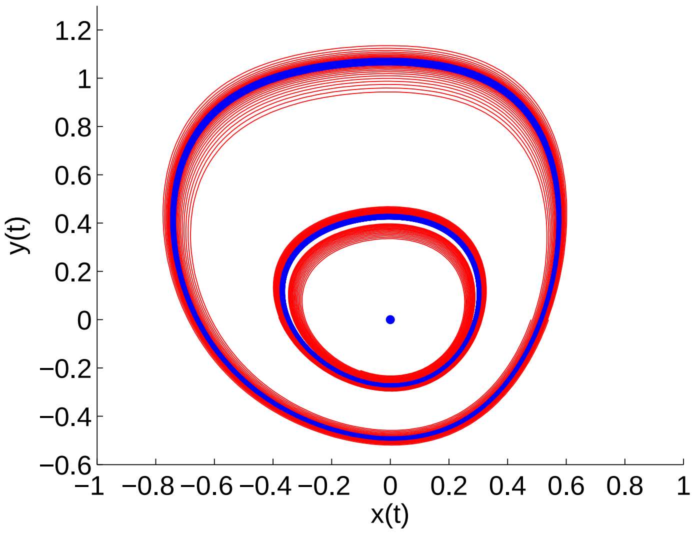
# 6G7Z3001: Advanced ODEs and Dynamical Systems

# Limit Cycles

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# Chapter 11

# Limit Cycles

## Aims and Objectives

- To give a brief historical background.
- To define features of phase plane portraits.
- To introduce the theory of planar limit cycles.
- To introduce perturbation methods.

On completion of this chapter the reader should be able to

- prove existence and uniqueness of a limit cycle;
- prove that certain systems have no limit cycles;
- interpret limit cycle behavior in physical terms;
- find approximate solutions for perturbed systems.

Limit cycles, or isolated periodic solutions, are the most common form of solution observed when modeling physical systems in the plane. Early investigations were concerned with mechanical and electronic systems, but periodic behavior is evident in all branches of science. Two limit cycles were plotted in Chapter 10 when considering the modeling of interacting species.

The chapter begins with a historical introduction, and then the theory of planar limit cycles is introduced.

## 11.1 Historical Background

**Definition 1.** A *limit cycle* is an isolated periodic solution.

Limit cycles in planar differential systems commonly occur when modeling both the technological and natural sciences. Most of the early history in the theory of limit cycles in the plane was stimulated by practical problems. For example, the differential equation derived by Rayleigh in 1877 [14], related to the oscillation of a violin string, is given by

$$\ddot{x} + \epsilon \left( \frac{1}{3}(\dot{x})^2 - 1 \right) \dot{x} + x = 0,$$

where $\ddot{x} = \frac{d^2x}{dt^2}$ and $\dot{x} = \frac{dx}{dt}$. Let $\dot{x} = y$. Then this differential equation can be written as a system of first-order autonomous differential equations in the plane

$$\dot{x} = y, \quad \dot{y} = -x - \epsilon \left( \frac{y^2}{3} - 1 \right) y. \tag{11.1}$$

A phase portrait is shown in Figure 11.1.

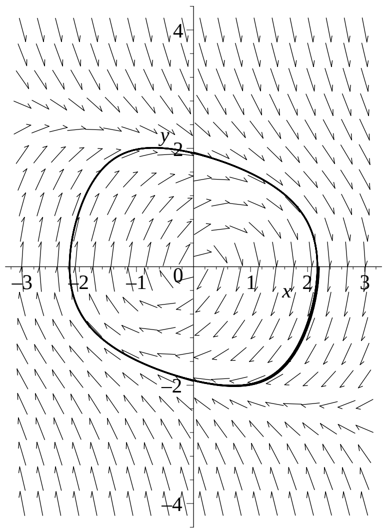



Figure 11.1: Periodic behavior in the Rayleigh system (11.1) when $\epsilon = 1.0$

Following the invention of the triode vacuum tube, which was able to produce stable self-excited oscillations of constant amplitude, van der Pol [17] obtained the following differential equation to describe this phenomenon

$$\ddot{x} + \epsilon \left( x^2 - 1 \right) \dot{x} + x = 0,$$

which can be written as a planar system of the form

$$\dot{x} = y, \quad \dot{y} = -x - \epsilon \left( x^2 - 1 \right) y. \tag{11.2}$$

A phase portrait is shown in Figure 11.2.

The basic model of a cell membrane is that of a resistor and capacitor in parallel. The equations used to model the membrane are a variation of the van der Pol equation. The famous Fitzhugh-Nagumo oscillator [3], [15], [9], used to model the action potential of a neuron is a two-variable simplification of the Hodgkin-Huxley equations [5] (see Chapter 20). The Fitzhugh-Nagumo model creates quite accurate

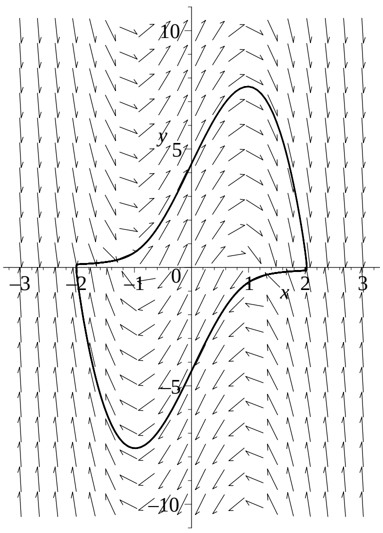



Figure 11.2: [MATLAB] Periodic behavior for system (11.2) when $\epsilon = 5.0$.

action potentials and models the qualitative behavior of the neurons. The differential equations are given by

$$\dot{u} = -u(u-\theta)(u-1) - v + \omega, \quad \dot{v} = \epsilon(u - \gamma v),$$

where $u$ is a voltage, $v$ is the recovery of voltage, $\theta$ is a threshold, $\gamma$ is a shunting variable, and $\omega$ is a constant voltage. For certain parameter values, the solution demonstrates a slow collection and fast release of voltage; this kind of behavior has been labeled integrate and fire. Note that, for biological systems, neurons cannot collect voltage immediately after firing and need to rest. Oscillatory behavior for the Fitzhugh-Nagumo system is shown in Figure 11.3. MATLAB command lines for producing Figure 11.3 are listed in Section 11.4.

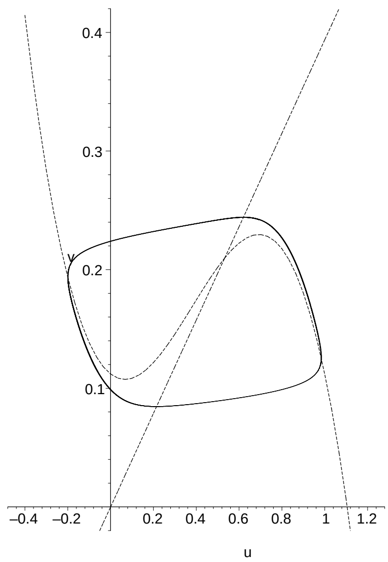



Figure 11.3: [MATLAB] A limit cycle for the Fitzhugh-Nagumo oscillator. In this case, $\gamma = 2.54$, $\theta = 0.14$, $\omega = 0.112$ and $\epsilon = 0.01$. The dashed curves are the isoclines, where the trajectories cross horizontally and vertically.

Note that when $\omega = \omega(t)$ is a periodic external input the system becomes nonautonomous and can display chaotic behavior [15]. The reader can investigate these systems via the exercises in Chapter 15.

Perhaps the most famous class of differential equations that generalize (11.2), are those first investigated by Liénard in 1928 [7],

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

or in the phase plane

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{11.3}$$

This system can be used to model mechanical systems, where $f(x)$ is known as the *damping* term and $g(x)$ is called the *restoring force* or *stiffness*. Equation (11.3) is also used to model resistor-inductor-capacitor circuits (see Chapter 8) with nonlinear circuit elements. Limit cycles of Liénard systems will be discussed in some detail in Chapters 16 and 17.

