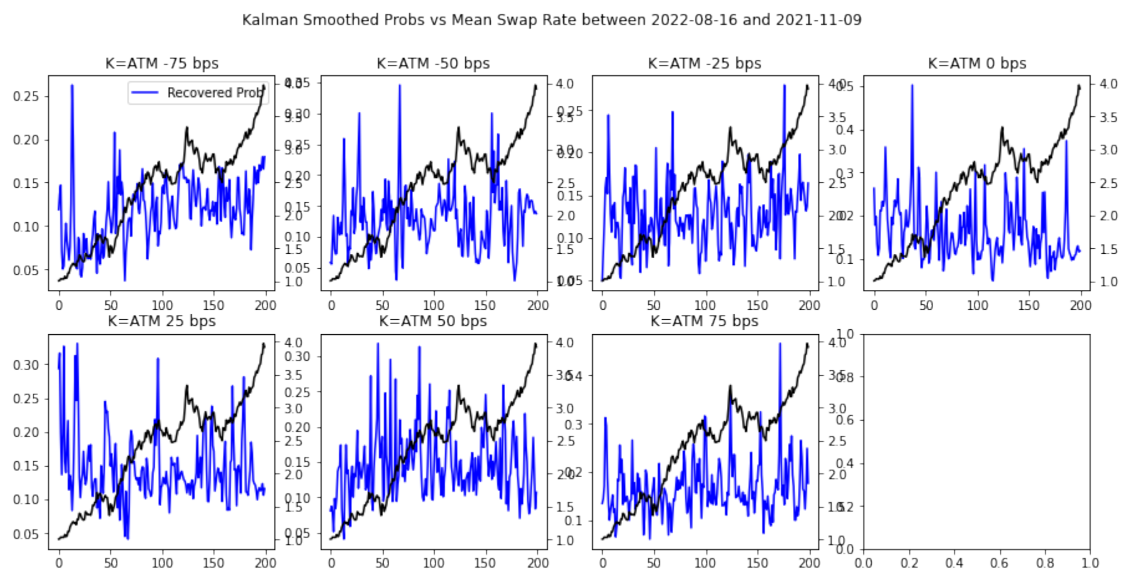
[1855]:
```python
significant_values = ['-75_K', '-25_K', '50_K', '75_K']
```

Let's look at the data:

[1856]:
```python
fig, ax = plt.subplots(figsize=(10, 5))
plt.plot(rec_prob_df.loc[:, '-75_K'], color='blue', label='Smoothed Rec Prob')


ax2 = ax.twinx()
ax2.plot(rec_prob_df.loc[:, 'mean_swap_fwd'], color='black', label='Mean Swap␣
↪Fwd')

ax.xaxis.set_major_locator(dates.MonthLocator())

fig.legend()
plt.title('Smoothed Recovered Prob vs Forward Swap, K=-75')
plt.show()
```



[1857]:
```python
fig, axes = plt.subplots(nrows=2, ncols=4, figsize=(15, 7))

fig.suptitle(f'Kalman Smoothed Probs vs Mean Swap Rate between {rec_prob_df.
 ↪index[-1]} and {rec_prob_df.index[0]}')
axes = axes.flatten()

for strike in rec_prob_df.iloc[:, :7].columns:
    idx = list(rec_prob_df.iloc[:, :-2].columns).index(strike)
    ax2 = axes[idx].twinx()
    axes[idx].plot(rec_prob_df.loc[:, f'{strike}_K'].values, label='Smoothed Rec␣
 ↪Prob', color='blue')
    ax2.plot(rec_prob_df.loc[:, 'mean_swap_fwd'].values, color='black')

    if idx == 0:
        axes[idx].legend(['Recovered Prob', 'Kalman Smoothed'])
    axes[idx].set_title(f'K=ATM {strike} bps')

plt.show();
```



