MANCHESTER METROPOLITAN UNIVERSITY

MATLAB-Based Examination

Mock Examination Two for
BSc Applicable Mathematics Combined Honours
BSc Mathematics
BSc Financial Mathematics

UNIT 63MA6302: Dynamical Simulation and Chaos

Instructions to Candidates

Answer **FIVE** questions.
This is a **CLOSED BOOK** exam.
The marks awarded for each question are shown in square brackets.
You must show **ALL** your working to obtain full marks.
MATLAB is available, but you are required to write the answers and all the working in your answer book.

Marks are awarded for the best five answers.

1. (a) Sketch a phase portrait for the following linear system showing all isoclines:

$$\frac{dx}{dt} = 2x - 4y,$$
$$\frac{dy}{dt} = 3x + 5y.$$

[8]

(b) Solve the differential equations

$$\dot{r} = r^2,$$
$$\dot{\theta} = 1.$$

given that $r(0) = 1$ and $\theta(0) = 0$. Hence show that the Poincaré return map, mapping points, say, r_n, on the positive x-axis to itself is given by

$$r_{n+1} = \frac{r_n}{1 - 2\pi r_n}.$$

[12]

2. (a) Determine the Hamiltonian of the system

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = x - x^2,$$

and plot a phase portrait.

[10]

(b) Sketch a phase portrait for the linear system

$$\dot{x} = -x,$$
$$\dot{y} = -y + 2z,$$
$$\dot{z} = -2y - z,$$

in three-dimensional space.

HINT: Uncouple the system.

[10]

3. (a) Plot phase portraits and a bifurcation diagram for the system

$$\frac{dx}{dt} = x^2 - \mu^2 x,$$
$$\frac{dy}{dt} = -y.$$

[14]

(b) Plot a bifurcation diagram for the system

$$\dot{r} = r(r-1)(r-\alpha),$$
$$\dot{\theta} = -1.$$

[6]

4. (a) Show that the system

$$\dot{x} = y + x^3 - x\left(x^2 + y^2\right)^2,$$
$$\dot{y} = -x + y^3 - y\left(x^2 + y^2\right)^2,$$

can be transformed to

$$\dot{r} = r^3\left(\cos^4\theta + \sin^4\theta - r^2\right),$$
$$\dot{\theta} = r^2 \sin\theta\cos\theta\left(\sin^2\theta - \cos^2\theta\right) - 1.$$

using the relations $r\dot{r} = x\dot{x} + y\dot{y}$ and $r^2\dot{\theta} = x\dot{y} - y\dot{x}$. Prove that $\dot{\theta} < 0$ for $\frac{1}{\sqrt{2}} \le r \le 1$, $\dot{r} < 0$ on $r = 1$, and $\dot{r} > 0$ on $r = \frac{1}{\sqrt{2}}$. For a 2-D system all bounded trajectories approach either a critical point or a limit cycle. What can you conclude?

[15]

(b) Determine the stability of the origin of the system

$$\dot{x} = -y - x^3,$$
$$\dot{y} = x - y^3,$$

using the Lyapunov function $V(x, y) = x^2 + y^2$.

[5]

5. (a) Find the fixed points of periods one and two for the complex mapping

$$z_{n+1} = z_n^2 - 3 .$$

and, if possible, determine the stability of the fixed points of period one.

[8]

(b) Prove that the circle of radius $\dfrac{AB}{1-B}$ centred at A is invariant for the system

$$E_{n+1} = A + BE_n e^{i|E_n|^2} ,$$

where $A > 0$ and $0 < B < 1$.

[12]

6. (a) Find the fixed points of periods one and two for the Hénon map

$$\begin{aligned} x_{n+1} &= 1.5 - x_n^2 + 0.2 y_n \\ y_{n+1} &= x_n \end{aligned} ,$$

and classify the fixed points of period one.

[10]

(b) Consider the map defined by $x_{n+1} = f(x_n)$, where $f(x)$ is defined by

$$f(x) = \begin{cases} \dfrac{7}{3}x & x \le \dfrac{1}{2} \\ \dfrac{7}{3}(1-x) & x > \dfrac{1}{2} \end{cases} .$$

Plot the function on graph paper. Consider the sets, S_n say, which remain in the interval [0,1] after n iterations. List the intervals in S_1 and S_2. The set of points that never escape from the interval [0,1] form a Cantor set. What is the fractal dimension of this Cantor set?

[10]

7. (a) Consider the blood cell iterative equation

$$c_{n+1} = (1-a)c_n + bc_n^r e^{-sc_n},$$

where c_n is the red cell count per unit volume in the nth time interval, $b = 1.1 \times 10^6$, $r = 8$ and $s = 16$. Determine the number and stability of the fixed points of period one when $a = 0.2$.

[15]

(b) Derive the inverse map of

$$x_{n+1} = \frac{3}{50} + \frac{9}{10}y_n - x_n^2$$
$$y_{n+1} = x_n \quad .$$

[5]

8. (a) The Lyapunov exponent, say λ, for the map $x_{n+1} = f(x_n)$, is defined by

$$\lambda = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right\}.$$

Use MATLAB to compute the Lyapunov exponent of the Gaussian map

$$x_{n+1} = e^{-6.2x_n^2} - 0.5,$$

when $x_0 = 0.1$. Write down the value of the Lyapunov exponent and the MATLAB code in your answer booklet. What type of solution is defined by (i) $\lambda < 0$, (ii) $\lambda = 0$ and (iii) $\lambda > 0$.

[10]

(b) Given that

$$\alpha_s = \frac{s\ln p_1 + (k-s)\ln p_2}{-k\ln 3} \qquad \text{and} \qquad -f_s = \frac{\ln\binom{k}{s}}{-k\ln 3},$$

write a short MATLAB program to plot the $f(\alpha)$ spectrum for the multifractal Cantor set constructed by removing the middle third segment at each stage and distributing the weight in the proportions $p_1 = \frac{1}{7}$ and $p_2 = \frac{6}{7}$. Sketch the $f(\alpha)$ curve and write down the MATLAB code in your answer booklet. What information does the width of the curve give?

HINT: Use the nchoosek command in MATLAB.

[10]

END