Possible physical interpretations for limit cycle behavior of certain dynamical systems are listed below:

- For an economic model, Bella [2] considers a Goodwin model of a class struggle and demonstrates emerging multiple limit cycles of different orientation.
- For predator-prey and epidemic models, the populations oscillate in phase with one another and the systems are robust (see Examples in Chapter 10, and Exercise 8 in Chapter 14).
- Periodic behavior is present in integrate and fire neurons (see Figure 11.3).
- For mechanical systems, examples include the motion of simple nonlinear pendula (see Section 15.3), wing rock oscillations in aircraft flight dynamics [11], and surge oscillations in axial flow compressors [1], for example.
- For periodic chemical reactions, examples include the Landolt clock reaction and the Belousov-Zhabotinski reaction (see Chapter 14).
- For electrical or electronic circuits, it is possible to construct simple electronic oscillators (Chua's circuit, for example) using a nonlinear circuit element; a limit cycle can be observed if the circuit is connected to an oscilloscope.

Limit cycles are common solutions for all types of dynamical systems. Sometimes it becomes necessary to prove the existence and uniqueness of a limit cycle, as described in the next section.

## 11.2 Existence and Uniqueness of Limit Cycles in the Plane

To understand the existence and uniqueness theorem, it is necessary to define some features of phase plane portraits. Assume that the existence and uniqueness theorem from Chapter 8 holds for all solutions considered here.

Definitions 1 and 2 in Chapter 9 can be extended to nonlinear planar systems of the form $\dot{x} = P(x, y), \dot{y} = Q(x, y)$, thus every solution, say, $\phi(t) = (x(t), y(t))$, can be represented as a curve in the plane and is called a trajectory. The phase portrait shows how the qualitative behavior is determined as $x$ and $y$ vary with $t$. The trajectory can also be defined in terms of the spatial coordinates $\mathbf{x}$, as in Definition 3 below. A brief look at Example 1 will help the reader to understand Definitions 1–7 in this section.

**Definition 2.** A *flow* on $\Re^2$ is a mapping $\pi : \Re^2 \to \Re^2$ such that

1. $\pi$ is continuous;
2. $\pi(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in \Re^2$;
3. $\pi(\pi(\mathbf{x}, t_1), t_2) = \pi(\mathbf{x}, t_1 + t_2)$.

**Definition 3.** Suppose that $I_{\mathbf{x}}$ is the maximal interval of existence. The *trajectory* (or *orbit*) through $\mathbf{x}$ is defined as $\gamma(\mathbf{x}) = \{\pi(\mathbf{x}, t) : t \in I_{\mathbf{x}}\}$.

The *positive semiorbit* is defined as $\gamma^+(\mathbf{x}) = \{\pi(\mathbf{x}, t) : t > 0\}$.

The *negative semiorbit* is defined as $\gamma^-(\mathbf{x}) = \{\pi(\mathbf{x}, t) : t < 0\}$.

**Definition 4** The *positive limit set* of a point $\mathbf{x}$ is defined as

$$\Lambda^+(\mathbf{x}) = \{\mathbf{y} : \text{there exists a sequence } t_n \to \infty \text{ such that } \pi(\mathbf{x}, t) \to \mathbf{y}\}.$$

The *negative limit set* of a point $\mathbf{x}$ is defined as

$$\Lambda^-(\mathbf{x}) = \{\mathbf{y} : \text{there exists a sequence } t_n \to -\infty \text{ such that } \pi(\mathbf{x}, t) \to \mathbf{y}\}.$$

In the phase plane, trajectories tend to a critical point, a closed orbit, or infinity.

**Definition 5.** A set $S$ is *invariant* with respect to a flow if $\mathbf{x} \in S$ implies that $\gamma(\mathbf{x}) \subset S$.

A set $S$ is *positively invariant* with respect to a flow if $\mathbf{x} \in S$ implies that $\gamma^+(\mathbf{x}) \subset S$.

A set $S$ is *negatively invariant* with respect to a flow if $\mathbf{x} \in S$ implies that $\gamma^-(\mathbf{x}) \subset S$.

A general trajectory can be labeled $\gamma$ for simplicity.

**Definition 6.** A limit cycle, say, $\Gamma$, is

- a *stable limit cycle* if $\Lambda^+(\mathbf{x}) = \Gamma$ for all $\mathbf{x}$ in some neighborhood; this implies that nearby trajectories are attracted to the limit cycle;
- an *unstable limit cycle* if $\Lambda^-(\mathbf{x}) = \Gamma$ for all $\mathbf{x}$ in some neighborhood; this implies that nearby trajectories are repelled away from the limit cycle;
- a *semistable limit cycle* if it is attracting on one side and repelling on the other.

The stability of limit cycles can also be deduced analytically using the Poincaré map (see Chapter 15). The following example will be used to illustrate each of the Definitions 1–6 above and 7 below.

**Definition 7.** The period, say, $T$, of a limit cycle is given by $\mathbf{x}(t) = \mathbf{x}(t+T)$, where $T$ is the minimum period. The period can be found by plotting a time series plot of the limit cycle (see the MATLAB command lines in Chapter 10).

**Example 1.** Describe some of the features for the following set of polar differential equations in terms of Definitions 1–7:

$$\dot{r} = r(1-r)(2-r)(3-r), \quad \dot{\theta} = -1. \tag{11.4}$$

**Solution.** A phase portrait is shown in Figure 11.4. There is a unique critical point at the origin since $\dot{\theta}$ is nonzero. There are three limit cycles that may be determined from the equation $\dot{r} = 0$. They are the circles of radii one, two, and three, all centered at the origin. Let $\Gamma_i$ denote the limit cycle of radius r=i.

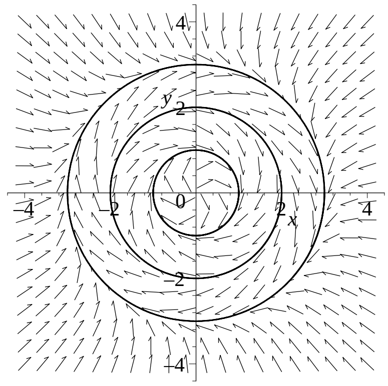



Figure 11.4: Three limit cycles for system (11.4).

There is one critical point at the origin. If a trajectory starts at this point, it remains there forever. A trajectory starting at $(1, 0)$ will reach the point $(-1, 0)$ when $t_1 = \pi$ and the motion is clockwise. Continuing on this path for another time interval $t_2 = \pi$, the orbit returns to $(1, 0)$. Using part 3 of Definition 2, one can write $\pi\left(\pi((1, 0), t_1), t_2\right) = \pi\left((1, 0), 2\pi\right)$ since the limit cycle is of period $2\pi$ (see below). On the limit cycle $\Gamma_1$, both the positive and negative semiorbits lie on $\Gamma_1$.

Suppose that $P = (\frac{1}{2}, 0)$ and $Q = (4, 0)$ are two points in the plane. The limit sets are given by $\Lambda^+(P) = \Gamma_1$, $\Lambda^-(P) = (0, 0)$, $\Lambda^+(Q) = \Gamma_3$, and $\Lambda^-(Q) = \infty$.

The annulus $A_1 = \{r \in \Re^2 : 0 < r < 1\}$, is positively invariant, and the annulus $A_2 = \{r \in \Re^2 : 1 < r < 2\}$ is negatively invariant.

If $0 < r < 1$, then $\dot{r} > 0$ and the critical point at the origin is unstable. If $1 < r < 2$, then $\dot{r} < 0$ and $\Gamma_1$ is a stable limit cycle. If $2 < r < 3$, then $\dot{r} > 0$ and $\Gamma_2$ is an unstable limit cycle. Finally, if $r > 3$, then $\dot{r} < 0$ and $\Gamma_3$ is a stable limit cycle.

Integrate both sides of $\dot{\theta} = -1$ with respect to time to show that the period of all of the limit cycles is $2\pi$.

**The Poincaré-Bendixson Theorem.** *Suppose that $\gamma^+$ is contained in a bounded region in which there are finitely many critical points. Then $\Lambda^+(\gamma)$ is either*

- *a single critical point*;
- *a single closed orbit*;

- *a* graphic—*critical points joined by heteroclinic orbits.*

A heteroclinic orbit connects two separate critical points and takes an infinite amount of time to make the connection; more detail is provided in Chapter 12.

