Mock Exam. Course: Stochastic Methods in Mathematical Modelling. MA06363. AY2022-2023. Skolkovo Institute of Science and Technology.

Problem 1. Distributions and their properties (5 marks in total)

1. Assume that n independent and identically distributed (**i.i.d**) variables are taken from a distribution $P(x) = \frac{1}{2\Gamma(5/4)}\exp(-x^4), -\infty < x < \infty$. What class of the limiting distributions the distribution of maximal values of $x_i \in \{x_1, ...x_n\}$ will belong to? (1 mark)

2. Assuming that the joint PDF of three random variables x, y and z reads $P(x,y,z) = \frac{1}{N}\exp(-x^2 - 3y^2 - 10z^2 - 2xy - 8yz - 4xz)$ compute **(a)** normalisation constant N, **(b)** marginal PDF $P(x)$ and **(c)** the averages $\langle xy \rangle$, $\langle x^2 z^4 \rangle$,$\langle xy^2 \rangle$ (2 marks).

3. Consider the following distribution $P(x) = \frac{1}{\pi\sqrt{(x(1-x))}}$, $x \in (0,1)$. **(a)** Compute its cumulative distribution function (CDF), mean, variance, skewness and kurtosis. **(b)** Describe the inverse CDF technique for sampling the distribution and derive the formula for sampling of $P(x)$ which uses variables drawn from the uniform distribution $U[0,1]$ as an input. **(c)** What class of the limiting distributions the distribution of maximal values of $x_i \in \{x_1, ...x_n\}$ will belong to? (2 marks).

Problem 2. Markov chains and convergence to stationary state (5 marks)

Consider the Markov chain depicted in Fig. 1

1. Write down the transition matrix (1 mark)
2. Check that the matrix is stochastic (values in every column add up to 1) (1 mark)
3. Find the eigenvectors and eigenvalues (1 mark)
4. Suppose that we start in the state "1", i.e. $\pi(0) = (1,0,0)^T$. Write the initial state as a linear combination of eigenvectors and find $\pi(t) = \pi^t$. (1 mark)
5. Check that the stationary distribution satisfies the detailed balance (1 mark)

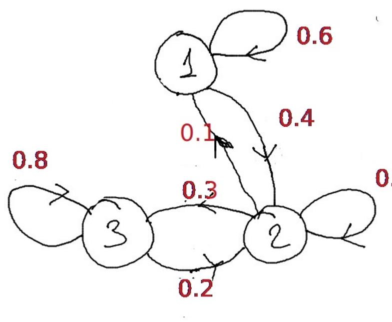

Figure 1: Markov chain

Problem 3. Itô's lemma and stochastic integration (5 marks)

Assume that W_t is a Wiener process, $f(X_t) = \log(X_t)$ and X_t is the solution of the following Langevin equation

$$\frac{dX_t}{X_t} = \sigma dW_t + \mu dt \tag{1}$$

1. Write down the Tailor expansion of $df(X_t)$ up to the second-order term $(dX_t)^2$. (1 mark)
2. Using the SDE (1) and Itô's lemma rewrite it as a function of dW and dt. (1 mark)
3. Integrate the equation from 0 to t using both Itô and Stratonovich interpretations of the integral. (2 marks)
4. Find the explicit form of X_t as a function of W_t and t. (1 mark)

Problem 4. The Harmonic Potential, Fokker-Planck and Langevin equations (5 marks)
The Fokker-Planck equation for the probability density $P(x,t)$ of a one-dimensional overdamped oscillator is given by

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\mu K x P(x,t) + \mu k_B T \frac{\partial}{\partial x} P(x,t)\right]$$

where K denotes the spring constant, μ the mobility, and $k_B T$ the thermal energy. The initial condition reads $P(x, t = 0) = \delta(x - x_0)$.

1. Find Langevin equation(s) (SDEs) which could produce this Fokker-Planck equation assuming they are in Itô and Stratonovich interpretation (1 mark).
2. Show that for the average

   $$\langle x(t)^n \rangle := \int dx\ x^n P(x,t)$$

   the following equation holds:

   $$\frac{\partial}{\partial t}\langle x(t)^n \rangle = -\mu K n \langle x(t)^n \rangle + \mu k_B T n(n-1)\langle x(t)^{n-2} \rangle\,.$$

   (1 mark)
3. Solve this equation for the first three moments. Determine the respective long-time behavior (1 mark).
4. Show that the centered moments $M_n(t) := \langle [x(t) - \langle x(t) \rangle]^n \rangle$ fulfil

   $$\frac{\partial}{\partial t} M_n(t) = -n\mu K M_n(t) + n(n-1)\mu k_B T M_{n-2}(t)\,.$$

   Conclude that all odd centered moments $M_{2k+1}(t)$ vanish. Show that for the even moments $M_{2k}(t) = C_k [M_2(t)]^k$ holds, if the combinatorial factor C_k is properly chosen (2 marks).