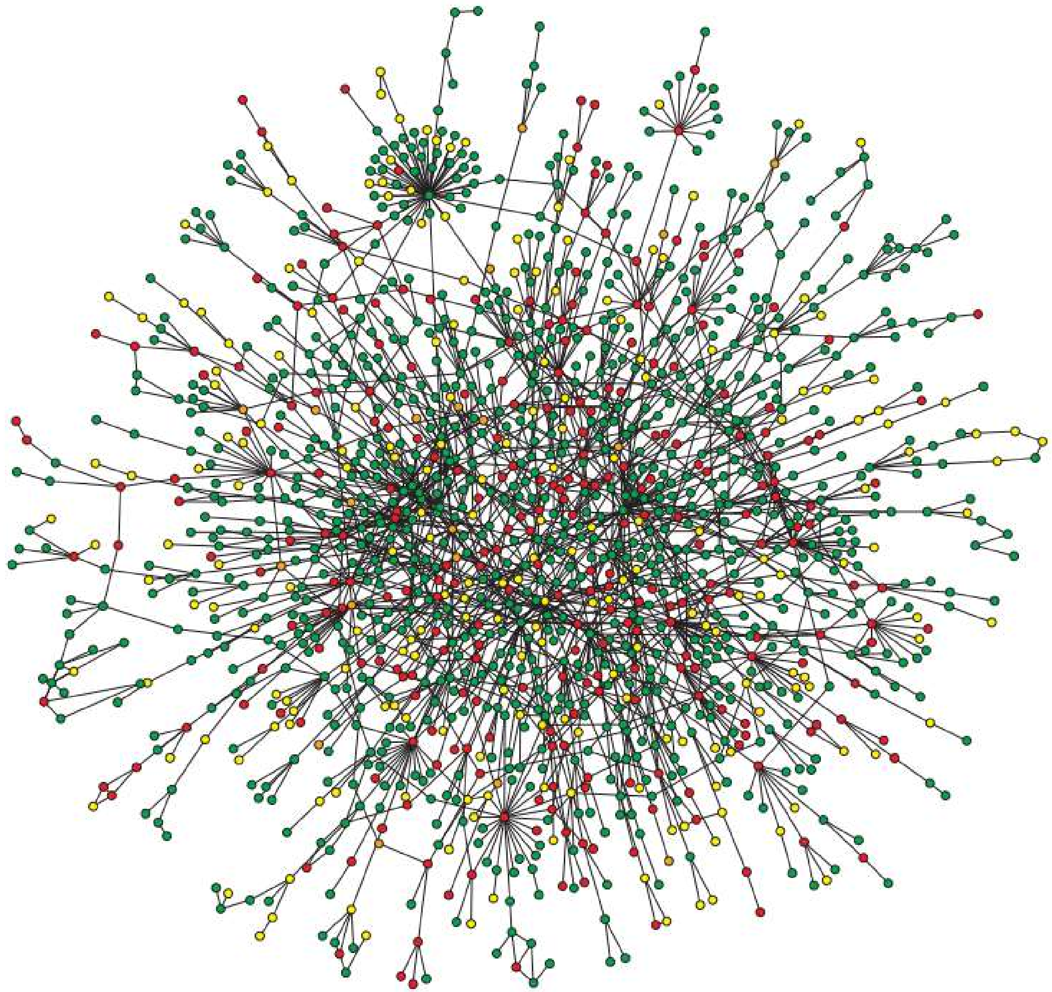
Collective Phenomena Emerging from the Interactions between Dynamical Processes in Multiplex Networks

