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Stochastic methods in Mathematical Modelling

Lecture 14. Anomalous random processes 2.

Normal vs Anomalous

Normal

$$\langle x^2(t)\rangle = 2D_1 t$$

Anomalous

$$\langle x^2(t)\rangle \neq 2K t$$

Mostly

$$\langle x^2(t)\rangle \sim K_\alpha t^\alpha$$

Superdiffusion $\alpha > 1$
Subdiffusion $\alpha < 1$

Could be more complicated (Sinai diffusion, for instance)

$$\langle x^2(t)\rangle \sim \log^\beta t$$

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Reasons for anomality

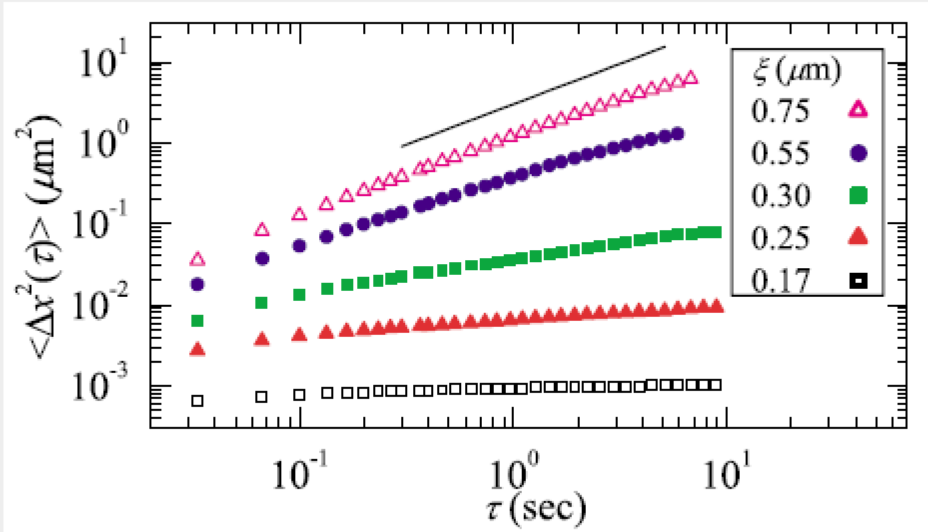
$$\langle x^2(t) \rangle \sim K_\alpha t^\alpha$$

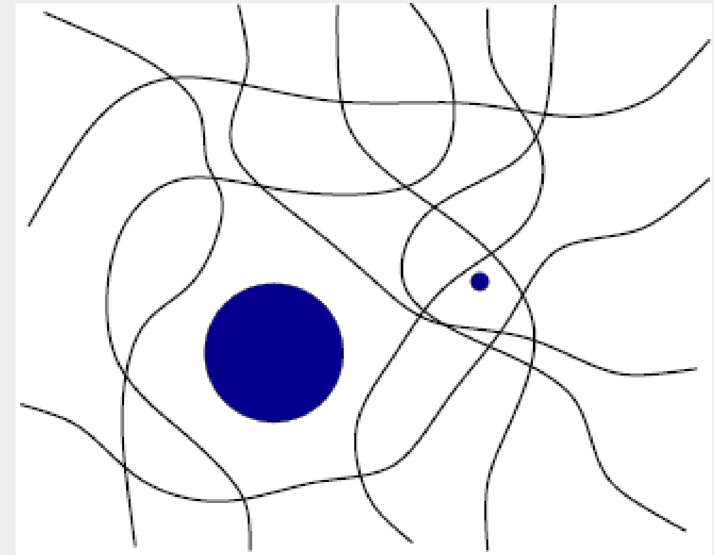
1. Specific power-law distributions for jump lengths/waiting times (diverging second or first moments)

