Hw3 Solvers

by me

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1 Problem

Consider the 2D Burgers' equation for u(x,y,t) and v(x,y,t):

$$u_t + uu_x + vu_y - \nu\Delta u = 0$$

$$v_t + uv_x + vv_y - \nu\Delta v = 0$$

$$x, y, t \in [0, 1]$$

With the ICs:

$$u(x, y, 0) = v(x, y, 0) = e^{-3x^2-3y^2}$$

And homogeneous Neumann conditions. Assume the viscosity ν to be 1E-2.

Generate a uniform triangular grid near 100x100 vertices in space and pick an appropriate time step, τ. Apply P1 finite-elements in space and the backward Euler scheme with the Newton's method in time (FEM-BE) and compute the FEM-BE solution within the segment 0¡t¡0.5. Let $N_t = \frac{1}{2\tau}$ be the number of time steps performed.

Next, compute the following two numerical solutions within the segment $0.5 \leq t \leq 1$:

1. The FEM-BE solution, i.e. simply continue time-stepping
2. The DMD solution, use the data from previous N_t steps to predict later values. Check different SVD truncation ranks, r.

Make a table of differences in L2 norms at times of 0.6, 0.8, 1. Also record CPU time. Compare plots at specified times. COmment the results.

2 Weak formualtion

We work out the formulation for u only, as it is analogous for v.

$$\int_\Omega u_t w + \int_\Omega uu_x w + \int_\Omega vu_y w - \nu \int_\Omega \Delta uw = 0$$

We employ the method of lines: first we apply the FEM method spatially, then the FD method temporally.

The only term we may fruitfully (e.g. with simplification of differentials) apply integration by parts to is the last one.

As we know,

$$\nabla \cdot (a\bar{b}) = \nabla a \cdot b + a\nabla \cdot b$$

We may recognize the last terms of our expression and this product rule to be analogous, with $a = w$ and $b = \nabla u$. Then:

$$\nu \int_\Omega \Delta u w = \nu(\int_\Omega \nabla \cdot (w\nabla u) - \int_\Omega \nabla w \cdot \nabla u)$$

$$\nu \int_\Omega \nabla \cdot (w\nabla u) = \nu \int_\Gamma w\nabla u \cdot n = 0$$

Where we have first invoked the divergence theorem and then recalled the homogeneous Neumann conditions.

We obtain, discretising backwards in time:

$$\int_\Omega u_t w + \int_\Omega u u_x w + \int_\Omega v u_y w + \nu \int_\Omega \nabla w \cdot \nabla u = 0$$

$$\int_\Omega w \frac{u^{n+1} - u^n}{\tau} + \int_\Omega u^{n+1} u_x^{n+1} w + \int_\Omega v^{n+1} u_y^{n+1} w + \nu \int_\Omega \nabla w \cdot \nabla u^{n+1} = 0$$

And now we make it suitable for formulating the FEniCS problem of $Au = L$:

$$A = \int_\Omega w u^{n+1} + \tau(\int_\Omega u^{n+1} u_x^{n+1} w + \int_\Omega v^{n+1} u_y^{n+1} w + \nu \int_\Omega \nabla w \cdot \nabla u^{n+1})$$

$$L = u^n$$

Or, writing in vector form,

$$A = \int_\Omega w u^{n+1} + \tau(\int_\Omega (u^{n+1}\nabla) u^{n+1} w + \nu \int_\Omega \nabla w \cdot \nabla u^{n+1})$$

$$L = u^n$$

3 Table

The time differences are absent in the table because they are extinct for BE times and very minor for DMD times.

L2 norm absolute	L2 norm relative	t_k	r	BE time,s	DMD time, s
29.9331	0.2985	0.6	1	13.4880	131.24
57.2981	0.4926	0.8	1	17.9841	371.24
90.6869	0.6904	1	1	22.4805	611.24
18.2019	0.1815	0.6	2	13.4880	131.24
56.1419	0.4827	0.8	2	17.9841	371.24
113.1327	0.8613	1	2	22.4805	611.24
5.7296	0.0571	0.6	5	13.4880	131.24
36.8276	0.3166	0.8	5	17.9841	371.24
95.6997	0.7286	1	5	22.4805	611.24
2.4653	0.0246	0.6	10	13.4880	131.24
24.5330	0.2109	0.8	10	17.9841	371.24
58.4914	0.4453	1	10	22.4805	611.24

