Articulation Points in Multiplex Networks

Artem Vergazov,[1] Vladimir V. Palyulin,[1] and Saeed Osat[2]
[1]*Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30, Moscow, 121205, Russia*
[2]*Max Planck Institute for Dynamics and Self-Organization (MPIDS), 37077 Göttingen, Germany*
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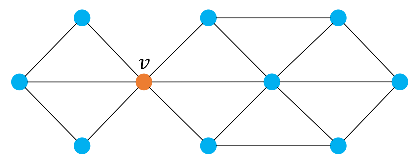
Many real systems can efficiently be represented by complex networks. Recently, robustness of these networks under random or targeted attacks has been of interest. Concentrating on node removal one of the strategies to attack a network is based on articulation points, nodes which removal can split the network into disconnected components. Here we generalise the definition of articulation points to multiplex networks. Inherent cascading failure within multiplex structures makes definition of articulation points evading; a node which can be considered as an articulation point considering a single layer of a monoplex loses this property due to removal by cascading failure. We report the results for random and real multiplex networks.

I. INTRODUCTION

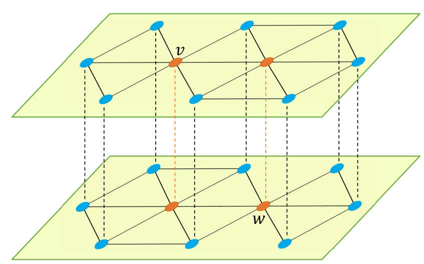
Complex Networks. Many of the real or man-made complex systems accept a network representation. The nodes stand for constituent building blocks of the system and the links between the nodes capture the relationships or interactions between nodes [1, 2]. Various real-world systems around us do not run in isolation but rather link with each other [3]. A multi-layer network is a framework developed to capture the complex structure of interacting networks [4]. A specific model of multilayer networks is called multiplex networks. In the latter each layer is composed of the same entities or nodes but the types of interactions or links are different, i.e. each layer corresponds to a specific flavour of interaction [5]. Such systems are ubiquitous with transportation, social and biological networks being prime examples.

Any large system is prone to failures of its components. To that end it is desirable to build it in a way that allows to withstand random shutdown events [6–9]. The concept of networks provides a handy framework to model failures in complex systems [10]. Some nodes are more important for the network integrity than others, as they disconnect the network when removed [9, 11–17]. One of the classes of such nodes is called as articulation points. An articulation point (AP) in a graph is a node that hinges two or more components, thus its removal breaks down a graph into several components (Fig. 1). To compare the behaviour of monoplex and multiplex networks under the circumstances of node failures we will analyse their structure via the articulation points distribution [17, 18].

Articulation Points. In an optimal percolation or graph dismantling the goal is to find the minimum number of nodes needed to break down the giant component of the network into sub-extensive small components. Among the most effective strategies of dismantling for both monoplex and multiplex networks is an articulation-based ranking of the nodes and their removal in an order which brings the highest damage to the network. In Ref. [17] it was shown that an articulation point-based attack can be effective specially if one is limited with available resources and has to increase the damage for a given node-removal budget.



(a) The node v is an articulation point of the presented network.



(b) Nodes v and w are both articulation points in their corresponding layers of the multiplex networks, which makes them both articulation points of the whole network according to the definition.

FIG. 1: Illustration of the concept of articulation points in monoplex (a) and multiplex (b) networks.

While applying the AP removal process to multiplex networks, one has to consider the effects of multiplexity. The failures in one layer can lead to a chain reaction of failures in other layers, with disastrous consequences. For instance, power grids are linked to communication networks, which rely on power grid networks, and disruptions in one layer can cause failures in others as happened during an electrical blackout in Italy in 2003 [19].This phenomenon is called cascading failure.

Hence, in order to define the APs in multiplex networks we use this cascade of failures process for each removal. This is necessary to account for the unique effect that arises from the interdependence and multiplexity of the network. In the Results section, we will explain how the presence of this effect impacted the critical values of mean degrees c that are required for the multiplex networks to remain functional after successive AP removal.