**Corollary.** *Let $D$ be a bounded closed set containing no critical points and suppose that $D$ is positively invariant. Then there exists a limit cycle contained in $D$.*

A proof to this theorem involves topological arguments and can be found in [13], for example.

**Example 2.** By considering the flow across the rectangle with corners at $(-1, 2)$, $(1, 2)$, $(1, -2)$, and $(-1, -2)$, prove that the following system has at least one limit cycle:

$$\dot{x} = y - 8x^3, \quad \dot{y} = 2y - 4x - 2y^3. \tag{11.5}$$

**Solution.** The critical points are found by solving the equations $\dot{x} = \dot{y} = 0$. Set $y = 8x^3$. Then $\dot{y} = 0$ if $x(1 - 4x^2 + 256x^8) = 0$. The graph of the function $y = 1 - 4x^2 + 256x^8$ is given in Figure 11.5(a). The graph has no roots and the origin is the only critical point.

Linearize at the origin in the usual way. It is not difficult to show that the origin is an unstable focus.

Consider the flow on the sides of the given rectangle:

- On $y = 2, |x| \leq 1$, $\dot{y} = -4x - 12 < 0$.
- On $y = -2, |x| \leq 1$, $\dot{y} = -4x + 12 > 0$.
- On $x = 1, |y| \leq 2$, $\dot{x} = y - 8 < 0$.
- On $x = -1, |y| \leq 2$, $\dot{y} = y + 8 > 0$.

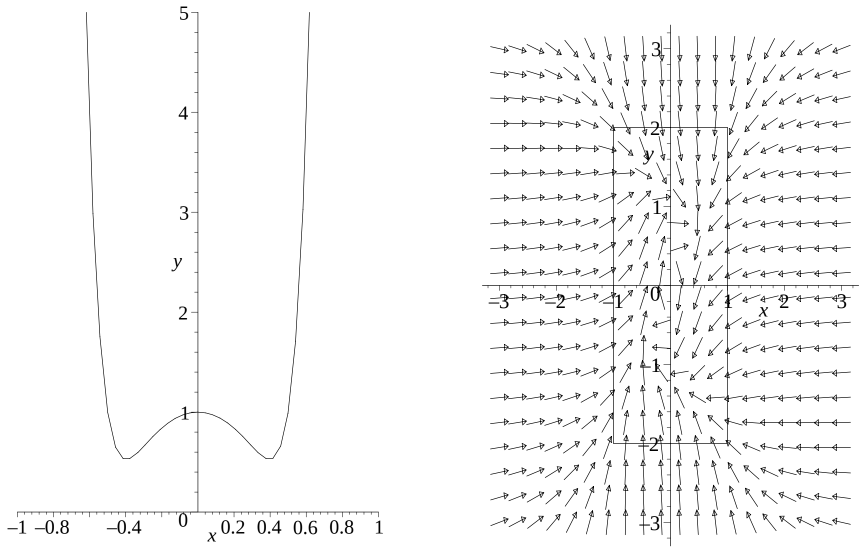
The flow is depicted in Figure 11.5(b). The rectangle is positively invariant and there are no critical points other than the origin, which is unstable. Consider a small deleted neighborhood, say, $N_\epsilon$, around this critical point. For example, the boundary of $N_\epsilon$ could be a small ellipse. On this ellipse, all trajectories will cross outwards. Therefore, there exists a stable limit cycle lying inside the rectangular region and outside of $N_\epsilon$ by the corollary to the Poincaré-Bendixson theorem.

**Definition 8.** A planar simple closed curve is called a *Jordan curve*.

Consider the system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{11.6}$$

where $P$ and $Q$ have continuous first-order partial derivatives. Let the vector field be denoted by $\mathbf{X}$ and let $\psi$ be a weighting factor that is continuously differentiable. Recall Green's Theorem, which will be required to prove the following two theorems.




(a) (b)

Figure 11.5: (a) Polynomial of degree 8. (b) Flow across the rectangle for system (11.5).

**Green's Theorem.** *Let $J$ be a Jordan curve of finite length. Suppose that $P$ and $Q$ are two continuously differentiable functions defined on the interior of $J$, say, $D$. Then*

$$\iint_D \left[\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right] dx\, dy = \oint_J P dy - Q dx.$$

**Dulac's Criteria.** *Consider an annular region, say, $A$, contained in an open set $E$. If*

$$\nabla.(\psi \mathbf{X}) = \mathrm{div}\,(\psi\mathbf{X}) = \frac{\partial}{\partial x}(\psi P)+\frac{\partial}{\partial y}(\psi Q)$$

*does not change sign in $A$, then there is at most one limit cycle entirely contained in $A$.*

**Proof.** Suppose that $\Gamma_1$ and $\Gamma_2$ are limit cycles encircling $K$, as depicted in Figure 11.6, of periods $T_1$ and $T_2$, respectively. Apply Green's Theorem to the region $R$ shown in Figure 11.6.

$$\iint_R \left[\frac{\partial(\psi P)}{\partial x}+\frac{\partial(\psi Q)}{\partial y}\right] dx\, dy = \oint_{\Gamma_2} \psi P dy - \psi Q dx +$$

$$\int_L \psi P dy - \psi Q dx - \oint_{\Gamma_1} \psi P dy - \psi Q dx - \int_L \psi P dy - \psi Q dx.$$

Now on $\Gamma_1$ and $\Gamma_2$, $\dot{x}=P$ and $\dot{y}=Q$, so

$$\iint_R \left[\frac{\partial(\psi P)}{\partial x}+\frac{\partial(\psi Q)}{\partial y}\right] dx\, dy$$

$$= \int_0^{T_2} (\psi PQ - \psi QP)\, dt - \int_0^{T_1} (\psi PQ - \psi QP)\, dt,$$

which is zero and contradicts the hypothesis that $\mathrm{div}(\psi\mathbf{X}) \neq 0$ in $A$. Therefore, there is at most one limit cycle entirely contained in the annulus $A$. □

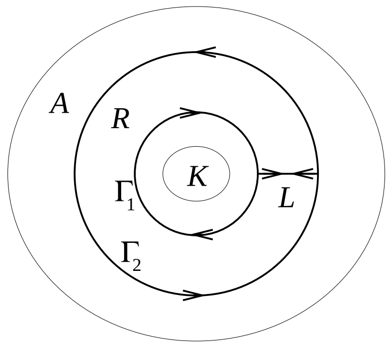



Figure 11.6: Two limit cycles encircling the region $K$.

**Example 3.** Use Dulac's criteria to prove that the system

$$\dot{x} = -y + x(1 - 2x^2 - 3y^2), \quad \dot{y} = x + y(1 - 2x^2 - 3y^2) \tag{11.7}$$

has a unique limit cycle in an annulus.

**Solution.** Convert to polar coordinates using the transformations

$$r\dot{r} = x\dot{x} + y\dot{y}, \quad r^2\dot{\theta} = x\dot{y} - y\dot{x}.$$

Therefore, system (11.7) becomes

$$\dot{r} = r(1 - 2r^2 - r^2 sin^2\theta), \quad \dot{\theta} = 1.$$

Since $\dot{\theta} = 1$, the origin is the only critical point. On the circle $r = \frac{1}{2}$, $\dot{r} = \frac{1}{2}(\frac{1}{2} - \frac{1}{4}\sin^2\theta)$. Hence $\dot{r} > 0$ on this circle. On the circle $r = 1$, $\dot{r} = -1 - \sin^2\theta$. Hence $\dot{r} < 0$ on this circle. If $r \geq 1$, then $\dot{r} < 0$, and if $0 < r \leq \frac{1}{2}$, then $\dot{r} > 0$. Therefore, there exists a limit cycle in the annulus $A = \{r : \frac{1}{2} < r < 1\}$ by the corollary to the Poincaré-Bendixson theorem.

Consider the annulus $A$. Now $\text{div}(\mathbf{X}) = 2(1 - 4r^2 - 2r^2\sin^2\theta)$. If $\frac{1}{2} < r < 1$, then $\text{div}(\mathbf{X}) < 0$. Since the divergence of the vector field does not change sign in the annulus $A$, there is at most one limit cycle in $A$ by Dulac's criteria.

A phase portrait is given in Figure 11.7.