[1858]:
```python
for strike in rec_prob_df.columns[:7]:
    X = rec_prob_df[[f'{strike}_K']]
#     X = sm.add_constant(X)
    y = rec_prob_df['mean_swap_fwd']

    model = sm.OLS(y, X)

    res = model.fit()
```

```python
print(res.summary())
```

```
                                 OLS Regression Results
================================================================================
=======
Dep. Variable:          mean_swap_fwd   R-squared (uncentered):
0.914
Model:                            OLS   Adj. R-squared (uncentered):
0.913
Method:                 Least Squares   F-statistic:
2110.
Date:                Tue, 18 Oct 2022   Prob (F-statistic):
6.90e-108
Time:                        00:12:12   Log-Likelihood:
-220.65
No. Observations:                 200   AIC:
443.3
Df Residuals:                     199   BIC:
446.6
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
-75_K         19.3090      0.420     45.939      0.000      18.480      20.138
==============================================================================
Omnibus:                       71.679   Durbin-Watson:                   0.596
Prob(Omnibus):                  0.000   Jarque-Bera (JB):              267.593
Skew:                          -1.401   Prob(JB):                     7.82e-59
Kurtosis:                       7.926   Cond. No.                         1.00
==============================================================================

Notes:
[1] R² is computed without centering (uncentered) since the model does not
contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly
specified.
                                 OLS Regression Results
================================================================================
=======
Dep. Variable:          mean_swap_fwd   R-squared (uncentered):
0.800
Model:                            OLS   Adj. R-squared (uncentered):
0.799
Method:                 Least Squares   F-statistic:
796.9
Date:                Tue, 18 Oct 2022   Prob (F-statistic):
1.63e-71
```

```
Time:                        00:12:12   Log-Likelihood:
-304.76
No. Observations:                 200   AIC:
611.5
Df Residuals:                     199   BIC:
614.8
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
-50_K         16.1228      0.571     28.230      0.000      14.997      17.249
==============================================================================
Omnibus:                       13.734   Durbin-Watson:                   0.485
Prob(Omnibus):                  0.001   Jarque-Bera (JB):               15.089
Skew:                          -0.565   Prob(JB):                     0.000529
Kurtosis:                       3.730   Cond. No.                         1.00
==============================================================================

Notes:
[1] R² is computed without centering (uncentered) since the model does not
contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly
specified.
                                 OLS Regression Results
================================================================================
=======
Dep. Variable:           mean_swap_fwd   R-squared (uncentered):
0.859
Model:                             OLS   Adj. R-squared (uncentered):
0.858
Method:                  Least Squares   F-statistic:
1213.
Date:                 Tue, 18 Oct 2022   Prob (F-statistic):
1.34e-86
Time:                         00:12:12   Log-Likelihood:
-269.88
No. Observations:                  200   AIC:
541.8
Df Residuals:                      199   BIC:
545.1
Df Model:                            1
Covariance Type:             nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
-25_K         17.3877      0.499     34.822      0.000      16.403      18.372
==============================================================================
```

```
Omnibus:                       12.693   Durbin-Watson:                   0.555
Prob(Omnibus):                  0.002   Jarque-Bera (JB):               13.332
Skew:                          -0.619   Prob(JB):                      0.00127
Kurtosis:                       3.256   Cond. No.                         1.00
==============================================================================

Notes:
[1] R² is computed without centering (uncentered) since the model does not
contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly
specified.
                                 OLS Regression Results
================================================================================
=======
Dep. Variable:          mean_swap_fwd   R-squared (uncentered):
0.729
Model:                            OLS   Adj. R-squared (uncentered):
0.727
Method:                 Least Squares   F-statistic:
534.0
Date:                Tue, 18 Oct 2022   Prob (F-statistic):
3.02e-58
Time:                        00:12:12   Log-Likelihood:
-335.42
No. Observations:                 200   AIC:
672.8
Df Residuals:                     199   BIC:
676.1
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
O_K           12.1335      0.525     23.108      0.000      11.098      13.169
==============================================================================
Omnibus:                       13.730   Durbin-Watson:                   0.349
Prob(Omnibus):                  0.001   Jarque-Bera (JB):               14.574
Skew:                          -0.644   Prob(JB):                     0.000684
Kurtosis:                       3.298   Cond. No.                         1.00
==============================================================================

Notes:
[1] R² is computed without centering (uncentered) since the model does not
contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly
specified.
                                 OLS Regression Results
================================================================================
```

```
=======
Dep. Variable:          mean_swap_fwd   R-squared (uncentered):
0.778
Model:                            OLS   Adj. R-squared (uncentered):
0.777
Method:                 Least Squares   F-statistic:
699.3
Date:                Tue, 18 Oct 2022   Prob (F-statistic):
4.74e-67
Time:                        00:12:12   Log-Likelihood:
-315.08
No. Observations:                 200   AIC:
632.2
Df Residuals:                     199   BIC:
635.5
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
25_K          14.2200      0.538     26.444      0.000      13.160      15.280
==============================================================================
Omnibus:                       32.906   Durbin-Watson:                   0.293
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               45.773
Skew:                          -0.985   Prob(JB):                     1.15e-10
Kurtosis:                       4.271   Cond. No.                         1.00
==============================================================================

Notes:
[1] R² is computed without centering (uncentered) since the model does not
contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly
specified.
                                 OLS Regression Results
================================================================================
=======
Dep. Variable:          mean_swap_fwd   R-squared (uncentered):
0.838
Model:                            OLS   Adj. R-squared (uncentered):
0.837
Method:                 Least Squares   F-statistic:
1031.
Date:                Tue, 18 Oct 2022   Prob (F-statistic):
1.20e-80
Time:                        00:12:12   Log-Likelihood:
-283.65
No. Observations:                 200   AIC:
569.3
```

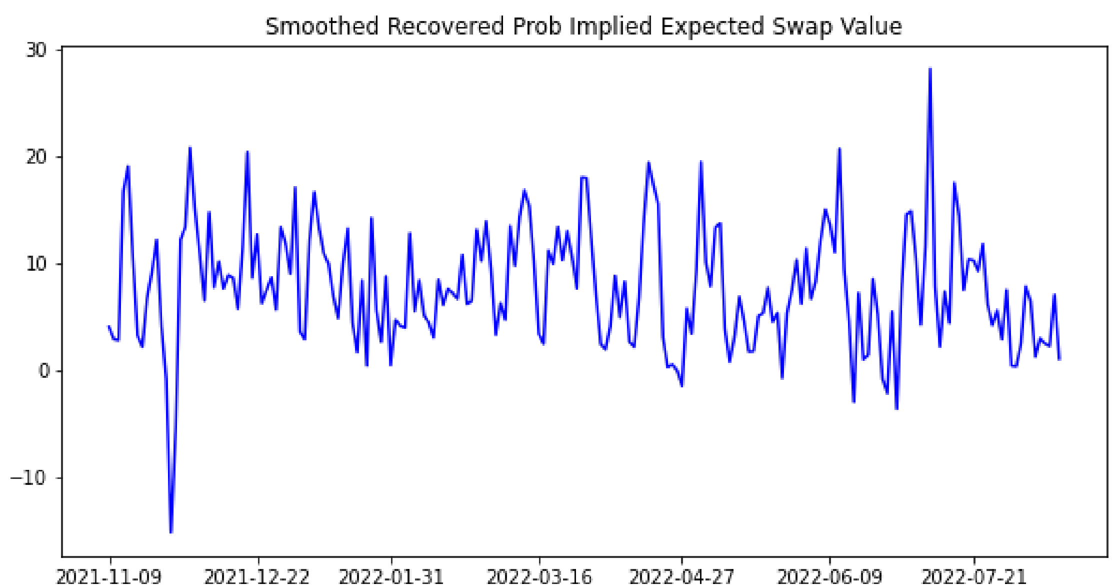
```
Df Residuals:                     199   BIC:
572.6
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
50_K          14.8860      0.464     32.109      0.000      13.972      15.800
==============================================================================
Omnibus:                        3.888   Durbin-Watson:                   0.571
Prob(Omnibus):                  0.143   Jarque-Bera (JB):                3.474
Skew:                          -0.278   Prob(JB):                        0.176
Kurtosis:                       3.330   Cond. No.                         1.00
==============================================================================

Notes:
[1] R² is computed without centering (uncentered) since the model does not
contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly
specified.
                                 OLS Regression Results
================================================================================
=======
Dep. Variable:          mean_swap_fwd   R-squared (uncentered):
0.852
Model:                            OLS   Adj. R-squared (uncentered):
0.852
Method:                 Least Squares   F-statistic:
1149.
Date:                Tue, 18 Oct 2022   Prob (F-statistic):
1.36e-84
Time:                        00:12:12   Log-Likelihood:
-274.53
No. Observations:                 200   AIC:
551.1
Df Residuals:                     199   BIC:
554.3
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
75_K          13.0966      0.386     33.890      0.000      12.335      13.859
==============================================================================
Omnibus:                       18.923   Durbin-Watson:                   0.601
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               22.703
Skew:                          -0.675   Prob(JB):                     1.17e-05
Kurtosis:                       3.951   Cond. No.                         1.00
```

```
==============================================================================

Notes:
[1] R² is computed without centering (uncentered) since the model does not
contain a constant.
[2] Standard Errors assume that the covariance matrix of the errors is correctly
specified.
```

