MANCHESTER METROPOLITAN UNIVERSITY

MATLAB-Based Examination

Manchester Metropolitan University

Mock Examination One for
BSc Applicable Mathematics Combined Honours
BSc Mathematics
BSc Financial Mathematics

UNIT 63MA6302: Dynamical Simulation and Chaos

Instructions to Candidates

Answer **FIVE** questions.
This is a **CLOSED BOOK** exam.
The marks awarded for each question are shown in square brackets.
You must show **ALL** your working to obtain full marks.
MATLAB is available, but you are required to write the answers and all the working in your answer book.

Marks are awarded for the best five answers.

1. (a) Sketch a phase portrait for the following linear system showing all isoclines:

$$\dot{x} = x + y,$$
$$\dot{y} = 4x - 2y.$$

[8]

(b) Solve the differential equations

$$\dot{r} = r - r^2,$$
$$\dot{\theta} = 1.$$

given that $r(0) = 2$ and $\theta(0) = 0$. Hence show that the Poincaré return map, mapping points, say, r_n, on the positive x-axis to itself is given by

$$r_{n+1} = \frac{r_n e^{2\pi}}{r_n e^{2\pi} - (r_n - 1)}.$$

[12]

2. (a) Determine the location and type of the five critical points of the Hamiltonian system

$$\dot{x} = y(x^2 - 1),$$
$$\dot{y} = x(1 - y^2)$$

Find the Hamiltonian function and sketch a phase portrait.

[14]

(b) Show that the origin of the following nonlinear system

$$\dot{x} = -2y + yz$$
$$\dot{y} = x - xz - y^3$$
$$\dot{z} = xy - z^3$$

is asymptotically stable using the Lyapunov function, $V = x^2 + 2y^2 + z^2$.

[6]

3. (a) Plot phase portraits and a bifurcation diagram for the system

$$\dot{x} = x(x-\mu)(x+\mu),$$
$$\dot{y} = -y.$$

[14]

(b) Plot a bifurcation diagram for the system

$$\dot{r} = r(r-\beta)(r-\beta^2),$$
$$\dot{\theta} = -1.$$

[6]

4. (a) Show that the system

$$\dot{x} = xy - x^2 y + y^3,$$
$$\dot{y} = y^2 + x^3 - xy^2,$$

can be transformed to

$$\dot{r} = r^2 \sin\theta,$$
$$\dot{\theta} = r^2(\cos\theta - \sin\theta)(\cos\theta + \sin\theta).$$

using the relations $r\dot{r} = x\dot{x} + y\dot{y}$ and $r^2\dot{\theta} = x\dot{y} - y\dot{x}$. Sketch a phase portrait for this system given that there is one non-hyperbolic critical point at the origin.

[12]

(b) Compute the eigenvalues of the three critical points of the Lorenz system

$$\dot{x} = y - x,$$
$$\dot{y} = 2x - y - xz,$$
$$\dot{z} = xy - 4z.$$

[8]

5. (a) Find the fixed points of periods one and two for the complex mapping

$$z_{n+1} = z_n^2 + (1+i),$$

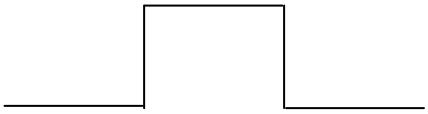
and, if possible, determine the stability of the fixed points of period one.

[8]

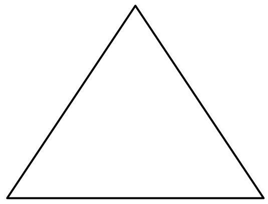
(b) Given the complex mapping $E_{n+1} = A + BE_n e^{i|E_n|^2}$, where $E_n = x_n + iy_n$, sketch real curves depicting the location of the fixed points of period one when *A=3* and *B=0.3*. Use the data cursor to approximate these co-ordinates.

[12]

6. (a) A variation of the Koch snowflake is constructed by adjoining the following motif (all lengths are equal) on to the outer edges of a unit length equilateral triangle:



Motif



Equilateral triangle

Determine the fractal dimension and the area bounded by the true fractal object.