**Example 4.** Plot a phase portrait for the Liénard system

$$\dot{x} = y, \; \dot{y} = -x - y(a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + a_{10}x^{10} + a_{12}x^{12} + a_{14}x^{14}),$$

where $a_2 = 90, a_4 = -882, a_6 = 2598.4, a_8 = -3359.997, a_{10} = 2133.34, a_{12} = -651.638$, and $a_{14} = 76.38$.

**Solution.** Not all limit cycles are convex closed curves as Figure 11.8 demonstrates.

## 11.3 Nonexistence of Limit Cycles in the Plane

**Bendixson's Criteria.** *Consider system (11.6) and suppose that $D$ is a simply connected domain (no holes in $D$) and that*

$$\nabla.(\psi\mathbf{X}) = \text{div}\,(\psi\mathbf{X}) = \frac{\partial}{\partial x}(\psi P) + \frac{\partial}{\partial y}(\psi Q) \neq 0$$

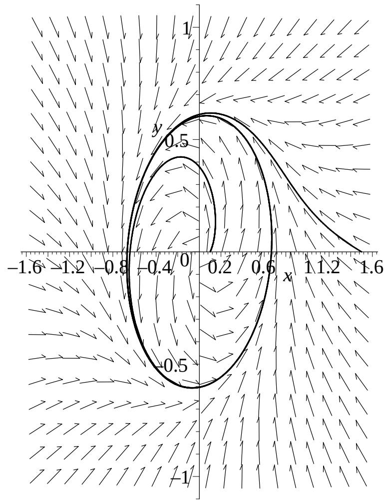



Figure 11.7: A phase portrait for system (11.7) showing the unique limit cycle.

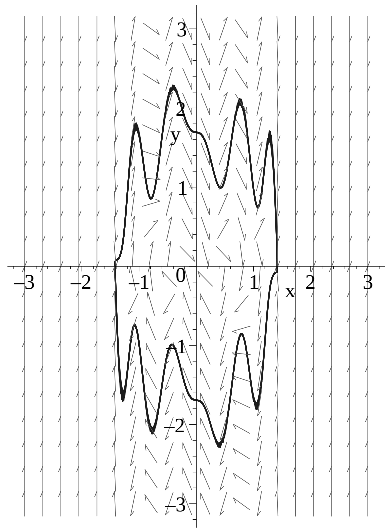



Figure 11.8: [MATLAB] A phase portrait for Example 4. The limit cycle is a nonconvex closed curve.

*in $D$. Then there are no limit cycles entirely contained in $D$.*

**Proof.** Suppose that $D$ contains a limit cycle $\Gamma$ of period $T$. Then from Green's Theorem

$$\iint_D \left[\frac{\partial(\psi P)}{\partial x} + \frac{\partial(\psi Q)}{\partial y}\right] dx\, dy = \oint_\Gamma (\psi P dy - \psi Q dx)$$
$$= \int_0^T \left(\psi P \frac{dy}{dt} - \psi Q \frac{dx}{dt}\right) dt = 0$$

since on $\Gamma$, $\dot{x} = P$ and $\dot{y} = Q$. This contradicts the hypothesis that $\mathrm{div}(\psi \mathbf{X}) \neq 0$, and therefore $D$ contains no limit cycles entirely. □

**Definition 9.** Suppose there is a compass on a Jordan curve $C$ and that the needle points in the direction of the vector field. The compass is moved in a counterclockwise direction around the Jordan curve by $2\pi$ radians. When it returns to its initial position, the needle will have moved through an angle, say, $\Theta$. The *index*, say, $I_{\mathbf{X}}(C)$, is defined as

$$I_{\mathbf{X}}(C) = \frac{\Delta\Theta}{2\pi},$$

where $\Delta\Theta$ is the overall change in the angle $\Theta$.

The above definition can be applied to isolated critical points. For example, the index of a node, focus, or center is $+1$ and the index of a col is $-1$. The following result is clear.

**Theorem 1.** *The sum of the indices of the critical points contained entirely within a limit cycle is* $+1$.

The next theorem then follows.

**Theorem 2.** *A limit cycle contains at least one critical point.*

When proving that a system has no limit cycles, the following items should be considered:

1. Bendixson's criteria;
2. indices;
3. invariant lines;
4. critical points.

**Example 5.** Prove that none of the following systems have any limit cycles:

(a) $\dot{x} = 1 + y^2 - e^{xy}, \quad \dot{y} = xy + cos^2 y.$

(b) $\dot{x} = y^2 - x, \quad \dot{y} = y + x^2 + yx^3.$

(c) $\dot{x} = y + x^3, \quad \dot{y} = x + y + y^3.$

(d) $\dot{x} = 2xy - 2y^4, \quad \dot{y} = x^2 - y^2 - xy^3.$

(e) $\dot{x} = x(2 - y - x), \quad \dot{y} = y(4x - x^2 - 3), \quad$ given $\quad \psi = \frac{1}{xy}$.

**Solutions.**

(a) The system has no critical points and hence no limit cycles by Theorem 2.

(b) The origin is the only critical point and it is a saddle point or col. Since the index of a col is $-1$, there are no limit cycles from Theorem 1.

(c) Find the divergence, div$\mathbf{X} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 3x^2 + 3y^2 + 1 \neq 0$. Hence there are no limit cycles by Bendixson's criteria.

(d) Find the divergence, div$\mathbf{X} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = -3x^2 y$. Now div$\mathbf{X} = 0$ if either $x = 0$ or $y = 0$. However, on the line $x = 0$, $\dot{x} = -2y^4 \leq 0$, and on the line $y = 0$, $\dot{y} = x^2 \geq 0$. Therefore, a limit cycle must lie wholly in one of the four quadrants. This is not possible since div$\mathbf{X}$ is nonzero here. Hence there are no limit cycles by Bendixson's criteria. Draw a small diagram to help you understand the solution.

(e) The axes are invariant since $\dot{x} = 0$ if $x = 0$ and $\dot{y} = 0$ if $y = 0$. The weighted divergence is given by div$(\psi\mathbf{X}) = \frac{\partial}{\partial x}(\psi P) + \frac{\partial}{\partial y}(\psi Q) = -\frac{1}{y}$. Therefore, there are no limit cycles contained entirely in any of the quadrants, and since the axes are invariant, there are no limit cycles in the whole plane.

**Example 6.** Prove that the system

$$\dot{x} = x(1 - 4x + y), \quad \dot{y} = y(2 + 3x - 2y)$$

has no limit cycles by applying Bendixson's criteria with $\psi = x^m y^n$.

**Solution.** The axes are invariant since $\dot{x} = 0$ on $x = 0$ and $\dot{y} = 0$ on $y = 0$. Now

$$\mathrm{div}(\psi\mathbf{X}) = \frac{\partial}{\partial x}\left(x^{m+1}y^n - 4x^{m+2}y^n + x^{m+1}y^{n+1}\right) + \frac{\partial}{\partial y}\left(2x^m y^{n+1} + 3x^{m+1}y^{n+1} - 2x^m y^{n+2}\right),$$

which simplifies to

$$\mathrm{div}(\psi\mathbf{X}) = (m + 2n + 2)x^m y^n + (-4m + 3n - 5)x^{m+1}y^n + (m - 2n - 3)x^m y^{n+1}.$$

Select $m = \frac{1}{2}$ and $n = -\frac{5}{4}$. Then

$$\mathrm{div}(\psi\mathbf{X}) = -\frac{43}{4}x^{\frac{3}{2}}y^{-\frac{5}{4}}.$$

Therefore, there are no limit cycles contained entirely in any of the four quadrants, and since the axes are invariant, there are no limit cycles at all.

## 11.4 Perturbation Methods

This section introduces the reader to some basic perturbation methods by means of example. The theory involves mathematical methods for finding series expansion approximations for perturbed systems. Perturbation theory can be applied to algebraic equations, boundary value problems, difference equations, Hamiltonian systems, ODEs, PDEs, and in modern times the theory underlies almost all of quantum field theory and quantum chemistry. There are whole books devoted to the study of perturbation methods and the reader is directed to the references [4], [10], and [16], for more detailed theory and more in-depth explanations.

