Skoltech

Percolation on complex networks and its applications

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Doctoral program: Computational and data science and engineering (CDSE)

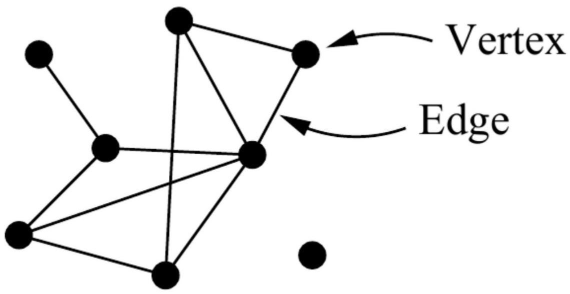
Date of the defense: 26.10.2021

Outline

- **Chapter 1. Introduction**
    - Networks
        - Definitions
        - Real and Random Networks
        - Monoplex vs Multiplex
    - Percolation
        - Definitions
        - Percolation Variants (site, bond, core, k-core, Gk-core, optimal)
- **Chapters 2-7**
    - Results
- **Conclusions**

Networks

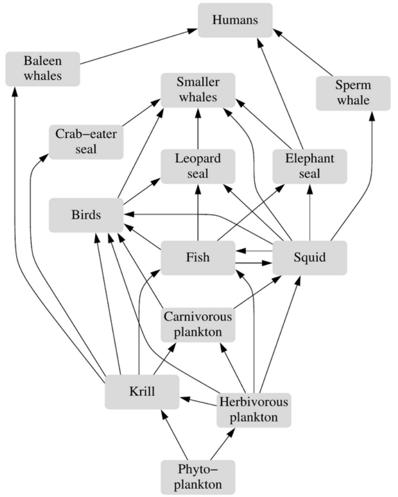
Graph/Network



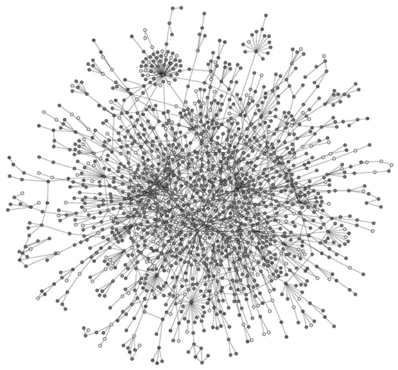
Social Network

https://medium.com/analytics-vidhya/social-network-analytics-f082f4e21b16

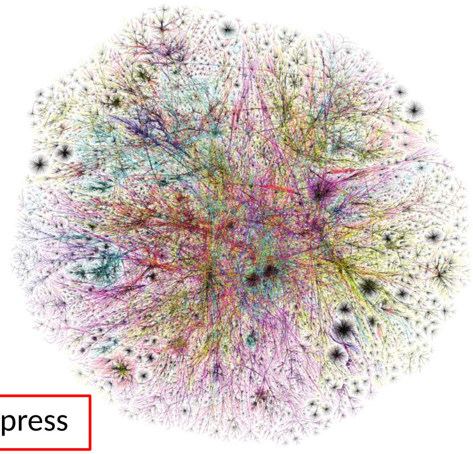
A food web of species in Antarctica



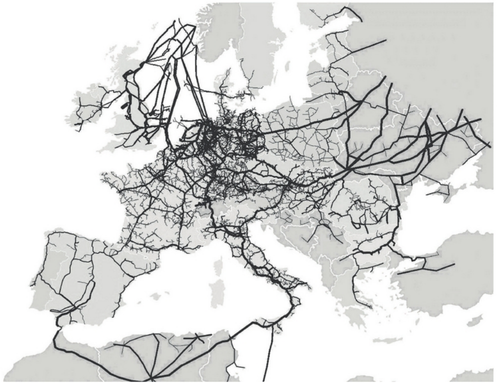
A protein–protein interaction network for yeast



The topology of the Internet



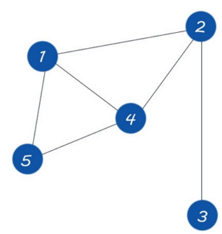
The network of natural gas pipelines in Europe



Networks, Mark Newman, Oxford university press

Networks: Definitions

Adjacency Matrix



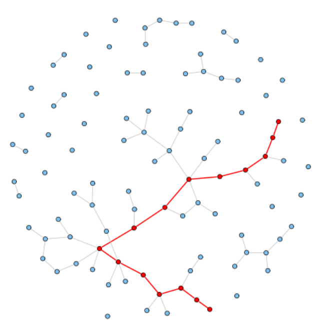
$$A = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \\ 5 & 1 & 0 & 0 & 1 & 0 \end{array}$$

Degree $$k_i = \sum_j A_{ij}$$

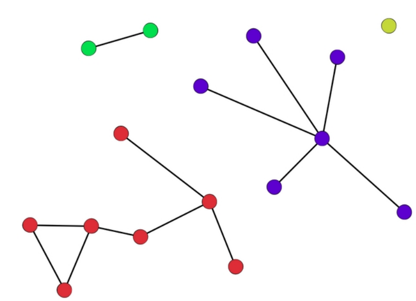
Degree Distribution $$p(k) = \frac{N_k}{N}$$

Clustering Coefficient $$c_i = \frac{N_{\Delta_i}}{\binom{k_i}{2}} \qquad \overline{C} = \frac{1}{N}\sum_i c_i$$

Path



Component & giant component



Networks: Real Networks

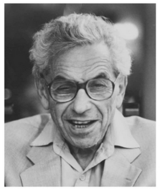
	Network	Type	n	m	c	S	ℓ	α	C	C_{WS}	r
Social	Film actors	Undirected	449 913	25 516 482	113.43	0.980	3.48	2.3	0.20	0.78	0.208
	Company directors	Undirected	7 673	55 392	14.44	0.876	4.60	–	0.59	0.88	0.276
	Math coauthorship	Undirected	253 339	496 489	3.92	0.822	7.57	–	0.15	0.34	0.120
	Physics coauthorship	Undirected	52 909	245 300	9.27	0.838	6.19	–	0.45	0.56	0.363
	Biology coauthorship	Undirected	1 520 251	11 803 064	15.53	0.918	4.92	–	0.088	0.60	0.127
	Telephone call graph	Undirected	47 000 000	80 000 000	3.16			2.1			
	Email messages	Directed	59 812	86 300	1.44	0.952	4.95	1.5/2.0		0.16	
	Email address books	Directed	16 881	57 029	3.38	0.590	5.22	–	0.17	0.13	0.092
	Student dating	Undirected	573	477	1.66	0.503	16.01	–	0.005	0.001	−0.029
	Sexual contacts	Undirected	2 810					3.2			
Information	WWW `nd.edu`	Directed	269 504	1 497 135	5.55	1.000	11.27	2.1/2.4	0.11	0.29	−0.067
	WWW AltaVista	Directed	203 549 046	1 466 000 000	7.20	0.914	16.18	2.1/2.7			
	Citation network	Directed	783 339	6 716 198	8.57			3.0/–			
	Roget's Thesaurus	Directed	1 022	5 103	4.99	0.977	4.87	–	0.13	0.15	0.157
	Word co-occurrence	Undirected	460 902	16 100 000	66.96	1.000		2.7		0.44	
Technological	Internet	Undirected	10 697	31 992	5.98	1.000	3.31	2.5	0.035	0.39	−0.189
	Power grid	Undirected	4 941	6 594	2.67	1.000	18.99	–	0.10	0.080	0.003
	Train routes	Undirected	587	19 603	66.79	1.000	2.16	–		0.69	−0.033
	Software packages	Directed	1 439	1 723	1.20	0.998	2.42	1.6/1.4	0.070	0.082	−0.016
	Software classes	Directed	1 376	2 213	1.61	1.000	5.40	–	0.033	0.012	−0.119
	Electronic circuits	Undirected	24 097	53 248	4.34	1.000	11.05	3.0	0.010	0.030	−0.154
	Peer-to-peer network	Undirected	880	1 296	1.47	0.805	4.28	2.1	0.012	0.011	−0.366
Biological	Metabolic network	Undirected	765	3 686	9.64	0.996	2.56	2.2	0.090	0.67	−0.240
	Protein interactions	Undirected	2 115	2 240	2.12	0.689	6.80	2.4	0.072	0.071	−0.156
	Marine food web	Directed	134	598	4.46	1.000	2.05	–	0.16	0.23	−0.263
	Freshwater food web	Directed	92	997	10.84	1.000	1.90	–	0.20	0.087	−0.326
	Neural network	Directed	307	2 359	7.68	0.967	3.97	–	0.18	0.28	−0.226

Networks, Mark Newman, Oxford university press

Networks: Random Networks

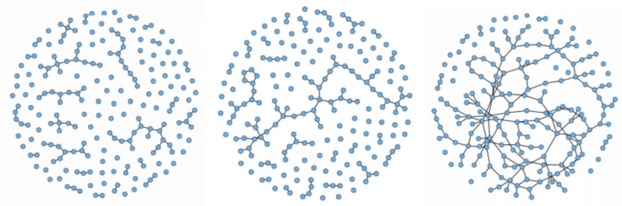
How can one design a random network mimicking a real one in terms of observed topological structures?

Erdős–Rényi Random Graph

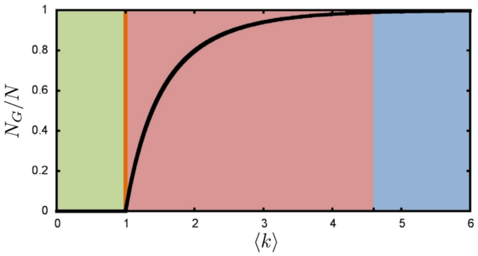


Pál Erdős (1913-1996)

Alfréd Rényi (1921-1970)



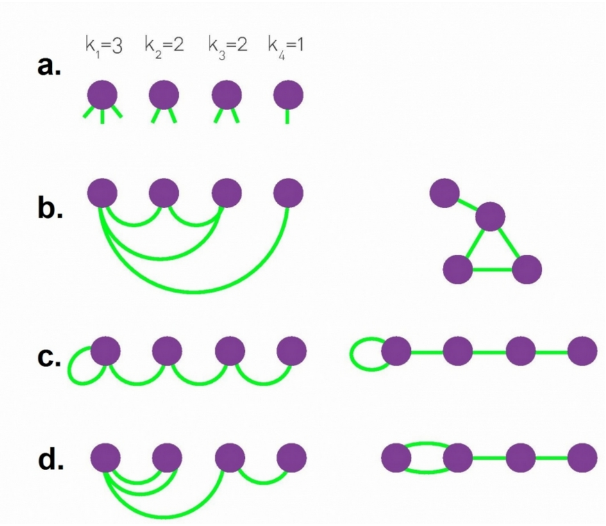
$p < p_c$ $p = p_c$ $p > p_c$



$$p_k = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

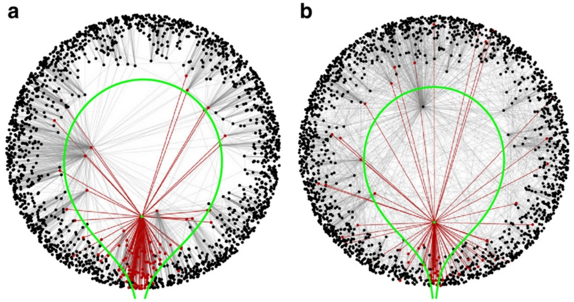
A.L. Barabasi Network Science

Configuration Model



$$p_{ij} = \frac{k_i k_j}{2L-1}$$

Hyperbolic Networks



$$N, \bar{k}, \gamma > 2, T \in [0.1)$$

$\mathbb{S}^1$ **model** κ_i, θ_i

$$\chi_{ij} = R\Delta\theta_{ij}/(\mu\kappa_i\kappa_j)$$

$$\mu = \sin T\pi/(2\bar{k}T\pi)$$

$$\Delta\theta_{ij} = \pi - |\pi - |\theta_i - \theta_j||$$

$$\rho(\kappa) = (\gamma - 1)\kappa_0^{\gamma-1}\kappa^{-\gamma}$$

$$p(\chi_{ij}) = \frac{1}{1 + \chi_{ij}^{1/T}}$$

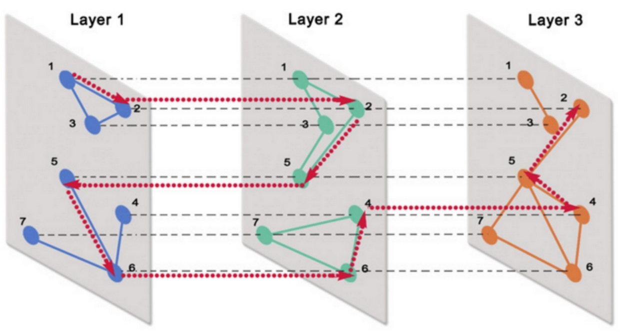
K.K Kleineberg,Nature communications 8 (1), 1-8