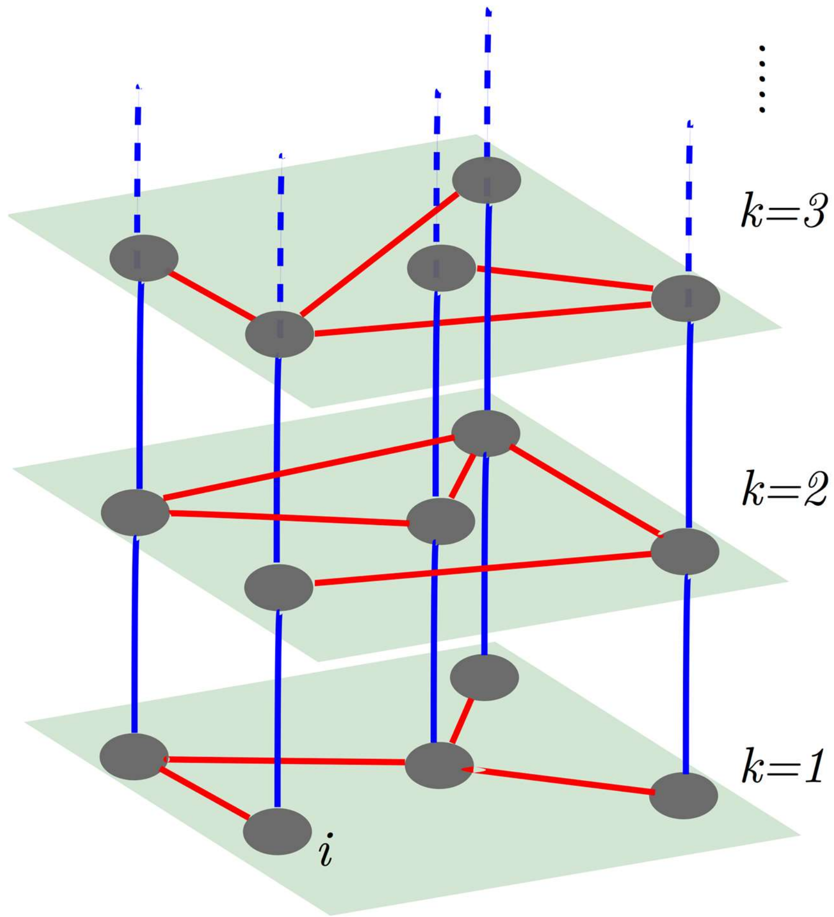
Paper by Vincenzo Nicosia, Per Sebastian Skardal, Alex Arenas, Vito Latora

