Scientific Computing Lecture 4

Integral Transforms and Integral Equations

Nikolay Koshev & Maxim Fedorov

October 11, 2021

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Structure of the lecture

Part 1. Convolution

Part 2. Fourier Transform

Part 3. Radon Transform

Part 4. Wavelet Transform

Part 5. Integral equations classification

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Part 1: Convolution

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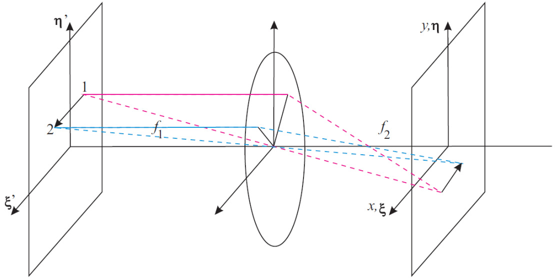
The distorted images reconstruction

- Smeared images
- Defocused images
- Electronic microscopy
- The convolution
- The equation

Smeared images



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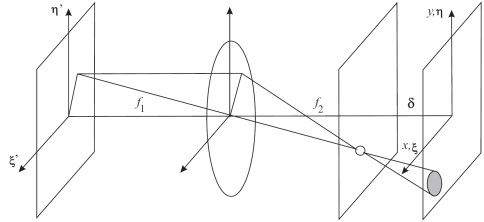
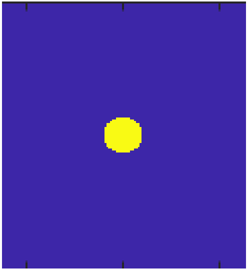
Defocused images





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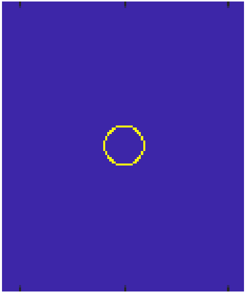

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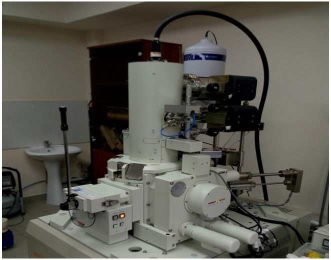
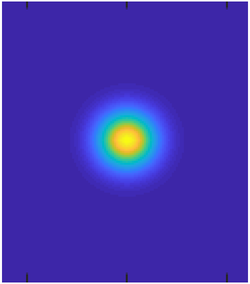
Defocused images





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The convolution

- Let the functions $k(\mathbf{x})$ and $f(\mathbf{x})$ are both defined for $\mathbf{x} \in \mathbb{R}^n$.
- The function $g(\mathbf{x})$ is called convolution of k and f if:

$$g(\mathbf{x}) = k * f \equiv \int\limits_{\mathbb{R}^n} k(\mathbf{x} - \xi) f(\xi) d\xi.$$

- The function $k(\mathbf{x})$ is the convolution kernel.
- The convolution means than each value $f(\mathbf{x})$ becomes some spot defined with $k(\mathbf{x})$. Thus, the function $k(\mathbf{x})$ is also sometimes called the **Point spread function**
- In the problems of image processing, the kernel k is often being defined by hardware function.
- If $\hat{.}$ denotes the Fourier transform of ., then:

$$k \hat{*} f = \hat{k} \hat{f}.$$

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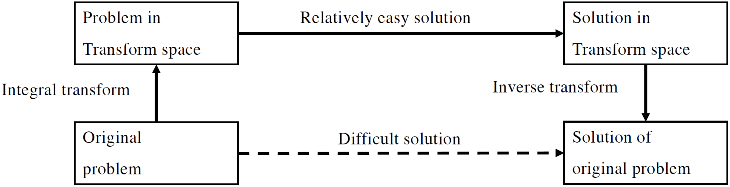
Integral transforms

$$g(\mathbf{x}) = \int_{\Omega} K(\mathbf{x}, \xi) f(\xi) d\xi, \quad \mathbf{x}, \xi \in \Omega \subset \mathbb{R}^n$$

- $K(\mathbf{x}, \xi)$ - the Kernel;
- Mapping a function $f(\mathbf{x})$ in $\mathbf{x}$-space into another function $g(\xi)$ in ξ-space
- Fourier, Wavelet, Z-transform, Laplace, Hilbert, Radon, etc



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Part 2: Fourier Transform

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Fourier Transform

1768-1830

- Each function defined in spatial or time domain (**real** domain) by some function $h(t)$, can be presented also in **frequency** or **reciprocal** domain with a function $H(f)$ of frequency f.
- One goes back and forth between these two representations by means of the Fourier transform,

$$H(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt; \quad h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(f) e^{2\pi i f t} df;$$

Or, using angular frequency $\omega = 2\pi f$:

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt; \quad h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega$$

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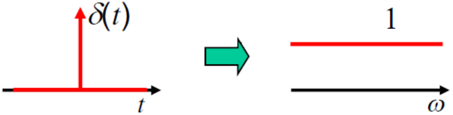
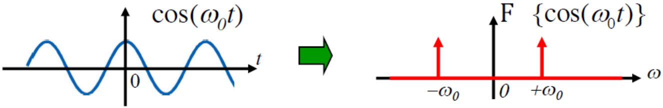
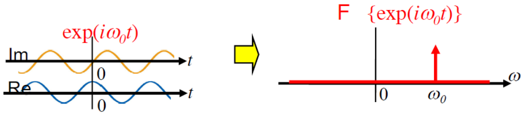
- FT of Dirac δ-function: $\int\limits_{-\infty}^{\infty} \delta(t) e^{-2\pi i f t} dt = 1$:



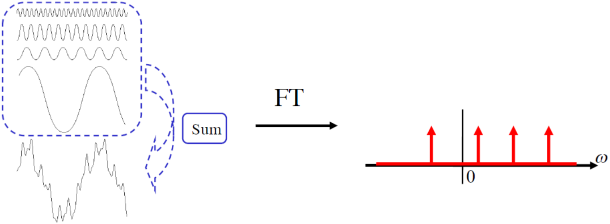
- FT of $cos(\omega_0 t)$:



- FT of $e^{\pi i \omega_0 t} = \sqrt{2\pi}\delta(\omega - \omega_0)$:



- The idea of FT: periodic functions could be represented as a weighted sum of wines and cosines:



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Let $f(\mathbf{x}) \in L_2(\mathbb{R}^n), \quad \mathbf{x} \in \mathbb{R}^n, n \geq 2.$ Then

$$F(\omega) = \frac{1}{(\sqrt{2\pi})^n} \int\limits_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\omega\cdot\mathbf{x}} d\mathbf{x}, \quad \omega \in \mathbb{R}^n$$

$$f(\mathbf{x}) = \frac{1}{(\sqrt{2\pi})^n} \int\limits_{\mathbb{R}^n} F(\omega) e^{i\omega\cdot\mathbf{x}} d\omega.$$