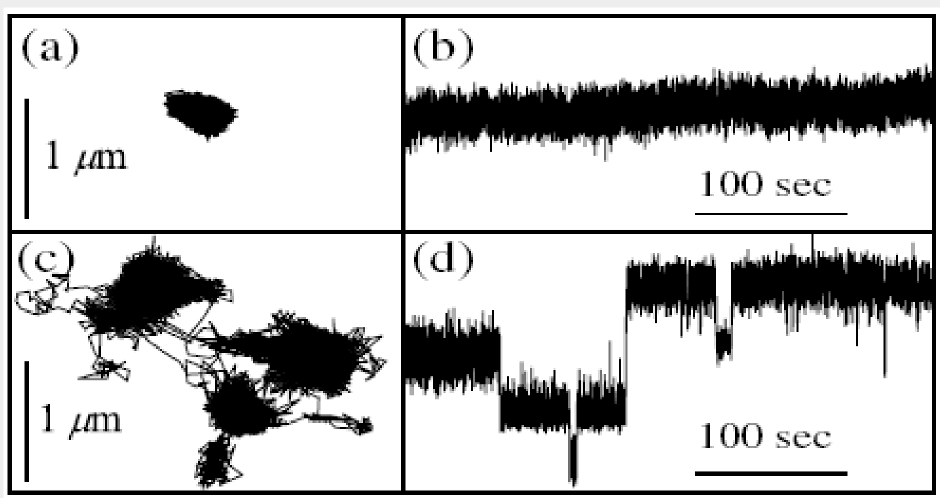
2. Correlations in jump directions

3. Specific geometry of the system (percolation cluster) or time evolution of parameters

Actin net trapping experiment







ξ is a typical cell size

The ball size is 0.25 micrometers

Jump probability P ~ $\tau^{-1-\gamma}$

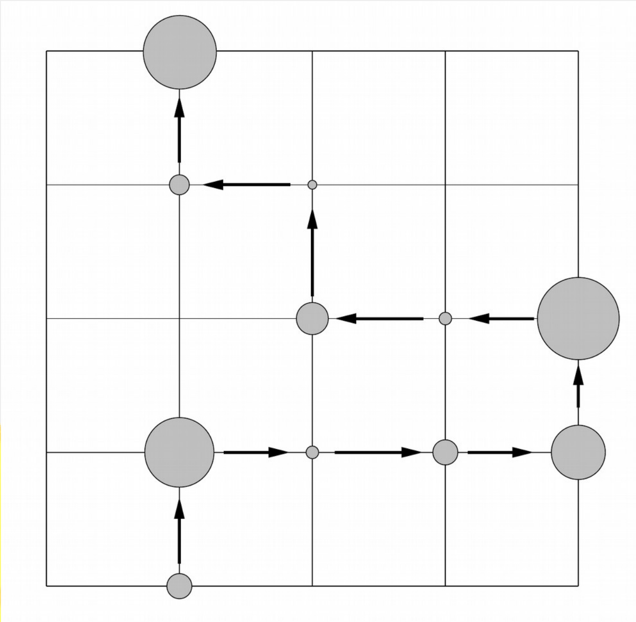
I.Y.Wong et al., Phys. Rev. Lett. 92, 178101 (2004)

Continuous-Time Random Walk model (CTRW)

$$X(t) = X_0 + \sum_{i=1}^{N(t)} \Delta X_i$$

The lengths and waiting times (durations) of the jumps are drawn from φ(x,t)

The concept of waiting times



Jump pdf $$\lambda(x) = \int dt\, \varphi(x,t)$$

Waiting time pdf $$w(t) = \int dx\, \varphi(x,t)$$

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Continuous-Time Random Walk model (CTRW)

$$X(t) = X_0 + \sum_{i=1}^{N(t)} \Delta X_i$$

Jump pdf

$$\lambda(x) = \int dt\, \varphi(x,t)$$

general jump-WT pdf $\varphi(x,t)$

Waiting time pdf

$$w(t) = \int dx\, \varphi(x,t)$$

General solution for the CTRW PDF

$$\boxed{P(k,s) = \frac{1-w(s)}{s} \frac{P_0(k)}{1-\varphi(k,s)}}$$

Long Rests: Fractional Diffusion Equation

$$\boxed{P(k,s)=\frac{1-w(s)}{s}\frac{P_0(k)}{1-\varphi(k,s)}}$$

Assume for *t* ->∞ $w(t)\sim A\left(\frac{\tau}{t}\right)^{1+\alpha}$

Assume that λ(x) has a finite variance

$$w(s)\sim 1-(s\,\tau)^{\alpha}$$

$$P(k,s)=\frac{P_0(k)/s}{1+K_\alpha k^2 s^{-\alpha}}$$

Long Rests:
Fractional Diffusion Equation

$$P(k,s)=\frac{P_0(k)/s}{1+K_\alpha k^2 s^{-\alpha}}$$

$$w(t)\sim A\left(\frac{\tau}{t}\right)^{1+\alpha}$$

λ(x) has
a finite variance

$$\frac{\partial P(x,t)}{\partial t}={}_0D_t^{1-\alpha}K_\alpha\frac{\partial^2}{\partial x^2}P(x,t),K_\alpha=\sigma^2/\tau^\alpha$$