Review by Artem Vergazov

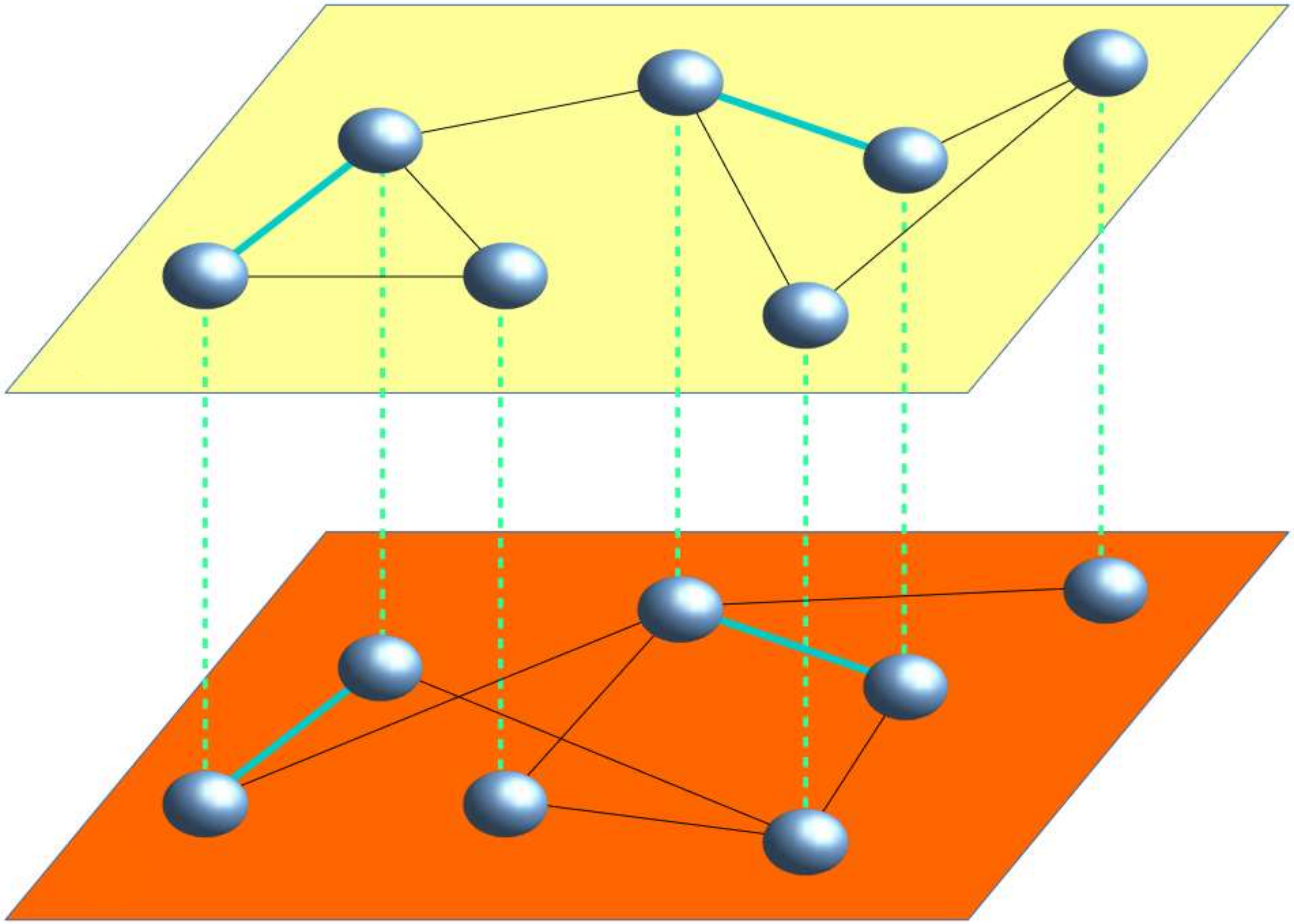
Networks



Multiplex networks



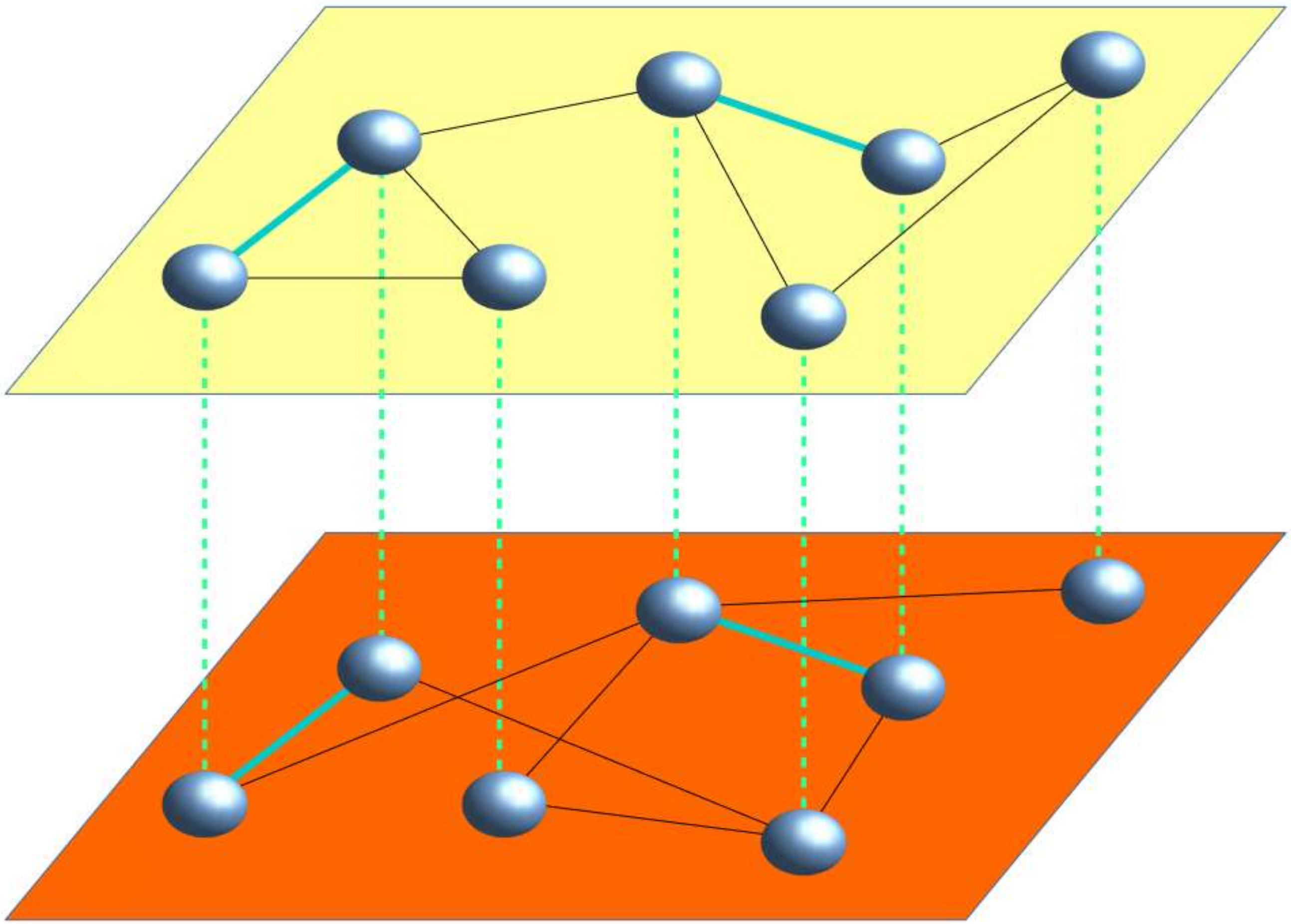
Dynamics on a multiplex network



$$\dot{x}_i = F_{\omega_i}(\boldsymbol{x}, A^{(1)})$$
$$\dot{\omega}_i = f(\omega_i, y_i)$$

$$\dot{y}_i = G_{\chi_i}(\boldsymbol{y}, A^{(2)})$$
$$\dot{\chi}_i = g(\chi_i, x_i)$$

Example: model of a human brain



Layer 1: neural activity

$$\dot{x}_i = F_{\omega_i}\left(\boldsymbol{x}, A^{(1)}\right)$$
$$\dot{\omega}_i = f(\omega_i, y_i)$$

Layer 2: energy transport

$$\dot{y}_i = G_{\chi_i}\left(\boldsymbol{y}, A^{(2)}\right)$$
$$\dot{\chi}_i = g(\chi_i, x_i)$$

Example: model of a human brain

$x_i(t) \in [0, 2\pi)$ – phase of oscillator i at time t

Layer 1: neural activity

Kuramoto model:

$$\dot{x}_i = F_{\omega_i}\left(\boldsymbol{x}, A^{(1)}\right)$$
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Example: model of a human brain

$x_i(t) \in [0, 2\pi)$ – phase of oscillator i at time t

λ – coupling strength (*control parameter*)

Layer 1: neural activity

Kuramoto model: $$\dot{x}_i = \omega_i + \lambda \sum_{j=1}^{N} a_{ij}^{(1)} \sin(x_j - x_i)$$