Fourier Transform: properties

- Correspondence between symmetries in the two domains:

$$h(t)) \text{ is real} \Rightarrow H(-f) = H^*(f);$$
$$h(t)) \text{ is imaginary} \Rightarrow H(-f) = -H^*(f);$$
$$h(t) \text{ is even} \Rightarrow H(-f) = H(f);$$
$$h(t) \text{ is odd} \Rightarrow H(-f) = -H(f);$$
$$h(t) \text{ is real and even} \Rightarrow H(-f) \text{ is real and even};$$
$$h(t) \text{ is real and odd} \Rightarrow H(-f) \text{ is imaginary and odd};$$
$$h(t) \text{ is imaginary and even} \Rightarrow H(-f) \text{ is imaginary and even};$$
$$h(t) \text{ is imaginary and odd} \Rightarrow H(-f) \text{ is real and odd};$$

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Fourier Transform: properties

- Scaling and shifting:

$$h(t) \Leftrightarrow H(f)\text{: Fourier Transform}$$
$$h(at) \Leftrightarrow \frac{1}{|a|}H(\tfrac{f}{a})\text{: time scaling}$$
$$\frac{1}{|b|}h(\tfrac{t}{b}) \Leftrightarrow H(bf)\text{: frequency scaling}$$
$$h(t-t_0) \Leftrightarrow H(f)e^{2\pi i f t_0}\text{: time shifting}$$
$$h(t)e^{-2\pi i f_0 t} \Leftrightarrow H(f-f_0)\text{: frequency shifting}$$

- Convolution theorem:

$$g * h \equiv \int_{-\infty}^{\infty} g(x)h(x-\xi)d\xi \Leftrightarrow G(f)H(f). \tag{1}$$

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Cross-correlation of two functions:

$$g \star h = \int_{-\infty}^{\infty} g(x+\xi)h(\xi)d\xi$$

- Correlation theorem; Autocorrelation, Wiener-Khinchin theorem::

$$g \star h \Leftrightarrow G(f)H^*(f), \quad \Rightarrow \quad g \star g = |G(f)|^2$$

- Parseval's theorem:

$$||h(t)||_{L^2} = ||H(f)||_{L^2} \Rightarrow \quad \int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |H(f)|^2 df$$

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- Derivatives:

$$\mathcal{F}\left[h'(t)\right] = -i\omega H(f); \quad \mathcal{F}\left[h^{(n)}\right] = (-i\omega)^n H(f).$$

- Application: D'Alambert equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{\nu^2}\frac{\partial^2 y(x,t)}{\partial t^2}, \quad y(x,0) = f(x)$$

- After application of FT to both sides of this equation:

$$(-i\omega)^2 Y(\omega,t) = \frac{1}{\nu^2}\frac{\partial^2 Y(\omega,t)}{\partial t^2}$$
$$F(\omega) = Y(\omega,0) = \mathcal{F}\left[f(x)\right]$$

Sampling theorem

- Suppose function $h(t)$ is sampled at evenly spaced intervals in time: $h_n = h(n\Delta), n = ..., -2, -1, 0, 1, 2, ...,$ $\frac{1}{\Delta}$ - sampling rate
- For any sampling interval Δ, there is a special frequency f_c, called **Nyquist critical frequency**, given by $f_c = 1/2\Delta$
  - ex: critical sampling of a sine wave of Nyquist critical frequency is two sample points per cycle.
- A function f is "bandwidth limited" if its Fourier transform is 0 outside of a finite interval $[-L, L]$;
- Sampling Theorem: If a continuous function $h(t)$, sampled at an interval Δ, is bandwidth limited to frequency smaller than f_c, i.e., $H(f) = 0$ for all $|f| > fc$, then the function $h(t)$ is completely determined by its samples h_n.

$$h(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{sin\big(2\pi f_c(t - n\Delta)\big)}{\pi(t - n\Delta)}.$$

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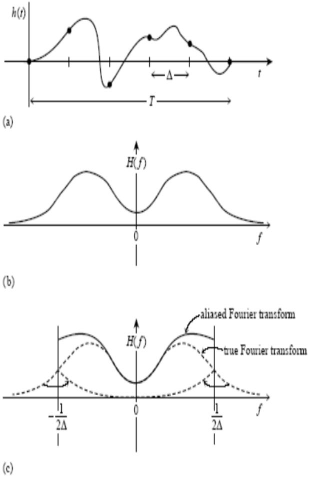
- For bandwidth limited signals, such as e.g. music in concert hall, sampling theorem tells us that the entire information content of the signal can be recorded by sampling rate Δ^{-1} equal to twice the maximum frequency pass by the amplifier
- For the function that is not bandwidth limited to less then the Nyquist critical frequency, frequency component that lies outside of the frequency range, $-f_c < f < f_c$ is spuriously moved into that range (aliasing).



Discrete Fourier Transform (DFT)

- Suppose we have N consecutive sampled values
$$h_k \equiv h(t_k), \quad t_k = \Delta k, \quad k = 0, 1, 2, ..., N-1,$$
where Δ is the sampling interval, and assume N is even.
- With N numbers of input, we can produce no more than N independent number of outputs. Therefore, we seek estimates only at the discrete values;
$$f_n \equiv \frac{n}{N\Delta}, \quad n = -\frac{N}{2}, ..., \frac{N}{2}$$
- Then, discrete Fourier Transform (DFT) is
$$H(f_n) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f_n t} dt \approx \sum_{k=0}^{N-1} h_k e^{2\pi i f_n t_k} \Delta = \Delta \sum_{k=0}^{N-1} h_k e^{2\pi i k n} \equiv \Delta H_n$$
- DFT maps N complex numbers (the h_k's) into N complex numbers (the H_n's).
- Trapezoidal approximation of the integral

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Discrete Fourier Transform (DFT)

$$H_{N-n} = \sum_{k=0}^{N-1} h_k e^{2\pi k(N-n)/N} = \sum_{k=0}^{N-1} h_k e^{2\pi k(-n)/N} e^{2\pi ikN/N} = H_{-n}$$

- **It's periodic in** n, with period N: $H_{-n} = H_{N-n}$.
- With this conversion, one lets the n in Hn vary from 0 to N-1. Then n and k vary exactly over the same range.
- With this convention
  - zero frequency: $n = 0$
  - positive frequencies, $0 < f < fc \Rightarrow 1 \leq n \leq N/2 - 1$
  - negative frequencies, $-f_c < f < 0 \Rightarrow N/2 + 1 \leq n \leq N - 1$
  - the value $n = N/2 \Rightarrow$ both $f = fc$ and $f = -fc$
- The DFT has symmetry properties almost exactly the same as the continuous Fourier transform.