Solution of subdiffusive FFPE

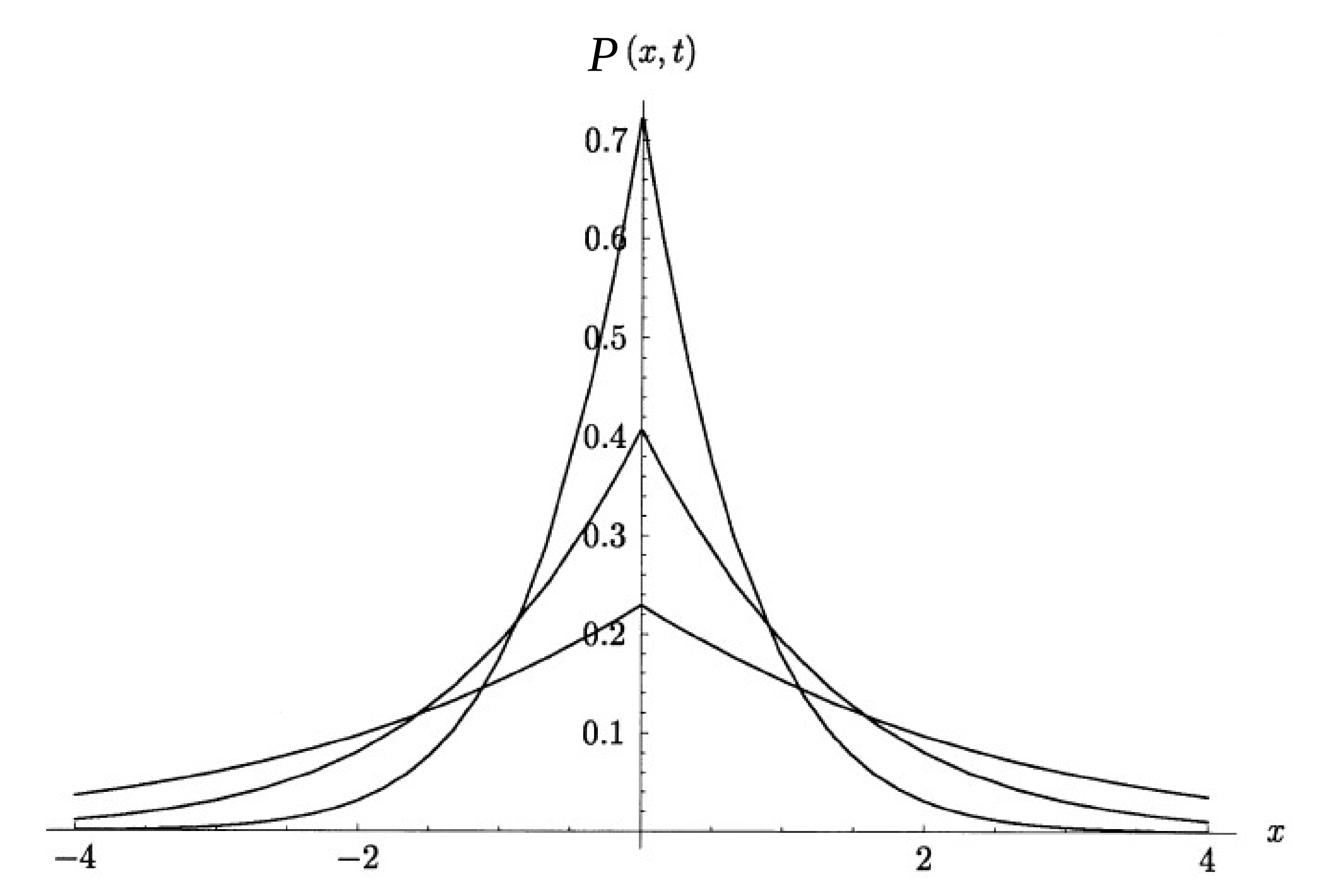
$$\frac{\partial P(x,t)}{\partial t} = {}_0D_t^{1-\alpha} K_\alpha \frac{\partial^2}{\partial x^2} P(x,t), K_\alpha = \sigma^2/\tau^\alpha$$