The main idea begins with the assumption that the solution to the perturbed system can be expressed as an *asymptotic expansion* of the form

$$x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots . \tag{11.8}$$

**Definition 10.** The sequence $f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \phi_n(\epsilon)$ is an asymptotic (or Poincaré) expansion of the continuous function $f(\epsilon)$ if and only if, for all $n \geq 0$,

$$f(\epsilon) = \sum_{n=0}^{N} a_n \phi_n(\epsilon) + O\left(\phi_{N+1}(\epsilon)\right) \quad \text{as}\, \epsilon \to 0, \tag{11.9}$$

where the sequence constitutes an asymptotic scale such that for every $n \geq 0$,

$$\phi_{n+1}(\epsilon) = o\left(\phi_n(\epsilon)\right) \text{ as } \quad \epsilon \to 0.$$

**Definition 11.** An asymptotic expansion (11.9) is said to be *uniform* if in addition

$$|R_N(x,\epsilon)| \leq K|\phi_{N+1}(\epsilon)|,$$

for $\epsilon$ in a neighborhood of 0, where the Nth remainder $R_N(x,\epsilon) = O(\phi_{N+1}(\epsilon))$ as $\epsilon \to 0$, and $K$ is a constant.

In this particular case, we will be looking for asymptotic expansions of the form

$$x(t,\epsilon) \sim \sum_k x_k(t)\delta_k(\epsilon),$$

where $\delta_k(\epsilon) = \epsilon^k$ is an asymptotic scale. It is important to note that the asymptotic expansions often do not converge, however, one-term and two-term approximations provide an analytical expression that is dependent on the parameter, $\epsilon$, and some initial conditions. The major advantage that the perturbation analysis has over numerical analysis is that a general solution is available through perturbation methods where numerical methods only lead to a single solution.

As a simple introduction consider the following first order ordinary differential equation:

**Example 7.** Suppose that for $x \geq 0$,

$$\frac{dx}{dt} + x - \epsilon x^2 = 0, \quad x(0) = 2. \tag{11.10}$$

**Solution.** This equation can be solved directly, use the `dsolve` command in MATLAB to solve the system analytically. Set

$$x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots ,$$

where in order to satisfy the initial condition $x(0) = 2$, we will have $x_1(0) = 0$, $x_2(0) = 0$, and so on. To compute to $O(\epsilon^2)$, substitute the first three terms into system (11.10) and collect powers of $\epsilon$ using the collect command in MuPAD. In the Command Window of MATLAB simply type mupad to open a MuPAD notebook. The commands are:

```
x(t):=x0(t)+eps*x1(t)+eps^2*x2(t):

p:= diff(x(t),t)+x(t)-eps*x(t)^2=0:

collect(p,eps)
```

one obtains:

$$\begin{array}{ll} \epsilon^0: & \dot{x}_0(t) + x_0(t) = 0 \\ \epsilon^1: & \dot{x}_1(t) + x_1(t) - x_0(t)^2 = 0, \\ \epsilon^2: & \dot{x}_2(t) + x_2(t) - 2x_0(t)x_1(t) = 0, \\ \vdots & \vdots \end{array}$$

and we solve at each order, applying the initial conditions as we proceed.

For $O(1)$:

$$\dot{x}_0(t) + x_0(t) = 0, \quad x_0(0) = 2,$$

and the solution using MATLAB:

```
>> dsolve('Dx+x=0','x(0)=2')
```

is $x_0(t) = 2 * exp(-t)$.

For $O(\epsilon)$:

$$\dot{x}_1(t) + x_1(t) = 4e^{-2t}, \quad x_1(0) = 0,$$

and the solution using MATLAB is $x_1(t) = 4\left(e^{-t} - e^{-2t}\right)$.

For $O(\epsilon^2)$:

$$\dot{x}_2(t) + x_2(t) = 4e^{-t}x_1(t), \quad x_2(0) = 0,$$

and the solution using MATLAB is $x_2(t) = 8\left(e^{-t} - 2e^{-2t} + e^{-3t}\right)$.

Therefore, the solution to second order is:

$$x(t) \approx 2e^{-t} + 4\epsilon\left(e^{-t} - e^{-2t}\right) + 8\epsilon^2\left(e^{-t} - 2e^{-2t} + e^{-3t}\right).$$

To keep the theory simple and in relation to other material in this chapter, the author has decided to focus on perturbed ODEs of the form

$$\ddot{x} + x = \epsilon f(x, \dot{x}), \tag{11.11}$$

where $0 \leq \epsilon \ll 1$ and $f(x, \dot{x})$ is an arbitrary smooth function. The unperturbed system represents a linear oscillator and when $0 < \epsilon \ll 1$, system (11.11) becomes a weakly nonlinear oscillator. Systems of this form include the Duffing equation

$$\ddot{x} + x = \epsilon x^3, \tag{11.12}$$

and the van der Pol equation

$$\ddot{x} + x = \epsilon \left(x^2 - 1\right) \dot{x}. \tag{11.13}$$

**Example 8.** Use perturbation theory to find a one-term and two-term asymptotic expansion of Duffing's equation (11.12) with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

**Solution.** Substitute (11.8) into (11.12) to get

$$\frac{d^2}{dt^2}(x_0 + \epsilon x_1 + \ldots) + (x_0 + \epsilon x_1 + \ldots) = \epsilon (x_0 + \epsilon x_1 + \ldots)^3.$$

Use the collect command in MuPAD to group terms according to powers of $\epsilon$, thus

$$[\ddot{x}_0 + x_0] + \epsilon \left[\ddot{x}_1 + x_1 - x_0^3\right] + O(\epsilon^2) = 0.$$

The order equations are

$$\begin{array}{llll} O(1): & \ddot{x}_0 + x_0 = 0, & x_0(0) = 1, & \dot{x}_0(0) = 0, \\ O(\epsilon): & \ddot{x}_1 + x_1 = x_0^3, & x_1(0) = 0, & \dot{x}_1(0) = 0. \\ \vdots & & \vdots & \end{array}$$

The $O(1)$ solution computed using the dsolve command in MATLAB is $x_0 = \cos(t)$. Let us compare this solution with the numerical solution, say, $x_N$, when $\epsilon = 0.01$. Figure 11.9 shows the time against the error, $x_N - x_0$, for $0 \leq t \leq 100$.

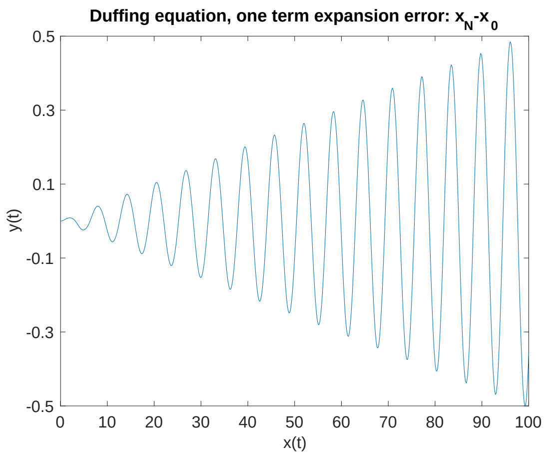



Figure 11.9: [MATLAB] The error between the numerical solution $x_N$ and the one-term expansion $x_0$ for the Duffing system (11.12) when $\epsilon = 0.01$.

Using dsolve and simplify in MATLAB, the $O(\epsilon)$ solution is computed to be

$$x_1 = \frac{3}{8} t \sin(t) + \frac{1}{8} cos(t) - \frac{1}{8} cos^3(t).$$

Using the combine with sincos command in MuPAD, we have

$$x \sim x_P = \cos(t) + \epsilon\left(\frac{1}{32}cos(t) - \frac{1}{32}cos(3t) + \frac{3}{8}tsin(t)\right),$$

where $x_P$ represents the Poincaré expansion up to the second term. The term $t\sin(t)$ is called a *secular* term and is an oscillatory term of growing amplitude. Unfortunately, the secular term leads to a nonuniformity for large $t$. Figure 11.10 shows the error for the two-term Poincaré expansion, $x_N - x_P$, when $\epsilon = 0.01$.