Given the reasonable theoretical justification let's construct the trading strategy, which works the following way: - If the expected value of the rate is higher than 0, we long the agregate swap curve -> Pay Fixed - If less than 0, we short -> Receive Fixed

[1859]:
```
rec_prob_df['exp_val'] = rec_prob_df[significant_values].values @ [-75, -25, 50,␣
↪75]
```

[1860]:
```
fig, ax = plt.subplots(figsize=(10, 5))
plt.plot(rec_prob_df.loc[:, 'exp_val'], color='blue', label='Smoothed Rec Prob')

ax.xaxis.set_major_locator(dates.MonthLocator())

plt.title('Smoothed Recovered Prob Implied Expected Swap Value')
plt.show()
```



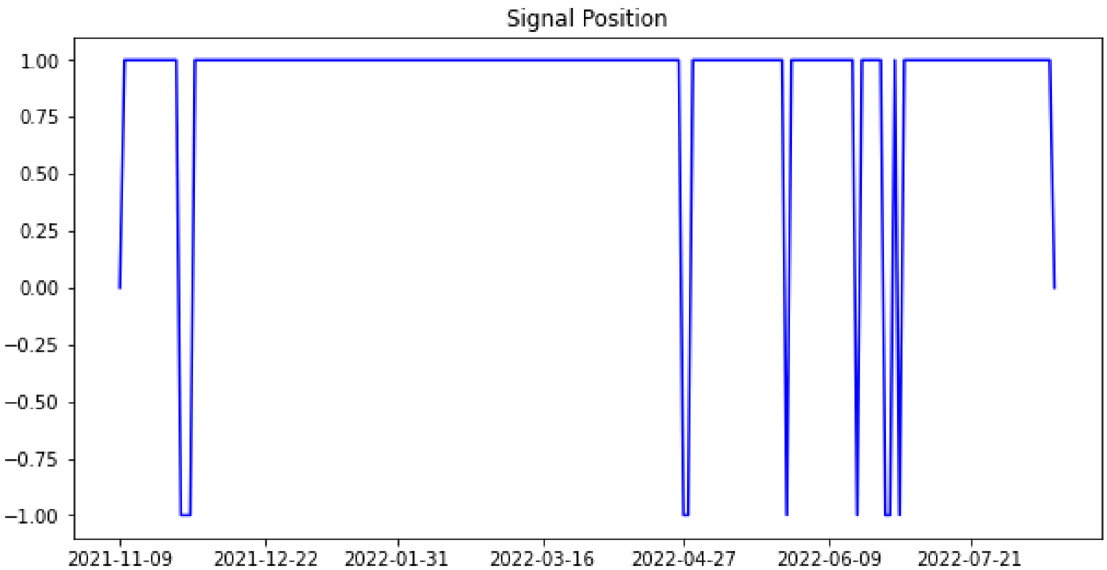
Construct the position

[1861]:
```
rec_prob_df['position'] = 0

for i in range(1, rec_prob_df.shape[0]-1):
    date = list(rec_prob_df.index)[i]
```

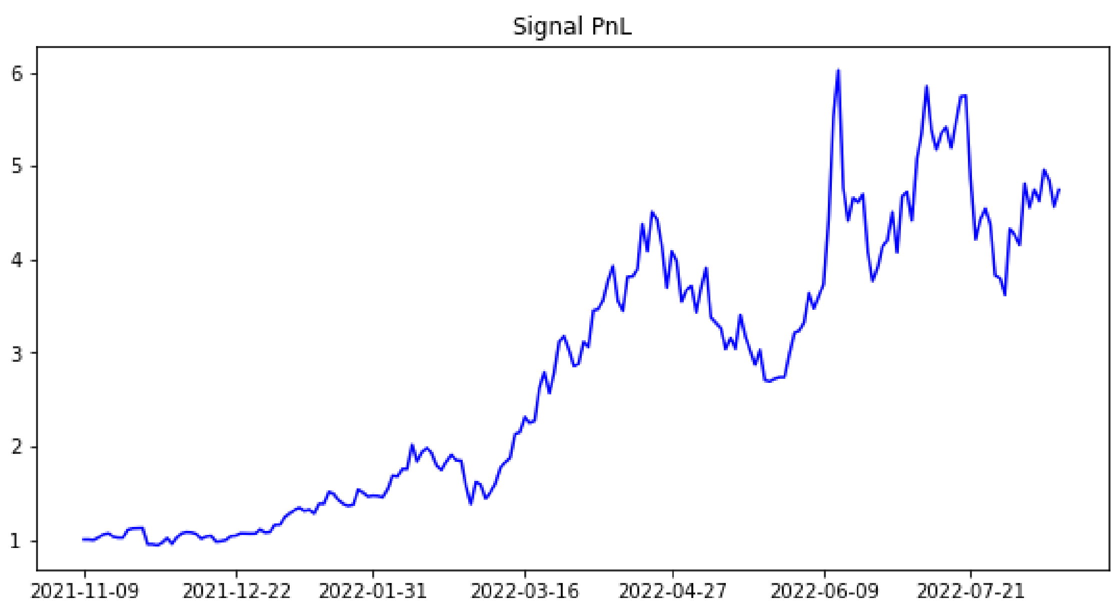
```
    date_fwd = list(rec_prob_df.index)[i+1]
    date_bwd = list(rec_prob_df.index)[i-1]

    if rec_prob_df.loc[date_bwd, 'exp_val'] > 0:
        rec_prob_df.loc[date, 'position'] = 1
        count +=1
    elif rec_prob_df.loc[date_bwd, 'exp_val'] < 0:
        rec_prob_df.loc[date, 'position'] = -1
        count +=1
    else:
        rec_prob_df.loc[date, 'position'] = 0

    if i == rec_prob_df.shape[0]-1:
        rec_prob_df.loc[date_fwd, 'position'] = 0

rec_prob_df['pnl'] = rec_prob_df['mean_swap'].diff().fillna(0) *␣
 ↪rec_prob_df['position'].shift(1).fillna(0)
rec_prob_df['pnl_agg'] = (1 + rec_prob_df['pnl']).cumprod()

print(f'Sharpe Ratio is {round(rec_prob_df["pnl"].mean()/ rec_prob_df["pnl"].
 ↪std(), 2)}')
```

```
Sharpe Ratio is 0.15
```