II. METHODS

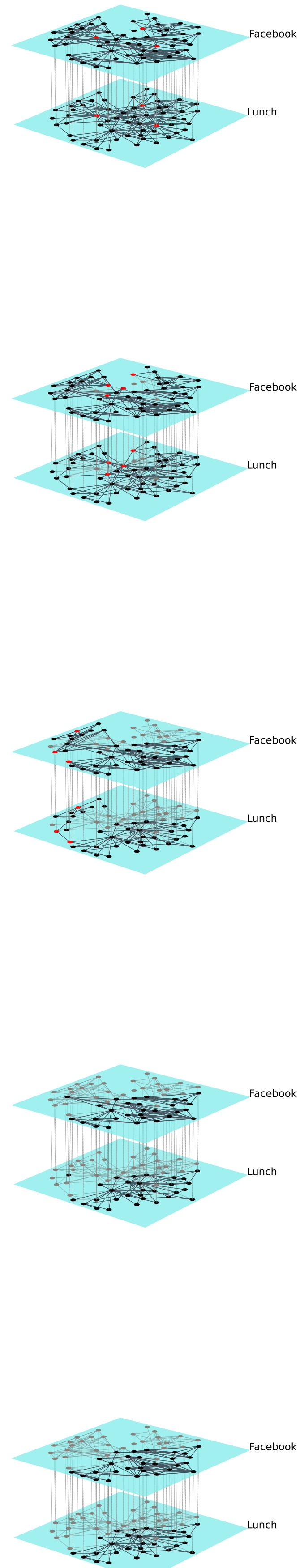
Characterizing articulation points in monoplex networks. In Ref. [17] the authors discuss strategies for attacking and breaking down networks, with a focus on the distribution of APs. They explain how removing APs can be used to analyse the resilience of a network to component failures. It is discovered that the process of AP removal offers new insights into the organisation of networks. For instance, in the terrorist communication network during the 9/11 attacks (Fig. **??**), each AP can be seen as a messenger for a specific subnetwork, which is usually the set of its neighboring nodes, as most paths in the network pass through APs. The AP removal process, as shown in Fig. **??**, involves identifying and removing all the APs in the network, which leads to the emergence of new APs in the remaining network. This process is repeated until no new APs appear.

Depending on the structure of the network, specifically the average degree of connectivity, the greedy AP removal process will either result in a completely decomposed network with disconnected subsets of nodes, each containing no more than three nodes, or a residual giant connected component (GCC) will emerge, occupying a finite portion of the network without any APs. This residual GCC represents the core of the network responsible for its integrity and functionality.

The simulations of AP removal process produce a few good measures of the network stability to dismantling. We focus on the fraction of APs n_{AP}, the size of the leftover residual giant connected component s_{GCC}, and the total number of steps T taken by the AP removal process. These quantities vary as functions of the average degree of the network c, and we will plot the corresponding curves $n_{AP}(c)$, $s_{GCC}(c)$, and $T(c)$.

The size of a monoplex network will be denoted by N, and the same notation will be used for the size of a layer of a multiplex network. For example, size of a two-layer multiplex network will be considered equal to N if the size of its single layer is N. We will also assume that the layers of a multiplex network have the same size.

In monoplex networks the fraction of APs n_{AP} is zero when $c \to 0$ or $c \to \infty$. It reaches a maximum value at a critical mean degree c_{AP}. The original paper shows



that for ER monoplex networks, this critical value is 1.41868..., and it increases with each step of the process, approaching a limiting value of 3.4... This critical value indicates the degree at which the network remains connected after the AP removal process, as networks with a degree greater than this critical value still have a non-zero residual GCC size even after 10 steps of the process (Fig. ??).

Cascade of node failures in multiplex networks. In Fig. ??, an illustration of this chain of failures is shown. After removing a single node, the remaining part of the network remains functional.