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- The **Discrete Inverse Fourier Transform** (DIFT) is:

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi ikn/N}$$

- Parseval's theorem:

$$\sum_{k=0}^{N-1} |h_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |H_n|^2$$

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Fast Fourier Transform (FFT)

- DFT appears to be an $O(N^2)$ process.
- Danielson and Lanczos; DFT of length N can be rewritten as the sum of two DFT of length $N/2$.

$$H_k = \sum_{j=0}^{N-1} e^{2\pi ijk/N} h_j = \sum_{j=0}^{N/2-1} e^{2\pi i(2j)n/N} h_{2j} + \sum_{j=0}^{N/2-1} e^{2\pi i(2j+1)k/N} =$$

$$\sum_{j=0}^{N/2-1} e^{2\pi ijk/(N/2)} h_{2j} + e^{2\pi ik/N} \sum_{k=0}^{N/2-1} e^{2\pi ijk/(N/2)} h_{2k+1} =$$

$$H_k^0 + e^{2\pi ik/N} H_k^1 = H_k^0 + W^k H_k^1, \quad W \equiv e^{2\pi i/N}. \quad (2)$$

- Such reduction may be applied at H_k^0.
- For $N = 2^R$ the reduction can be applied multiple times until we subdivide the data into the R transforms of length 1.
- For every pattern of $log_2 N$ number of 0's and 12's, there is one-point transformation that is just one of the input number h_n.

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Main Applications of FFT

- Signal & Image Analysis (e.g. finding periodicities, feature extraction, correlation analysis etc)
- Signal & Image Processing (e.g. compression, de-noising etc)
- Convolution operations (mind the the convolution theorem and $O(NlogN)$ scaling)
- Numerical Solution of Integral Equations (discretization in the reciprocal domain)
- Treatment of long-range potential problem in N-body simulations (electrostatic and gravitational interactions)

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Part 3: Radon Transform

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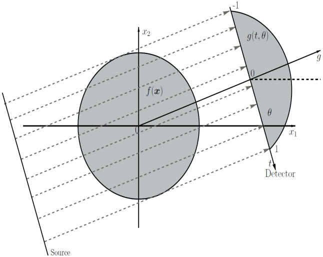
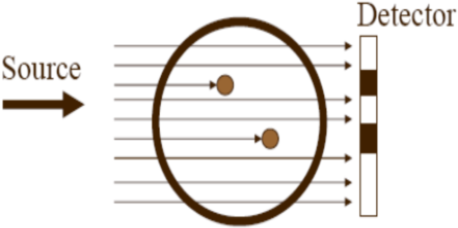
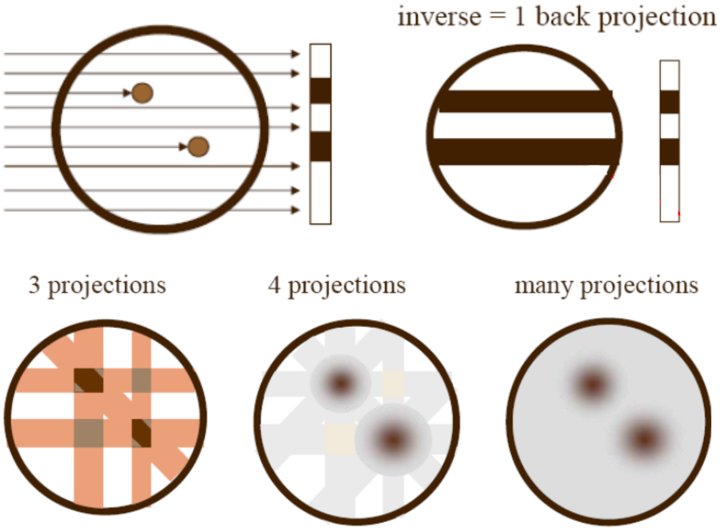




- Measurements: attenuation of X-rays from many different angles
- Possibility of reconstruction the inner structure of the object of study in a series of cross sections or planes
- Combination of X-ray pictures of many planes (cross sections) allows the reconstruction of 3D structures
- The reconstruction also can be considered as a convolution

The X-ray tomography



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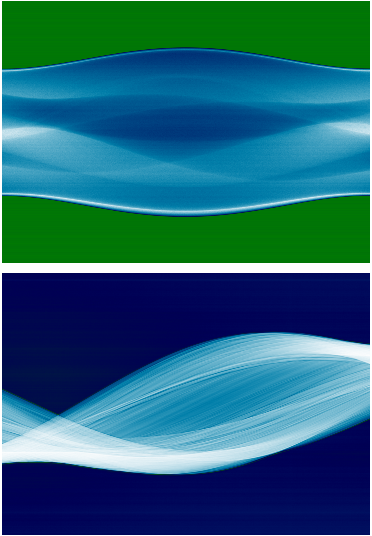
The X-ray tomography

Source



Sinogram

Johann Karl August Radon

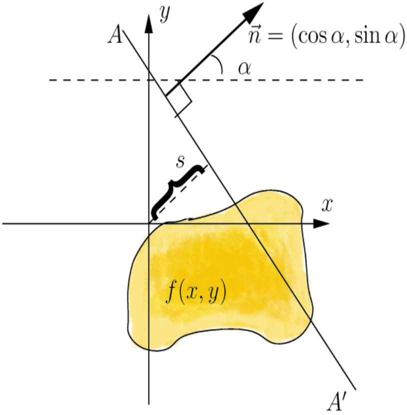
- Born in Decin (Austrian monarchy, now North Bohemia, CZ) in 1887
- Austrian mathematician living in Vienna
- Discover the transform and its inversion in 1917 as pure theoretical result
- No practical applications during his life
- Died in 1956 in Vienna

Actual applications of inverse Radon Transform

- CT - Computer Tomography
- MRI - Magnetic Resonance Imaging
- PET - Positron Emission Tomography
- SPECT - Single Photon Emission Computer Tomography

Johann Karl August Radon

- Input space coordinates x, y
- Input function $f(x, y)$
- Output space coordinates α, s
- Output function $F(\alpha, s)$



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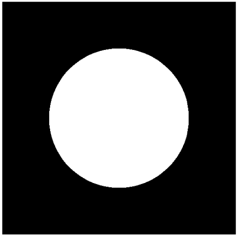
- **Radon Transform**

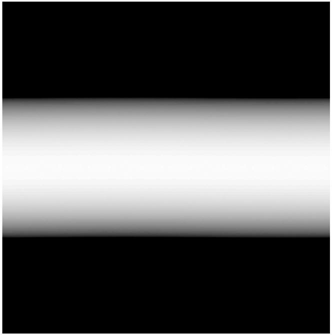
$$F(\alpha, s) = \int_{-\infty}^{\infty} f(t \cdot sin\alpha + s \cdot cos\alpha, -t \cdot cos\alpha + s \cdot sin\alpha) dt$$