$$\langle x^2(t) \rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)} t^\alpha$$

$$P(x,t) = \frac{1}{\sqrt{4K_\alpha t^\alpha}} H_{1,1}^{1,0}\left[\frac{|x|}{\sqrt{K_\alpha t^\alpha}} \middle| \begin{matrix} (1-\alpha/2, \alpha/2) \\ (0,1) \end{matrix} \right]$$

Numerical results for subdiffusive FFPE

$$\frac{\partial P(x,t)}{\partial t} = {}_0D_t^{1-\alpha} K_\alpha \frac{\partial^2}{\partial x^2} P(x,t), K_\alpha = \sigma^2/\tau^\alpha \quad \alpha = 0.5; t = 0.1, 1, 10$$



R. Metzler, J. Klafter, Phys. Rep. 2000

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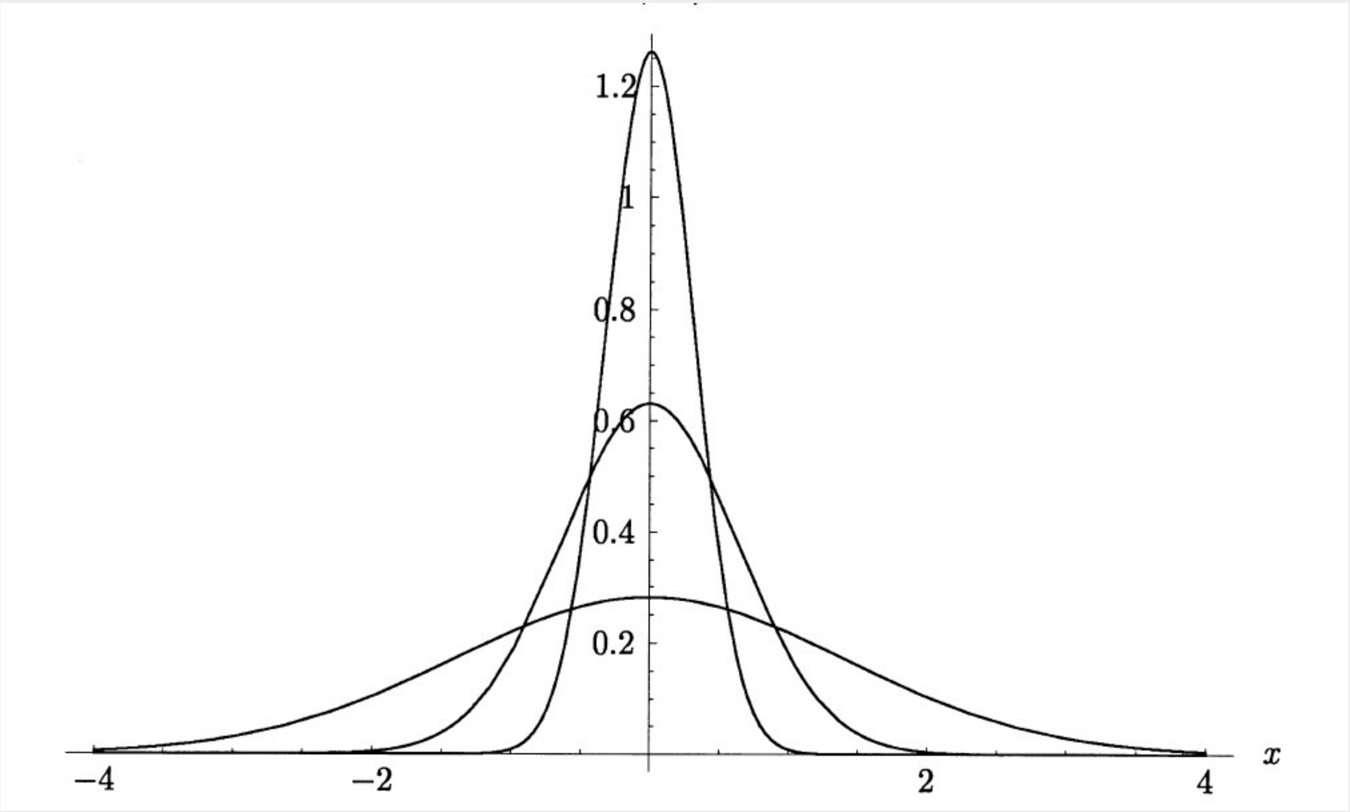
Reminder: Numerical results for normal FPE