Networks: Monoplex vs Multiplex

Multiplexity

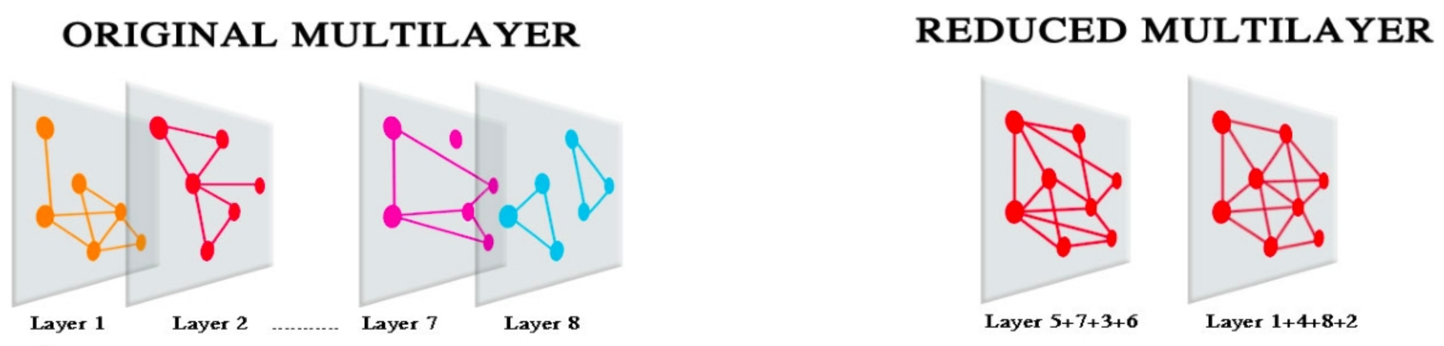
Arises in complex systems where elementary units have different types of interactions. (Most of the real-world networks)

Transportation Multiplex



De Domenico, Manlio, et al.
PNAS 111.23 (2014): 8351-8356

To aggregate or not to aggregate?!

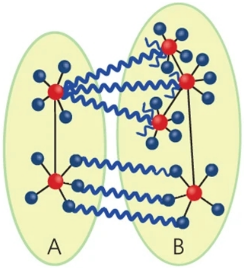


De Domenico, Manlio, et al.
Nature communications 6.1 (2015): 1-9

Networks: Monoplex vs Multiplex

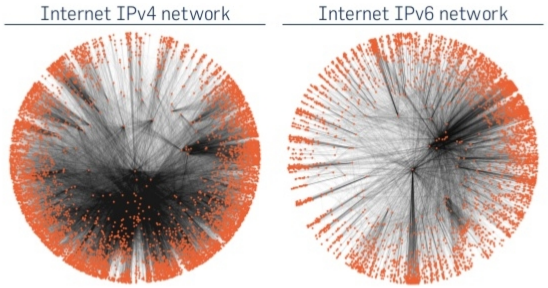
Main observed inter-layer correlations in real multiplex networks

Degree Correlation



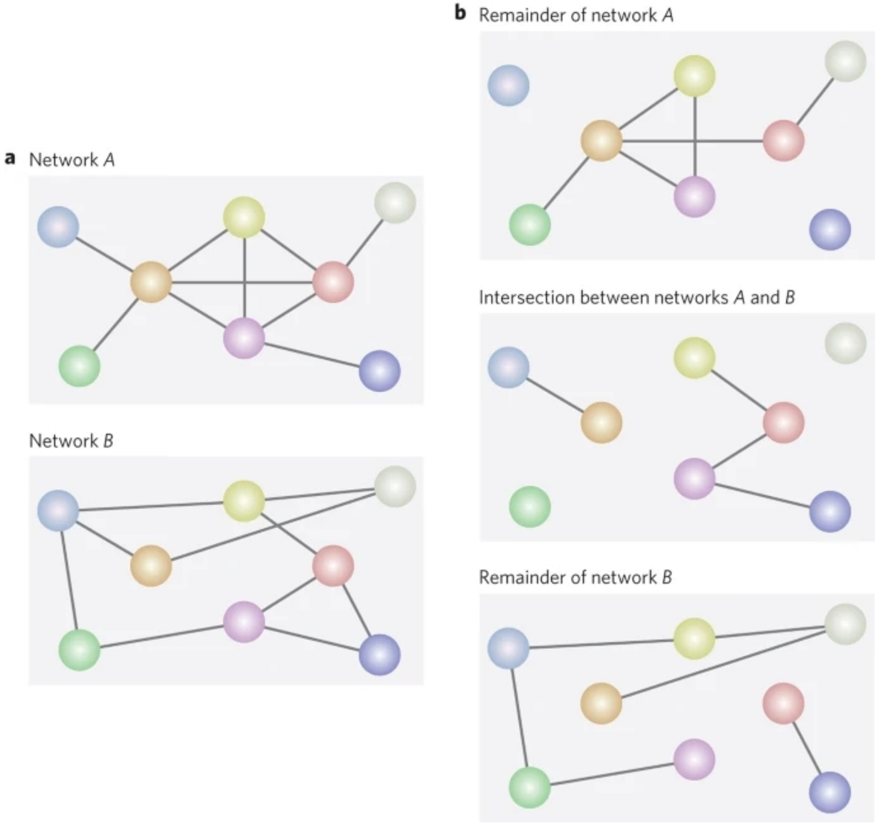
Reis, Saulo DS, et al.
Nature Physics 10.10 (2014): 762-767

Hidden geometric correlations



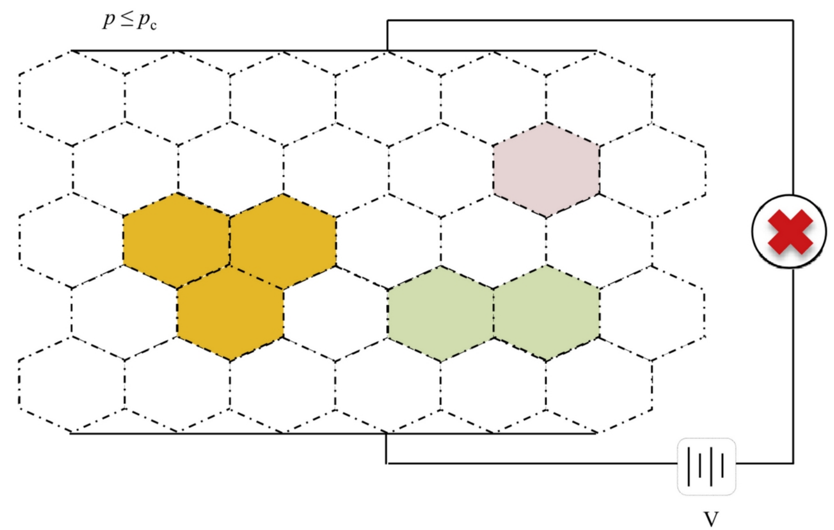
Kleineberg, Kaj-Kolja, et al.
Nature Physics 12.11 (2016): 1076-1081

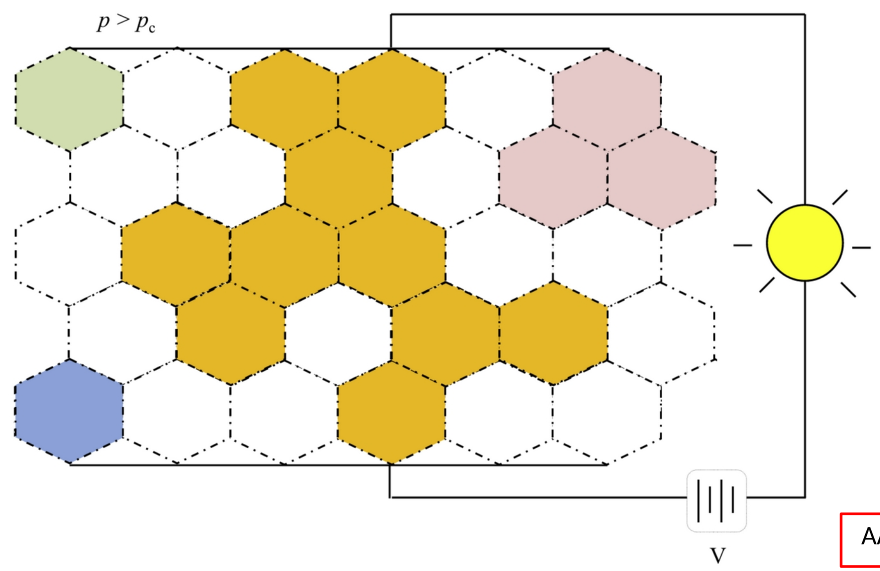
Edge Overlap

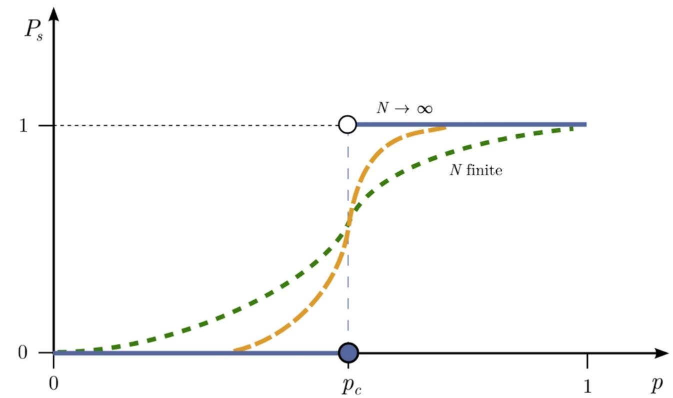


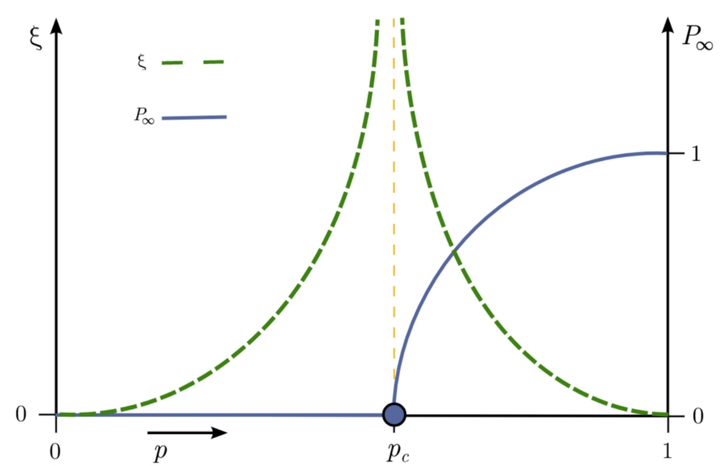
Radicchi, Filippo.
Nature Physics 11.7 (2015): 597-602

Percolation: Definitions



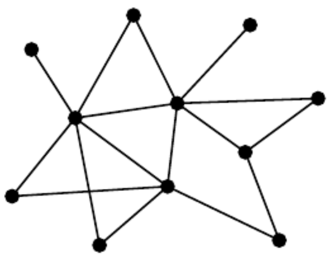




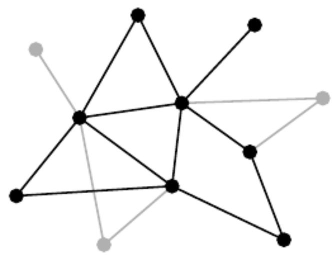


AA Saberi, Physics Reports 578, 1-32

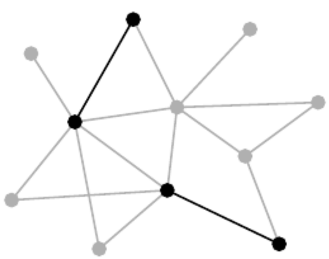
Percolation: Definitions



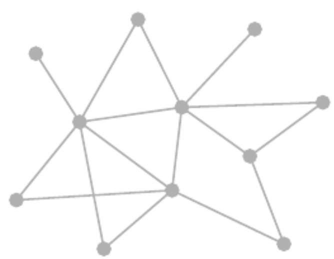
(a) p=1



(b) p=0.7



(c) p=0.3



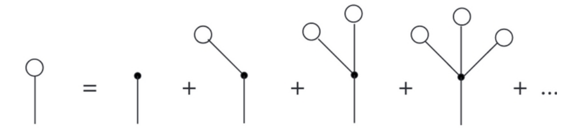
(d) p=0

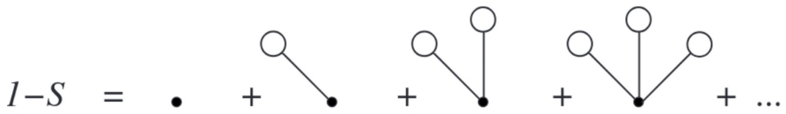
Networks, Mark Newman, Oxford university press