[15]

(b) Derive the inverse map of

$$E_{n+1} = A + BE_n e^{i|E_n|^2} \quad .$$

[5]

7. (a) Find the fixed points of periods one and two for the Hénon map

$$x_{n+1} = \frac{3}{50} + \frac{9}{10} y_n - x_n^2 ,$$
$$y_{n+1} = x_n$$

and classify the fixed points of period one.

[10]

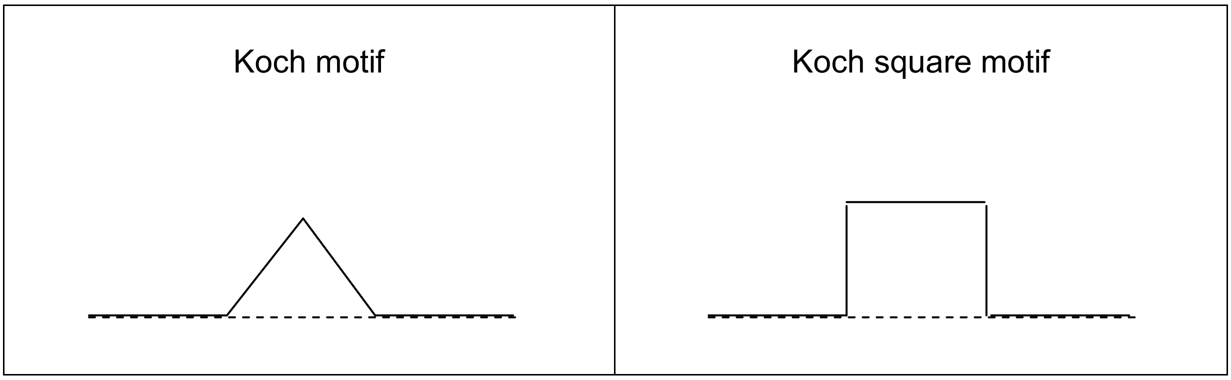
(b) Consider the map defined by $x_{n+1} = f(x_n)$, where $f(x)$ is defined by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 3x & 0 \leq x \leq \frac{1}{2} \\ 3 - 3x & \frac{1}{2} < x \leq 1 \\ 0 & x > 1 \end{cases} .$$

Plot the function on graph paper. Which two sets of points remain bounded within the interval (0,1) on the first iteration? Which four sets of points remain bounded within (0,1) after two iterations? The set of points which never escape the region (0,1) is a Cantor set. What is the fractal dimension of this Cantor set?

[10]

8. (a) Motifs for the Koch curve and Koch square curve are shown below:



Below is the MATLAB program for plotting the Koch curve up to stage 7. Modify the program (copy into the answer booklet) to plot the Koch square curve up to stage 7.

```
% Plot the Koch curve up to stage k=7
clear
k=7;
mmax=4^k;
x=zeros(1,mmax);y=zeros(1,mmax);segment=zeros(1,mmax);
h=3^(-k);
x(1)=0;y(1)=0;
angle(1)=0;angle(2)=pi/3;angle(3)=-pi/3;angle(4)=0;
    for a=1:mmax
    m=a-1;ang=0;
        for b=1:k
            segment(b)=mod(m,4);
                m=floor(m/4);
            r=segment(b)+1;
            ang=ang+angle(r);
        end
    x(a+1)=x(a)+h*cos(ang);
    y(a+1)=y(a)+h*sin(ang);
    end

plot(x,y,'b');
axis equal
```

[10]

(b) Given that

$$\alpha_s = \frac{s \ln p_1 + (k-s)\ln p_2}{-k \ln 3} \qquad \text{and} \qquad -f_s = \frac{\ln\binom{k}{s}}{-k \ln 3},$$

write a short MATLAB program to plot the $f(\alpha)$ spectrum for the multifractal Cantor set constructed by removing the middle third segment at each stage and distributing the weight in the proportions $p_1 = 0.2$ and $p_2 = 0.8$. Sketch the $f(\alpha)$ curve and write down the MATLAB code in your answer booklet. What information does the local maxima give?

HINT: Use the nchoosek command in MATLAB.

[10]

END