- **Inverse Radon Transform**

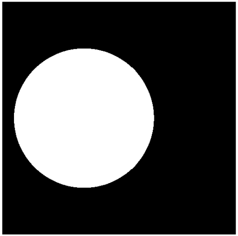
$$f(x, y) = \int_{0}^{2\pi} F(\alpha, x \cdot cos\alpha + y \cdot sin\alpha) d\alpha.$$

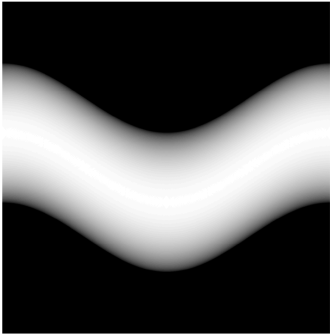
Radon Transform example



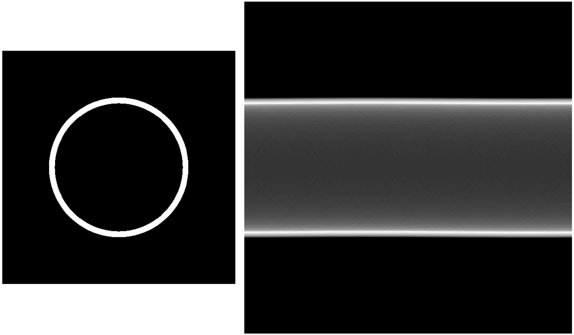


Radon Transform example





Radon Transform example



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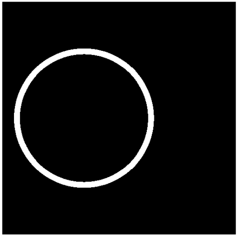
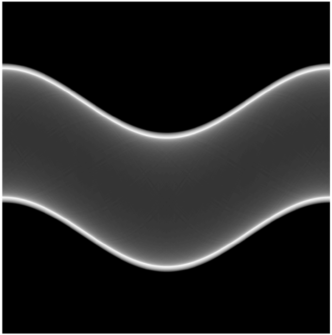
Radon Transform example





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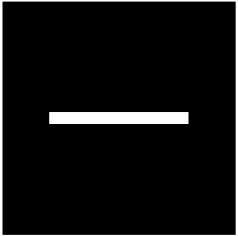

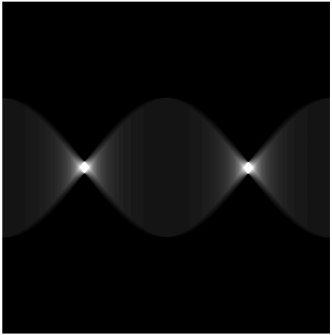

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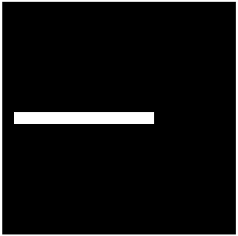
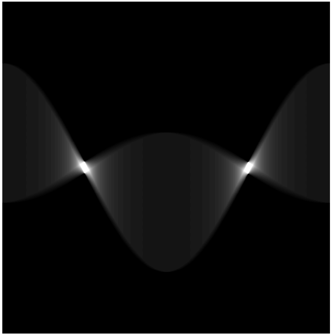
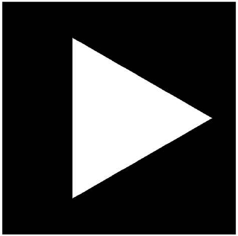
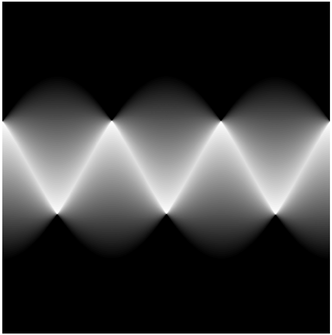
Radon Transform example



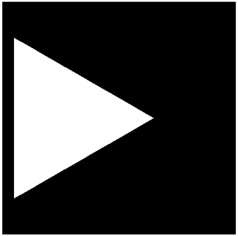


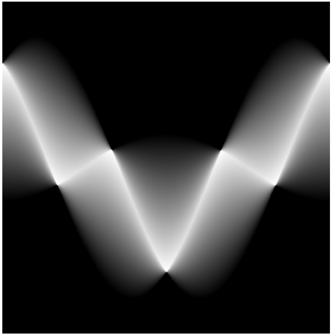
Radon Transform example



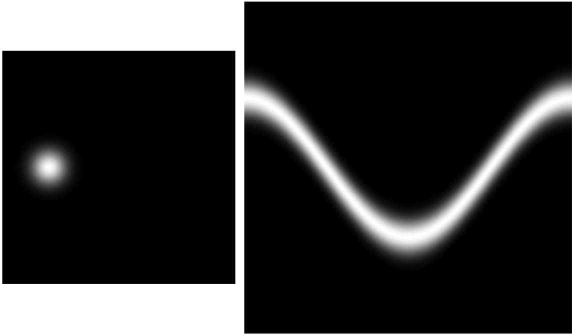


Radon Transform example





Radon Transform example



Natural transform as result of measurement:

- Gamma ray decay from local density map
- Extinction from local concentration map
- Total radioactivity from local concentration map
- Total echo from local nuclei concentration map
- 3D reality is investigated via 2D slices

Artificial realization:

- Noise - RT - noise - IRT simulations
- Image decryption as a fun
- TSR invariant recognition of objects

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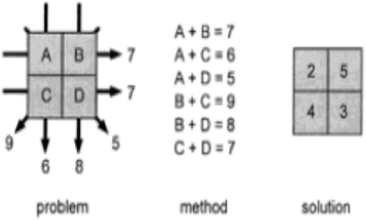
- Image of any $f + g$ is $F + G$
- Image of cf is cF for any real c
- Rotation of f causes translation of F in α
- Scaling of f in (x, y) causes scaling of F in s
- Image of a point (2D Dirac function) is sine wave line
- Image of n points is a set of n sine wave lines

- Consider the basic 2×2 image known only by projection values:

- Given are sums, we have to reconstruct values of pixels A, B, C and D

ART or Algebraic Reconstruction Technique



$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{bmatrix} 7 \\ 6 \\ 5 \\ 9 \\ 8 \\ 7 \end{bmatrix}$$

Over-determined non-square matrix K
Multiply by K^T first
Invert square matrix

$$K\ x = b$$

$$K^T K\ x = K^T b$$

$$x = [K^T K]^{-1} K^T b$$

Larger problems must be solved iteratively using standard methods for solving large matrix operation problems.