$$\frac{\partial P(x,t)}{\partial t} = K_2 \frac{\partial^2}{\partial x^2} P(x,t) \qquad t = 0.1,\ 1,\ 10$$



R. Metzler, J. Klafter, Phys. Rep. 2000

Modes of subdiffusive FFPE

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$$\frac{\partial P(x,t)}{\partial t} = {}_0D_t^{1-\alpha} K_\alpha \frac{\partial^2}{\partial x^2} P(x,t), K_\alpha = \sigma^2/\tau^\alpha$$

In k-space:

$$\frac{dP(k,t)}{dt} = -{}_0D_t^{1-\alpha} K_\alpha k^2 P(k,t)$$

$$P(k,t) = E_\alpha(-K_\alpha k^2 t^\alpha)$$

Mittag-Leffler function

$$E_\alpha(-\lambda_n t^\alpha) \equiv \sum_{j=0}^{\infty} \frac{(-\lambda_n t^\alpha)^j}{\Gamma(1+\alpha j)} \sim \begin{cases} \exp\left(-\frac{\lambda_n t^\alpha}{\Gamma(1+\alpha)}\right) & t \ll \lambda^{1/\alpha} \\ (\lambda_n t^\alpha \Gamma(1-\alpha))^{-1} & t \gg \lambda^{1/\alpha} \end{cases}$$

Modes of FFPE

$$\frac{dP}{dt} = - {}_0D_t^{1-\alpha} K_\alpha k^2 P(k,t)$$

$$P(k,s) = \frac{P_0(k)/s}{1 + K_\alpha k^2 s^{1-\alpha}}$$

$$\mathcal{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-at^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha + a}$$

'Mittag-Leffler function

$$P(k,t) = \underline{E_\alpha(-K_\alpha k^2 t^\alpha)}$$

$$E_1(z) = e^z$$

$$E_2(z) = \cosh(\sqrt{z})$$

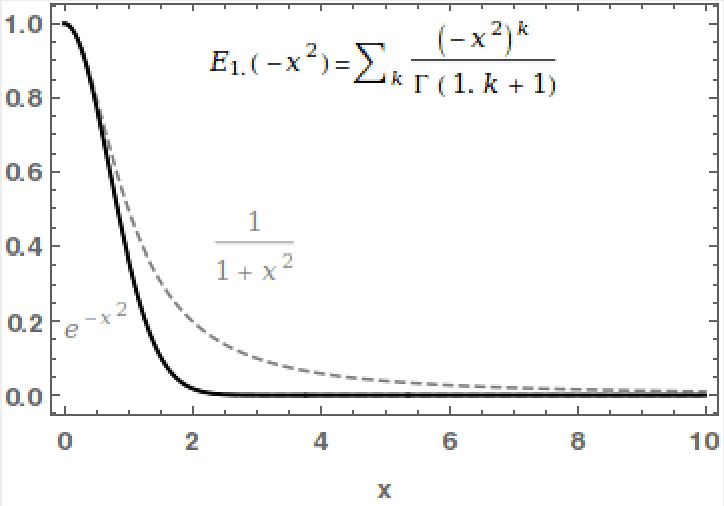
$$E_\alpha(z) = E_{\alpha,\beta=1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$$

$$E_\alpha(-\lambda_n t^\alpha) = \sum_{j=0}^{\infty} \frac{(-\lambda_n t^\alpha)^j}{\Gamma(1+\alpha j)} \sim \begin{cases} \exp\left(-\frac{\lambda_n t^\alpha}{\Gamma(1+\alpha)}\right), & t \ll \lambda^{1/\alpha} \\ (\lambda_n t^\alpha \Gamma(1-\alpha))^{-1}, & t \gg \lambda^{1/\alpha} \end{cases}$$