[1862]:
```
fig, ax = plt.subplots(figsize=(10, 5))
plt.plot(rec_prob_df.loc[:, 'position'], color='blue', label='Smoothed Rec Prob')

ax.xaxis.set_major_locator(dates.MonthLocator())

plt.title('Signal Position')
plt.show()
```



[1863]:
```
fig, ax = plt.subplots(figsize=(10, 5))
plt.plot(rec_prob_df.loc[:, 'pnl_agg'], color='blue', label='Smoothed Rec Prob')

ax.xaxis.set_major_locator(dates.MonthLocator())

plt.title('Signal PnL')
plt.show()
```



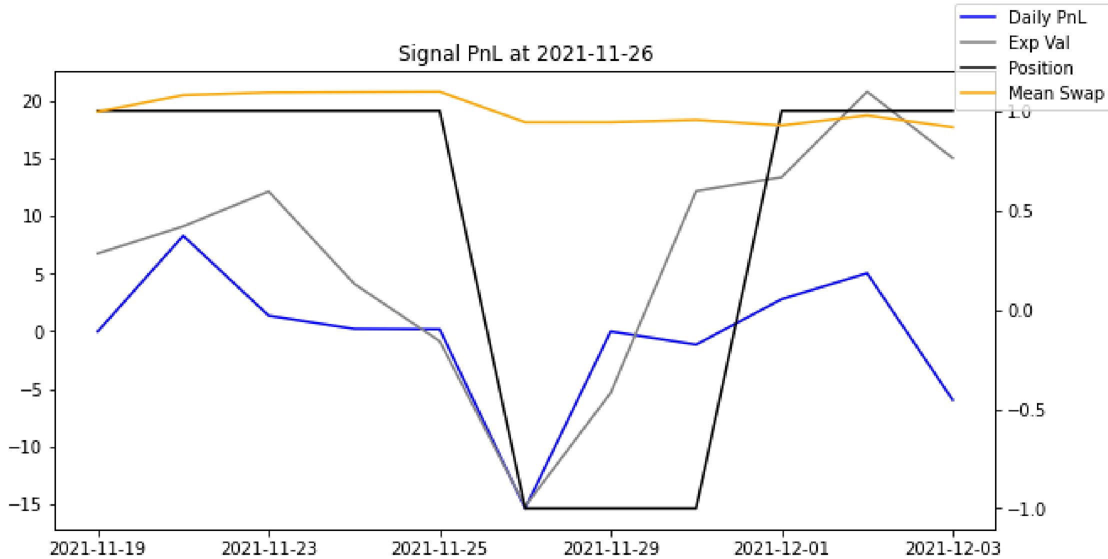
Let's investigate the cases when the position goes short:

[1873]:
```
print(f'Mean PnL for shorts is {round(rec_prob_df[rec_prob_df.position == -1].
↪pnl.mean(), 4)}')
```

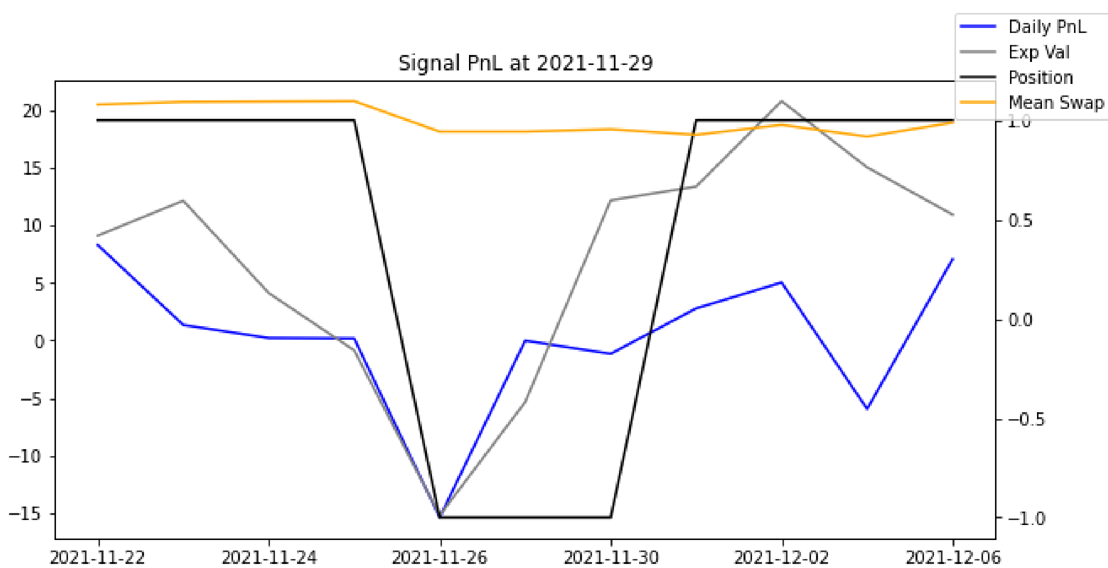
```
Mean PnL for shorts is -0.0045
```

[1865]:
```
for i in rec_prob_df[rec_prob_df.position == -1].index:
    date = list(rec_prob_df.index).index(i)
    date_fwd = list(rec_prob_df.index)[date+5]
    date_bwd = list(rec_prob_df.index)[date-5]

    fig, ax = plt.subplots(figsize=(10, 5))
    display(rec_prob_df.loc[date_bwd:date_fwd,
                ['pnl', 'mean_swap', 'position', 'exp_val']])

    plt.plot(rec_prob_df.loc[date_bwd:date_fwd, 'pnl']*100, color='blue',␣
↪label='Daily PnL')
    plt.plot(rec_prob_df.loc[date_bwd:date_fwd, 'exp_val'], color='grey',␣
↪label='Exp Val')

    ax2 = ax.twinx()
    ax2.plot(rec_prob_df.loc[date_bwd:date_fwd, 'position'], color='black',␣
↪label='Position')
    ax2.plot(rec_prob_df.loc[date_bwd:date_fwd, 'mean_swap'], color='orange',␣
↪label='Mean Swap')


    fig.legend()
    ax.xaxis.set_major_locator(dates.DayLocator(interval=2))

    plt.title(f'Signal PnL at {i}')
    plt.show()
```