Locally Tree-Like & N→ ∞

u= 

$q(k) = \frac{(k+1)p(k+1)}{<k>}$





$$G_0(z) = \sum_k p(k) z^k \qquad G_1(z) = \sum_k q(k) z^k$$

$$\boxed{\begin{aligned} u &= 1 - p + pG_1(u) \\ S &= p(1 - G_0(u)) \end{aligned}}$$

$$p_c = \frac{< k >}{< k^2 > - < k >}$$

Rev. Mod. Phys., Vol. 80, No. 4,2008

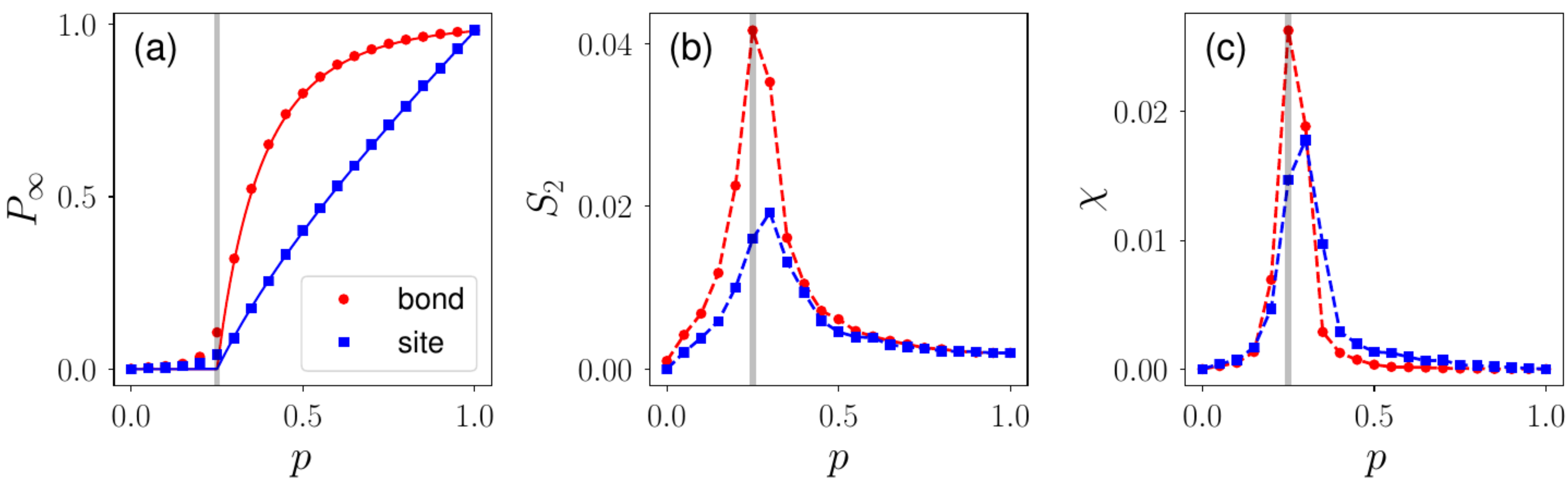
Percolation on Monoplex Networks

Poisson Random Graph: $G_0(z) = G_1(z) = e^{<k>(z-1)}$

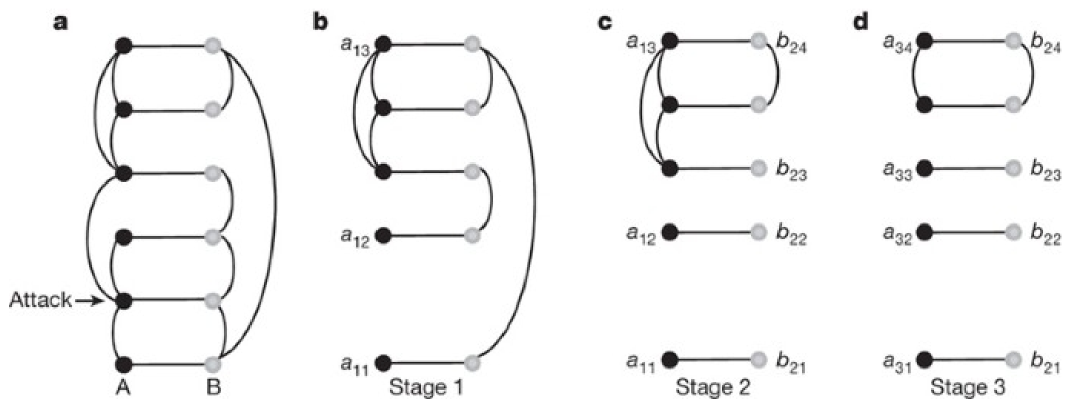
$$S_{SP} = p(1 - e^{-<k>S})$$

$$S_{BP} = 1 - e^{-<k>Sp}$$

$$\chi = \frac{< S^2 > - < S >^2}{< S >}$$



Percolation on Multiplex Networks



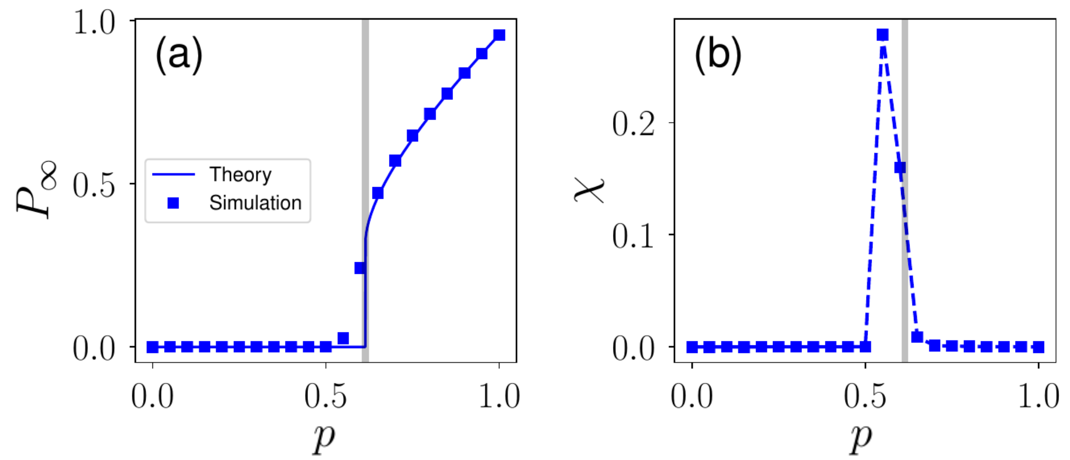
$$S = p[1 - G_0^{[1]}(u_1)][1 - G_0^{[2]}(u_2)]$$

$$u_1 = p[1 - G_1^{[1]}(u_1)][1 - G_0^{[2]}(u_2)]$$

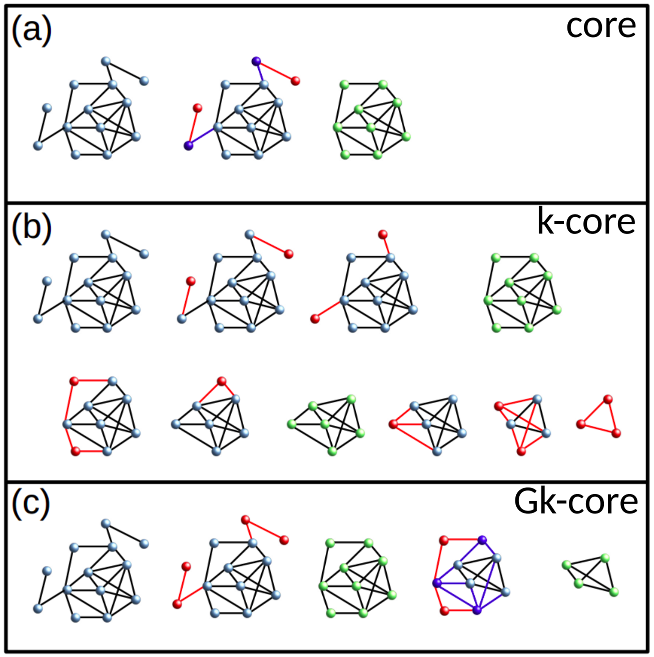
$$u_2 = p[1 - G_0^{[1]}(u_1)][1 - G_1^{[2]}(u_2)]$$

Buldyrev, Sergey V., et al.
Nature 464.7291 (2010): 1025-1028

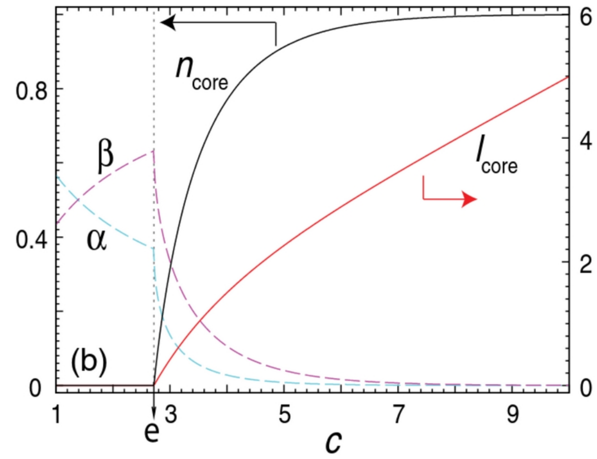
$$S = p(1 - e^{-<k>S})^2$$



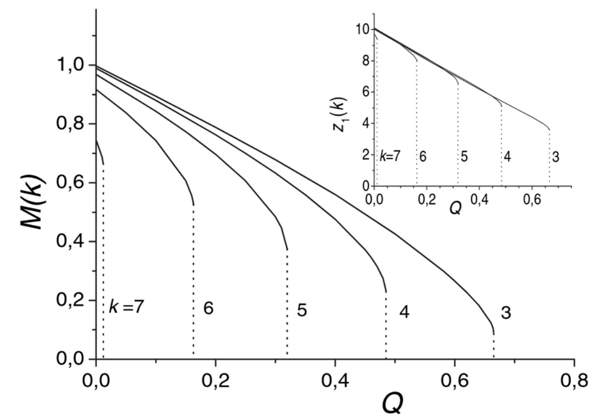
Percolation: Percolation Variants



PRL 109, 205703 (2012)

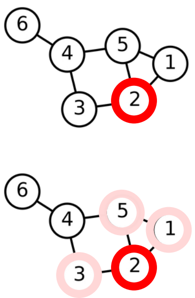


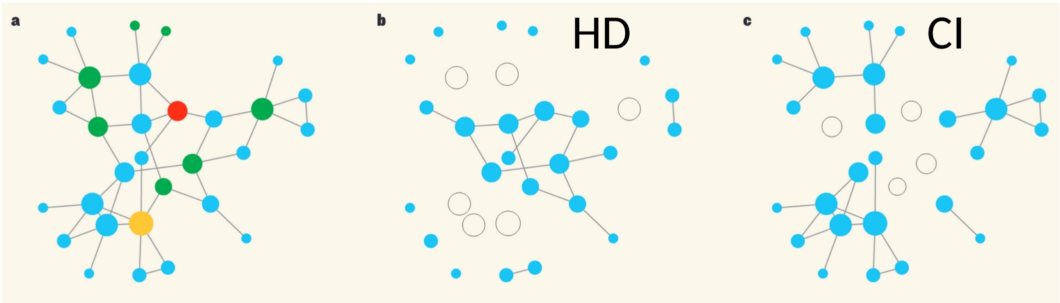
PRL 96, 040601 (2006)

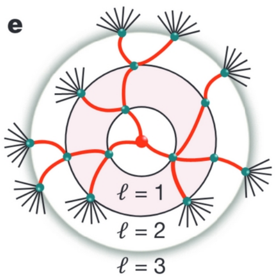


$$\mathrm{CI}_\ell(i) = (k_i - 1) \sum_{j \in \partial \mathrm{Ball}(i,\ell)} (k_j - 1)$$

Observability







Optimal percolation

Nature volume 524, pages 38–39 (2015)

Goals

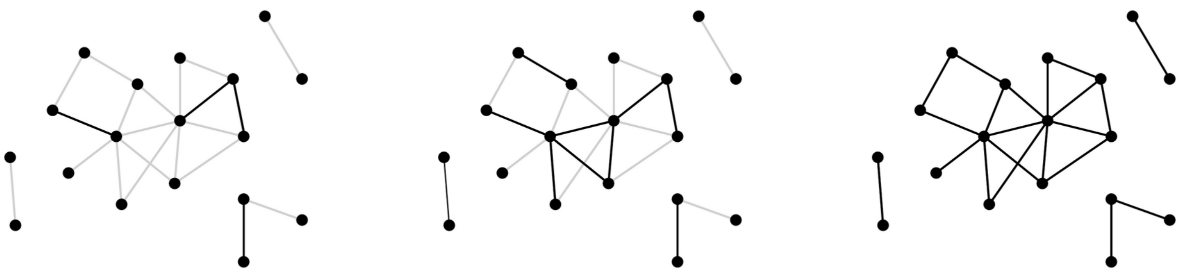
- Using an independent cascade model (ICM) to model avalanches in neuronal systems to shed light on the relation between power-law and criticality.
- Define Gk-core based on a new leaf-removal algorithm to obtain new layout for networks.
- Generalize observability transition to synthetic and real multiplex networks and study its phase transition.
- Reveal the correspondence between hyperbolic embedding and community structure of networks and confirm previously known results in this new framework.
- Generalize optimal percolation problem to multiplex networks, Desing dismantling algorithms and demonstrate the necessity of preserving multiplexity.
- Quantify k-core structure of monoplex and multiplex networks in the framework of hyperbolic networks, reveal the role of inter-layer correlations on emergence of rich k-core structures in real multiplex networks.