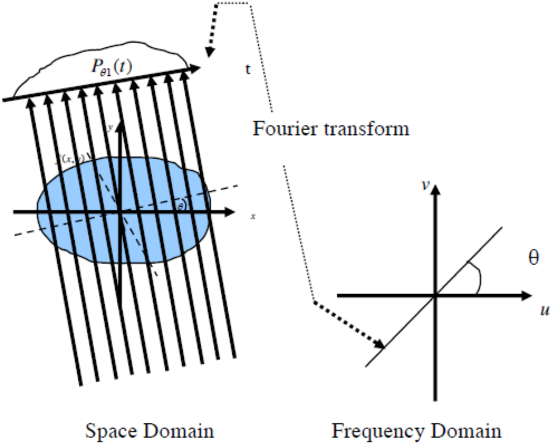
Fourier Slice Theorem

- Denote the projection at angle $\alpha = P_\alpha(s)$
- $S_\alpha(\omega) = \hat{P}_\alpha(\omega)$ - 1D Fourier image of the projection.
- $\hat{f}(\omega, \nu)$ - the Fourier image of the feature (object) function

The Fourier Slice Theorem:

$$\hat{f}(\omega, \nu) = S_{\alpha=\nu}(\omega).$$

The FST maps the 1D Fourier image of the projection at angle α into the radial line with angle α in 2D Fourier image of the feature function



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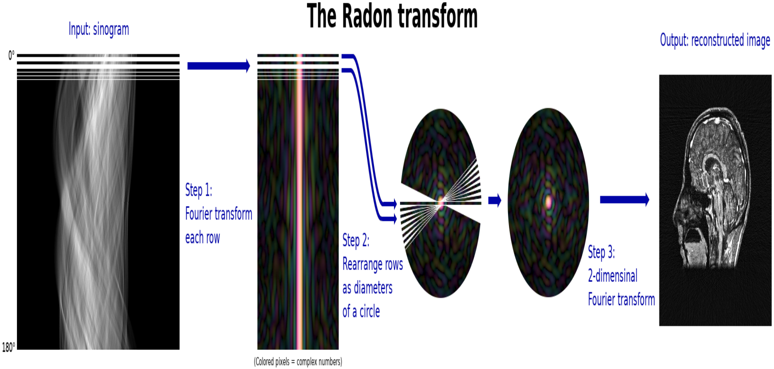

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Let $g = Rf$, and $b = Bg \equiv BRf$. The operator B is called **the Backprojection operator**:

$$\int_{\mathbb{R}_+} Rf(s,\alpha)g(s,\alpha)dsd\alpha = \int_{\mathbb{R}^2} f(x)Bg(x)dx.$$

- The FST-based convolution:

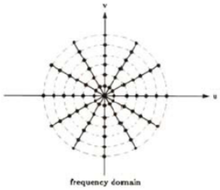
$$b = BRg = g * \left(\frac{1}{|x|}\right)$$

- The Log-Polar backprojection:

$$[Bg]_L(s,\alpha) = [g]_L * [K]_L(s,\alpha), \quad \text{where } [K]_L(s,\alpha) = \delta(1 - e^s cos\alpha).$$

Filtered Backprojection

- We have to account for the non-uniform sampling.
- The ideal filter for this is the pie shaped wedges shown with Fig. (a).
- Unfiltered processing would simply use response (b)
- To emulate the response (a), people have used the filter response (c).
  - Let we have K projections over 180 degrees. Width of the wedge will be $2\pi|\omega|$ (a).
  - The ramp filter in (c) corresponds that width.



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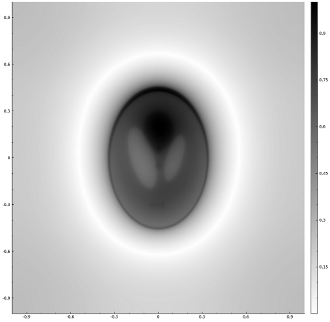
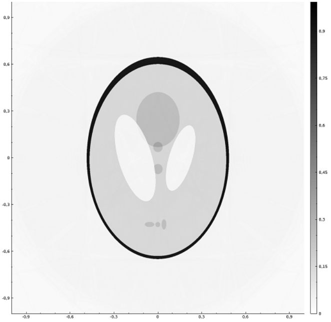
Filtered Backprojection



Unfiltered



Filtered

Radon Transform

- 2D FFT of original
- Resampling to polar coordinates (2D interpolation)
- Line-by-line inverse 1D FFT brings result

Inverse Radon Transform

- Line-by-line 1D FFT of original
- Resampling to rectangular coordinates (2D interpolation)
- 2D LF filtering in frequency domain
- Inverse 2D FFT brings result

Comparison on usage of direct methods and FST

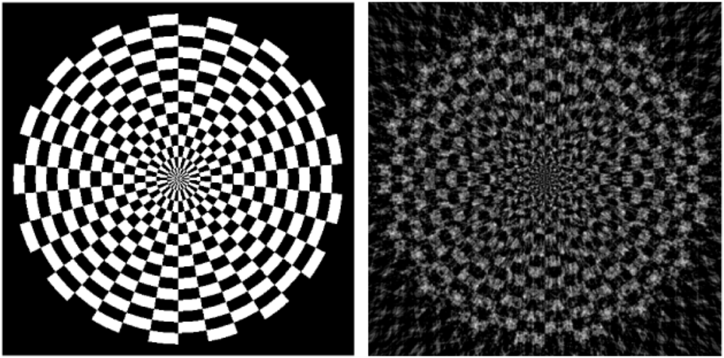
FBP

- Computationally cheap
- Clinically usually 500 projections per slice
- problematic for noisy projections

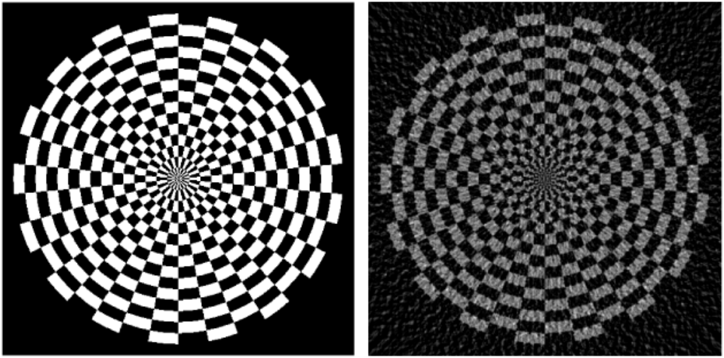
ART

- Still slow
- BEtter quality for fewer projections
- better quality for non-uniform project
- "guided" reconstruct (initial guess!)