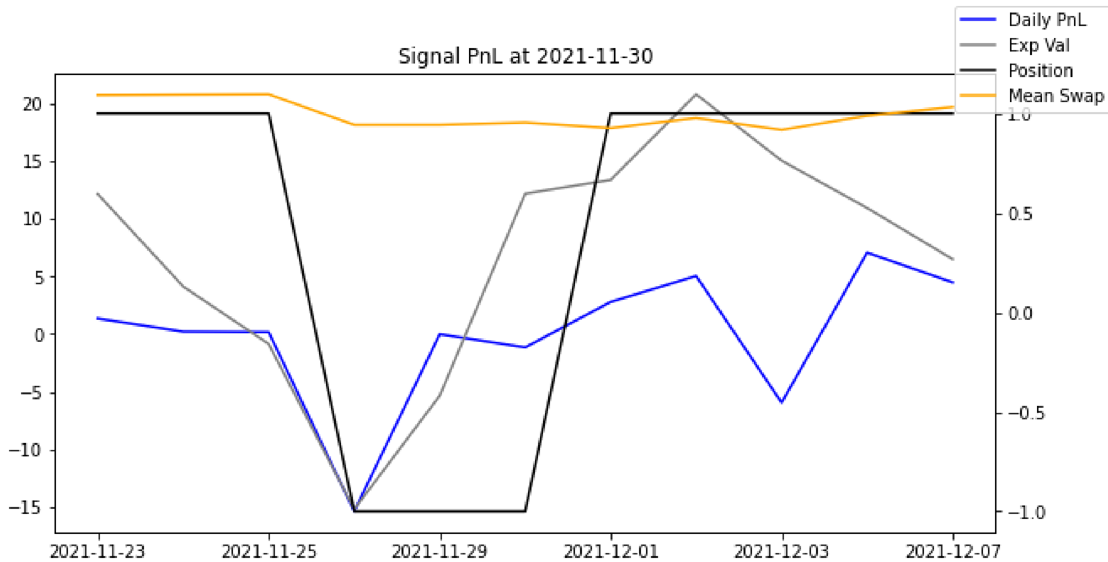
	pnl	mean_swap	position	exp_val
2021-11-19	-0.00011	0.99572	1	6.753158
2021-11-22	0.08289	1.07861	1	9.105183
2021-11-23	0.01356	1.09217	1	12.140045
2021-11-24	0.00212	1.09429	1	4.121661
2021-11-25	0.00173	1.09602	1	-0.858330
2021-11-26	-0.15358	0.94244	-1	-15.183108
2021-11-29	-0.00012	0.94256	-1	-5.341180
2021-11-30	-0.01156	0.95412	-1	12.172705
2021-12-01	0.02790	0.92622	1	13.365422
2021-12-02	0.05038	0.97660	1	20.784968
2021-12-03	-0.05937	0.91723	1	15.033549



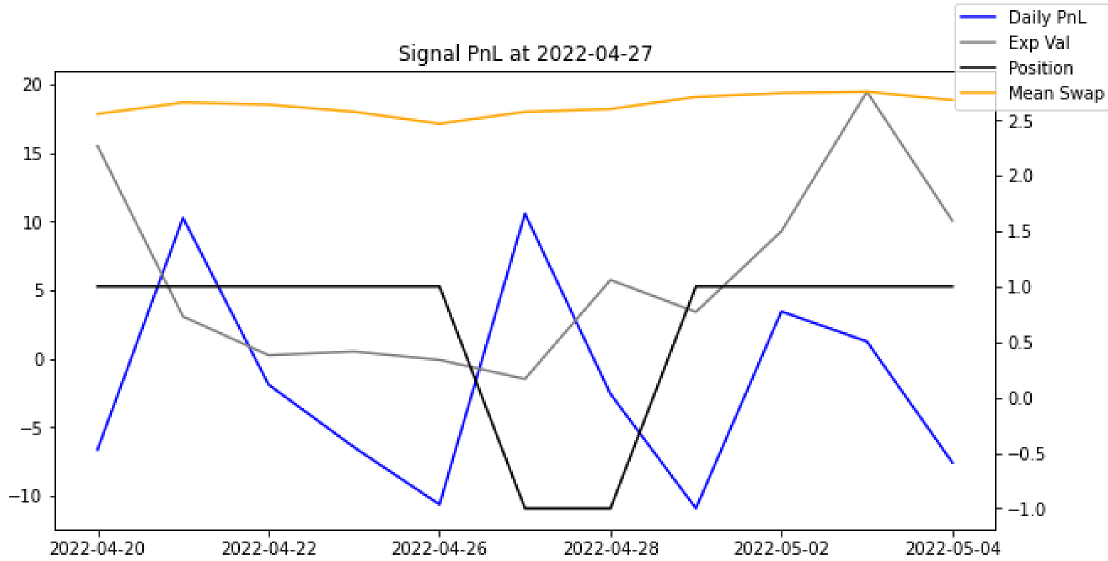
	pnl	mean_swap	position	exp_val
2021-11-22	0.08289	1.07861	1	9.105183
2021-11-23	0.01356	1.09217	1	12.140045
2021-11-24	0.00212	1.09429	1	4.121661
2021-11-25	0.00173	1.09602	1	-0.858330
2021-11-26	-0.15358	0.94244	-1	-15.183108
2021-11-29	-0.00012	0.94256	-1	-5.341180
2021-11-30	-0.01156	0.95412	-1	12.172705
2021-12-01	0.02790	0.92622	1	13.365422
2021-12-02	0.05038	0.97660	1	20.784968
2021-12-03	-0.05937	0.91723	1	15.033549
2021-12-06	0.07060	0.98783	1	10.932793