Results:
Emergence of power laws in noncritical neuronal systems

A. Faqeeh, S. Osat, F. Radicchi, and J.P. Gleeson

Phys. Rev. E 100, 010401(R)

Challenge: Does a power-law always indicate criticality?

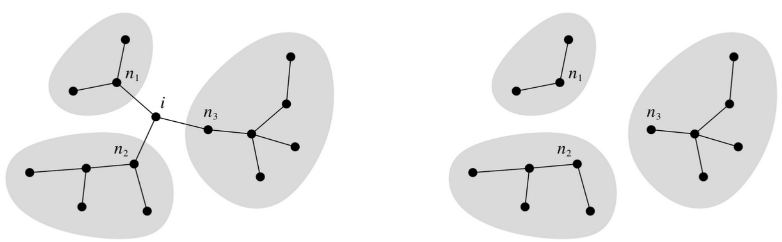


(a) $\phi = 0.2$ (b) $\phi = 0.5$ (c) $\phi = 1$

π_s distribution of avalanche sizes

ρ_s the probability that following an edge of the network we reach an avalanche (cluster) with size s

$$H_0(z) = \sum_{s=1}^{\infty} \pi_s z^s$$
$$H_1(z) = \sum_{s=0}^{\infty} \rho_s z^s$$



Locally Tree-Like

Product Property of GFs

$$\boxed{\begin{aligned} H_1(z) &= zG_1[1 - p + pH_1(z)], \\ H_0(z) &= zG_0[1 - p + pH_1(z)]. \end{aligned}}$$

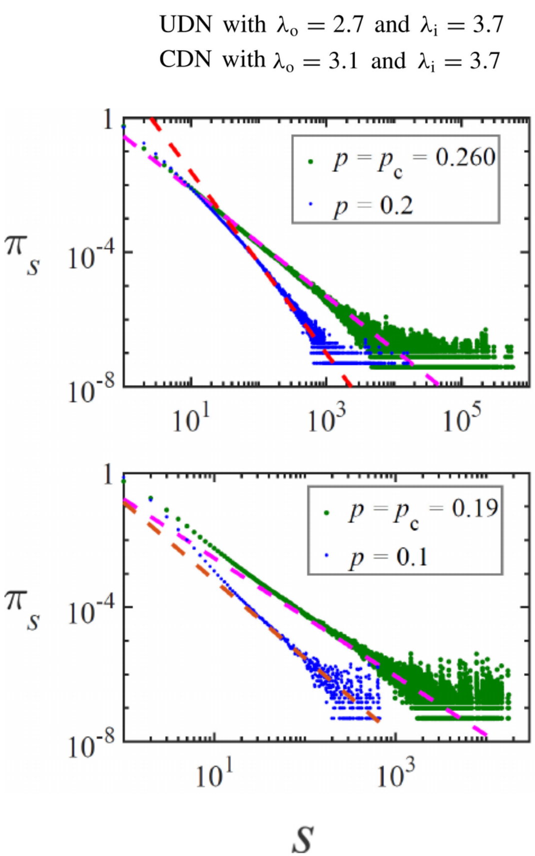
Networks, Mark Newman, Oxford university press

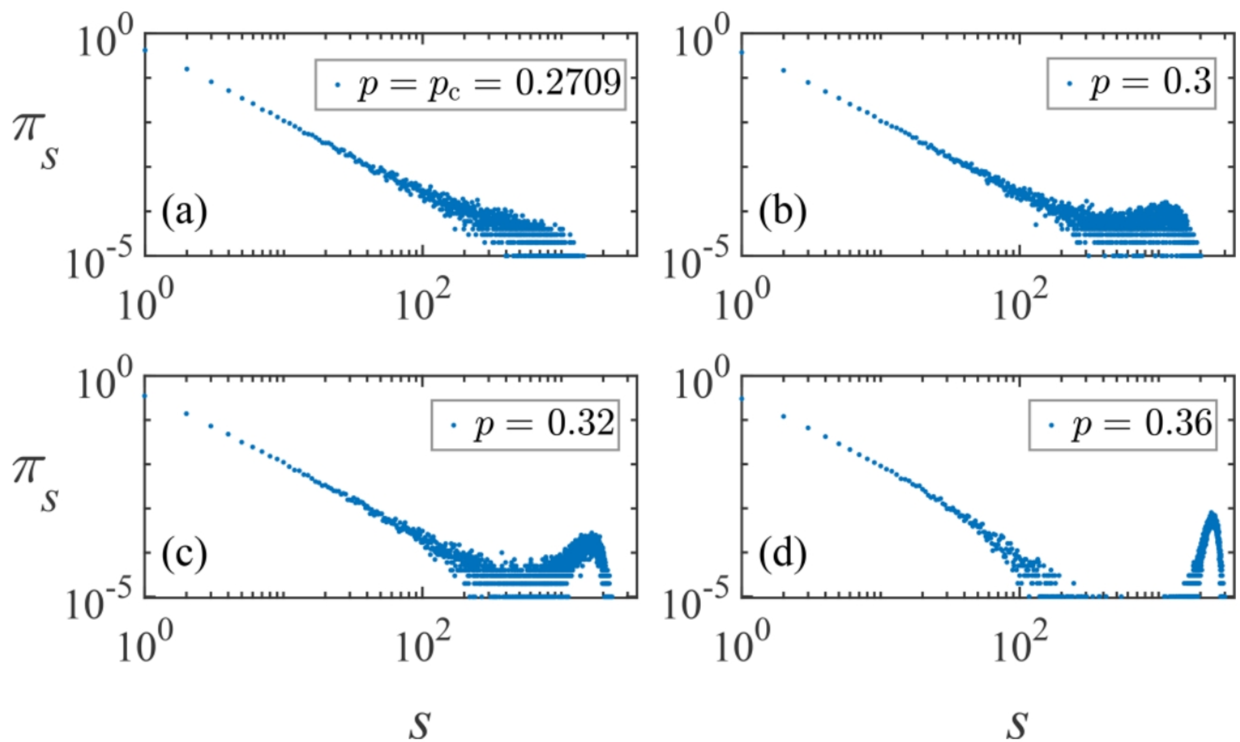
Results:
Emergence of power laws in noncritical neuronal systems

A. Faqeeh, S. Osat, F. Radicchi, and J.P. Gleeson

Phys. Rev. E 100, 010401(R)

The presence of power-law distributions for the size of neuronal avalanches is not a sufficient condition for the system to operate near criticality.



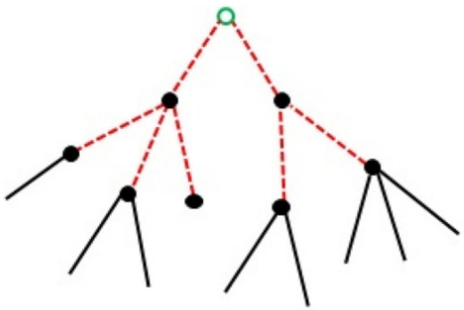


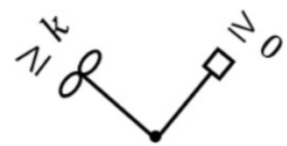
Results:
Generalization of core percolation on complex networks

N. Azimi-Tafreshi, S. Osat, and S. N. Dorogovtsev

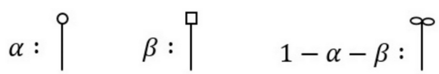
Phys. Rev. E 99, 022312

Challenge: What is the new lay-out if we combine core and k-core?

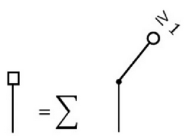




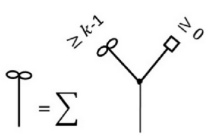
$$n_{kc} = \sum_{q>k} P(q) \sum_{s=k}^{q} \binom{q}{s} (1-\alpha-\beta)^s \beta^{q-s}$$



$$G(x) \equiv \sum_q P(q) x^q$$



$$\beta = 1 - \sum_q \frac{qP(q)}{\langle q \rangle} (1-\alpha)^{q-1}$$



$$1-\alpha-\beta = \sum_q \frac{qP(q)}{\langle q \rangle} \times \sum_{s=k-1}^{q-1} \binom{q-1}{s} (1-\alpha-\beta)^s \beta^{q-1-s}$$

$$\alpha = \sum_q \frac{qP(q)}{\langle q \rangle} \sum_{s=0}^{k-2} \binom{q-1}{s} (1-\alpha-\beta)^s \beta^{q-1-s}$$

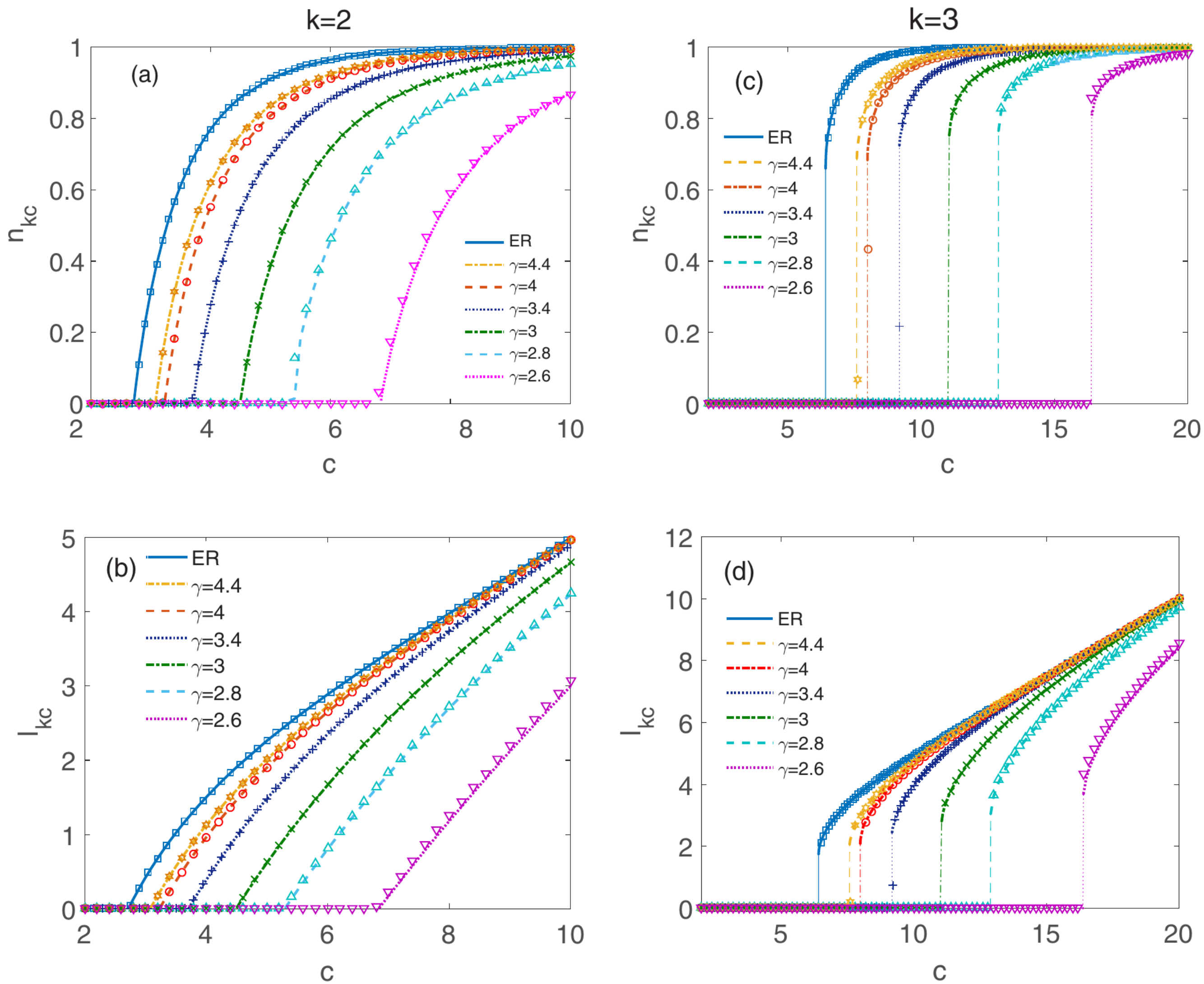
$$\boxed{\begin{aligned} \alpha &= \frac{1}{\langle q \rangle} \sum_{s=0}^{k-2} \frac{(1-\alpha-\beta)^s}{s!} G^{(s+1)}(\beta), \\ \beta &= 1 - \frac{G^{(1)}(1-\alpha)}{\langle q \rangle}, \\ n_{kc} &= G(1-\alpha) - \sum_{s=0}^{k-1} \frac{(1-\alpha-\beta)^s}{s!} G^{(s)}(\beta) \end{aligned}}$$

$$\boxed{l_{kc} = \frac{c}{2}(1-\alpha-\beta)^2}$$

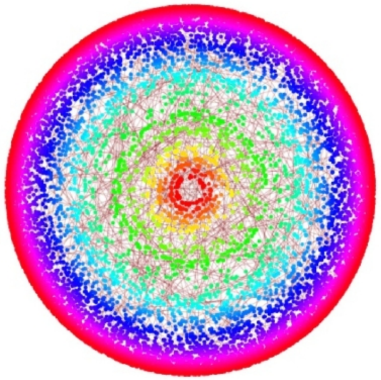
Results:
Generalization of core percolation on complex networks

