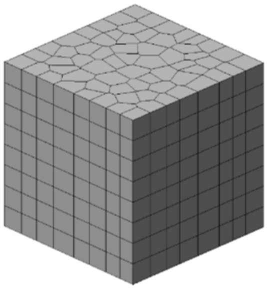
Advanced Solvers for Numerical Partial Differential Equations (PDEs)



Lecture 6

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Part 1
Finite Volume Method

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Example of a Non-conservative Scheme

Consider the BVP for an ODE,

$$-\frac{d}{dx}(K\frac{du}{dx}) = 0, \qquad u(0) = 1 \quad u(1) = 0$$

$$K(x) = \begin{cases} 1 & x < 1/2 \\ 5 & x > 1/2 \end{cases}$$

Notice

$u(x)$ are $W = -K(x)u_x(x)$ continuous.

Naively performing differentiation,

$$-Ku_{xx} - K_x u_x = 0$$

we receive a finite-difference scheme,

$$-K_i \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{K_{i+1} - K_{i-1}}{2h} \cdot \frac{u_{i+1} - u_{i-1}}{2h} = 0$$

$$u_0 = 1 \quad u_n = 0$$

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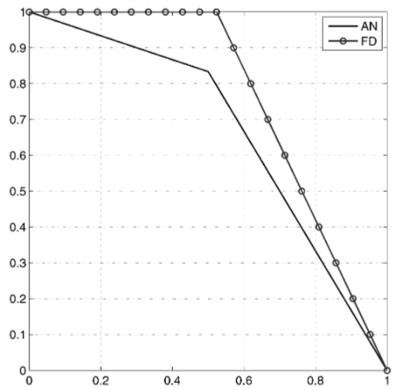
Example of a Non-conservative Scheme

$u(x)$



Divergence!
Flux continuity is violated!
We received an extra energy source!

How to design conservative schemes,
i.e. the schemes respecting energy conservation?

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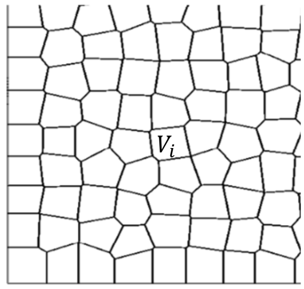
Conservative Schemes

$K(x,y) > 0$



Consider the diffusion equation,

$$-div(K\nabla u) = f \quad \text{в } V,$$
$$u = \varphi \quad \text{на } \Gamma$$

Define W, heat flux , following Fourier law,

$$\boldsymbol{W} = -K\nabla u$$

$$div(\boldsymbol{W}) = f$$

(*) $$\int_{\partial V_i} \boldsymbol{W} \cdot \boldsymbol{n}_i \, dl = \int_{V_i} f \, dS$$

Integrating, we receive energy conservation equation within each cell.

The schemes respecting it are referred to as **conservative**.

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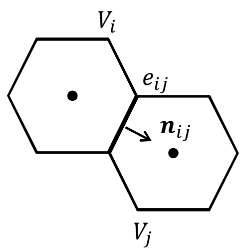
Finite Volume Method

e_{ij} edge between V_i and V_j

$\boldsymbol{n}_{ij}$ normal to e_{ij} towards V_j

W_{ij} averaged normal flux through e_{ij}

$$W_{ij} = \frac{1}{|e_{ij}|} \int_{e_{ij}} \boldsymbol{W} \cdot \boldsymbol{n}_{ij} dl$$



Defining,

$$f_i = \frac{1}{|V_i|} \int_{V_i} f dS \quad u_i = \frac{1}{|V_j|} \int_{V_i} u dS$$

Energy conservation equation takes the form,

(**) $$\sum_{c} |e_{ij}| W_{ij} = |V_i| \, f_i$$

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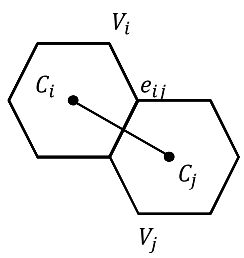
Finite Volume Method

Let us express W_{ij} via u_i, using $\boldsymbol{W} = -K\nabla u$:

$$\frac{W_{ij}}{K} = -\nabla u \cdot \boldsymbol{n}_{ij}$$

Integrate this equality between points C_i and C_j :

$$W_{ij} \int_{C_iC_j} \frac{1}{K} dl = -\int_{C_iC_j} \nabla u \cdot \boldsymbol{n}_{ij} dl$$



Segment C_iC_j has length h_{ij} and is divided by e_{ij} into two parts

K constant within V_i and equal to K_i

This ultimately gives us,

$$W_{ij} = (u_i - u_j)\frac{2}{h_{ij}}\frac{K_iK_j}{K_i + K_j}$$

Finite Volume Method

Combining, we receive

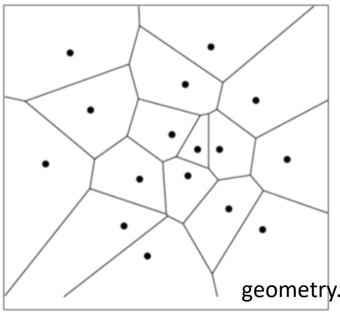
$$\sum_j (u_i - u_j)\frac{|e_{ij}|}{h_{ij}}\frac{2K_iK_j}{K_i + K_j} = |V_i|\, f_i$$