Reconstruction from 32 angles



Reconstruction from 64 angles



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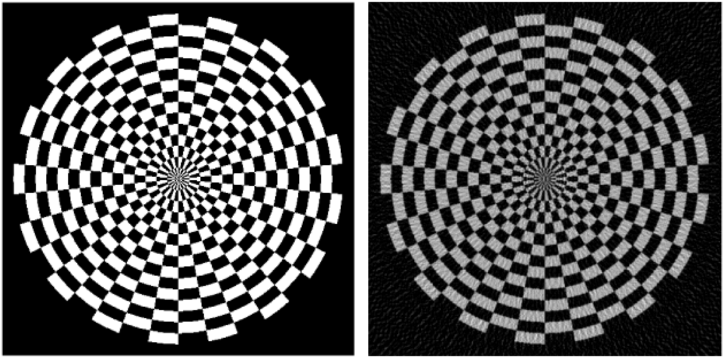
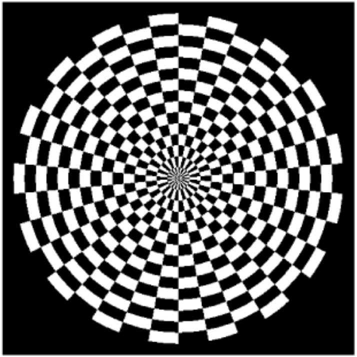
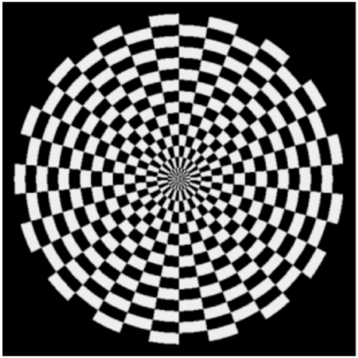
Reconstruction from 128 angles



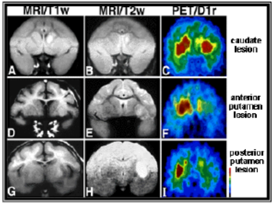
Reconstruction from 512 angles



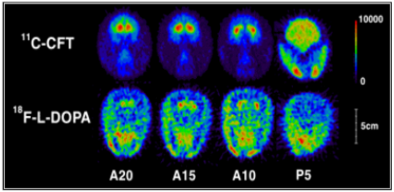


Applications & Types of Tomography

Medical Applications	Type of Tomography
Full body scan	X-ray
Respiratory, digestive systems, brain scanning	PET Positron Emission Tomography
Respiratory, digestive systems.	Radio-isotopes
Mammography	Ultrasound
Whole Body	Magnetic Resonance (MRI, NMR)



MRI and PET showing lesions in the brain.



PET scan on the brain showing **Parkinson's Disease**

Scientific Computing
Lecture 5

Part 4: Wavelet Transform

Nikolay Koshev & Maxim Fedorov

October 11, 2021

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Or: What's wrong with Fourier?

- By using Fourier Transform , we loose the time information : WHEN did a particular event take place ?
- FT can not locate drift, trends, abrupt changes, beginning and ends of events, etc.
- Calculating use complex numbers.

Time and Space definition

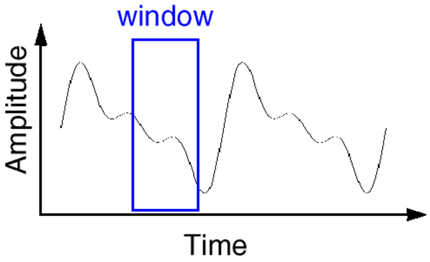
- Time - for one dimension waves we start point shifting from source to end in time scale .
- Space - for image point shifting is two dimensional .
- Here they are synonyms .

Kronneker function

$$\psi_k(t) = \delta_k(t) = \begin{cases} 1, & k == t \\ 0, & k \neq t \end{cases}$$

Can exactly show the time of appearance but have not information about frequency and shape of signal.

In order to analyze small section of a signal, Denis Gabor (1946), developed a technique, based on the FT and using windowing : STFT



STFT or Gabor Transform

- A compromise between time-based and frequency-based views of a signal.
- both time and frequency are represented in limited precision.
- The precision is determined by the size of the window.
- Once you choose a particular size for the time window - it will be the same for all frequencies.

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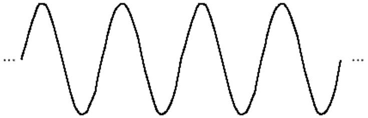
Why wavelets?

What's wrong with Gabor?

- Many signals require a more flexible approach - so we can vary the window size to determine more accurately either time or frequency.

What is Wavelet Analysis?

- And what is a wavelet...?



Sine Wave

Wavelet (db10)

- A wavelet is a waveform of effectively limited duration that has an average value of zero.
- Short time localized waves with zero integral value.
- Possibility of time shifting.
- Flexibility.

For easier understanding, consider the 1D wavelets in a time domain.

- $x(t)$ - time-evolving signal.
- The Mother Wavelet: $\Psi(t)$.
- Daughter Wavelets: $\Psi(\frac{t-b}{a})$ - shifted and scaled versions of the MW.
- Continuous Wavelet Transform (CWT):

$$X(a,b) = \frac{1}{|a|^{1/2}} \int_{\mathbb{R}} dt x(t) \Psi^*\left(\frac{t-b}{a}\right)$$

- Continuous Inverse Wavelet Transform:

$$x(t) = \frac{1}{C_\Psi} \iint_{\mathbb{R}^2} X(a,b) \frac{1}{|a|^{1/2}} \tilde{\Psi}\left(\frac{t-b}{a}\right) db \frac{da}{a^2}$$

- C_Ψ - the admission constant; $\tilde{\Psi}$ - dual wavelet.

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- A mathematical representation of the Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

- Meaning: the sum over all time of the signal $f(t)$ multiplied by a complex exponential, and the result is the Fourier coefficients $F(\omega)$

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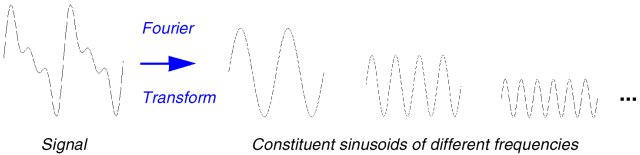
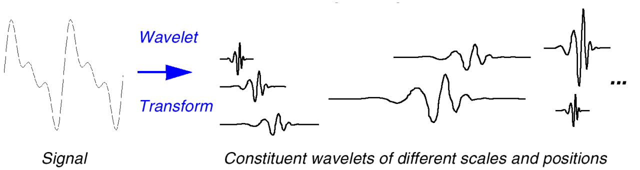
- Those coefficients, when multiplied by a sinusoid of appropriate frequency ω yield the constituent sinusoidal component of the original signal:



- And the result of the CWT are Wavelet coefficients .
- Multiplying each coefficient by the appropriately scaled and shifted wavelet yields the constituent wavelet of the original signal:



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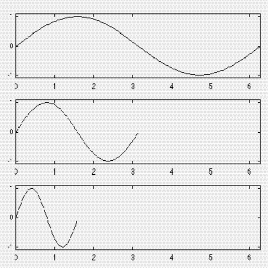
Scaling