III. RESULTS

Defining articulation points in multiplex networks. A two-layer multiplex network, a duplex, is composed of two layers where each node is representative or replica of the same unit or entity in each of the layers. Patterns of interaction between nodes in the first and second layers are captured by $A^{[1]}$ and $A^{[2]}$, respectively. The Adjacency matrix A encodes the interactions, i.e., $A_{i,j} = 1$ if there is a link between the nodes i and j, and $A_{i,j} = 0$ otherwise. In monoplex networks, a connected component is a proxy of the functionality of the nodes belonging to that component. Connected components in multiplex networks further should contain the dependency inherent in the multiplexity. A mutually connected component (MCC) in a multiplex network is a maximal subset where each pair of nodes in this set are connected at least by a path in the first layer and a path in the second layer with a condition that paths should be inclusive, in that all the nodes and links in the path should also belong to that MCC [5, 19]. At this point, we can generalize the definition of an articulation point to multiplex networks as follows. In analogy with articulation points in monoplex networks, we define an articulation point of a multiplex network as a node whose removal increases the number of mutually connected components of the multiplex network. Having the definition of an MCC it is easy to see that a node is an articulation point of a multiplex network if it is at least an articulation point in the subgraph of the first layer or second layer (where the subgraphs consist only of nodes within the MCC), Fig We need a schematic to show AP in Monoplex. It is important to note that when considering layers separately, one should only focus on the nodes that remain after the removal of non-MCC nodes. This is because, due to cascading failures in multiplex networks, some nodes are removed, and only the nodes within MCCs retain relevance.

Articulation points of a random multiplex network. We initiate the systematic analysis of articulation points in multiplex networks and their structural roles as outlined below. To unveil the structure of a given multiplex network, we employ an iterative mechanism. Initially, at t=0, we identify the set of articulation points of the multiplex network, denoted as AP_0. We define $n_{AP}(0) = |AP_0|/N$ and the $GCC = |GMCC|/N$. Subsequently, we remove all nodes belonging to the set AP_0 from both layers of the multiplex network. We then determine the new MCCs after the removal stage and proceed to obtain the new sets and measurements at $t = 1$ and so on. By the end of this iterative process, either all nodes will have been removed, or a fraction of nodes will remain that do not contain any articulation points. We refer to this remaining GCC as the residual biconnected component [17]. The term ”biconnected” signifies that nodes within this set are connected through two independent paths in both layers.

In Fig. 3, the fraction of APs n_{AP}, the size of the GCC s_{GCC}, and the total number of AP removal steps T are plotted as functions of the mean degree of the network c. Similarly to the monoplex network case (fig:ER monoplex), each discrete step of the AP removal process produces a new AP distribution and the GCC size function which are plotted in the same figures as separate curves. In this case, we will as well limit ourselves to $t = 10$ curves, as this is enough to define the limiting behavior of the curves, although the process itself might take more steps for some values of c.

Let us begin the discussion of the plots with the similarities they share with the monoplex network case plots. Like we have observed in the monoplex network example, n_{AP} is zero when $c \to 0$ or $c \to \infty$. It is explained trivially: in case when $c \to 0$ the network has not yet grown enough connections and consists mainly of disconnected tree-like components with no APs, and in case when $c \to \infty$ a major part of the network is occupied by the GCC, in which any pair of nodes is connected by at least two paths and no APs exist either. Furthermore, the AP distribution curves, too, have peaks. In general, qualitatively Fig. 3 is similar to Fig. ??, because the coordinates of the peaks of the n_{AP} curves coincide with the coordinates of the fastest growth points of the s_{GCC} curves. This means that emergence of the GCC produces the largest numbers of APs, but when the GCC has not yet emerged ($c \to 0$) or has already occupied most of the network ($c \to \infty$), the fraction of APs is not high in both monoplex and multiplex cases.

However, fundamental differences can be found between Fig. ?? and Fig. 3. First, the coordinates of the AP curves peaks, as well as corresponding to them coordinates of the fastest growth points of s_{GCC}, shift to the right compared to the monoplex network case, and the percolation phase transition now happens at larger values of c. For ER monoplex networks, we found that the value of the degree the network was required to have in order to stay connected after the AP removal process was equal to 3.4... For the multiplex network case, this value is now 5.1... (the coordinate of the red curve peak, assuming the red curve is close enough to the limiting curve). This is the result of the cascade of failures effect which emerges during the AP removal process. Due to this effect, multiplex networks require more connections to stay functional; Fig. 3b shows that, after 10 steps of the AP removal process, the network does not have a GCC if its initial mean degree was less than 5.1.