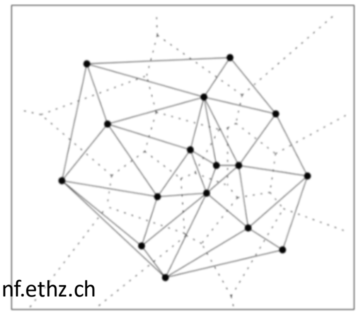
- In case rectangular grid and constant K, we receive exactly the FD system.
- More geometrically flexible than the FD.
- Discretization error $O(h^2)$
- Energy conservation is fulfilled with machine precision.

What if K is a tensor rather than a scalar?
So what were the assumptions on geometry?

Voronoi Grids

- The requirements on the grid are satisfied on Voronoi grids.





geometry.inf.ethz.ch

Assume we have a set of points S_k. The every cell is formed a set of points whose distance to S_k is less than or equal to its distance to any other S_k.

Delone triangulation

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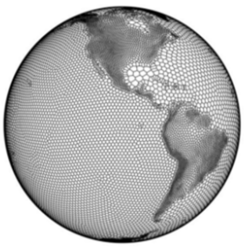
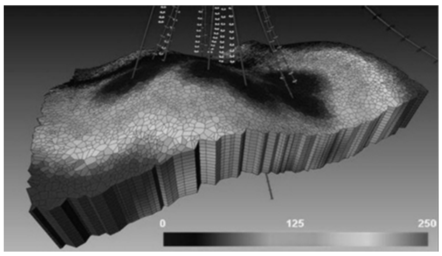
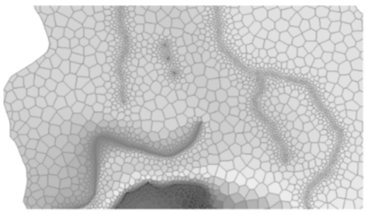
Voronoi Grids Examples



Weather prediction
Voronoi grid refinement for terrain height
T. O. Roomi, 2021



Petroleum engineering



Hydrogeology
Grid refined around pumping wells and river boundaries.
waterloohydrogeologic.com

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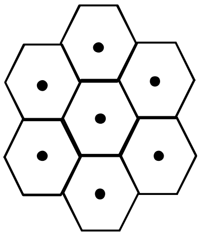
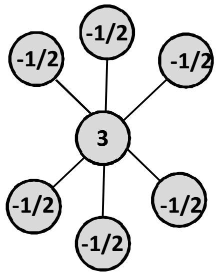
Example 1 – Hexagonal Grid

1)

$$-\Delta u = f$$

$$|e_{ij}| = \frac{h_{ij}}{2} \qquad K_{ij} = 1$$





Example 2

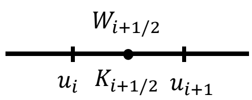
2)

$$-\frac{d}{dx}\left(K(x)\frac{du}{dx}\right) = 0$$

$$u(0) = 1 \quad u(1) = 0$$

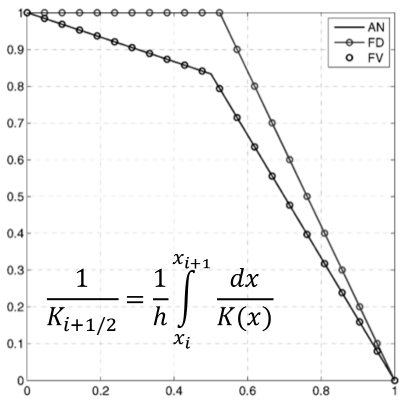
$$K(x) = \begin{cases} 1 & x < 1/2 \\ 5 & x > 1/2 \end{cases}$$

$$W = -K(x)\,u_x \quad W_x = 0$$



$$W_{i+1/2} = -K_{i+1/2}\frac{u_{i+1} - u_i}{h}$$

$$\frac{W_{i+1/2} - W_{i-1/2}}{h} = 0$$



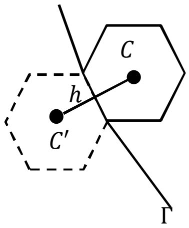
$$\frac{1}{K_{i+1/2}} = \frac{1}{h}\int_{x_i}^{x_{i+1}} \frac{dx}{K(x)}$$