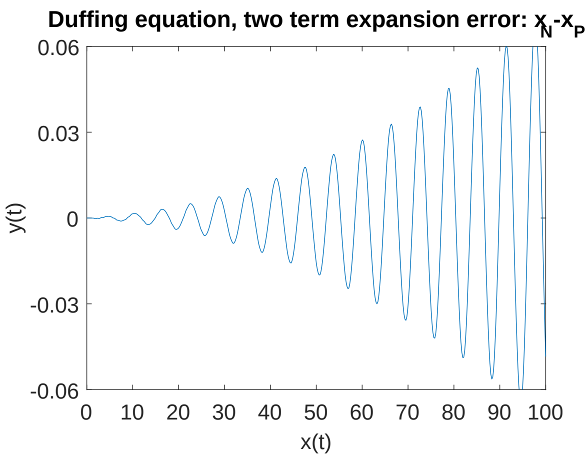



Figure 11.10: [MATLAB] The error between the numerical solution $x_N$ and the two-term expansion $x_P$ for the Duffing system (11.12) when $\epsilon = 0.01$.

By introducing a strained coordinate, the nonuniformity may be overcome and this is the idea behind the Lindstedt-Poincaré technique for periodic systems. The idea is to introduce a straining transformation of the form

$$\frac{\tau}{t} = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots, \tag{11.14}$$

and seek values $\omega_1, \omega_2, \ldots$, that avoid secular terms appearing in the expansion.

**Example 9.** Use the Lindstedt-Poincaré technique to determine a two-term uniform asymptotic expansion of Duffing's equation (11.12) with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

**Solution.** Using the transformation given in (11.14)

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \left(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots\right)\frac{d}{d\tau},$$

$$\frac{d^2}{dt^2} = \left(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots\right)^2\frac{d^2}{d\tau^2}.$$

Applying the transformation to equation (11.12) leads to

$$\left(1 + 2\epsilon\omega_1 + \epsilon^2\left(\omega_1^2 + 2\omega_2\right) + \cdots\right)\frac{d^2x}{d\tau^2} + x = \epsilon x^3,$$

where $x$ is now a function of the strained variable $\tau$. Assume that

$$x(\tau, \epsilon) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \ldots. \tag{11.15}$$

Substituting (11.15) into (11.12) using MATLAB gives the following order equations:

$$
\begin{aligned}
O(1): \quad & \frac{d^2x_0}{d\tau^2} + x_0 = 0, \\
& x_0(\tau = 0) = 1, \quad \frac{dx_0}{d\tau}(\tau = 0) = 0, \\
O(\epsilon): \quad & \frac{d^2x_1}{d\tau^2} + x_1 = x_0^3 - 2\omega_1 \frac{d^2x_0}{d\tau^2}, \\
& x_1(0) = 0, \quad \frac{dx_1}{d\tau}(0) = 0, \\
O(\epsilon^2): \quad & \frac{d^2x_2}{d\tau^2} + x_2 = 3x_0^2 x_1 - 2\omega_1 \frac{d^2x_1}{d\tau^2} - (\omega_1^2 + 2\omega_2)\frac{d^2x_0}{d\tau^2}, \\
& x_2(0) = 0, \quad \frac{dx_2}{d\tau}(0) = 0.
\end{aligned}
$$

The $O(1)$ solution is $x_0(\tau) = \cos(\tau)$. Using MATLAB, the solution to the $O(\epsilon)$ equation is

$$x_1(\tau) = \frac{1}{8}\sin(\tau)\left(3\tau + 8\omega_1\tau + \cos(\tau)\sin(\tau)\right).$$

To avoid secular terms, select $\omega_1 = -\frac{3}{8}$, then the $O(\epsilon)$ solution is

$$x_1(\tau) = \frac{1}{8}\sin^2(\tau)\cos(\tau).$$

Using MATLAB, the $O(\epsilon^2)$ solution is

$$x_2(\tau) = \frac{1}{512}\sin(\tau)\left(42\tau + 512\omega_2\tau + 23\sin(2\tau) - \sin(4\tau)\right),$$

and selecting $\omega_2 = -\frac{21}{256}$ avoids secular terms.

The two-term uniformly valid expansion of equation (11.12) is

$$x(\tau, \epsilon) \sim x_{LP} = \cos(\tau) + \frac{\epsilon}{8}\sin^2(\tau)\cos(\tau),$$

where

$$\tau = t\left(1 - \frac{3}{8}\epsilon - \frac{21}{256}\epsilon^2 + O(\epsilon^3)\right),$$

as $\epsilon \to 0$. Note that the straining transformation is given to a higher order than the expansion of the solution. The difference between the two-term uniform asymptotic expansion and the numerical solution is depicted in Figure 11.11.

Unfortunately, the Lindstedt-Poincaré technique does not always work for oscillatory systems. An example of its failure is provided by the van der Pol equation (11.13).

**Example 10.** Show that the Lindstedt-Poincaré technique fails for the ODE (11.13) with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

**Solution.** Substituting (11.15) into (11.13) using MATLAB gives the following order equations:

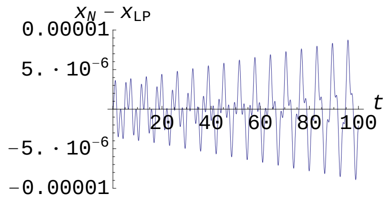



Figure 11.11: The error between the numerical solution $x_N$ and the two-term Linstedt-Poincaré expansion $x_{LP}$ for the Duffing system (11.12) when $\epsilon = 0.01$.

$$\begin{aligned}
O(1): \quad & \frac{d^2x_0}{d\tau^2} + x_0 = 0, \\
& x_0(\tau = 0) = 1, \quad \frac{dx_0}{d\tau}(\tau = 0) = 0, \\
O(\epsilon): \quad & \frac{d^2x_1}{d\tau^2} + x_1 = \frac{dx_0}{d\tau} - x_0^2\frac{dx_0}{d\tau} - 2\omega_1\frac{d^2x_0}{d\tau^2}, \\
& x_1(0) = 0, \quad \frac{dx_1}{d\tau}(0) = 0,
\end{aligned}$$

The $O(1)$ solution is $x_0(\tau) = \cos(\tau)$. Using MATLAB, the solution to the $O(\epsilon)$ equation can be simplified to

$$x_1(\tau) = \frac{1}{16}\left(6\tau\cos(\tau) - (5 - 16\tau\omega_1 + \cos(2\tau))\sin(\tau)\right)$$

or

$$x_1(\tau) = \frac{1}{16}\left(\{6\tau\cos(\tau) + 16\tau\omega_1\sin(\tau)\} - (5 + \cos(2\tau))\sin(\tau)\right).$$

To remove secular terms set $\omega_1 = -\frac{3}{8}\cot(\tau)$, then

$$x(\tau, \epsilon) = \cos(\tau) + O(\epsilon),$$

where

$$\tau = t - \frac{3}{8}\epsilon t\cot(t) + O(\epsilon^2).$$

This is invalid since the cotangent function is singular when $t = n\pi$, where $n$ is an integer. Unfortunately, the Lindstedt-Poincaré technique does not work for all ODEs of the form (11.11); it cannot be used to obtain approximations that evolve aperiodically on a slow time scale.

Consider the van der Pol equation (11.13), Figure 11.12 shows a trajectory starting at $x(0) = 0.1, \dot{x}(0) = 0$ for $\epsilon = 0.05$ and $0 \leq t \leq 800$. The trajectory spirals around the origin and it takes many cycles for the amplitude to grow substantially. As $t \to \infty$, the trajectory asymptotes to a limit cycle of approximate radius two. This is an example of a system whose solutions depend simultaneously on widely different scales. In this case there are two time scales: a fast time scale for the sinusoidal oscillations $\sim O(1)$, and a slow time scale over which the amplitude grows $\sim O(\frac{1}{\epsilon})$. The *method of multiple scales* introduces new slow-time variables for each time scale of interest in the problem.

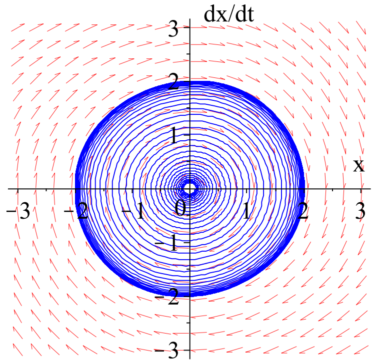



Figure 11.12: A trajectory for the van der Pol equation (11.13) when $\epsilon = 0.05$.