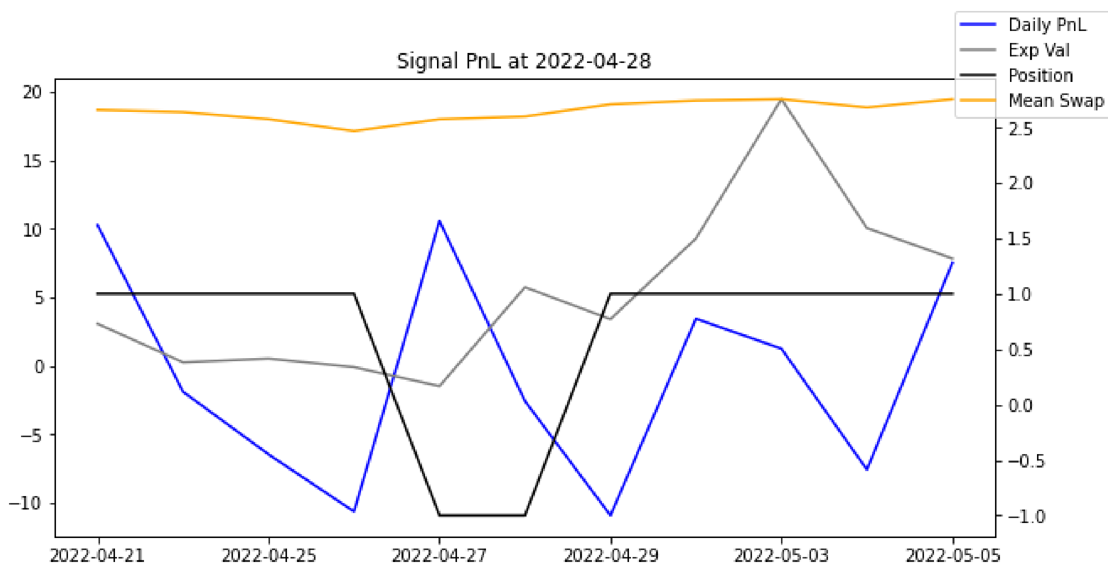
	pnl	mean_swap	position	exp_val
2021-11-23	0.01356	1.09217	1	12.140045
2021-11-24	0.00212	1.09429	1	4.121661
2021-11-25	0.00173	1.09602	1	-0.858330
2021-11-26	-0.15358	0.94244	-1	-15.183108
2021-11-29	-0.00012	0.94256	-1	-5.341180
2021-11-30	-0.01156	0.95412	-1	12.172705
2021-12-01	0.02790	0.92622	1	13.365422
2021-12-02	0.05038	0.97660	1	20.784968
2021-12-03	-0.05937	0.91723	1	15.033549
2021-12-06	0.07060	0.98783	1	10.932793
2021-12-07	0.04485	1.03268	1	6.496641



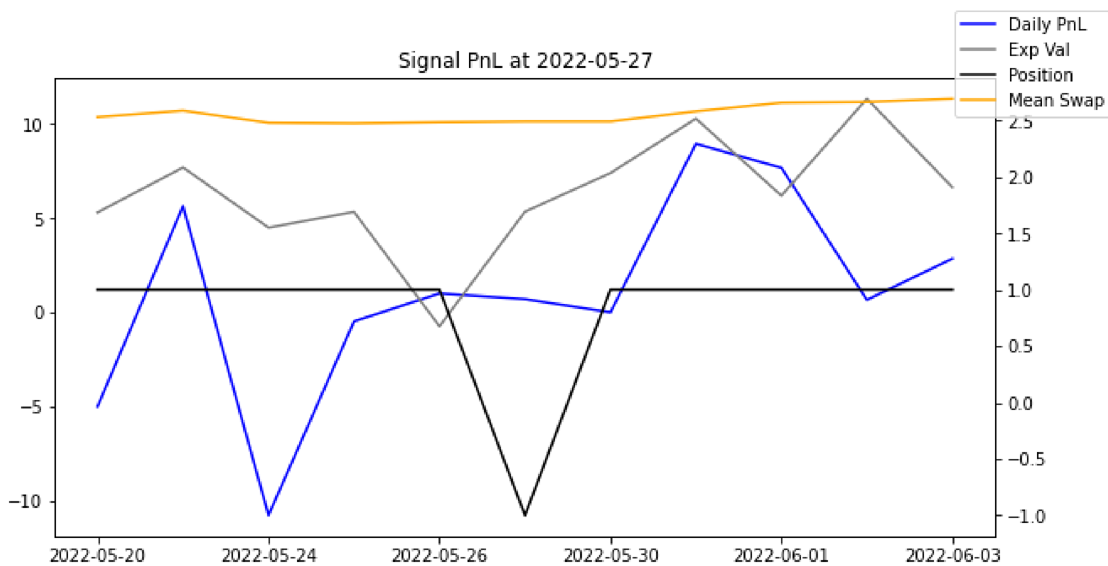
	pnl	mean_swap	position	exp_val
2022-04-20	-0.06622	2.55440	1	15.504003
2022-04-21	0.10267	2.65707	1	3.076724
2022-04-22	-0.01885	2.63822	1	0.256691
2022-04-25	-0.06453	2.57369	1	0.528700
2022-04-26	-0.10630	2.46739	1	-0.090780
2022-04-27	0.10570	2.57309	-1	-1.477216
2022-04-28	-0.02567	2.59876	-1	5.738454
2022-04-29	-0.10910	2.70786	1	3.408595
2022-05-02	0.03445	2.74231	1	9.293731
2022-05-03	0.01253	2.75484	1	19.462155
2022-05-04	-0.07567	2.67917	1	10.057492



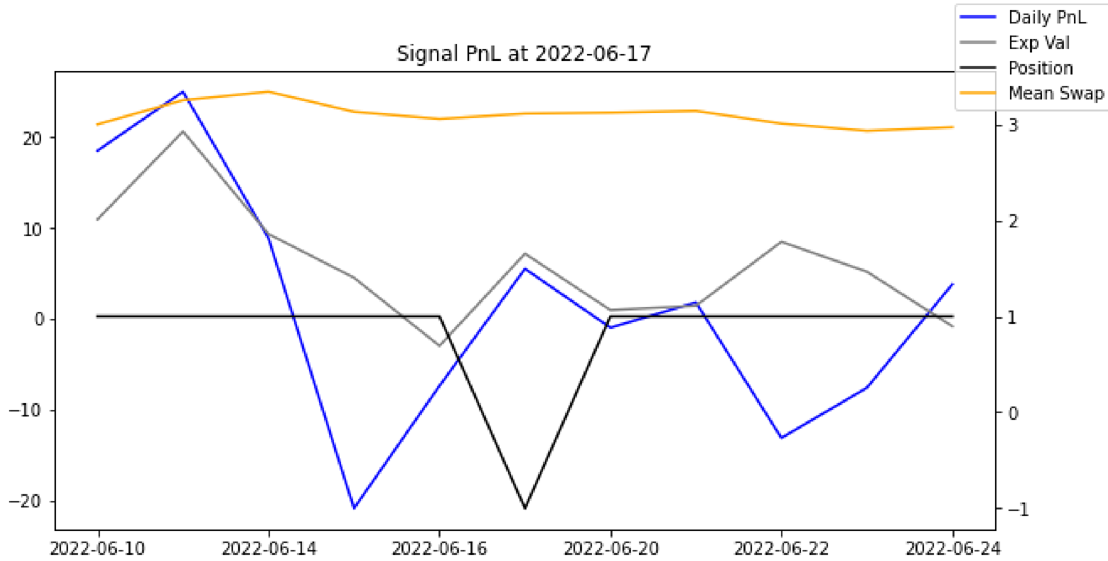
	pnl	mean_swap	position	exp_val
2022-04-21	0.10267	2.65707	1	3.076724
2022-04-22	-0.01885	2.63822	1	0.256691
2022-04-25	-0.06453	2.57369	1	0.528700
2022-04-26	-0.10630	2.46739	1	-0.090780
2022-04-27	0.10570	2.57309	-1	-1.477216
2022-04-28	-0.02567	2.59876	-1	5.738454
2022-04-29	-0.10910	2.70786	1	3.408595
2022-05-02	0.03445	2.74231	1	9.293731
2022-05-03	0.01253	2.75484	1	19.462155
2022-05-04	-0.07567	2.67917	1	10.057492
2022-05-05	0.07516	2.75433	1	7.832490