4 Plots

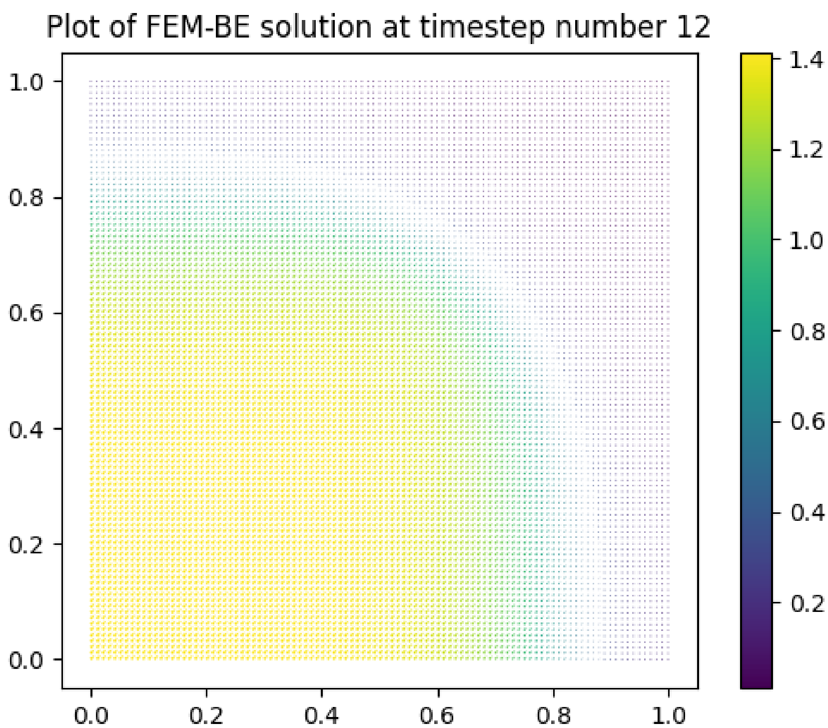

Figure 1: FEM-BE solution, $t = 0.6$ s

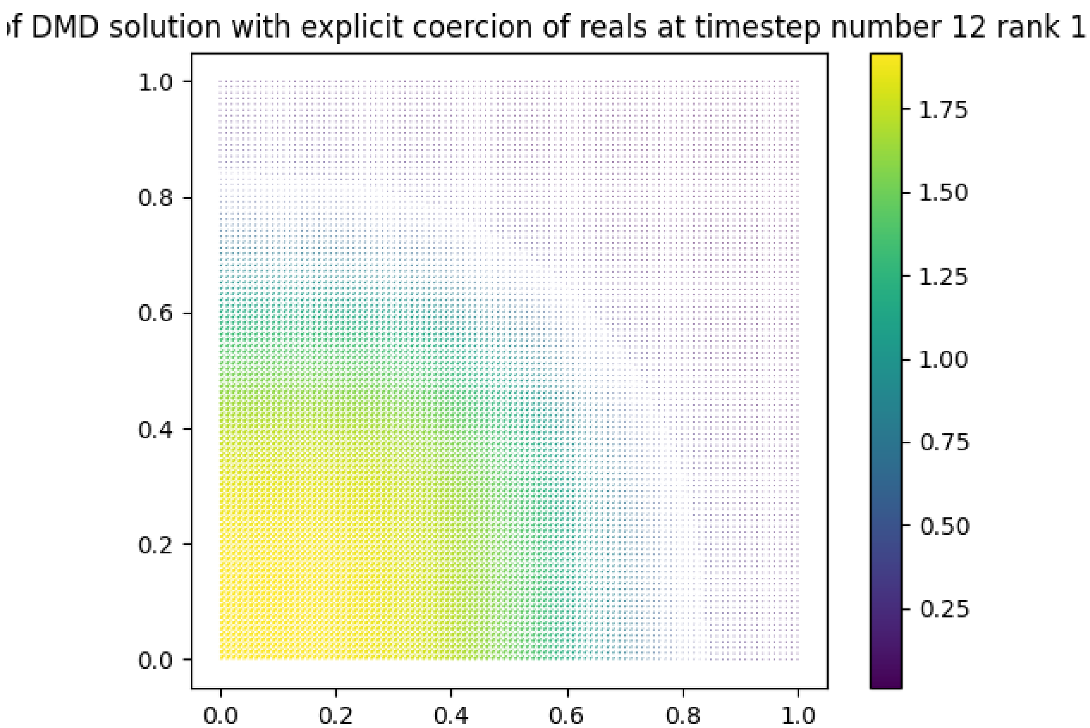

Figure 2: DMD solution, $t = 0.6$ s, $r = 1$

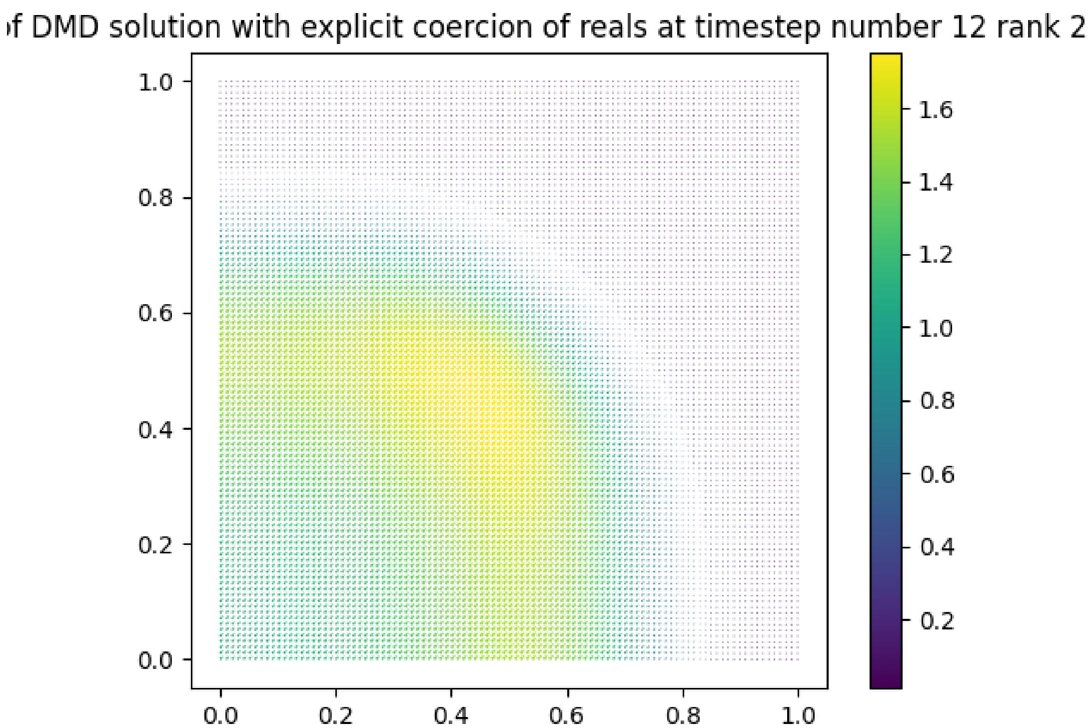

Figure 3: DMD solution, $t = 0.6$ s, $r = 2$

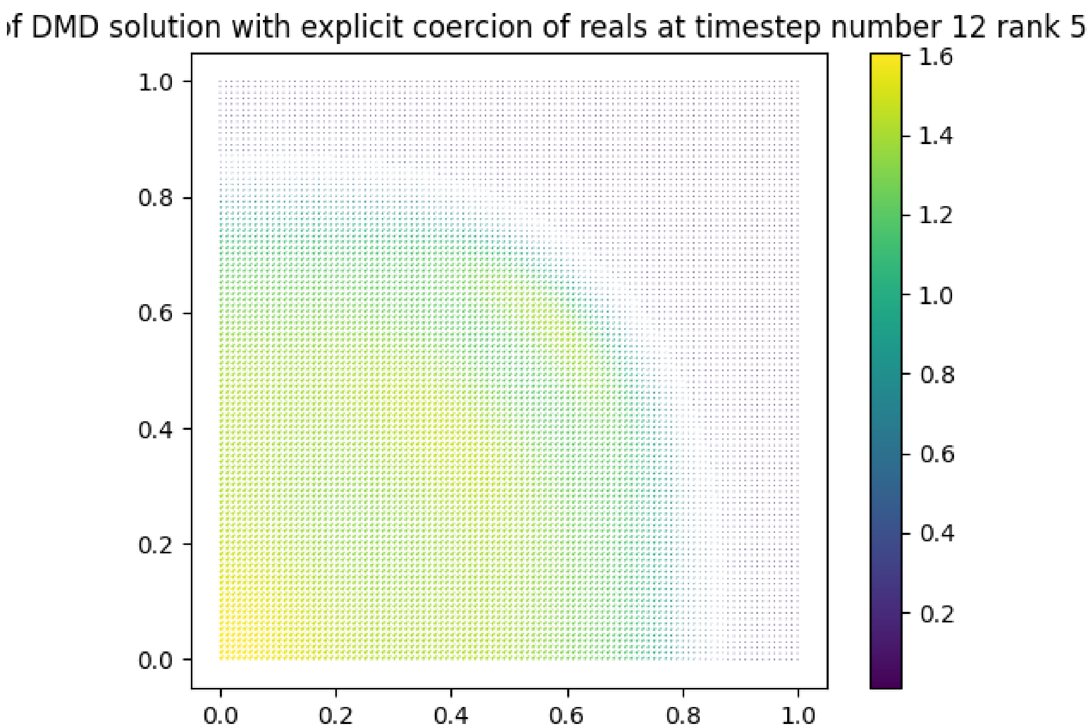

Figure 4: DMD solution, $t = 0.6$ s, $r = 5$

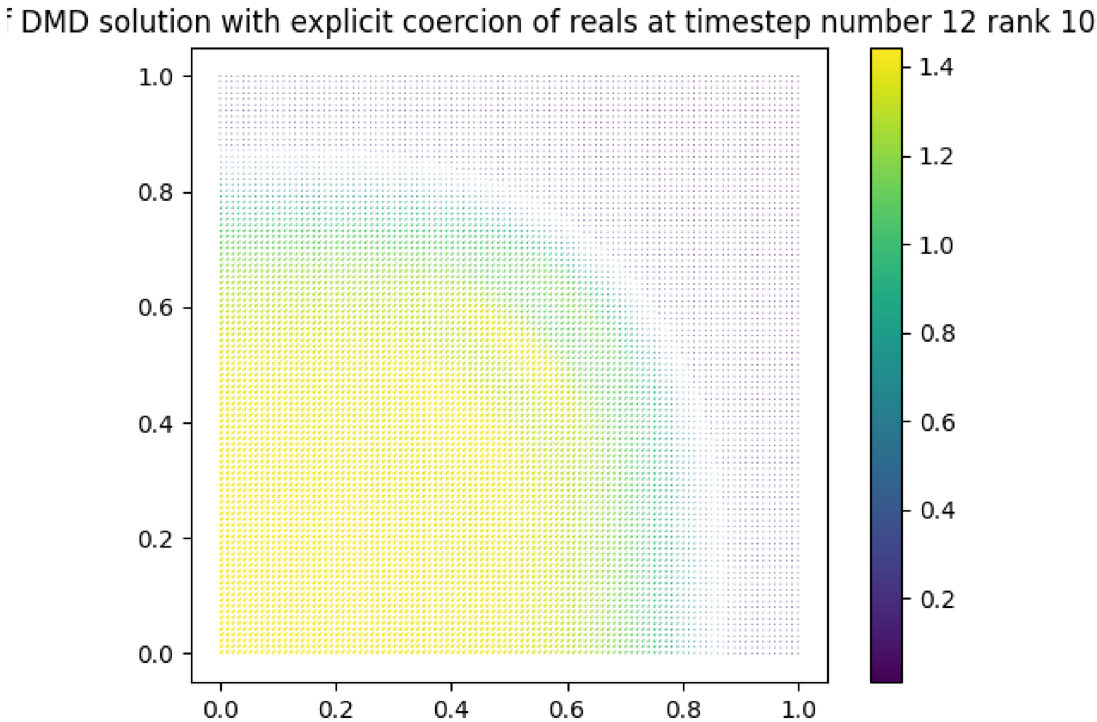

Figure 5: DMD solution, $t = 0.6$ s, $r = 10$

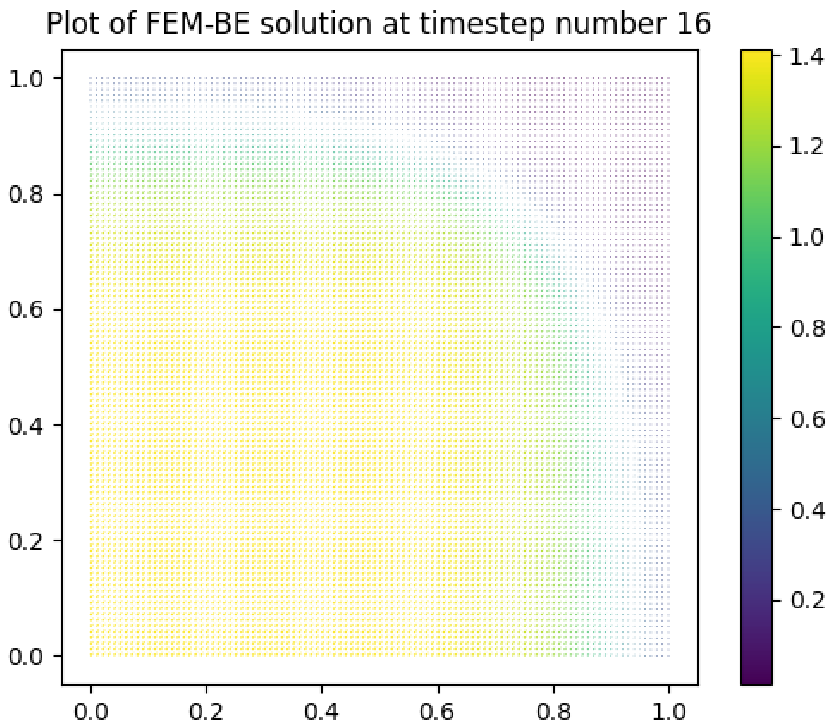

Figure 6: FEM-BE solution, $t = 0.8$ s

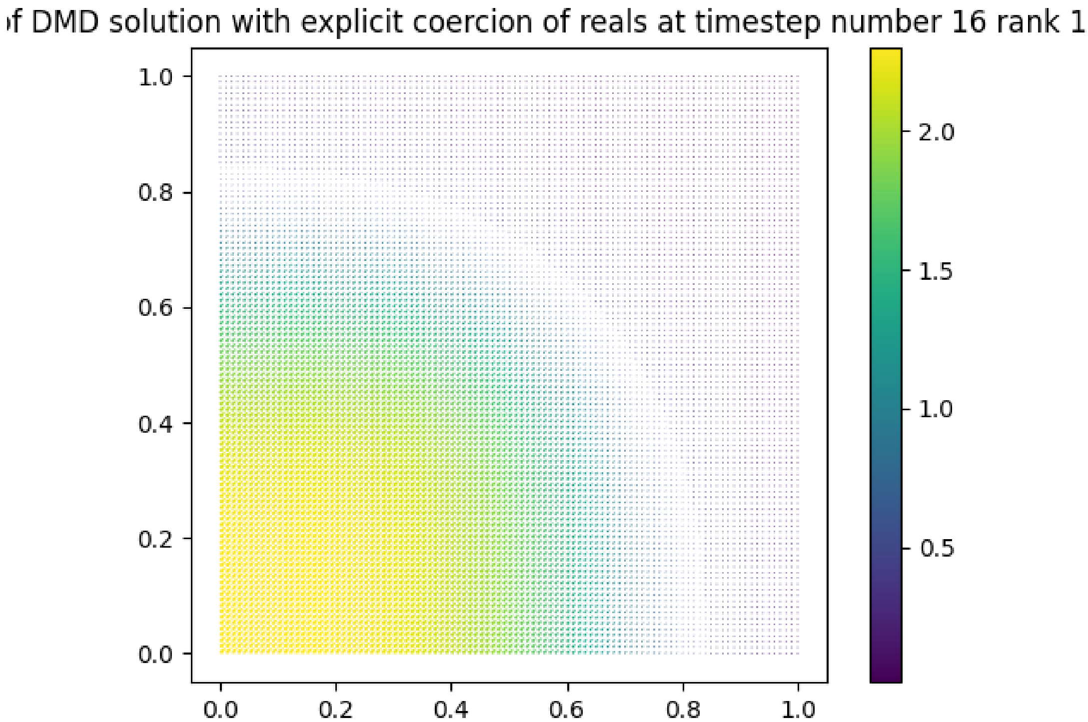