## The Method of Multiple Scales.

Introduce new time scales, say, $\tau_0 = t$ and $\tau_1 = \epsilon t$, and seek approximate solutions of the form

$$x(t,\epsilon) \sim x_0(\tau_0,\tau_1) + \epsilon x_1(\tau_0,\tau_1) + \cdots . \tag{11.16}$$

Substitute into the ODE and solve the resulting PDEs. An example is given below.

**Example 11.** Use the method of multiple scales to determine a uniformly valid one-term expansion for the van der Pol equation (11.13) with initial conditions $x(0) = a$ and $\dot{x}(0) = 0$.

**Solution.** Substitute equation (11.16) into (11.13) using MATLAB gives the following order equations:

$$\begin{aligned} O(1): &\quad \frac{\partial^2 x_0}{\partial \tau_0^2} + x_0 = 0, \\ O(\epsilon): &\quad \frac{\partial^2 x_1}{\partial \tau_0^2} + x_1 = -2\frac{\partial x_0}{\partial \tau_0 \tau_1} - \left(x_0^2 - 1\right)\frac{\partial x_0}{\partial \tau_0}. \end{aligned}$$

The general solution to the $O(1)$ PDE may be found using MATLAB,

$$x_0(\tau_0,\tau_1) = c_1(\tau_1)\cos(\tau_0) + c_2(\tau_1)\sin(\tau_0)$$

which using trigonometric identities can be expressed as

$$x_0(\tau_0,\tau_1) = R(\tau_1)\cos(\tau_0 + \theta(\tau_1)), \tag{11.17}$$

where $R(\tau_1)$ and $\theta(\tau_1)$ are the slowly-varying amplitude and phase of $x_0$, respectively. Substituting (11.17), the $O(\epsilon)$ equation becomes

$$\begin{aligned} \frac{\partial^2 x_1}{\partial \tau_0^2} + x_1 = &-2\left(\frac{dR}{d\tau_1}\sin(\tau_0 + \theta(\tau_1)) + R(\tau_1)\frac{d\theta}{d\tau_1}\cos(\tau_0 + \theta(\tau_1))\right) \\ &- R(\tau_1)\sin(\tau_0 + \theta(\tau_1))\left(R^2(\tau_1)\cos^2(\tau_0 + \theta(\tau_1)) - 1\right). \end{aligned} \tag{11.18}$$

In order to avoid resonant terms on the right-hand side which lead to secular terms in the solution it is necessary to remove the linear terms $\cos(\tau_0 + \theta(\tau_1))$ and $\sin(\tau_0 +$

$\theta(\tau_1)$) from the equation. Use the combine command in MATLAB to reduce an expression to a form linear in the trigonometric function. Equation (11.18) then becomes

$$\frac{\partial^2 x_1}{\partial \tau_0^2} + x_1 = \left\{-2\frac{dR}{d\tau_1} + R - \frac{R^3}{4}\right\}\sin(\tau_0 + \theta(\tau_1))$$
$$\left\{-2R\frac{d\theta}{d\tau_1}\right\}\cos(\tau_0 + \theta(\tau_1)) - \frac{R^3}{4}\sin(3\tau_0 + 3\theta(\tau_1)).$$

To avoid secular terms set

$$-2\frac{dR}{d\tau_1} + R - \frac{R^3}{4} = 0 \quad \text{and} \quad \frac{d\theta}{d\tau_1} = 0. \tag{11.19}$$

The initial conditions are $x_0(0,0) = a$ and $\frac{\partial x_0}{\partial \tau_0} = 0$ leading to $\theta(0) = 0$ and $R(0) = \frac{a}{2}$. The solutions to system (11.19) with these initial conditions are easily computed with MATLAB, thus

$$R(\tau_1) = \frac{2}{\sqrt{1 + \left(\frac{4}{a^2} - 1\right)e^{-\tau_1}}} \quad \text{and} \quad \theta(\tau_1) = 0.$$

Therefore, the uniformly valid one-term solution is

$$x_0(\tau_0, \tau_1) = \frac{2\cos(\tau_0)}{\sqrt{1 + \left(\frac{4}{a^2} - 1\right)e^{-\tau_1}}} + O(\epsilon)$$

or

$$x(t) = \frac{2\cos(t)}{\sqrt{1 + \left(\frac{4}{a^2} - 1\right)e^{-\epsilon t}}} + O(\epsilon).$$

As $t \to \infty$, the solution tends asymptotically to the limit cycle $x = 2\cos(t) + O(\epsilon)$, for all initial conditions. Notice that only the initial condition $a = 2$ gives a periodic solution.

Figure 11.13 shows the error between the numerical solution and the one-term multiple scale approximation, say, $x_{MS}$, when $\epsilon = 0.01$, and $x(0) = 1, \dot{x}(0) = 0$.

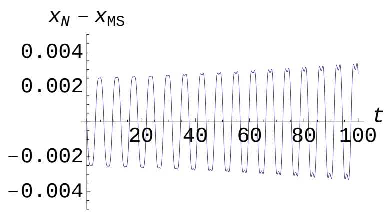



Figure 11.13: The error between the numerical solution $x_N$ and the one-term multiple scale expansion $x_{MS}$ for the van der Pol equation (11.13) when $\epsilon = 0.01$, and $x(0) = 1, \dot{x}(0) = 0$.

## 11.5 MATLAB Commands

See Chapter 9 for help with plotting phase portraits.

```
% Programs 11a - Phase portrait (Fig. 11.2).
% Limit cycle of a van der Pol system.
% IMPORTANT - vectorfield.m must be in the same folder.
clear
hold on
sys=@(t,x) [x(2);-x(1)-5*x(2)*((x(1))^2-1)];
vectorfield(sys,-3:.3:3,-10:1.3:10);
[t,xs] = ode45(sys,[0 30],[2 1]);
plot(xs(:,1),xs(:,2))
hold off
axis([-3 3 -10 10])
fsize=15;
set(gca,'xtick',-3:1:3,'FontSize',fsize)
set(gca,'ytick',-10:5:10,'FontSize',fsize)
xlabel('x(t)','FontSize',fsize)
ylabel('y(t)','FontSize',fsize)
hold off
```

---

```
% Programs 11b - Limit cycle, Fitzhugh-Nagumo system.
% Chapter 11 - Limit Cycles.
% Copyright Springer 2014. Stephen Lynch.
% Limit cycle of a Fitzhugh-Nagumo system (Figure 11.3).
% This is a simple model of an integrate and fire neuron.
clear
hold on
epsilon=0.01;omega=0.112;theta=0.14;gamma=2.54;
sys = @(t,x) [-x(1)*(x(1)-theta)*(x(1)-1)-x(2)+omega;
epsilon*(x(1)-gamma*x(2))];
vectorfield(sys,-1:.2:2,1:.2:3);
[t,xs] = ode45(sys,[0 100],[1 .1]);
plot(xs(:,1),xs(:,2));
x=-0.4:0.01:1.2;
plot(x,x./gamma,'r',x,omega-x.*(x-theta).*(x-1),'m')
hold off
axis([-0.4 1.2 0 0.4])
fsize=15;
set(gca,'XTick',-0.4:0.2:1.2,'FontSize',fsize)
set(gca,'YTick',0:0.1:0.4,'FontSize',fsize)
xlabel('u(t)','FontSize',fsize)
ylabel('v(t)','FontSize',fsize)
hold off
```

---

```
% Programs 11c - Non-convex limit cycle of a Lienard system (Figure 11.8).
clear
hold on
sys1=@(t,x) [x(2);-x(1)-x(2)*(90*(x(1))^2-882*(x(1))^4+2598.4*(x(1))^6
-3359.997*(x(1))^8+2133.34*(x(1))^10-651.638*(x(1))^12+76.38*(x(1))^14)];
[t,xt] = ode45(sys1,[0 30],[1.4 0]);
plot(xt(:,1),xt(:,2))
hold off
axis([-3 3 -3 3])
```

```
fsize=15;
set(gca,'xtick',[-3:1:3],'FontSize',fsize)
set(gca,'ytick',[-3:1:3],'FontSize',fsize)
xlabel('x(t)','FontSize',fsize)
ylabel('y(t)','FontSize',fsize)
hold off
```