$$\dot{\omega}_i = f(\omega_i, y_i)$$

Layer 2: energy transport

$$\dot{y}_i = G_{\chi_i}\left(\boldsymbol{y}, A^{(2)}\right)$$

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Layer 2: energy transport

Continuous-time RW: $$\dot{y}_i = G_{\chi_i}\left(\boldsymbol{y}, A^{(2)}\right)$$

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$y_i(t) \in [0, 1]$ – fraction of RWs at node i at time t

$\pi_{ij} = \frac{a_{ji}^{(2)} \chi_i^\alpha}{\sum_{l=1}^{N} a_{jl}^{(2)} \chi_l^\alpha}$ – transition probability from j to i

χ_i – node bias for RW

α – bias exponent (*control parameter*)

Layer 2: energy transport

Continuous-time RW: $$\dot{y}_i = \frac{1}{\tau_y} \sum_{j=1}^{N} (\pi_{ij} - \delta_{ij}) y_j$$

$$\dot{\chi}_i = g(\chi_i, x_i)$$

Coupling the neural dynamics and the diffusion of nutrients

$x_i(t) \in [0, 2\pi)$ – phase of oscillator i at time t

λ – coupling strength (*control parameter*)

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higher frequency requires more energy $$\dot{\omega}_i = \frac{1}{\tau_\omega}(N y_i(t) - \omega_i)$$