N. Azimi-Tafreshi, S. Osat, and S. N. Dorogovtsev

Phys. Rev. E 99, 022312



astrophysics

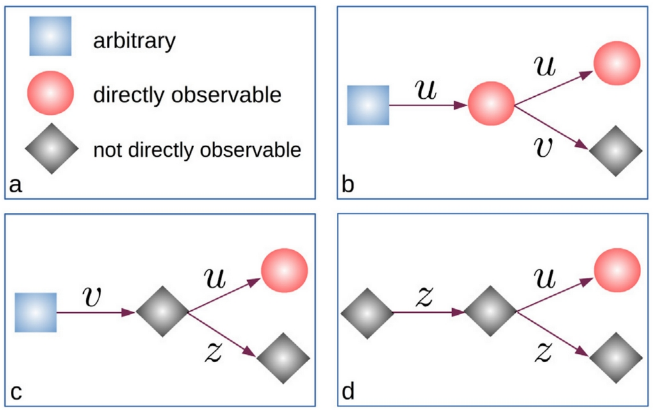


Results:
Observability transition in multiplex networks

S. Osat, F. Radicchi

Physica A, 503, 745-761

Challenge: How to generalize observability problem to multiplexes?



$$q^{[\alpha]} = \sum_k P^{[\alpha]}(k) q_k^{[\alpha]} = 1 - G_0^{[\alpha]}\left(1 - \phi u^{[\alpha]} - (1-\phi) v^{[\alpha]}\right)$$

$$r^{[\alpha]} = \sum_k P^{[\alpha]}(k) r_k^{[\alpha]} =$$
$$1 - G_0^{[\alpha]}\left(1 - \phi u^{[\alpha]} - (1-\phi) z^{[\alpha]}\right)$$
$$+ G_0^{[\alpha]}\left((1-\phi)(1 - z^{[\alpha]})\right) - G_0^{[\alpha]}(1-\phi)$$

$$\begin{aligned}
u^{[\alpha]} = \quad & \left[1 - G_1^{[\alpha]}(1 - \phi u^{[\alpha]} - (1-\phi) v^{[\alpha]})\right] \times \\
& \left[1 - G_0^{[\beta]}(1 - \phi u^{[\beta]} - (1-\phi) v^{[\beta]})\right],
\end{aligned}$$

$$\begin{aligned}
v^{[\alpha]} = \quad & \left[1 - G_1^{[\alpha]}(1 - \phi u^{[\alpha]} - (1-\phi) z^{[\alpha]})\right] \times \\
& \left[1 - G_0^{[\beta]}(1 - \phi u^{[\beta]} - (1-\phi) z^{[\beta]}) + \right. \\
& \left. - G_0^{[\beta]}(1-\phi) + G_0^{[\beta]}((1-\phi)(1 - z^{[\beta]}))\right],
\end{aligned}$$

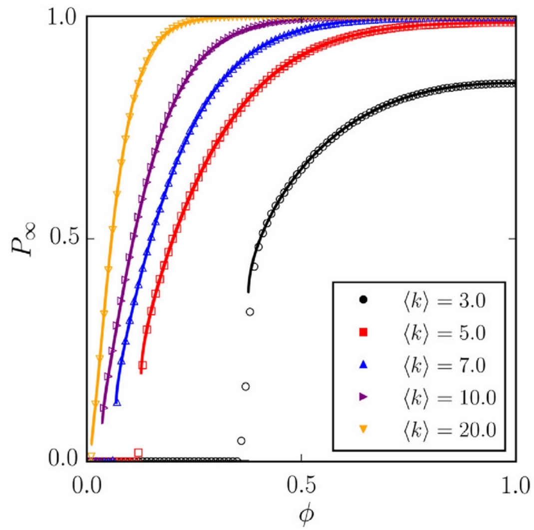
$$\begin{aligned}
z^{[\alpha]} = \quad & \left[1 - G_1^{[\alpha]}(1 - \phi u^{[\alpha]} - (1-\phi) z^{[\alpha]}) + \right. \\
& \left. - G_1^{[\alpha]}(1-\phi) + G_1^{[\alpha]}((1-\phi)(1 - z^{[\alpha]}))\right] \times \\
& \left[1 - G_0^{[\beta]}(1 - \phi u^{[\beta]} - (1-\phi) z^{[\beta]}) + \right. \\
& \left. - G_0^{[\beta]}(1-\phi) + G_0^{[\beta]}((1-\phi)(1 - z^{[\beta]}))\right].
\end{aligned}$$

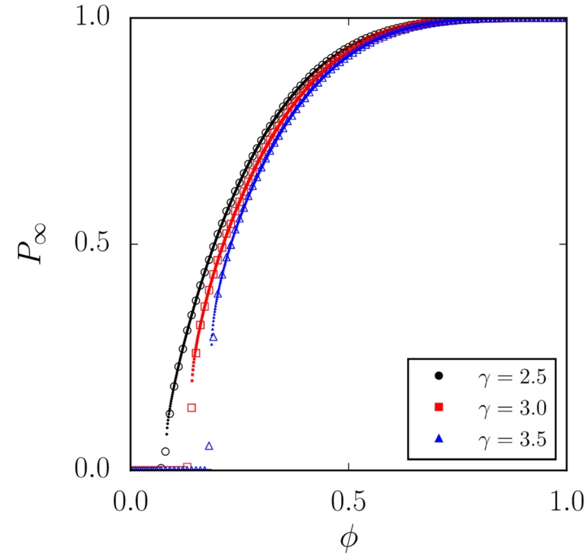
$$\boxed{\begin{aligned} P_\infty &= q + r \\ r &= (1-\phi)\, r^{[\alpha]}\, r^{[\beta]} \\ q &= \phi\, q^{[\alpha]}\, q^{[\beta]} \end{aligned}}$$

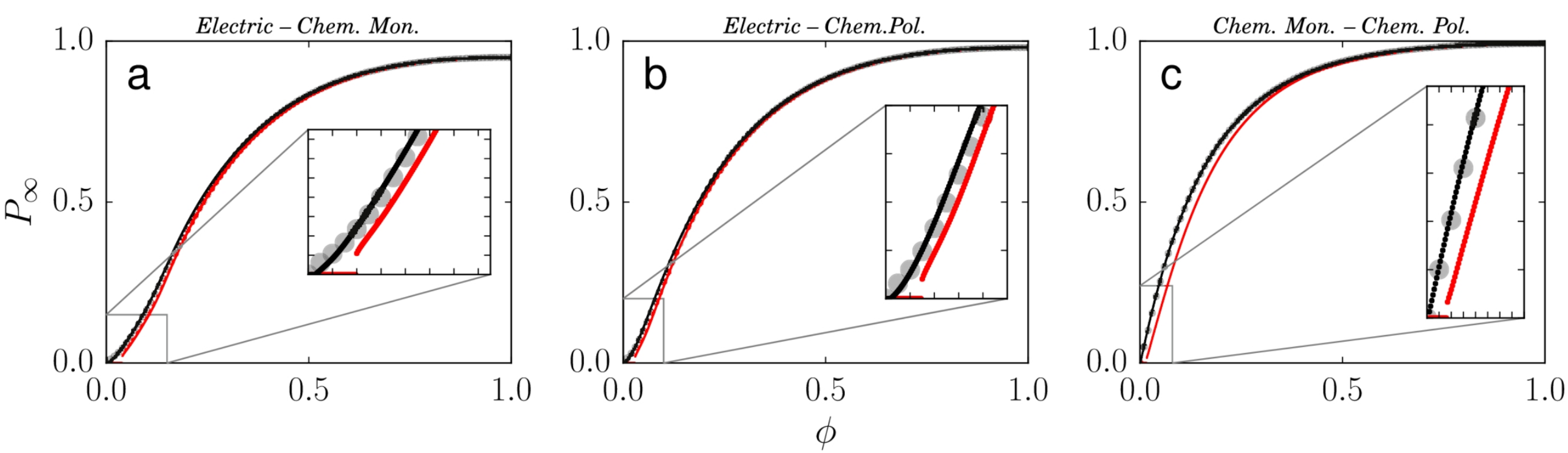
Results:
Observability transition in multiplex networks

S. Osat, F. Radicchi

Physica A, 503, 745-761





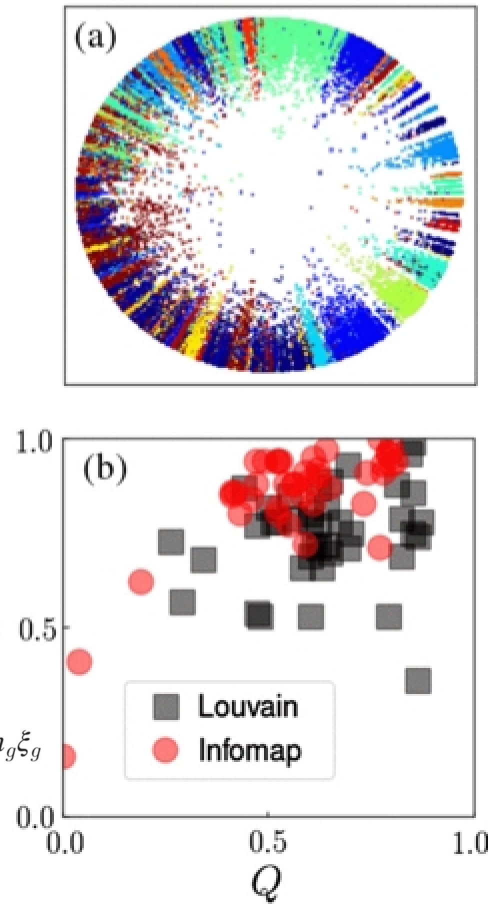


Results: Characterizing the analogy between hyperbolic embedding and community structure of complex networks

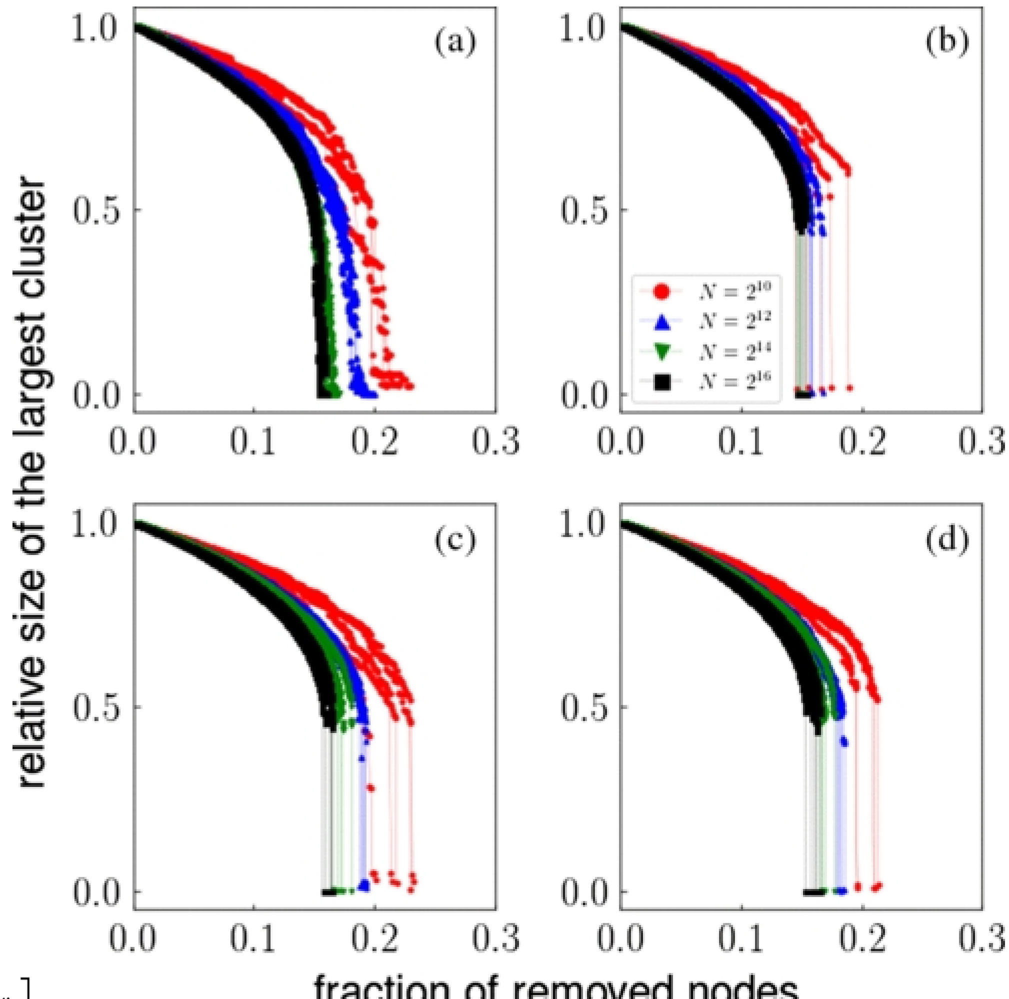
A. Faqeeh, S. Osat, and F. Radicchi

Phys. Rev. Lett. 121, 098301

Challenge: How to connect community structure and embedding of networks?