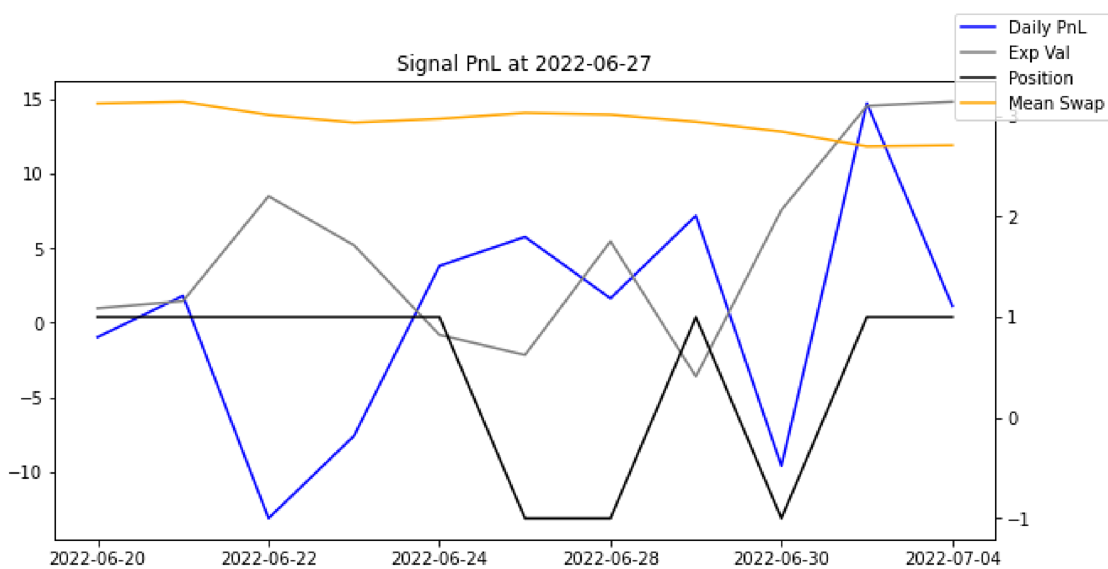
	pnl	mean_swap	position	exp_val
2022-05-20	-0.05009	2.52983	1	5.307967
2022-05-23	0.05648	2.58631	1	7.695054
2022-05-24	-0.10800	2.47831	1	4.499782
2022-05-25	-0.00485	2.47346	1	5.337023
2022-05-26	0.01000	2.48346	1	-0.746280
2022-05-27	0.00697	2.49043	-1	5.347120
2022-05-30	-0.00002	2.49045	1	7.398284
2022-05-31	0.08958	2.58003	1	10.295527
2022-06-01	0.07678	2.65681	1	6.198352
2022-06-02	0.00658	2.66339	1	11.352957
2022-06-03	0.02848	2.69187	1	6.640405



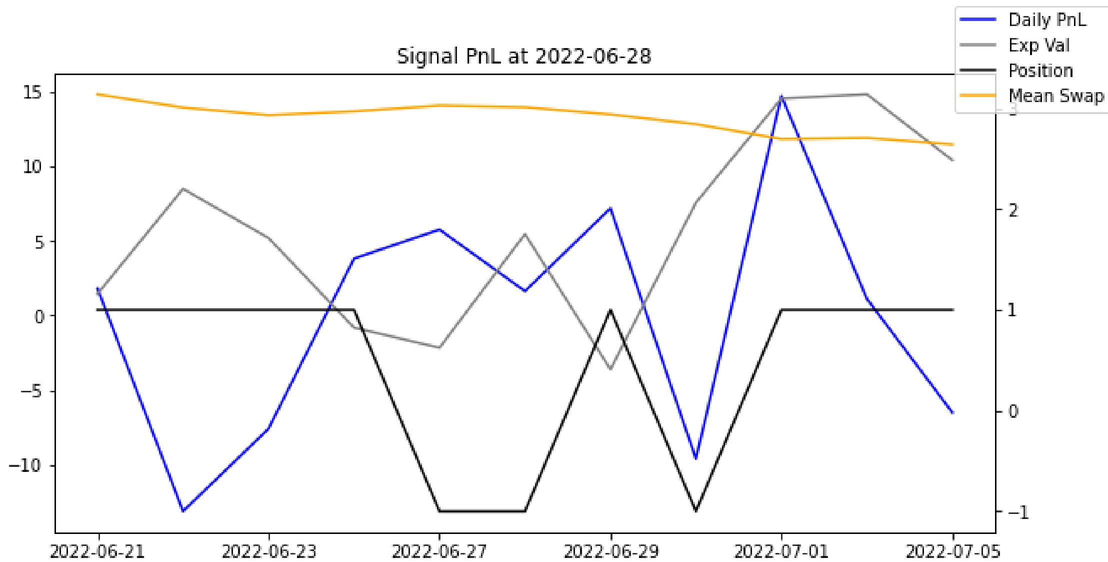
	pnl	mean_swap	position	exp_val
2022-06-10	0.18549	2.99937	1	10.974319
2022-06-13	0.25048	3.24985	1	20.671028
2022-06-14	0.08884	3.33869	1	9.349077
2022-06-15	-0.20910	3.12959	1	4.530647
2022-06-16	-0.07394	3.05565	1	-2.984454
2022-06-17	0.05515	3.11080	-1	7.185208
2022-06-20	-0.00974	3.12054	1	0.960761
2022-06-21	0.01802	3.13856	1	1.442416
2022-06-22	-0.13145	3.00711	1	8.491379
2022-06-23	-0.07606	2.93105	1	5.192505
2022-06-24	0.03810	2.96915	1	-0.822617