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Continuous-time RW: $$\dot{y}_i = \frac{1}{\tau_y} \sum_{j=1}^{N} (\pi_{ij} - \delta_{ij}) y_j$$

higher electrical activity requires blood inflow $$\dot{\chi}_i = \frac{1}{\tau_\chi}(s_i - \chi_i)$$

s_i – dynamic strength of node i

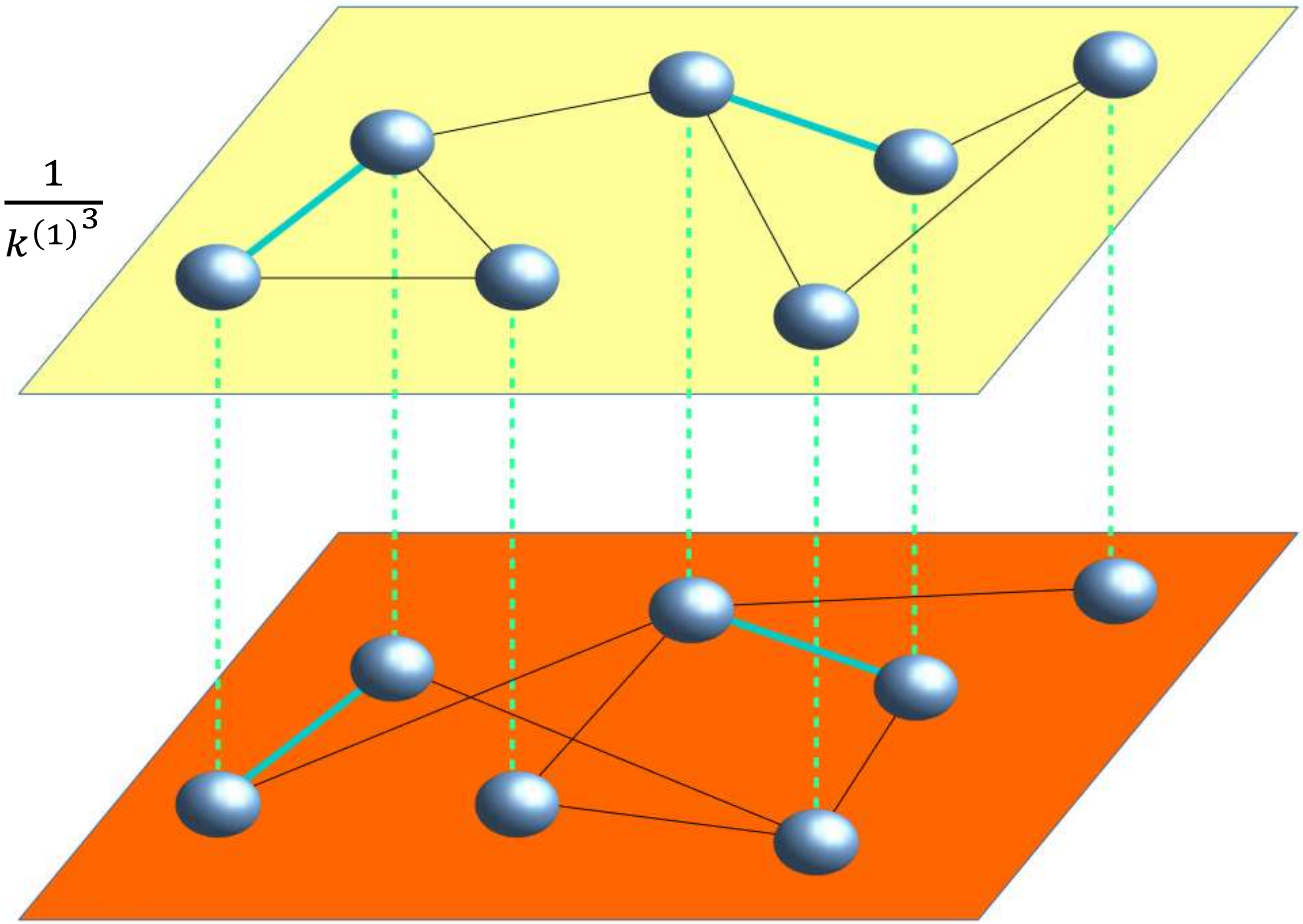
Modeling

Scale-free graph

Degree distribution $P\left(k^{(1)}\right) \propto \frac{1}{{k^{(1)}}^3}$

Erdős–Rényi graph

Poisson degree distribution



Fast relaxation approximation

$$\dot{x}_i = \omega_i + \lambda \sum_{j=1}^{N} a_{ij}^{(1)} \sin(x_j - x_i)$$

$$\dot{\omega}_i = \frac{1}{\tau_\omega}(N y_i(t) - \omega_i)$$

$$\tau_y \to 0$$
$$\tau_\omega \to 0$$
$$\tau_\chi \to 0$$

$$\dot{y}_i = \frac{1}{\tau_y} \sum_{j=1}^{N} (\pi_{ij} - \delta_{ij}) y_j$$

$$\dot{\chi}_i = \frac{1}{\tau_\chi}(s_i - \chi_i)$$

Fast relaxation approximation

$$\dot{x}_i = \omega_i + \lambda \sum_{j=1}^{N} a_{ij}^{(1)} \sin(x_j - x_i)$$

$$\omega_i(t + dt) = w_i(t) + \frac{dt}{\tau_\omega}(N y_i(t) - \omega_i)$$

$$y_i(t + dt) = y_i(t) + \frac{dt}{\tau_y} \sum_{j=1}^{N} \left(\pi_{ij} - \delta_{ij}\right) y_j$$

$$\chi_i(t + dt) = \chi_i(t) + \frac{dt}{\tau_\chi}(s_i - \chi_i)$$

$$\tau_y \to 0$$
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Fast relaxation approximation

$$\dot{x}_i = \omega_i + \lambda \sum_{j=1}^{N} a_{ij}^{(1)} \sin(x_j - x_i)$$

$$\omega_i(t+dt) = w_i(t) + \frac{\cancel{dt}}{\cancel{\tau_\omega}}(Ny_i(t) - \omega_i)$$

$$y_i(t+dt) = y_i(t) + \frac{\cancel{dt}}{\cancel{\tau_y}} \sum_{j=1}^{N} \left(\pi_{ij} - \delta_{ij}\right) y_j$$

$$\chi_i(t+dt) = \chi_i(t) + \frac{\cancel{dt}}{\cancel{\tau_\chi}}(s_i - \chi_i)$$

$$\tau_y \to 0$$
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Fast relaxation approximation