Figure 7: DMD solution, $t = 0.8$ s, $r = 1$

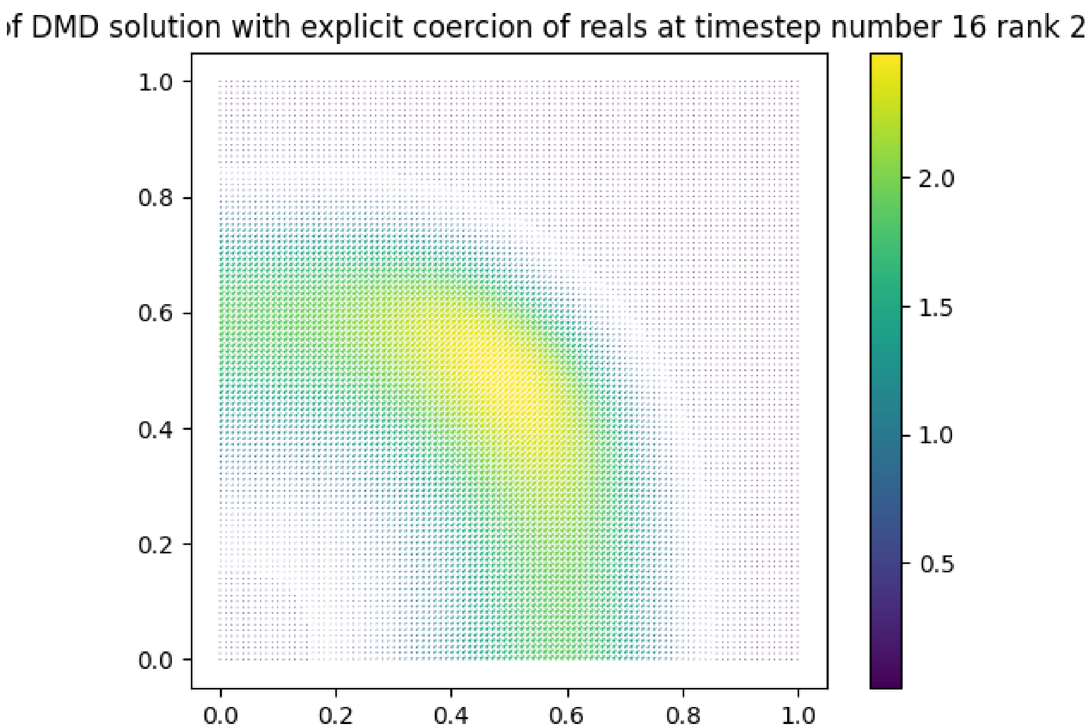

Figure 8: DMD solution, $t = 0.8$ s, $r = 2$

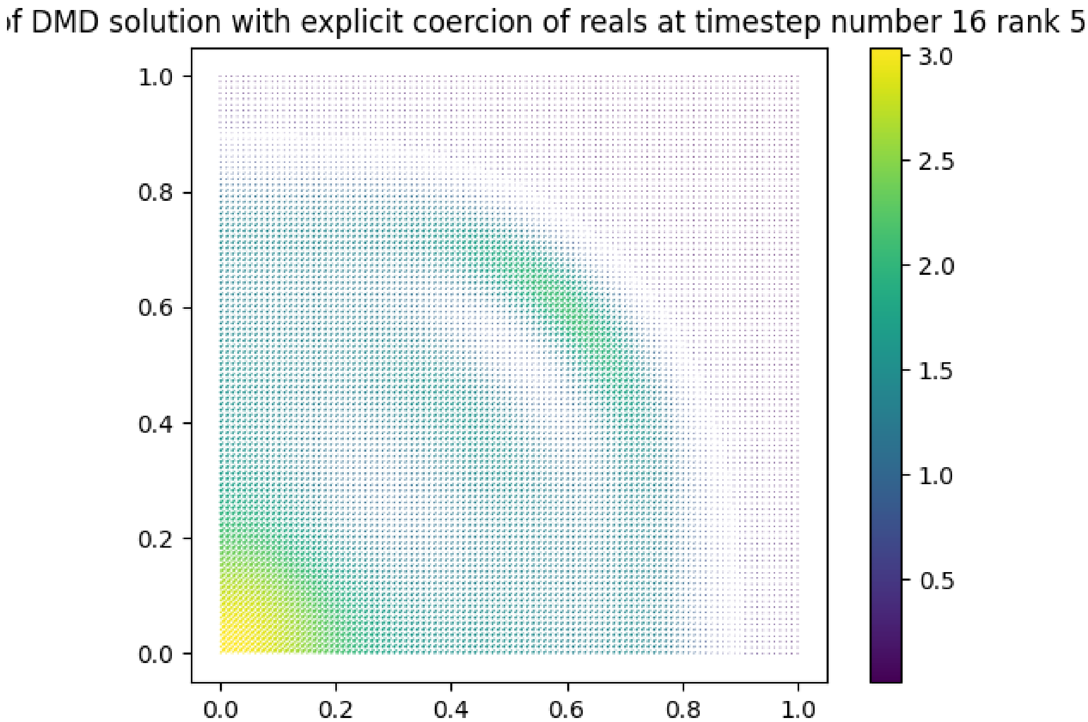

Figure 9: DMD solution, $t = 0.8$ s, $r = 5$

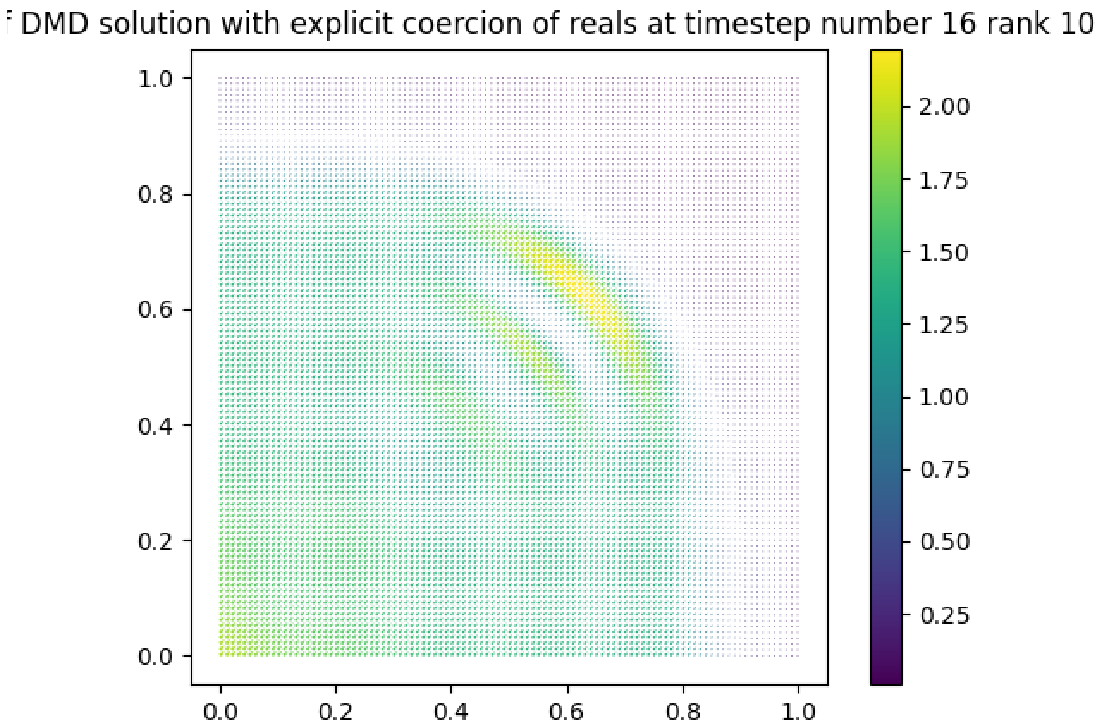

Figure 10: DMD solution, $t = 0.8$ s, $r = 10$

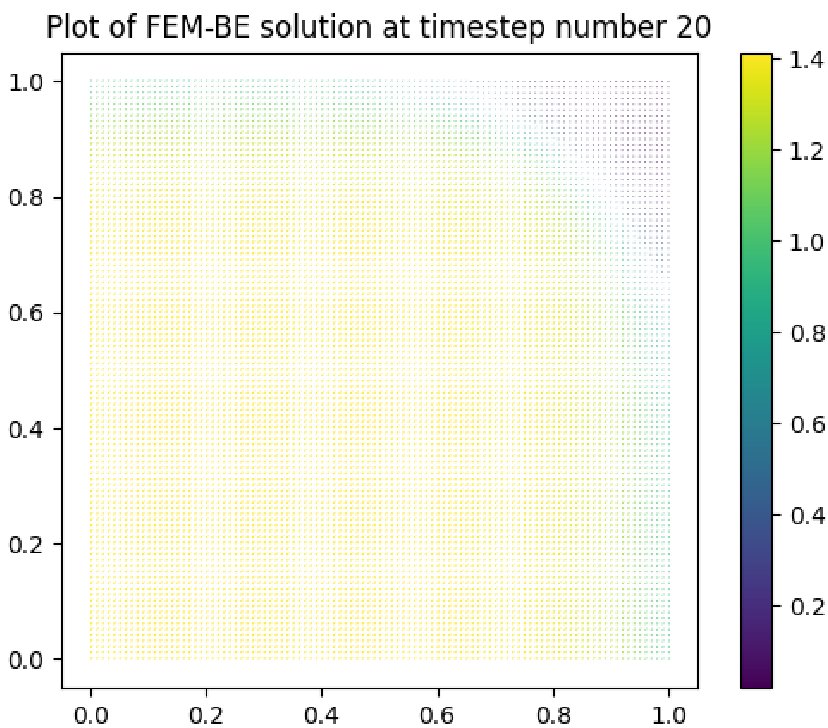

Figure 11: FEM-BE solution, $t = 1$ s

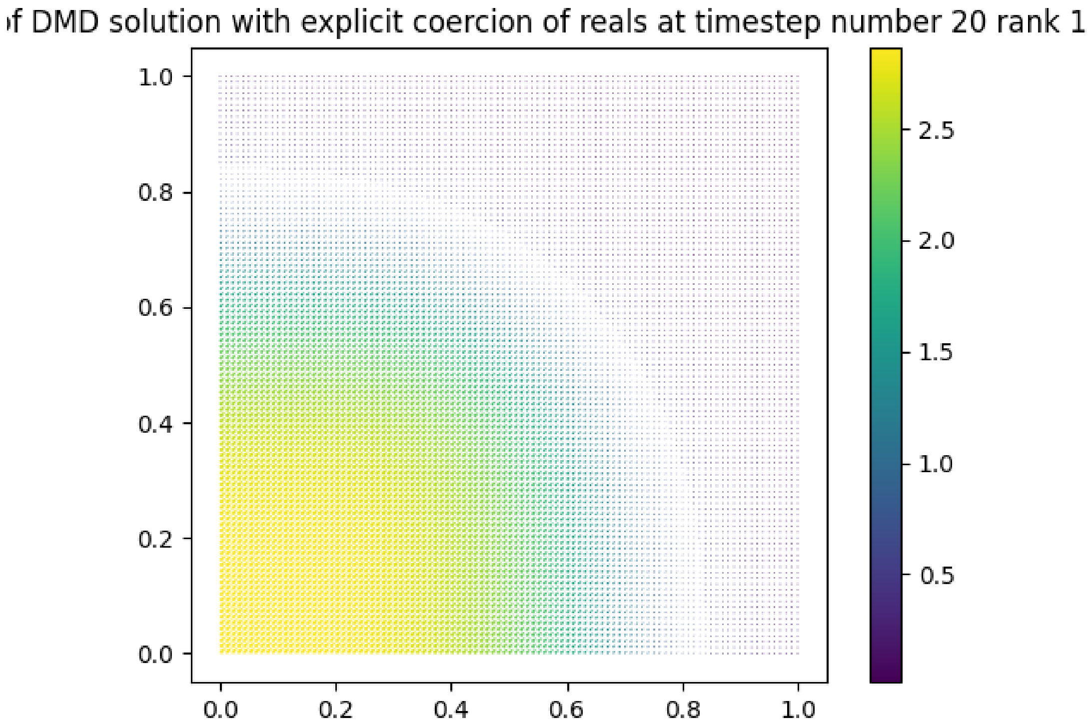

Figure 12: DMD solution, $t = 1$ s, $r = 1$

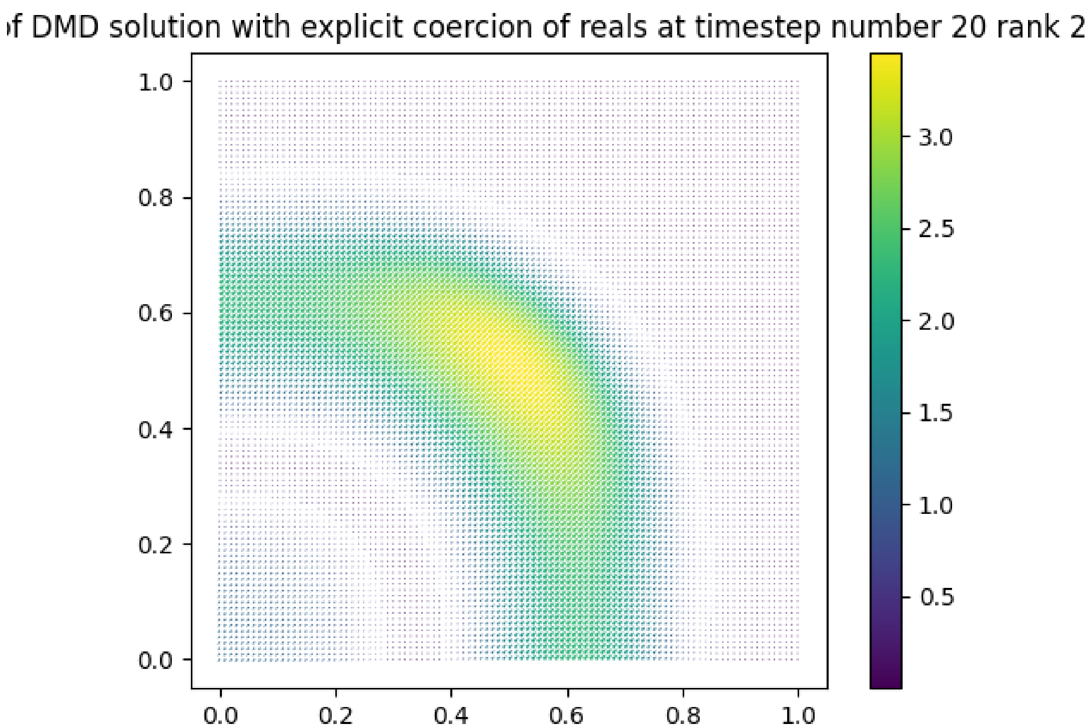

Figure 13: DMD solution, $t = 1$ s, $r = 2$

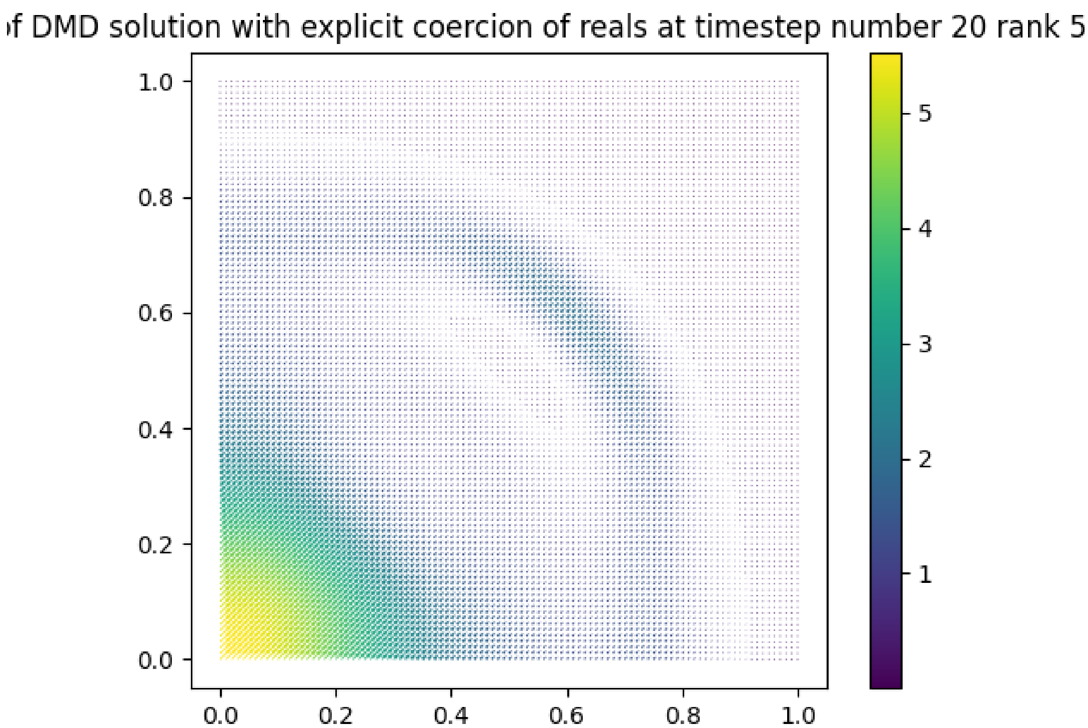