Dirichlet BC in the FVM

Dirichlet BC

$$u = \varphi \quad \text{на } \Gamma$$

Is implemented with a ghost cell:



$$\varphi = \frac{u_C + u_{C'}}{2} + O(h^2)$$

$$W_{CC'} = -K \frac{u_C - u_{C'}}{h}$$

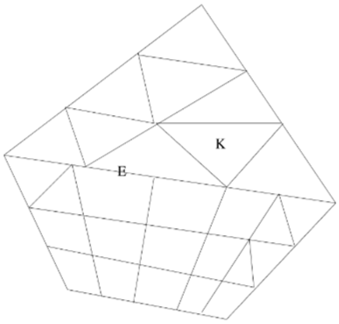
Eliminating $u_{C'}$:

$$W_{CC'} = -K \frac{2u_C - 2\varphi}{h}$$

Part 2
Discontinuous Galerkin Method

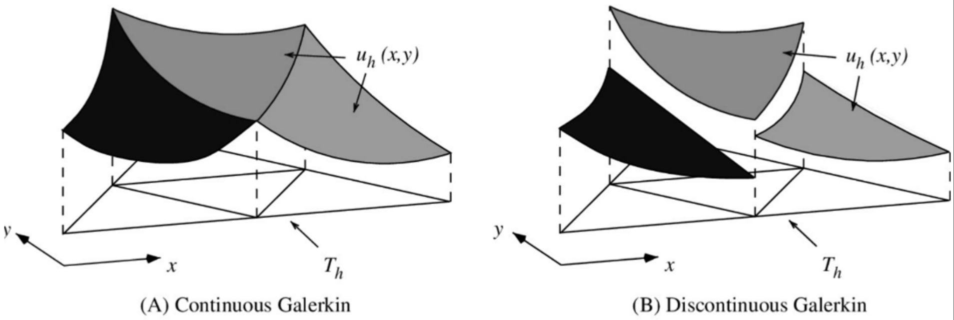
Limitation of Conventional FEM

(a.k.a. continuous Galerkin, CG)



Non-matching grids cannot be used with the conventional FEM (P1, Q1, etc),
Yet they are on high demand – they can be easily adapted where needed.

Discontinuous Galerkin Method Idea

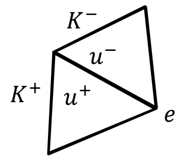


K.J. Fidkowski & G.Chen, 2021

Discontinuous Galerkin Formulation

$-\Delta u = f$ in Ω

$\frac{\partial u}{\partial n} = 0$ on Γ



$[v] = v^+ - v^-$

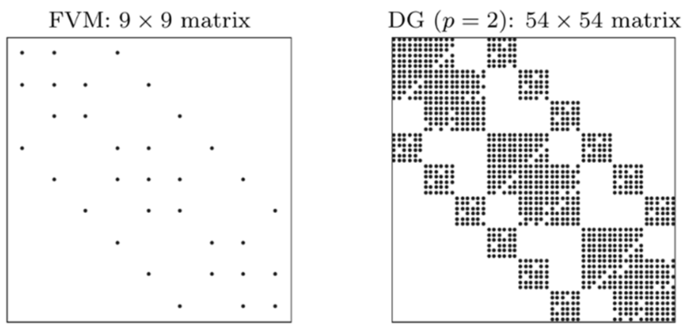
$\{v\} = \frac{1}{2}(v^+ + v^-)$

With some work,
and using $ac - bd = \frac{1}{2}(a+b)(c-d) + \frac{1}{2}(a-b)(c+d)$
we receive:

$$\sum_K \int_K \nabla u \cdot \nabla v \, dx - \underbrace{\sum_e \int_e \left\{\frac{\partial u}{\partial n}\right\} [v] \, ds}_{\text{Consistency term}} - \underbrace{\sum_e \int_e [u] \left\{\frac{\partial v}{\partial n}\right\} ds}_{\text{Symmetricity term}} + \underbrace{\sum_e \int_e \frac{\sigma}{|e|} [u][v] \, ds}_{\text{Interior penalty term}} = \sum_K \int_K f \, v \, dx$$

Symmetric interior penalty method (SIPG)

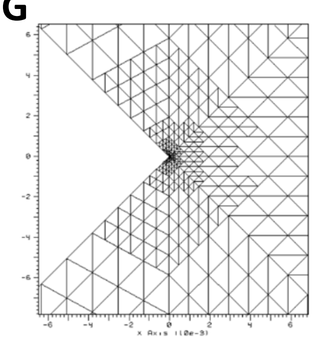
Matrix structure



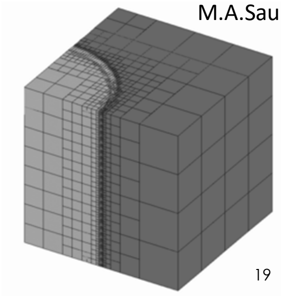
System matrices for a 3x3 grid

Summary on DG

- Requires estimation of σ
- Since SIPG produces a symmetric, positive definite linear system to solve, CG and PCG can be used
- Straightforward for adaptivity
  Easily handles complex geometry and nonmatching grids
- Due to the natural level based tree hierarchy produced, multigrid can also be used.
- Locally conservative
- Handles advection dominated flow regimes efficiently



M.A.Saum



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