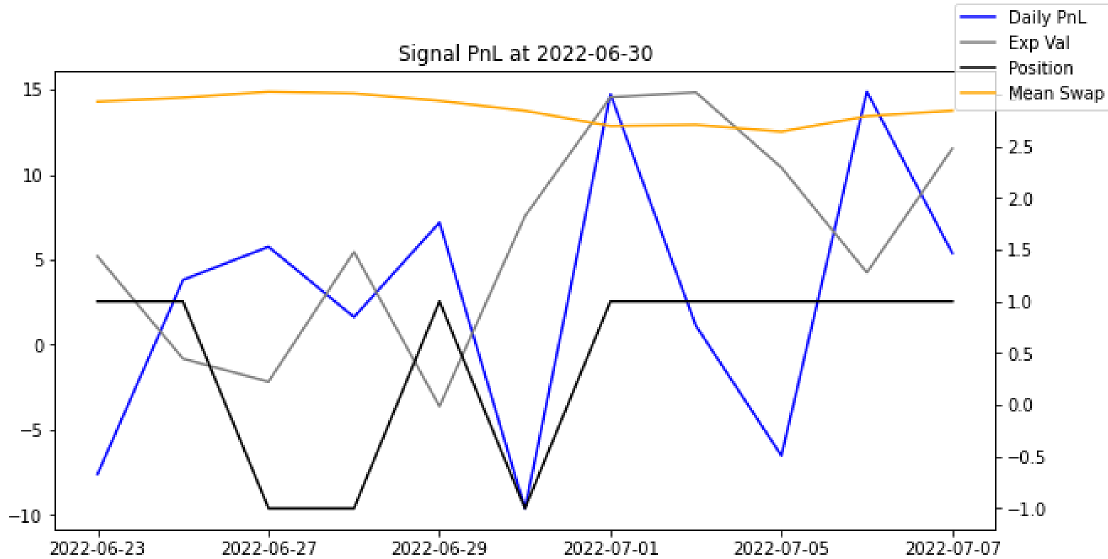
	pnl	mean_swap	position	exp_val
2022-06-20	-0.00974	3.12054	1	0.960761
2022-06-21	0.01802	3.13856	1	1.442416
2022-06-22	-0.13145	3.00711	1	8.491379
2022-06-23	-0.07606	2.93105	1	5.192505
2022-06-24	0.03810	2.96915	1	-0.822617
2022-06-27	0.05757	3.02672	-1	-2.175269
2022-06-28	0.01626	3.01046	-1	5.455327
2022-06-29	0.07183	2.93863	1	-3.621882
2022-06-30	-0.09621	2.84242	-1	7.560011
2022-07-01	0.14724	2.69518	1	14.550637
2022-07-04	0.01115	2.70633	1	14.832692



	pnl	mean_swap	position	exp_val
2022-06-21	0.01802	3.13856	1	1.442416
2022-06-22	-0.13145	3.00711	1	8.491379
2022-06-23	-0.07606	2.93105	1	5.192505
2022-06-24	0.03810	2.96915	1	-0.822617
2022-06-27	0.05757	3.02672	-1	-2.175269
2022-06-28	0.01626	3.01046	-1	5.455327
2022-06-29	0.07183	2.93863	1	-3.621882
2022-06-30	-0.09621	2.84242	-1	7.560011
2022-07-01	0.14724	2.69518	1	14.550637
2022-07-04	0.01115	2.70633	1	14.832692
2022-07-05	-0.06518	2.64115	1	10.412682



	pnl	mean_swap	position	exp_val
2022-06-23	-0.07606	2.93105	1	5.192505
2022-06-24	0.03810	2.96915	1	-0.822617
2022-06-27	0.05757	3.02672	-1	-2.175269
2022-06-28	0.01626	3.01046	-1	5.455327
2022-06-29	0.07183	2.93863	1	-3.621882
2022-06-30	-0.09621	2.84242	-1	7.560011
2022-07-01	0.14724	2.69518	1	14.550637
2022-07-04	0.01115	2.70633	1	14.832692
2022-07-05	-0.06518	2.64115	1	10.412682
2022-07-06	0.14873	2.78988	1	4.242454
2022-07-07	0.05380	2.84368	1	11.535816



8 Results

[1795]:
```
rec_prob_df.index = pd.to_datetime(rec_prob_df.index)
```

[1796]:
```
rec_prob_df_month = rec_prob_df.resample('M').mean()
rec_prob_df_month_std = rec_prob_df.resample('M').std()
rec_prob_df_month['time'] = rec_prob_df_month.index
```

[1908]:
```
rec_prob_df_month[['pnl']]
```

[1908]:

	pnl
2021-11-30	-0.032880
2021-12-31	0.080452
2022-01-31	0.188303
2022-02-28	0.068299
2022-03-31	0.355205
2022-04-30	0.152754
2022-05-31	-0.069747
2022-06-30	0.228595
2022-07-31	-0.000406
2022-08-31	0.261360

[1907]:
```
rec_prob_df_month_std[['pnl']]
```

[1907]:

	pnl
2021-11-30	0.164449
2021-12-31	0.117357

```
2022-01-31  0.158898
2022-02-28  0.232051
2022-03-31  0.275474
2022-04-30  0.265297
2022-05-31  0.224846
2022-06-30  0.348432
2022-07-31  0.287125
2022-08-31  0.289670
```

[1909]:
```
rec_prob_df_month[['pnl']] / rec_prob_df_month_std[['pnl']]
```

[1909]:
```
                 pnl
2021-11-30 -0.057718
2021-12-31  0.197897
2022-01-31  0.342095
2022-02-28  0.084965
2022-03-31  0.372228
2022-04-30  0.166215
2022-05-31 -0.089547
2022-06-30  0.189390
2022-07-31 -0.000408
2022-08-31  0.260462
```

[1899]:
```
print(f'Mean Daily PnL is {round(rec_prob_df["pnl"].mean(), 4)}')
```

```
Mean Monthly PnL is 0.0102
```

[1903]:
```
print(f'Mean Annual PnL is {round(rec_prob_df["pnl"].mean() * 252, 4)}')
```

```
Mean Annual PnL is 2.5805
```

[1902]:
```
print(f'Std Daily PnL is {round(rec_prob_df["pnl"].std(), 4)}')
```

```
Std Daily PnL is 0.0701
```

[1904]:
```
print(f'Std Annual PnL is {round(rec_prob_df["pnl"].std() * np.sqrt(252), 4)}')
```

```
Std Annual PnL is 1.1126
```

[1905]:
```
print(f'Sharpe Ratio is {round(rec_prob_df["pnl"].mean()*252/ rec_prob_df["pnl"].
 ↪std(), 2) / np.sqrt(252)}')
```

```
Sharpe Ratio is 2.3194419826999577
```

[ ]:
```

```