$$\bar{\xi} = \frac{1}{N} \sum_{g=1}^{C} n_g \xi_g$$



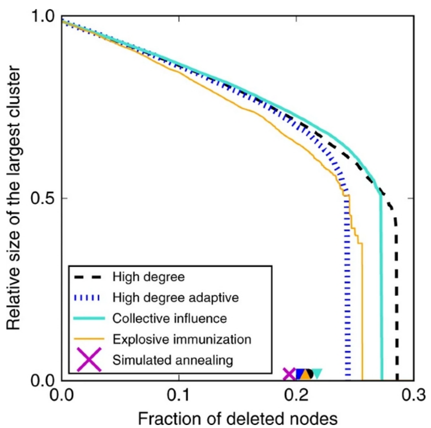
$$\xi_g e^{\mathrm{i}\,\phi_g} = \frac{1}{n_g} \sum_{j=1}^{N} \delta_{\sigma_j, g} e^{\mathrm{i}\,\theta_j} \qquad Q = \frac{1}{(2m)} \sum_{vw} \left[A_{vw} - \frac{k_v k_w}{(2m)} \right] \delta(c_v, c_w)$$

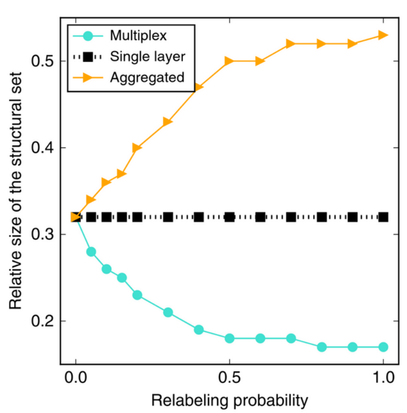
Results:
Optimal percolation on multiplex networks

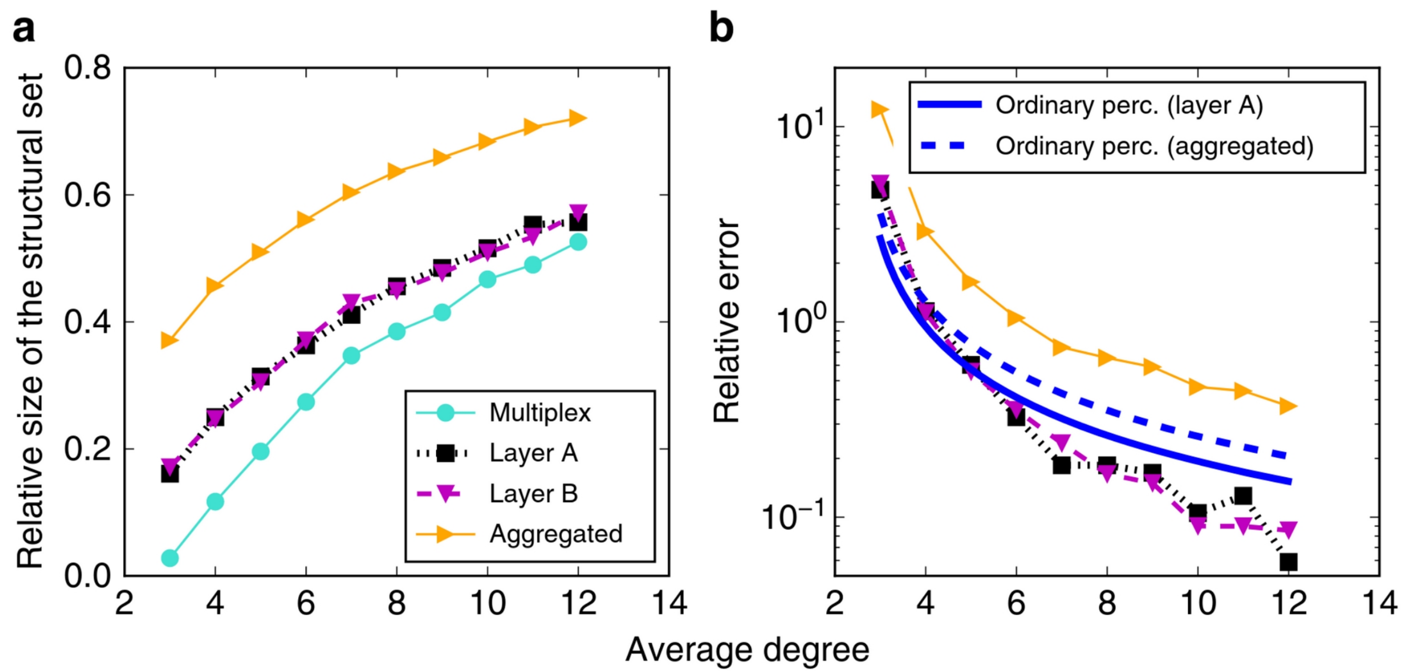
S. Osat, A. Faqeeh and F. Radicchi

Nature Communications 8 1540

Challenge: Find minimum number of nodes which their removal breaks down the network into small (sub-extensive) components.



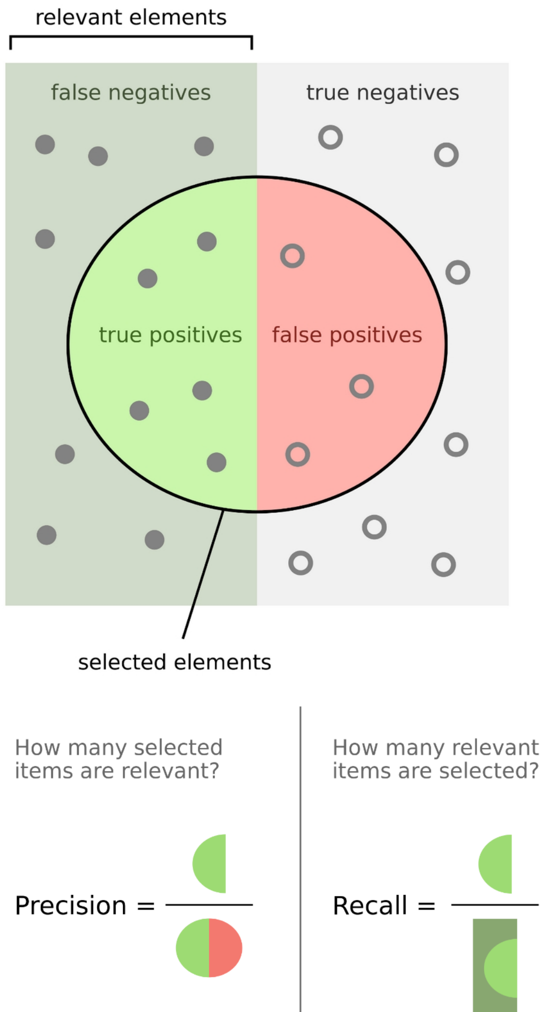




Results:
Optimal percolation on multiplex networks

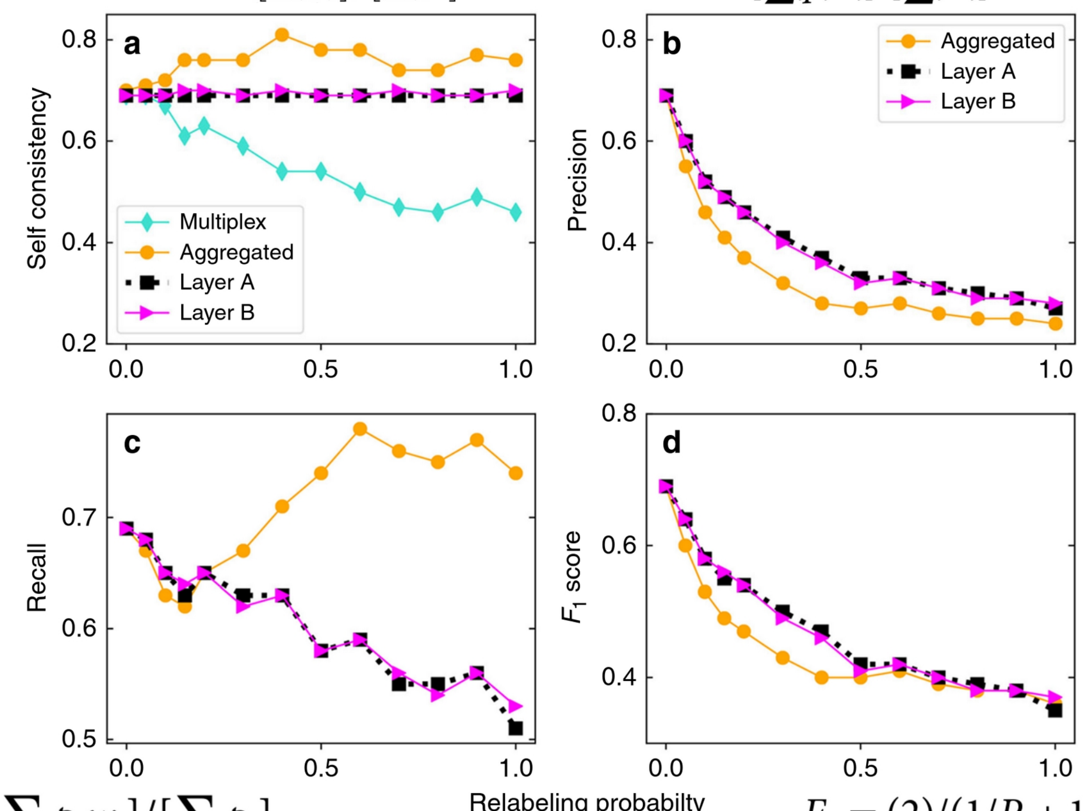
S. Osat, A. Faqeeh and F. Radicchi

Nature Communications 8 1540



$S = [\sum_i p_i^2]/[\sum_i p_i]$

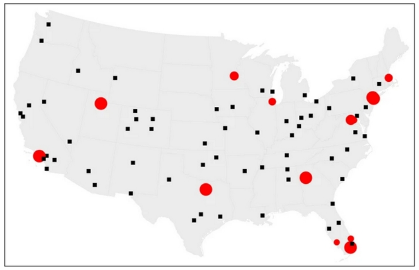
$P = [\sum_i p_i w_i]/[\sum_i w_i]$



$R = [\sum_i p_i w_i]/[\sum_i p_i]$

$F_1 = (2)/(1/P + 1/R)$

Multiplex network of US domestic flights operated in January 2014 by American Airlines and Delta