$$\dot{x}_i = \omega_i + \lambda \sum_{j=1}^{N} a_{ij}^{(1)} \sin(x_j - x_i)$$

$$\omega_i(t + dt) = N y_i(t)$$

$$\tau_y \to 0$$
$$\tau_\omega \to 0$$
$$\tau_\chi \to 0$$

$$y_i(t + dt) = \sum_{j=1}^{N} \pi_{ij} y_j(t)$$

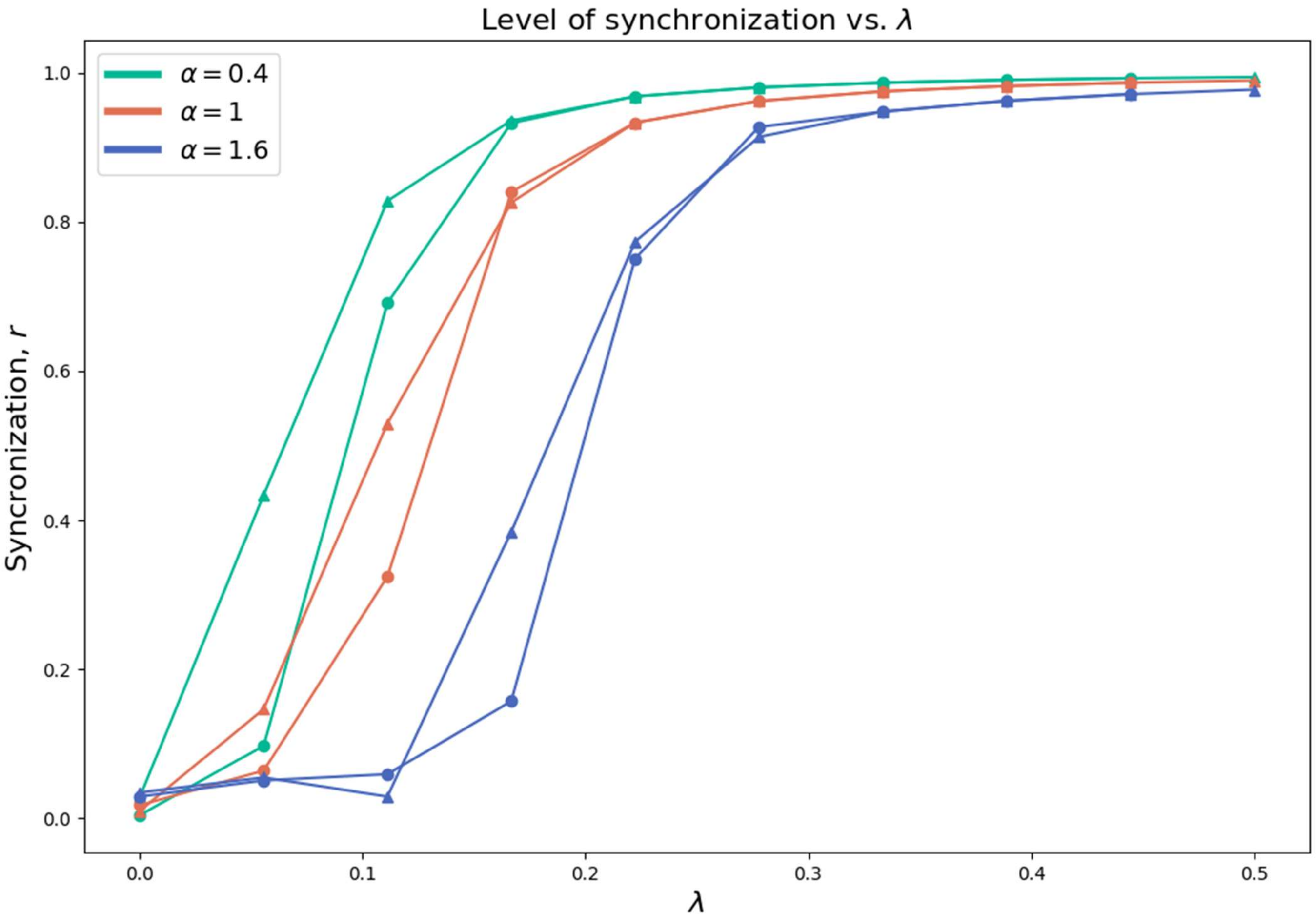
$$\chi_i(t + dt) = s_i(t)$$

Metric

$$r = \left| \sum_{i=1}^{N} e^{i x_i} \right|$$

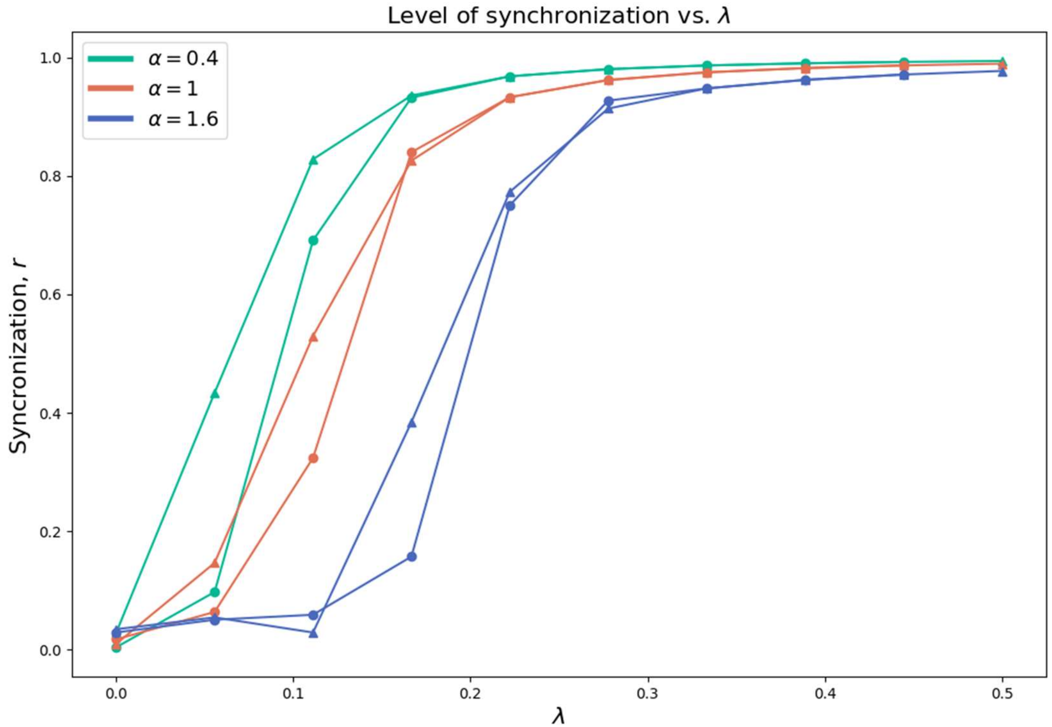
Kuramoto order parameter – degree of the global synchronization