Figure 14: DMD solution, $t = 1$ s, $r = 5$

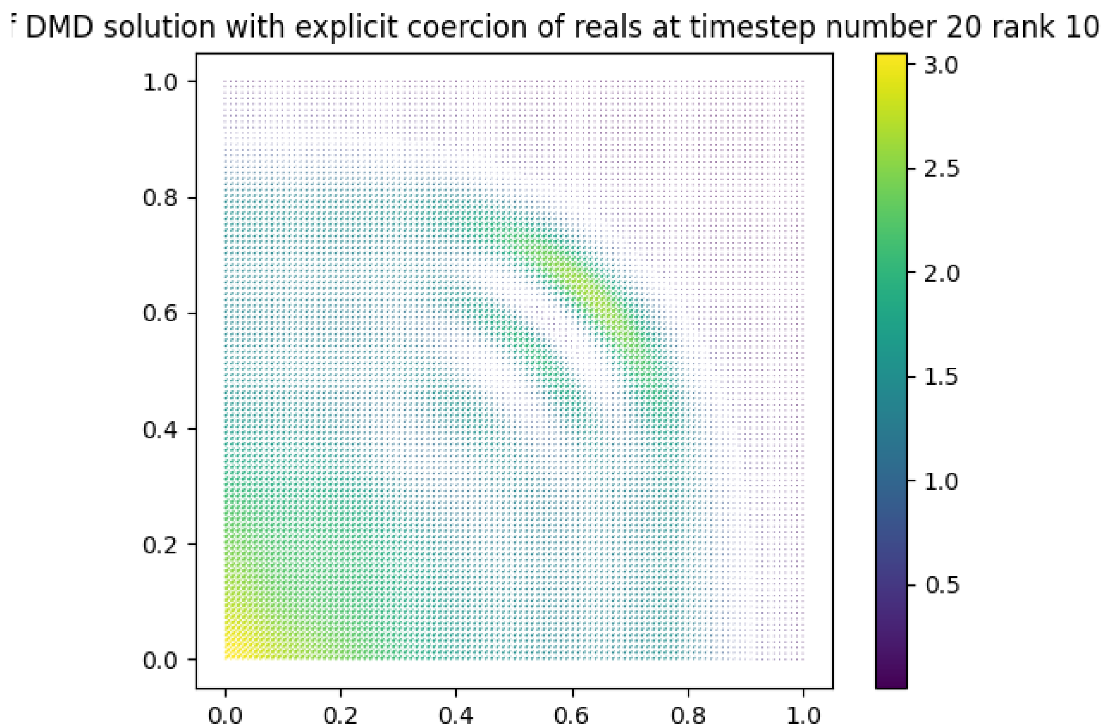

Figure 15: DMD solution, $t = 1$ s, $r = 10$

5 Discussion

Firstly we are obliged to note that the comparison of times is not particularly fruitful: FEniCS computations and DMD were done on different machines, as one lacks the RAM to do DMD, and the other lacks FEniCS. The time inefficiency of DMD here is scarcely surprising: we are obliged to perform a number of multiplications with high-dimension matrices that are not necessarily sparse and do not enjoy FEniCS's further optimisations. The worst