Another difference between monoplex and multiplex cases is found in the order of phase transition that the networks undergo. In the monoplex network case (Fig. ??), the GCC curves are continuous suggesting a continuous phase transition that is common in monoplex networks percolation. However, in Fig. 3, we see AP distribution curves having a discontinuous jump and GCC curves having a jump in their first order derivatives, which suggests a first order percolation phase transition. This, once again, emphasizes that multiplex networks are fundamentally different from monoplex networks because of the interlayer dependencies.

It is important to note that the obtained critical value of c and the order of phase transition are only true for ER two-layer multiplex networks. The analogous results for SF networks will be presented in the next subsection.

AP removal in SF multiplex network. In this subsection, we calculate n_{AP} and s_{GCC} curves and obtain the plots analogous to fig:ER for another canonical random graph model: scale free model with power law degree distribution $P(k) \sim k^{-\lambda}$. λ is called the degree exponent. In Fig. 4, AP distribution curves, GCC size curves, and the total number of steps of the AP removal process curve are presented for degree exponent $\lambda = 3$. Comparative analysis of n_{AP} curves for SF networks with various exponents can be found in the upcoming subsection.

We will focus on comparison between ER multiplex case (Fig. 3) and SF multiplex case (Fig. 4). To begin with, the curves in all three figures are stretched to the right compared to Fig. 3. Apparently, ER networks come to steadiness at $c \sim 7-8$, and in SF networks it only happens at $c \sim 10-12$. This might be a consequence of SF networks having power law degree distribution. For the same reason, the value of the degree the network was required to have in order to stay connected after the AP removal process for SF networks is $c = 8$ (red curve peak in Fig. 4a), and for ER networks it was $c = 5.1$.

Interestingly, the AP distribution curves in Fig. 4a and GCC size curves in Fig. 4b appear to be continuous and indicate a continuous percolation phase transition, which is more typical for ER monoplex networks rather than multiplex networks, in which we observed a clear discontinuous first order percolation phase transition. This might be due to the chosen value of the degree exponent $\lambda = 3$, which is a known borderline value for SF networks: SF networks with $2 < \lambda < 3$ have infinite second moment of the degree distribution.

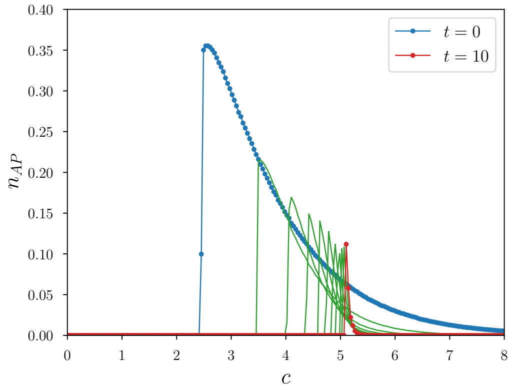
SF models comparison. Finally, AP distributions of scale free two layer networks with degree exponents 3 and 4 are presented in Fig. 5. AP distribution of ER two layer network is presented as well. In the previous subsection, we made a hypothesis that the probable cause for continuous phase transition in SF networks (Fig. 4) is explained by a borderline value of the degree exponent $\lambda = 3$. Fig. 5 confirms our hypothesis: we see a jump in the first order derivative of the red curve that corresponds to SF networks with $\lambda = 4$, as we could have expected based on first order transition of the ER multiplex networks. We once again remind the difference between the two cases: SF networks with the degree exponent of 3 have infinite second moment of their degree distribution, meanwhile networks with the exponent of 4 have finite second moment. These results suggest that there could be a connection between the network's degree distribution and the order of the phase transition it undergoes during AP removal process. This leaves avenue for further research.

IV. CONCLUSIONS

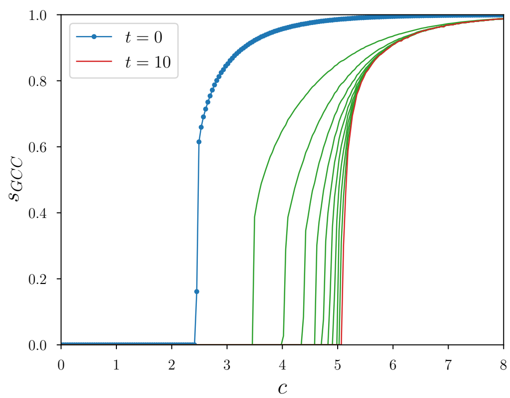
Appendix A: Application to Real Data

Articulation points removal in real multiplex networks. The plots presented below (Fig. 6) indicate low resilience of real-world multiplex infrastructures to articulation points removal process. Indeed, in the result of this process, all examined networks end up with close to zero n_{GMCC} which in real life scenario would mean complete network dismantling and require more careful network construction.