Simulation



Level of synchronization vs. λ
α = 0.4
α = 1
α = 1.6
Syncronization, r
λ
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.1
0.2
0.3
0.4
0.5

Simulation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Syncronization, r
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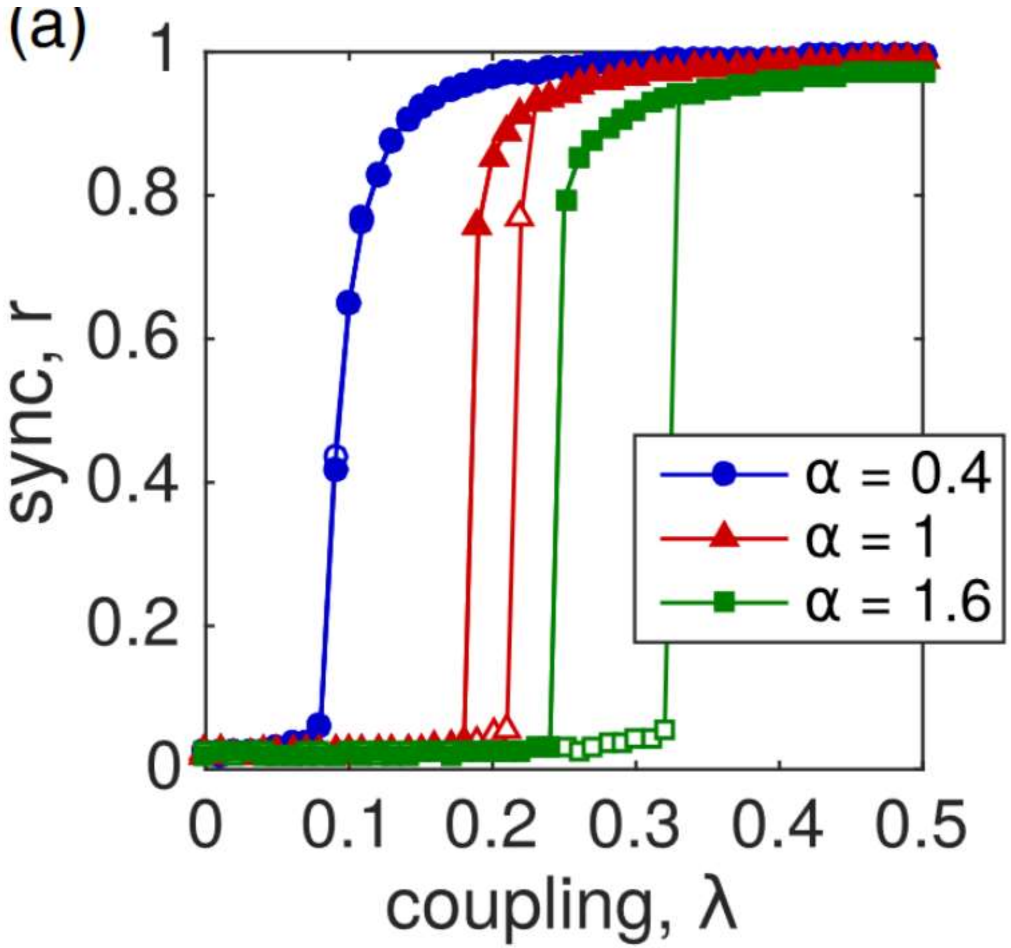
(a)
sync, r
coupling, λ
α = 0.4
α = 1
α = 1.6
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5

References

- Vincenzo Nicosia, Per Sebastian Skardal, Alex Arenas, and Vito Latora. Phys. Rev. Lett. 118, 138302 (2017)