contributors here are the relative simplicity of the equation's structure and the low dimensionality: when I as a part of my research did approximations of parametric PDEs with a distinct algorithm that nonetheless combined all parts that DMD has (SVD and matrix multiplications), I observed very significant performance gains, but they were primarily on parameterised Navier-Stokes equations with high dimensionality (of millions or at least hundreds of thousands; we have about twenty thousand DoFs). In fact, in some cases two-fold increase of performance was observed (I am unsure how much data I can share from a paper pending acceptance). My conjecture regarding DMD times is as follows: the dimensionality was too low, and the equation itself required only one subproblem per timestep (as opposed to, say, three subproblems per timestep in NS).

Now the errors are to be discussed. The trend of increasing error towards higher times is unsurprising, as fewer ground truth snapshots are used, and the errors introduced before engender further deviations. From this we see that the Burgers equation, as we know, is nonlinear. Much more peculiar is the behavior of error at $t_k = 1$ when looked at as a function of rank. One would expect a uniform decrease from this, but instead it experiences a sharp increase first. We can make a conjecture regarding it by looking at the plots. From the plots we see that the rank-1 approximation keeps only the constant term and therefore engenders milder than expected error; in comparison, higher-rank approximations produce a succession of distinct waves (which is not unexpected given the underlying idea of the DMD of representing the next-time solution as a linear combination of previous-time solutions) and likely simply differ from the baseline more substantially at a greater area than at rank 1. This is supported by the observation that if the constant-term outlier was to be removed, the progression would be monotonic, and by the fact that at lower times, when the qualitatively different behavior of higher-rank approximations was not yet given enough time to increase the divergence from the baseline, increase of rank produces an expected increase of accuracy (which is a generally manifested feature of DMD, consistent with the usual linear approximations, e.g. the truncated power-series approximation).