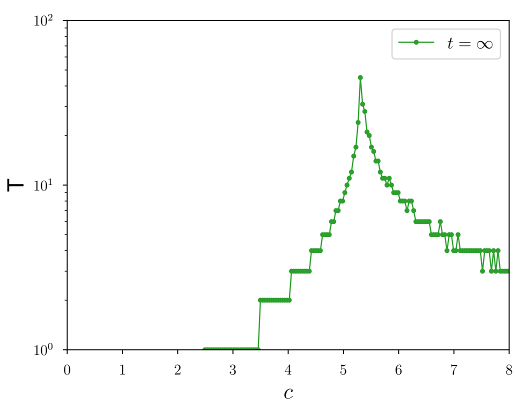
Real networks with random node relabeling. In this subsection we planned to show how random relabeling destroys inter-layer correlations, thus reduces number of hubs and number of APs as well, which in general improves n_{AP} vs. n_{GMCC} dynamics. Also it would be a good exmaple of something you cannot do in monoplex networks. HOWEVER: It actually made things worse. See the graph. POSSIBLE EXPLANATION: With random relabeling, new hubs (and thus APs) were introduced because of the definition of APs as being the union of APs in each layer. Overall, networks are dismantled now even faster. Need larger datasets to prove this hypothesis.



(a) Fraction of APs n_{AP}.

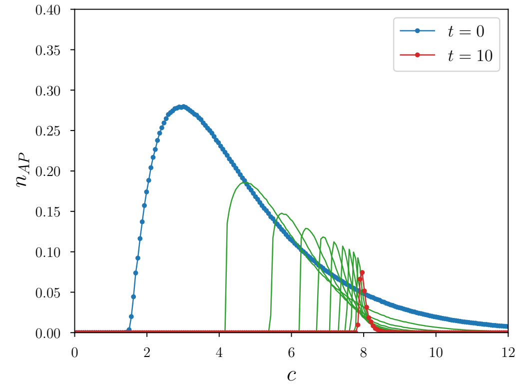


(b) Size of the residual GCC s_{GCC}.

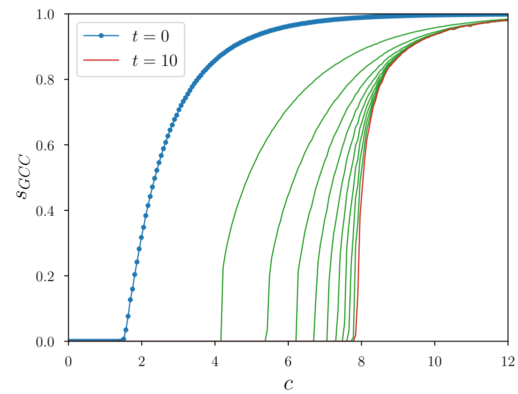


(c) Total number of AP removal process steps T.

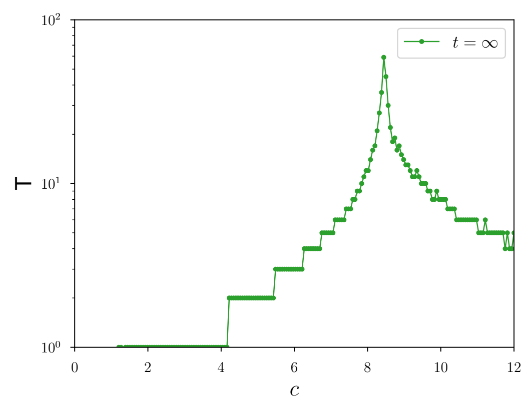
FIG. 3: Network parameters (fraction of APs n_{AP} (a), size of the residual GCC s_{GCC} (b), and the total number of AP removal process steps T) during phase transition of AP removal process in two-layer ER networks as functions of the mean degree c. $N = 10^5$.