- Wavelet analysis produces a time-scale view of the signal.
- Scaling means stretching or compressing of the signal.
- scale factor (a) for sine waves:



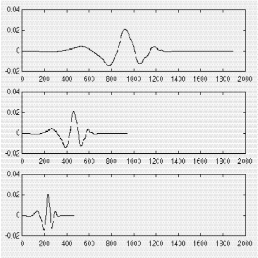
$$f(t) = sin(t); a = 1$$

$$f(t) = sin(2t); a = \frac{1}{2}$$

$$f(t) = sin(4t); a = \frac{1}{4}$$

Scaling

- Scale factor works exactly the same with wavelets:



$$f(t) = \Psi(t);\, a = 1$$
$$f(t) = \Psi(2t);\, a = \frac{1}{2}$$
$$f(t) = \Psi(4t);\, a = \frac{1}{4}$$

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The Wavelet function

$$\Psi_{a,b}(x) = \frac{1}{\sqrt{a}}\Psi\Big(\frac{x-b}{a}\Big),$$

here

- b - shift coefficient
- a - scale coefficient

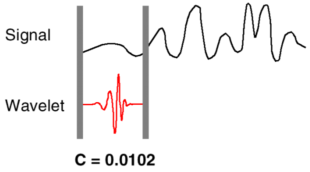
2D function:

$$\Psi_{a,b_x,b_y}(x,y) = \frac{1}{|a|}\Psi\Big(\frac{x-b_x}{a}, \frac{y-b_y}{a}\Big),$$

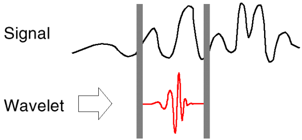
- Reminder: The CWT Is the sum over all time of the signal, multiplied by scaled and shifted versions of the wavelet function

Step 1: Take a Wavelet and compare it to a section at the start of the original signal.

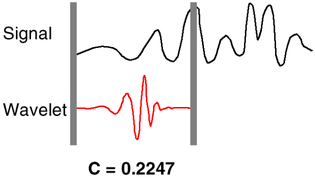
Step 2: Calculate a number, C, that represents how closely correlated the wavelet is with this section of the signal. The higher C is, the more the similarity.



Step 3: Shift the wavelet to the right and repeat steps 1-2 until you've covered the whole signal

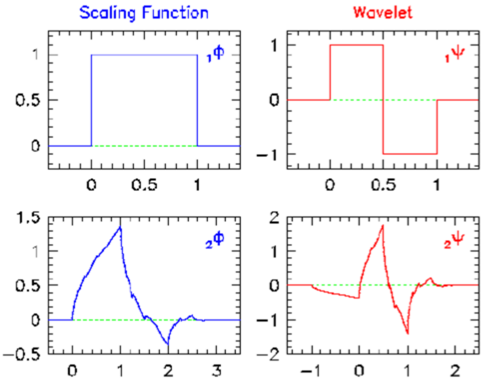


Step 4: Scale (stretch) the wavelet and repeat steps 1-3



Wavelet functions examples

- Haar function
- Daubechies function



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Scientific Computing
Lecture 5

Part 5: Integral Equations

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October 11, 2021

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The linear integral equation generally is

$$g(\mathbf{x})y(\mathbf{x}) - Ay(\mathbf{x}) = f(\mathbf{x})$$

Here A is an integral operator, which can be expressed as follows:

$$Ay(\mathbf{x}) = \lambda \int\limits_{\Omega} K(\mathbf{x}, \xi) y(\xi) d\xi$$

- Ω is some domain (may represent the whole space).
- $K(\mathbf{x}, \xi)$ is a Kernel.
- $y(\mathbf{x})$ is unknown function to be found.
- $f(\mathbf{x})$ is a right-hand side.

The linear integral equation generally is

$$g(\mathbf{x})y(\mathbf{x}) - Ay(\mathbf{x}) = f(\mathbf{x})$$

- **The integral equation of the first kind:** $g(\mathbf{x}) \equiv 0$.

$$Ay(\mathbf{x}) = f(\mathbf{x})$$

  These equations are commonly ill-posed.
- **The integral equation of the second kind:**

$$y(\mathbf{x}) - \lambda Ay(\mathbf{x}) = f(\mathbf{x}).$$

  These equations are commonly well-posed.

The linear integral equation generally is

$$g(\mathbf{x})y(\mathbf{x}) - Ay(\mathbf{x}) = f(\mathbf{x})$$

- Homogeneous equation: $f(\mathbf{x}) \equiv 0$.
  Homogeneous IE with $g(\mathbf{x}) \equiv 1$ are eigenvalues problems.
- Heterogeneous equation: $f(\mathbf{x}) \neq 0$.

Volterra and Fredholm IE

The linear integral equation generally is

$$g(\mathbf{x})y(\mathbf{x}) - Ay(\mathbf{x}) = f(\mathbf{x})$$

Here A is an integral operator, which can be expressed as follows:

$$Ay(\mathbf{x}) = \lambda \int\limits_{\Omega} K(\mathbf{x}, \xi)y(\xi)d\xi$$

- **Volterra IE:** the domain Ω is changeable domain. The 1D example:

$$y(x) - \int\limits_{0}^{x} K(x, s)y(s)ds = f(x)$$

- **Fredholm IE:** the domain Ω is a constant domain.

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Consider the integration problem:

$$f'(x) = g(x), 0 \leq x \leq 1$$

In order to calculate antiderivative, we have to integrate the latter equation with Volterra differential operator:

$$f(x) = f(0) + \int\limits_{0}^{x} g(\xi) d\xi.$$

or

$$f(x) = f(1) - \int\limits_{x}^{1} g(\xi) d\xi.$$

Classical example of the ill-posed problem in image processing task

Ill-posed problem classical example: FT and convolution

- The Fourier Transform (FT):

$$\tilde{f}(\omega) = F[f](\omega) = \frac{1}{(2\pi)^{n/2}} \int\limits_{-\infty}^{\infty} f(x) e^{-ix\cdot\omega} dx, \quad (x, \omega) \in \mathbb{R}^n.$$

- The Inverse Fourier Transform (IFT):

$$f(x) = F^{-1}[\tilde{f}(\omega)](x) = \frac{1}{(2\pi)^{n/2}} \int\limits_{-\infty}^{\infty} \tilde{f}(\omega) e^{ix\cdot\omega} d\omega$$

- The convolution:

$$(f * g)(x) = \int\limits_{\mathbb{R}^n} f(\xi) g(x - \xi) d\xi$$

- The Convolution Theorem:

$$\widetilde{f * g}(\omega) = \tilde{f}(\omega)\tilde{g}(\omega).$$

$$Af \equiv K{*}f = b, \quad (K{*}f)(\mathbf{x}) \equiv \int\limits_P f(\xi)K(\mathbf{x}-\xi)d\xi, P = \{\xi : |\xi| < 1\}$$