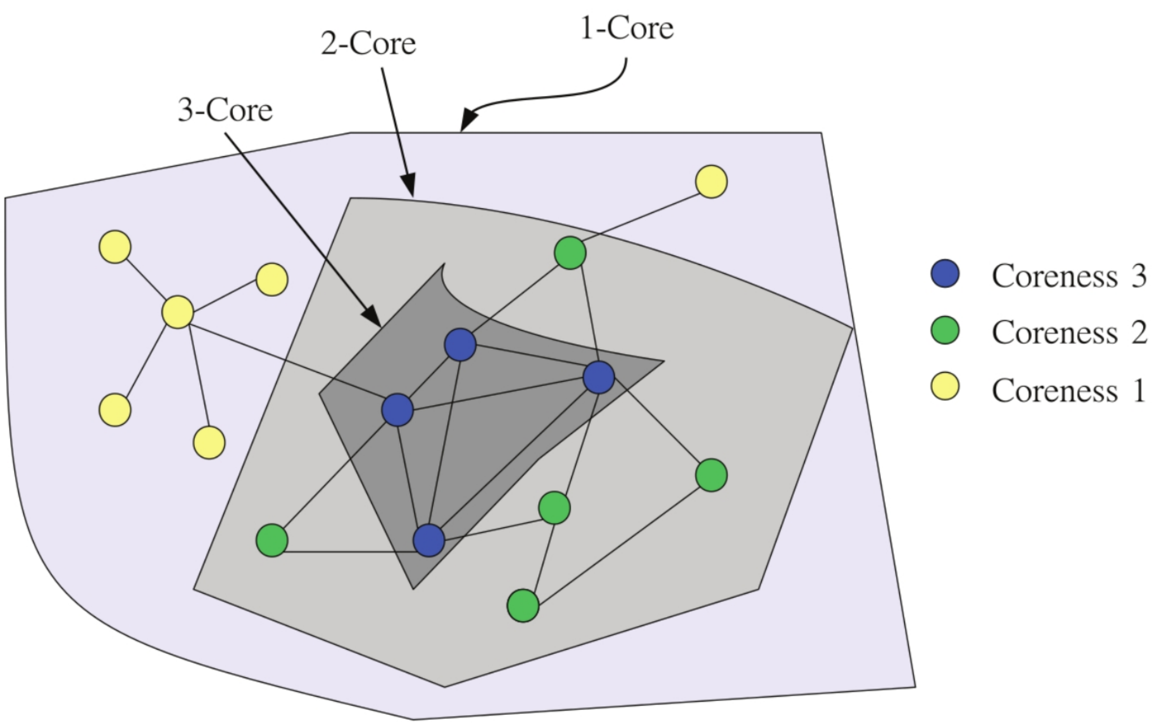


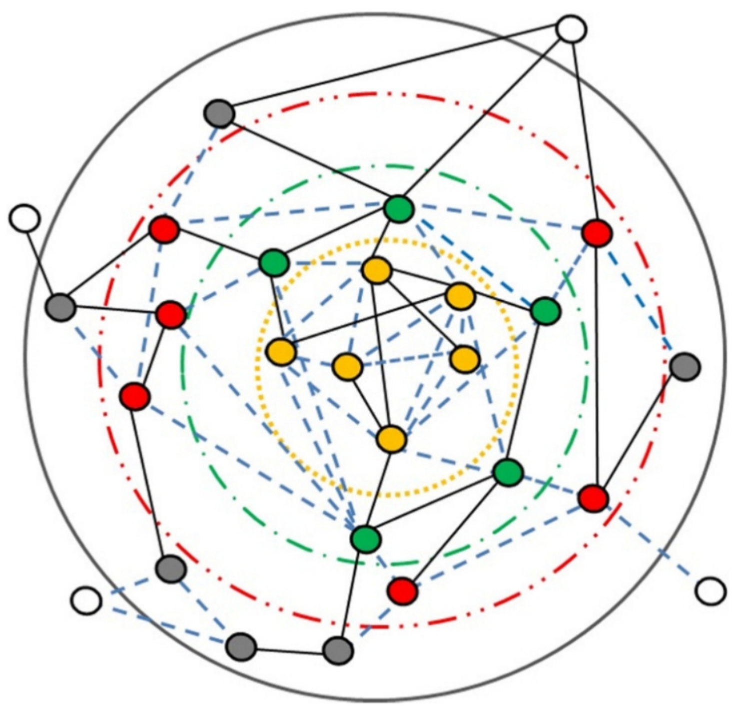
Result:
k-core structure of real multiplex networks

S. Osat, F. Radicchi, and F. Papadopoulos

Phys. Rev. Research 2, 023176

Challenge: How to characterize k-core structure of real networks?





Azimi-Tafreshi, et al.
Physical Review E 90.3 (2014): 032816

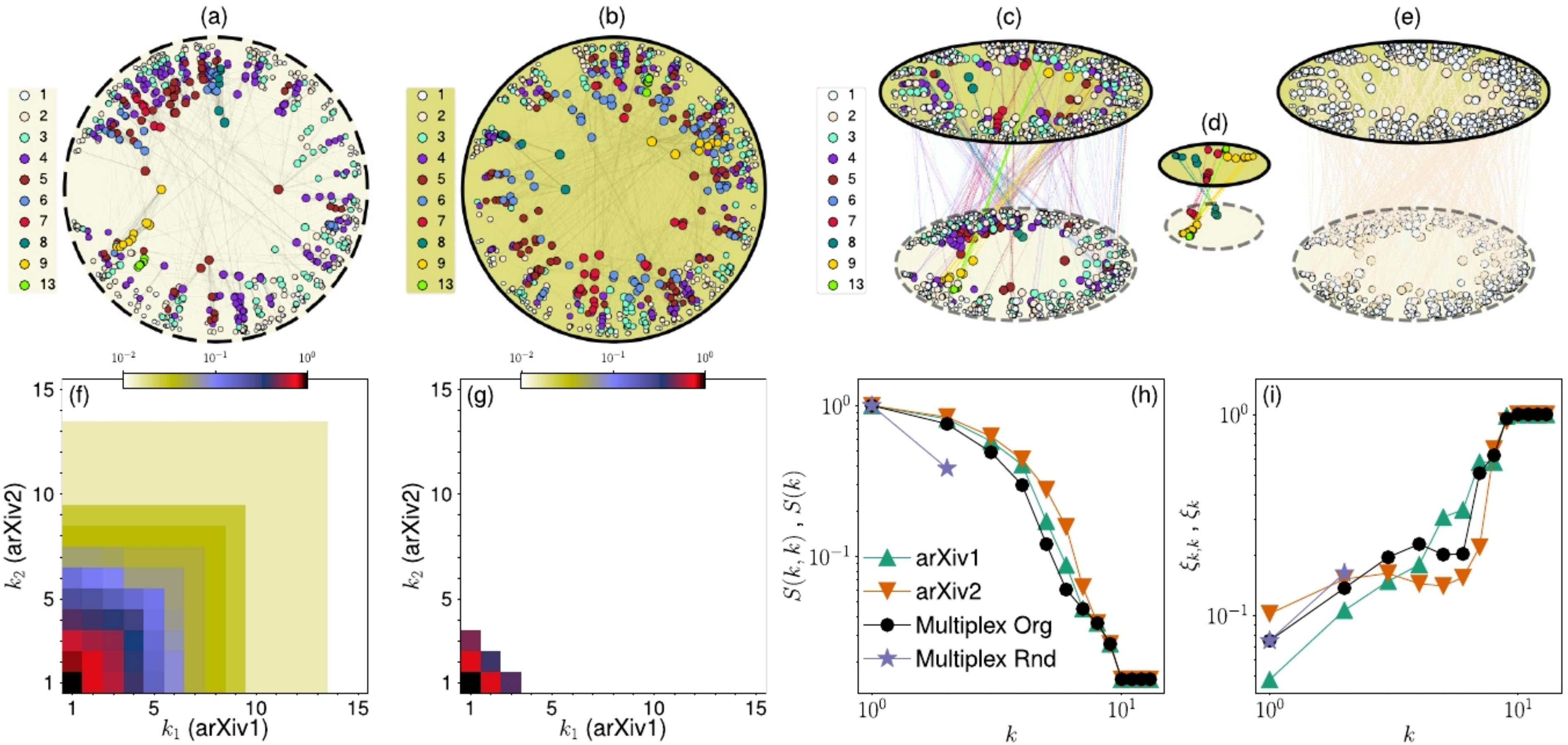
Result:
k-core structure of real multiplex networks

S. Osat, F. Radicchi, and F. Papadopoulos

Phys. Rev. Research 2, 023176

$$\xi_k e^{i\phi_k} = \frac{1}{N_k} \sum_{j \in k\text{-core}} e^{i\theta_j}$$

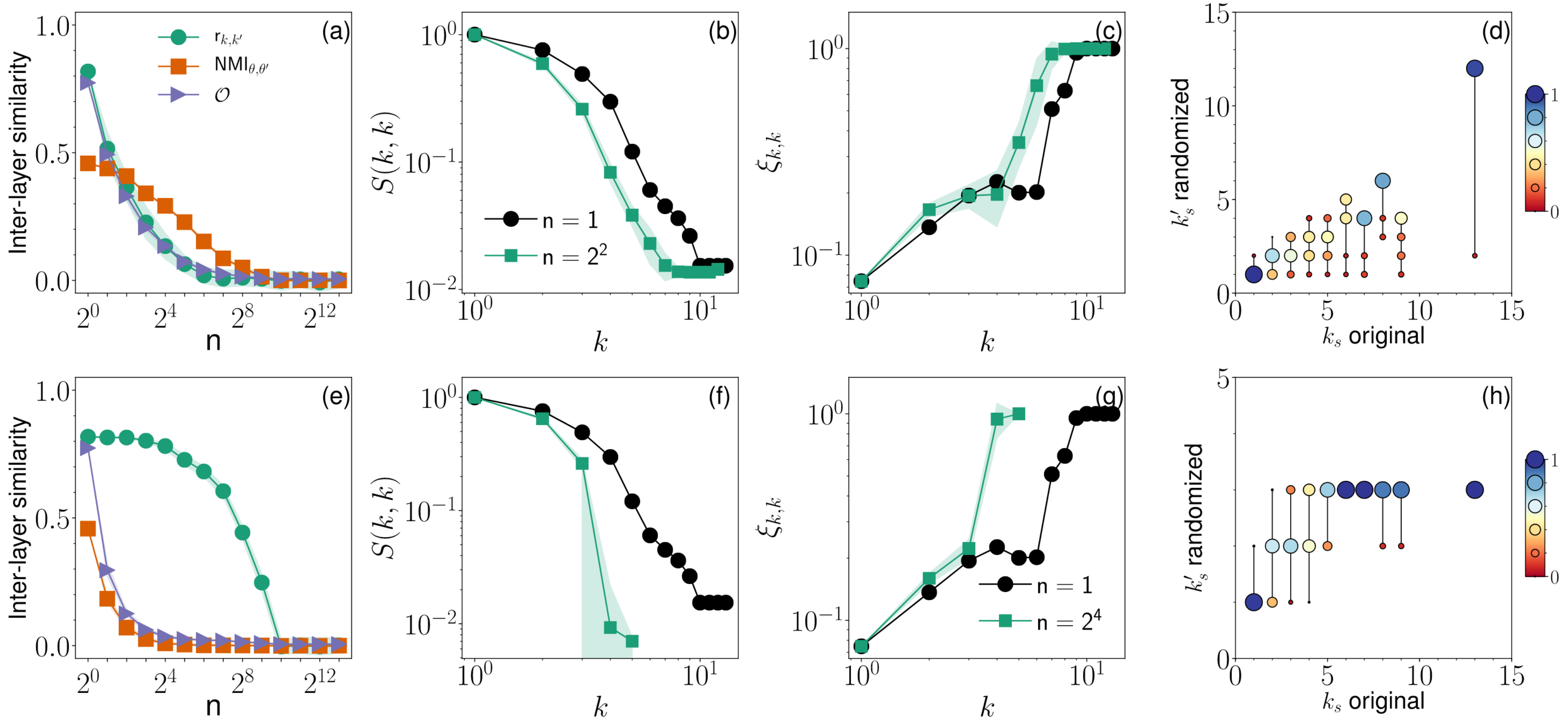
$$\xi_{k,k} e^{i\phi_{k,k}} = \frac{1}{2} \sum_{\ell=1}^{2} \left(\frac{1}{N_{k,k}} \sum_{j \in (k,k)\text{-core}} e^{i\theta_j^\ell} \right)$$



Result:
k-core structure of real multiplex networks

S. Osat, F. Radicchi, and F. Papadopoulos

Phys. Rev. Research 2, 023176



Conclusions

- **One should keep the multiplexity of the system when considering the optimal percolation problem.**

- **Hyperbolic embedding and community structure are two different but equivalent descriptions of structure of networks.**

- **Earlier it was believed that the power-laws are indicators of criticality. We show that the topology of a network can induce a power-law without system being at critical state.**

- **Depending on the degree distribution, different inter-layer correlations change the k-core structure of multiplex networks.**

Publications

- **Emergence of power laws in noncritical neuronal systems**
  A. Faqeeh, S. Osat, F. Radicchi, and J.P. Gleeson
  Phys. Rev. E 100, 010401
- **Generalization of core percolation on complex networks**
  N. Azimi-Tafreshi, S. Osat, and S. N. Dorogovtsev
  Phys. Rev. E 99, 022312
- Observability transition in multiplex networks
  S. Osat, F. Radicchi
  Physica A, 503, 745-761
- **Characterizing the analogy between hyperbolic embedding and community structure of complex networks**
  A. Faqeeh, S. Osat, and F. Radicchi
  Phys. Rev. Lett. 121, 098301
- **Optimal percolation on multiplex networks**
  S. Osat, A. Faqeeh and F. Radicchi
  Nature Communications 8 1540
- **k-core structure of real multiplex networks**
  S. Osat, F. Radicchi, and F. Papadopoulos
  Phys. Rev. Research 2, 023176
- **Deep Learning Super-Diffusion in Multiplex Networks (collaboration out of thesis)**
  V.M Leli, S. Osat, T. Tlyachev, D. Dylov, J. Biamonte,
  Journal of Physics: Complexity, 2, 035011, (2021)