(a) Fraction of APs n_{AP}.



(b) Size of the residual GCC s_{GCC}.



(c) Total number of AP removal process steps T.

FIG. 4: Network parameters (fraction of APs n_{AP} (a), size of the residual GCC s_{GCC} (b), and the total number of AP removal process steps T) during phase transition of AP removal process in two-layer SF networks as functions of the mean degree c. $N = 10^5$. Degree exponent $\lambda = 3$.

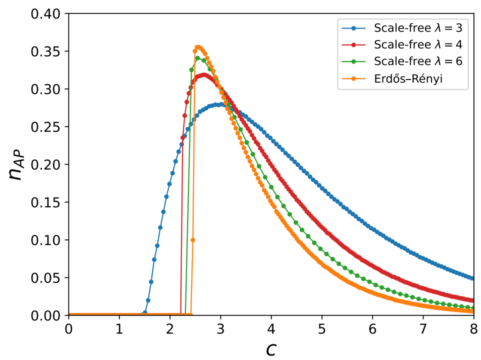


FIG. 5: SF networks with different degree exponents λ.

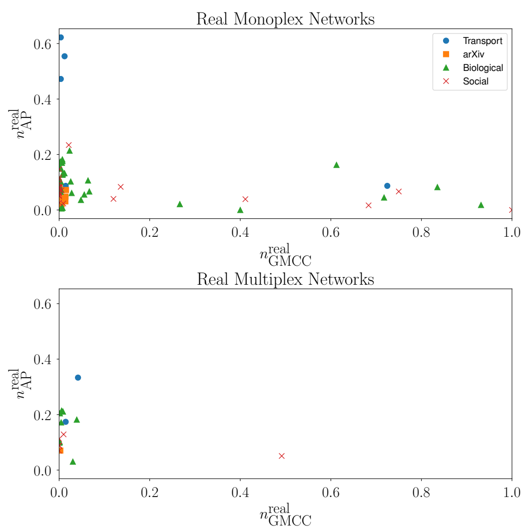


FIG. 6: Fraction of articulation points n_{AP}^{real} vs. relative size of the GMCC n_{GMCC}^{real} in real monoplex (top plot) and multiplex (lower plot) networks. Almost all of the networks presented in the second plot have rather small residual GMCC which indicates lack of real-world structures resilience to articulation points removal.

Network	Layers	N	Type	Source
Air/train	Indian airport network Indian train network	69	Transport	[21]
Arabidopsis GPI	Direct interaction Physical association Additive genetic interaction defined by inequality Suppressive genetic interaction defined by inequality Synthetic genetic interaction defined by inequality Association Colocalization	1449	Biological	[22]
ArXiv	physics.bioph category cond-mat.disnn category physics.soc-ph category physics.data-an category math.OC category cond-mat.stat-mech category q-bio.MN category cs.SI category	3896	Collaboration	[23]
Bos GPI	Physical association Association Direct interaction Colocalization	33	Biological	[22]
C. elegans	Electric Chemical monadic Chemical polyadic	237	Biological	[24]
Cannes 2013	Retweets Mentions Replies	135571	Social	[25]
CS-Aarhus	Facebook Leisure Work Co-authorship Lunch	59	Social	[20]
Drosophila	Suppressive genetic interaction Additive genetic interaction	557	Biological	[22]
Human brain	Structural network Functional network	77	Biological	[26]
London transport	Underground Overground DLR	24	Transport	[27]
MLKing 2013	Retweets Mentions Replies	40148	Social	[25]
Moscow Athletics 2013	Retweets Mentions Replies	32732	Social	[25]
NY Climate March 2014	Retweets Mentions Replies	42212	Social	[25]
Obama in Israel 2013	Retweets Mentions Replies	438392	Social	[25]
Physicians	Advice Discussion Friendship	202	Social	[28]
Rattus	Physical association Direct interaction	454	Biological	[22]
SacchCere GPI	Physical association Suppressive genetic interaction defined by inequality Direct interaction Synthetic genetic interaction defined by inequality Association Colocalization Additive genetic interaction defined by inequality	4634	Biological	[22]

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