- Convolution theorem: $\tilde{K}\tilde{f} = \tilde{b}$, where $\tilde{*}$ denotes the Fourier transform.
- Why cannot we calculate just a $f = F^{-1}\left(\frac{\tilde{b}}{\tilde{K}}\right)$?
- The Fourier image of the finite function (the function with bounded support area) is not a finite function.
- Since both functions b and K are defined only inside the frame P **(finite!!!)**, we can't predict the behavior of integral:

$$f = F^{-1}\left(\frac{\tilde{b}}{\tilde{K}}\right) \equiv \iint\limits_{-\infty}^{\infty} \frac{\tilde{b}(\omega)}{\tilde{K}(\omega)} e^{i\omega\cdot x} d\omega.$$

The cost Tikhonov's functional:

$$M_\alpha[u] = ||Au - f||^2 + \alpha\Omega[u],$$

where the functional $\Omega[u]$ is a stabilizer.

1. $\Omega[u] = ||u||^2_{H^1}$. The solution of a quadratic functional can be found analytically.
2. $\Omega[u] = ||u||_{TV}$. In this case, the cost functional is not quadratic and requires iterative minimization. The minimization was provided using the method of Conjugated Gradients Projections (MCGP).

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Case of Sobolev's space regularization

Let the functional Ω be a norm in Sobolev's space H^1

$$\Omega(u) = ||u||^2_{H^1} = ||u||^2_{L^2} + \sum_{i=1}^{n} ||\frac{\partial u}{\partial x_i}||^2_{L^2}$$

Due to Parseval's theorem, $||f|| = ||\hat{f}||$. In terms of Fourier images, the cost functional takes the form:

$$M(u) = ||\hat{k}\hat{u} - \hat{f}||^2_{L^2} + \alpha||\hat{u}||^2_{H^1}.$$

Consider some uniform mesh and approximate the functional $M_\alpha(u)$ on this mesh using the Riemann sum, we obtain the following approximate cost functional:

$$\tilde{M}(u_1 1, u_1 2, ..., u_N M) = \sum_{i,j=0}^{N,M} (\hat{k}_{ij}\hat{u}_{ij} - \hat{f}_{ij})^2 + \sum_{i,j=0}^{N,M} \hat{u}^2_{ij} + \sum_{i,j=0}^{N,M} \Big(\frac{\hat{u}_{i+1j} - \hat{u}_{i-1j}}{2h} + ...$$

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The functional is linear; thus, in order to find its minimizer, we can equal to zero its gradient. The latter can be easily found directly, allowing us to write the equation:

$$\alpha(\hat{u}_{ij} + \omega^2 \hat{u}_{ij} + \lambda^2 \hat{u}_{ij}) + |\hat{k}_{ij}|^2 \hat{u}_{ij} - \hat{f}_{ij} \hat{k}^*_{ij} = 0.$$

From the equation above we find the approximate solution:

$$\hat{u}_{ij} = \frac{\hat{f}_{ij} \hat{k}^*_{ij}}{|\hat{k}_{ij}|^2 + \alpha(1 + \omega^2 + \lambda^2)}.$$

Applying to the last result the inverse Fourier transform, we obtain the approximate solution (deconvolution result). **NOTE:** This result can be obtained much easier in terms of the Frechét derivative of infinite cost functional.

Results: Sobolev space regularization for defocused photos

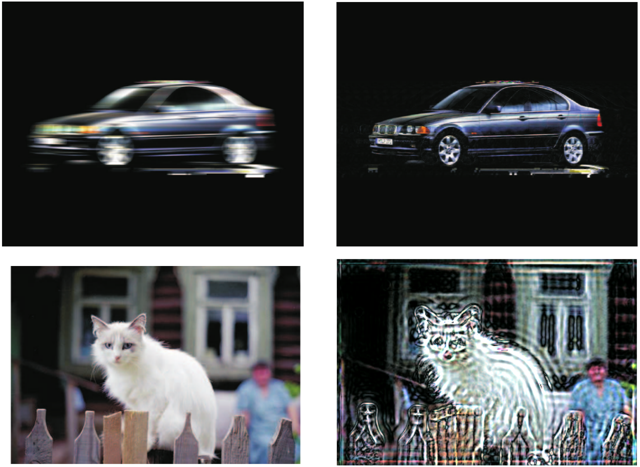

Figure: Smoothed car, Defocused woman and Scary cat

Case of VH regularization

Let the function $f(x,y)$ is defined in some box $B=[0,a]\times[0,b]$. Introduce in B some mesh $s=\{x_i,y_j,i=0,...,N-1,j=0,...,M-1\}$. The norm in Total Variation functional space is given in the simplest case by:

$$VH(u)=\sup_{S}\sum_{i,j=0}^{N-2,M-2}|u_{i+1,j+1}-u_{ij+1}-u_{i+1j}+u_{ij}|.$$

The cost Tikhonov's functional with uniform $N\times N$ mesh takes the form:

$$M(f)=\frac{1}{N^2}\sum_{i,j=0}^{N-1}\left(\frac{1}{N^2}\sum_{p,l=0}^{N-1}k_{i-p,j-l}u_{pl}-f_{ij}\right)+\alpha\sum_{i,j=0}^{N-2}|u_{i+1,j+1}-u_{ij+1}-u_{i+1j}+u_{ij}|.$$

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Results on image restoration in SEM BE

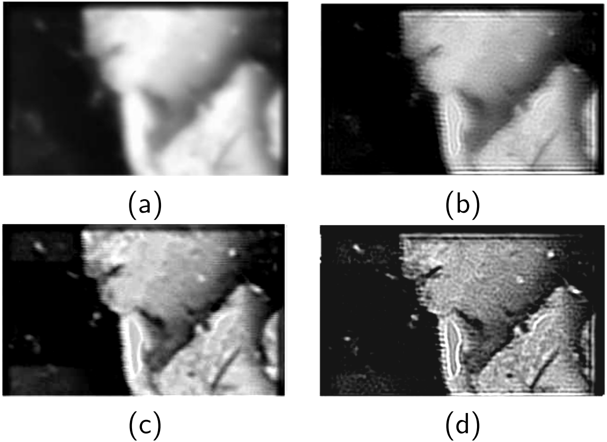

Figure: Results of image reconstruction in SEM BE: (a) - the enter data, (b) - reconstruction on H^1 Sobolev's space, analytical solution, (c) - reconstruction on TV Bounded Total Variations functional space, (MCGP, zero first approximation), (d) - reconstruction on TV with the result on H^1 taken as a first approximation for MCGP

Thank you for your attention!