---

```
% Programs 11d - Two limit cycles in an economic Kaldor model.
% Chapter 11 - Limit Cycles.
% Copyright Springer 2014. Stephen Lynch.

% J. Grasman and J.J. Wentzel, Co-existence of a limit cycle and an
% equilibrium in Kaldor's business cycle model and its consequences,
% J. Economic Behavior and Organization 24, 369-377 (1994).
% NOTE: There is an error in the equations in the paper - corrected below.

clear;figure;
hold on
% Plot the isoclines.
ezplot('25*2^(-1/(0.015*x+0.00001)^2)+0.05*x+5*(320/y)^3-0.282*x',
[10 100 200 500])
ezplot('25*2^(-1/(0.015*x+0.00001)^2)+0.05*x+5*(320/y)^3-0.05*y',
[10 100 200 500])
alpha = 3; delta = 0.05; s = 0.282;

% The Kaldor business cycle model.
sys = @(t,x)[alpha*(25*2^(-1/(0.015*x(1)+0.00001)^2)+0.05*x(1)+
5*(320/x(2))^3-s*x(1)); 25*2^(-1/(0.015*x(1)+0.00001)^2)+0.05*x(1)
+5*(320/x(2))^3-delta*x(2)];

% Plot the stable limit cycle.
[~,xs] = ode23s(sys,[0 100],[25 300]);
plot(xs(:,1),xs(:,2),'r')
% Plot the unstable limit cycle.
[t,xb] = ode23s(sys,[0 -100],[60 355]);
plot(xb(:,1),xb(:,2),'b')
hold off

axis([0 100 200 500])
fsize=15;
set(gca,'XTick',0:20:100,'FontSize',fsize)
set(gca,'YTick',200:50:400,'FontSize',fsize)
title('Two limit cycles in a Kaldor economic model','FontSize',fsize)
xlabel('y','FontSize',fsize)
ylabel('k','FontSize',fsize)
hold off
```

---

```
% Programs 11e - Perturbation Methods
% Chapter 11 - Limit Cycles.
% Copyright Springer 2014. Stephen Lynch.
```

```
% See Example 8.
% Solve the Order epsilon ODE.
deq1=dsolve('D2x+x=cos(t)^3','x(0)=0','Dx(0)=0')
O_epsilon=simplify(deq1)
% Use MuPAD to simplify again (See combine, sincos).

% Numerical solution of the Duffing equation.
epsilon=0.01;
deq2=@(t,x) [x(2);-x(1)+epsilon*x(1)^3];
[t,xa]=ode45(deq2,[0,100],[1,0]);

% Plot x_N-x_0 and x_N-x_P: see Figures 11.9 and 11.10.
subplot(2,1,1)
plot(t,xa(:,1)-cos(t))
title('Duffing equation, one term expansion error: x_N-x_0',
'FontSize',fsize)
ylim=0.5;
axis([0 100 -ylim ylim])
fsize=15;
set(gca,'XTick',0:10:100,'FontSize',fsize)
set(gca,'YTick',-ylim:0.2:ylim,'FontSize',fsize)
xlabel('x(t)','FontSize',fsize)
ylabel('y(t)','FontSize',fsize)
subplot(2,1,2)
plot(t,xa(:,1)-cos(t)-epsilon*(cos(t)/8 - cos(t).^3/8 +
(3*t.*sin(t))/8))
title('Duffing equation, two term expansion error: x_N-x_P',
'FontSize',fsize)
ylim=0.18;
axis([0 100 -ylim ylim])
fsize=15;
set(gca,'XTick',0:10:100,'FontSize',fsize)
set(gca,'YTick',-ylim:0.09:ylim,'FontSize',fsize)
xlabel('x(t)','FontSize',fsize)
ylabel('y(t)','FontSize',fsize)

% End of Programs 11e.
```

---

```
% Programs 11f.mn - Perturbation Methods
% Chapter 11 - Limit Cycles.
% Copyright Springer 2014. Stephen Lynch.
% See Example 8.

reset():
x(t):=x0(t)+eps*x1(t)+eps^2*x2(t):
p:= diff(x(t),t,t)+x(t)-eps*x(t)^3=0:
collect(p,eps)

reset():
x0:=ode::solve({x''(t)+x(t)=0,x(0)=1,x'(0)=0},x(t))
x1:=ode::solve({x''(t)+x(t)=cos(t)^3,x(0)=0,x'(0)=0},x(t))
```

```
x1:=simplify(x1)
combine(x1,sincos)

% End of Programs 11f.
```

---

## 11.6 Exercises

1. Prove that the system

$$\dot{x} = y + x\left(\frac{1}{2} - x^2 - y^2\right), \quad \dot{y} = -x + y\left(1 - x^2 - y^2\right)$$

   has a stable limit cycle. Plot the limit cycle.

2. By considering the flow across the square with coordinates $(1,1)$, $(1,-1)$, $(-1,-1)$, $(-1,1)$, centered at the origin, prove that the system

$$\dot{x} = -y + x\cos(\pi x), \quad \dot{y} = x - y^3$$

   has a stable limit cycle. Plot the vector field, limit cycle, and square.

3. Prove that the system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$

   has a unique limit cycle.

4. Prove that the system.

$$\dot{x} = y + x(\alpha - x^2 - y^2), \quad \dot{y} = -x + y(1 - x^2 - y^2),$$

   where $0 < \alpha < 1$, has a limit cycle and determine its stability.

5. For which parameter values does the Holling-Tanner model

$$\dot{x} = x\beta\left(1 - \frac{x}{k}\right) - \frac{rxy}{(a + ax)}, \quad \dot{y} = by\left(1 - \frac{Ny}{x}\right)$$

   have a limit cycle?

6. Plot phase portraits for the Liénard system

$$\dot{x} = y - \mu(-x + x^3), \quad \dot{y} = -x,$$

   when (a) $\mu = 0.01$, and (b) $\mu = 10$.

7. Prove that none of the following systems have limit cycles:

   (a) $\dot{x} = y, \quad \dot{y} = -x - (1 + x^2 + x^4)y;$
   (b) $\dot{x} = x - x^2 + 2y^2, \quad \dot{y} = y(x + 1);$
   (c) $\dot{x} = y^2 - 2x, \quad \dot{y} = 3 - 4y - 2x^2y;$
   (d) $\dot{x} = -x + y^3 - y^4, \quad \dot{y} = 1 - 2y - x^2y + x^4;$

(e) $\dot{x} = x^2 - y - 1, \quad \dot{y} = y(x-2);$

(f) $\dot{x} = x - y^2(1 + x^3), \quad \dot{y} = x^5 - y;$

(g) $\dot{x} = 4x - 2x^2 - y^2, \quad \dot{y} = x(1 + xy).$

8. Prove that neither of the following systems have limit cycles using the given multipliers:

   (a) $\dot{x} = x(4 + 5x + 2y), \quad \dot{y} = y(-2 + 7x + 3y), \quad \psi = \frac{1}{xy^2}.$

   (b) $\dot{x} = x(\beta - \delta x - \gamma y), \quad \dot{y} = y(b - dy - cx), \quad \psi = \frac{1}{xy}.$

   In case (b), prove that there are no limit cycles in the first quadrant only. These differential equations were used as a general model for competing species in Chapter 10.

9. Use the Lindstedt-Poincaré technique to obtain a one-term uniform expansion for the ODE
$$\frac{d^2x}{dt^2} + x = \epsilon x \left(1 - \left(\frac{dx}{dt}\right)^2\right),$$
   with initial conditions $x(0) = a$ and $\dot{x}(0) = 0$.

10. Using the method of multiple scales, show that the one-term uniform valid expansion of the ODE
$$\frac{d^2x}{dt^2} + x = -\epsilon \frac{dx}{dt},$$
   with initial conditions $x(0) = b$, $\dot{x}(0) = 0$ is
$$x(t, \epsilon) \sim x_{MS} = be^{-\frac{\epsilon t}{2}} \cos(t),$$
   as $\epsilon \to 0$.

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