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- Jury Members
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- SKOLTECH → Dormitory Team
- Friends: Bakhodor, Mohammad and Vito

Thank you for your attention

I am happy to answer your questions

Appendix

Rate Equation approach for Gk-core

$$
\begin{aligned}
N(q, t+\Delta t) - N(q,t) &= \dot{P}(q,t) \\
&= -\frac{\theta(k-q)P(q,t)}{\sum_q \theta(k-q)P(q,t)} + \delta_{q,0}\left[1 + \frac{\sum_q q\theta(k-q)P(q,t)}{\sum_q \theta(k-q)P(q,t)}\right] - \frac{\sum_q q\theta(k-q)P(q,t)}{\sum_q \theta(k-q)P(q,t)} \frac{qP(q,t)}{\langle q \rangle_t} \\
&\quad + \frac{\sum_q q\theta(k-q)P(q,t)}{\sum_q \theta(k-q)P(q,t)} \frac{\sum_q q(q-1)P(q,t)}{\sum_q qP(q,t)} \left[\frac{(q+1)P(q+1,t) - qP(q,t)}{\langle q \rangle_t}\right].
\end{aligned}
$$

$$
\frac{\dot{L}(t)}{N} = -\frac{\langle q^2 \rangle_t}{\langle q \rangle_t} \frac{\sum_q q\theta(k-q)P(q,t)}{\sum_q \theta(k-q)P(q,t)}
$$

$$
N_{kc} = N[1 - P(0, t_k^*)],
$$

$$
L_{kc} = L(t_k^*).
$$

Appendix

Observability transition in real multiplex networks

1. $u_{j\to i}^{[\alpha]}$ is the probability that node j is in the LMOC, irrespective of whether node i is in the LMOC or not, given that node j is directly observable.
2. $v_{j\to i}^{[\alpha]}$ is the probability that node j is in the LMOC, irrespective of whether node i is in the LMOC or not, given that node j is not directly observable.
3. $z_{j\to i}^{[\alpha]}$ is the probability that node j is in the LMOC, irrespective of whether node i is in the LMOC or not, given that neither j nor i are directly observable.

$$P_\infty = \frac{1}{N}\sum_i [q_i + r_i]$$

$$q_i = \phi\, q_i^{[\alpha]} q_i^{[\beta]}$$

$$r_i = (1-\phi)\, r_i^{[\alpha]} r_i^{[\beta]}$$

$$q_i^{[\alpha]} = \left[1 - \prod_{j\in\partial_i^{[\alpha]}} (1 - \phi u_{j\to i}^{[\alpha]} - (1-\phi) v_{j\to i}^{[\alpha]})\right]$$

$$r_i^{[\alpha]} = \left[1 - \prod_{j\in\partial_i^{[\alpha]}} (1 - \phi u_{j\to i}^{[\alpha]} - (1-\phi) z_{j\to i}^{[\alpha]}) + \right.$$
$$\left. - (1-\phi)^{k_i^{[\alpha]}} \left[1 - \prod_{j\in\partial_i^{[\alpha]}} (1 - z_{j\to i}^{[\alpha]})\right]\right].$$

$$u_{j\to i}^{[\alpha]} = \left[1 - \prod_{k\in\partial_j^{[\alpha]}\setminus i} (1 - \phi u_{k\to j}^{[\alpha]} - (1-\phi) v_{k\to j}^{[\alpha]})\right] \times$$
$$\left[1 - \prod_{k\in\partial_j^{[\beta]}} (1 - \phi u_{k\to j}^{[\beta]} - (1-\phi) v_{k\to j}^{[\beta]})\right],$$

$$v_{j\to i}^{[\alpha]} = \left[1 - \prod_{k\in\partial_j^{[\alpha]}\setminus i} (1 - \phi u_{k\to j}^{[\alpha]} - (1-\phi) z_{k\to j}^{[\alpha]})\right] \times$$
$$\left[1 - \prod_{k\in\partial_j^{[\beta]}} (1 - \phi u_{k\to j}^{[\beta]} - (1-\phi) z_{k\to j}^{[\beta]}) + \right.$$
$$\left. - (1-\phi)^{k_j^{[\beta]}} \left[1 - \prod_{k\in\partial_j^{[\beta]}} (1 - z_{k\to j}^{[\beta]})\right]\right],$$

$$z_{j\to i}^{[\alpha]} = \left[1 - \prod_{k\in\partial_j^{[\alpha]}\setminus i} (1 - \phi u_{k\to j}^{[\alpha]} - (1-\phi) z_{k\to j}^{[\alpha]}) + \right.$$
$$\left. - (1-\phi)^{k_j^{[\alpha]}-1} \left[1 - \prod_{k\in\partial_j^{[\alpha]}\setminus i} (1 - z_{k\to j}^{[\alpha]})\right]\right] \times$$
$$\left[1 - \prod_{k\in\partial_j^{[\beta]}} (1 - \phi u_{k\to j}^{[\beta]} - (1-\phi) z_{k\to j}^{[\beta]}) + \right.$$
$$\left. - (1-\phi)^{k_j^{[\beta]}} \left[1 - \prod_{k\in\partial_j^{[\beta]}} (1 - z_{k\to j}^{[\beta]})\right]\right],$$

Appendix

Why hyperbolic space?!

Property	Euclidean	Spherical	Hyperbolic
Curvature K	0	>0	<0
Parallel lines	1	0	∞
Triangles are	Normal	Thick	Thin
Shape of triangles			
Sum of angles in triangles	π	$>\pi$	$<\pi$
Circle length	$2\pi r$	$2\pi \sin \zeta r$	$2\pi \sinh \zeta r$
Disk area	$2\pi r^2/2$	$2\pi(1-\cos \zeta r)$	$\boxed{2\pi(\cosh \zeta r-1)}$

We need exponentially growing space to fit the network in!

We assume that hyperbolic geometry underlies these networks, and we show that with this assumption, heterogeneous degree distributions and strong clustering in complex networks emerge naturally as simple reflections of the negative curvature and metric property of the underlying hyperbolic geometry. Conversely, we show that if a network has some metric structure, and if the network degree distribution is heterogeneous, then the network has an effective hyperbolic geometry underneath.

PHYSICAL REVIEW E 82, 036106 (2010)

Appendix

Hyperbolic Networks

$\mathbb{S}^1$ model

Each node i in the $\mathbb{S}^1$ model has hidden variables κ_i, θ_i. The hidden variable κ_i is the node expected degree in the resulting network, while θ_i is the angular (similarity) coordinate of the node on a circle of radius $R = N/(2\pi)$, where N is the total number of nodes. To construct a network with the $\mathbb{S}^1$ model that has size N, average node degree $\bar{k}$, power law degree distribution with exponent $\gamma > 2$, and temperature $T \in [0, 1)$, we perform the following steps.

(i) Sample the angular coordinates of nodes θ_i, $i = 1, 2, \ldots, N$, uniformly at random from $[0, 2\pi]$, and their hidden variables κ_i, $i = 1, 2, \ldots, N$, from the probability density function

$$\rho(\kappa) = (\gamma - 1)\kappa_0^{\gamma-1}\kappa^{-\gamma},$$

where $\kappa_0 = \bar{k}(\gamma - 2)/(\gamma - 1)$ is the expected minimum node degree.

(ii) Connect every pair of nodes i, j with probability

$$p(\chi_{ij}) = \frac{1}{1 + \chi_{ij}^{1/T}},$$

where $\chi_{ij} = R\Delta\theta_{ij}/(\mu\kappa_i\kappa_j)$ is the effective distance between i and j, $\Delta\theta_{ij} = \pi - |\pi - |\theta_i - \theta_j||$ is the angular distance, and $\mu = \sin T\pi/(2\bar{k}T\pi)$ is derived from the condition that the expected degree in the network is indeed $\bar{k}$.

$\mathbb{H}^2$ model

The $\mathbb{S}^1$ model is isomorphic to hyperbolic geometric graphs ($\mathbb{H}^2$ model) after transforming the expected node degrees κ_i to radial coordinates r_i via

$$r_i = R_{\mathrm{H}} - 2\ln\frac{\kappa_i}{\kappa_0},$$

where R_{H} is the radius of the hyperbolic disk where all nodes reside,

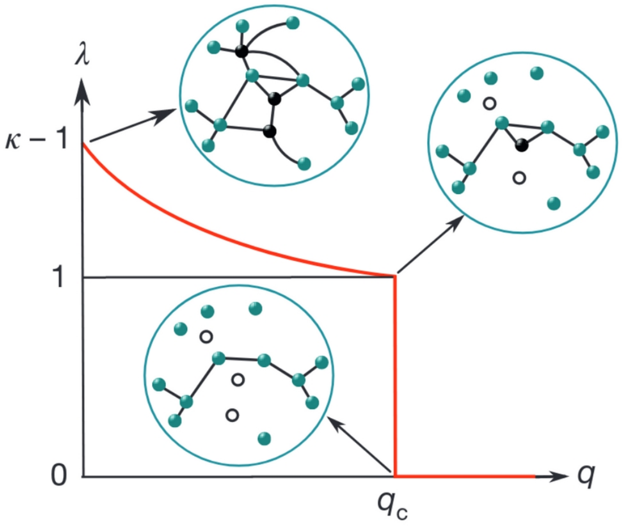
$$R_{\mathrm{H}} = 2\ln\frac{N}{c} \qquad c = \bar{k}\frac{\sin T\pi}{2T}\left(\frac{\gamma-2}{\gamma-1}\right)^2$$

$$p(x_{ij}) = \frac{1}{1 + e^{\frac{1}{2T}(x_{ij} - R_{\mathrm{H}})}}$$

$$x_{ij} = r_i + r_j + 2\ln(\Delta\theta_{ij}/2)$$

Appendix

Hamiltonian of dismantling



$\hat{\mathcal{M}} : |\boldsymbol{w}_\ell(\boldsymbol{n})| = \langle \boldsymbol{w}_\ell | \boldsymbol{w}_\ell \rangle^{\frac{1}{2}} = |\hat{\mathcal{M}}^\ell \boldsymbol{w}_0| = \left\langle \boldsymbol{w}_0 \middle| (\hat{\mathcal{M}}^\ell)^\dagger \hat{\mathcal{M}}^\ell \middle| \boldsymbol{w}_0 \right\rangle^{\frac{1}{2}} \sim e^{\ell \log \lambda(\boldsymbol{n})}$.

The largest eigenvalue is then calculated by the power method:

$$\lambda(\boldsymbol{n}) = \lim_{\ell \to \infty} \left[\frac{|\boldsymbol{w}_\ell(\boldsymbol{n})|}{|\boldsymbol{w}_0|} \right]^{1/\ell}$$

$$\min_{\mathbf{n}:\langle n \rangle = 1 - q_c} \lambda_0'(\mathbf{n}) = 1$$

$$\mathrm{CI}_\ell(i) = (k_i - 1) \sum_{j \in \partial \mathrm{Ball}(i,\ell)} (k_j - 1)$$

Appendix

von Neumann entropy of a network

density matrix $\boldsymbol{\rho}$

$$\boldsymbol{\rho} = \frac{e^{-\tau \mathbf{L}}}{Z}, \qquad Z = \mathrm{Tr} e^{-\tau \mathbf{L}}$$

$$\mathbf{L} = \mathbf{D} - \mathbf{A}$$

Hermitian and positive semidefinite matrix

$$Z = \sum_{i=1}^{N} e^{-\beta \lambda_i(\mathbf{L})}$$

spectral decomposition $\boldsymbol{\rho} = \sum_{i=1}^{N} \lambda_i |\phi_i\rangle\langle\phi_i|$

$$S(\boldsymbol{\rho}) = -\sum_{i=1}^{N} \lambda_i(\boldsymbol{\rho}) \log_2 \lambda_i(\boldsymbol{\rho})$$

$$\lambda_i(\boldsymbol{\rho}) = Z^{-1} e^{-\beta \lambda_i(\mathbf{L})}$$

$$S(\boldsymbol{\rho}) = -\mathrm{Tr}(\boldsymbol{\rho} \log_2 \boldsymbol{\rho}) = -\sum_{i=1}^{N} \lambda_i \log_2 \lambda_i$$

SUBADDITIVITY

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

$$S_A + S_B \geq S_C$$

PHYSICAL REVIEW X 6, 041062 (2016)