Mittag-Leffler function

$$E_{\alpha,\beta}(-z) = H^{1;1}_{1;2}\left[z \middle| \begin{matrix} (0,1) \\ (0,1),(1-\beta,\alpha) \end{matrix} \right] = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\beta+\alpha j)}$$

$$t^{\beta-1}E_{\alpha,\beta}(-at^{\alpha}) \div \frac{s^{\alpha-\beta}}{s^{\alpha}+a}$$



Review reference: Mittag-Leffler Functions and Their Applications, H.J. Haubold, A. M. Mathai, and R.K. Saxena, Journal of Applied Mathematics, 2011, Article ID 298628, doi:10.1155/2011/29862

For sampling you could look into the section 19

Mittag-Leffler function practical example

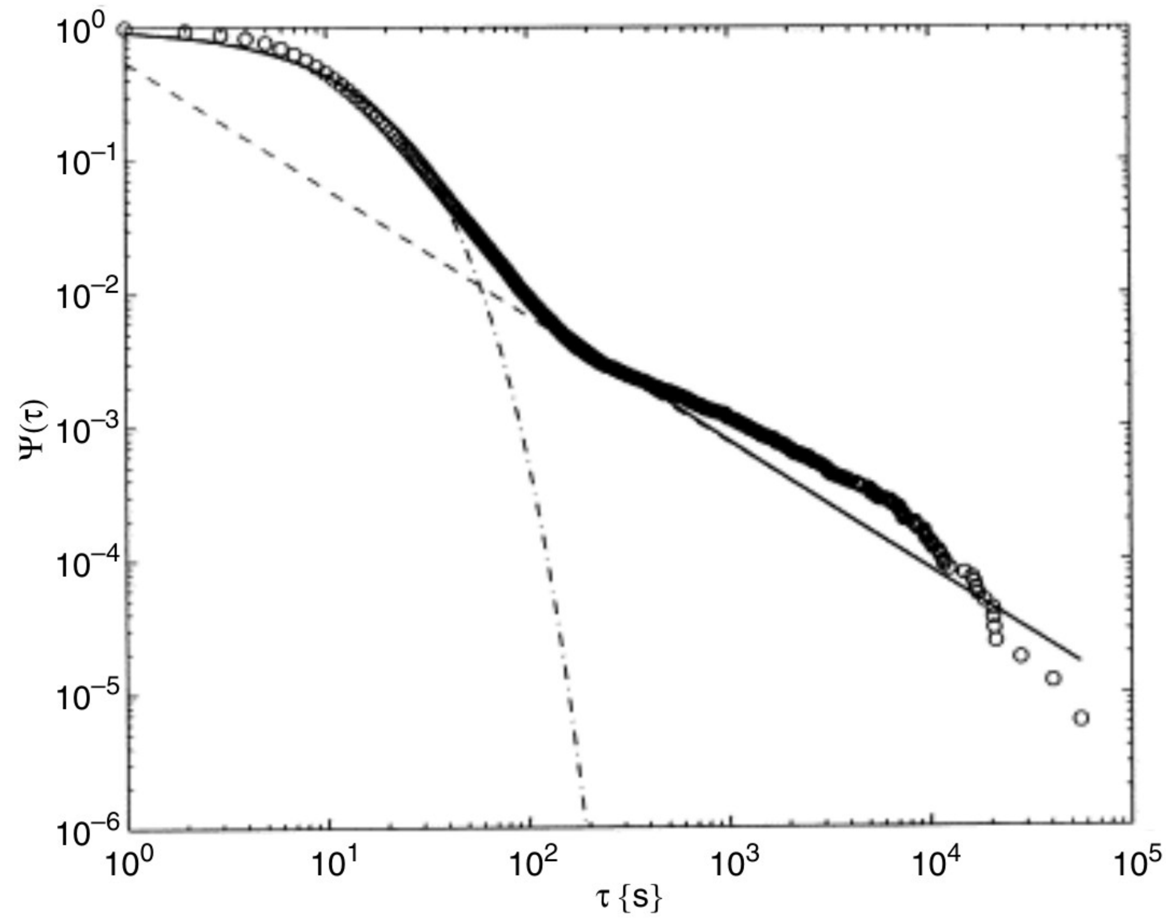
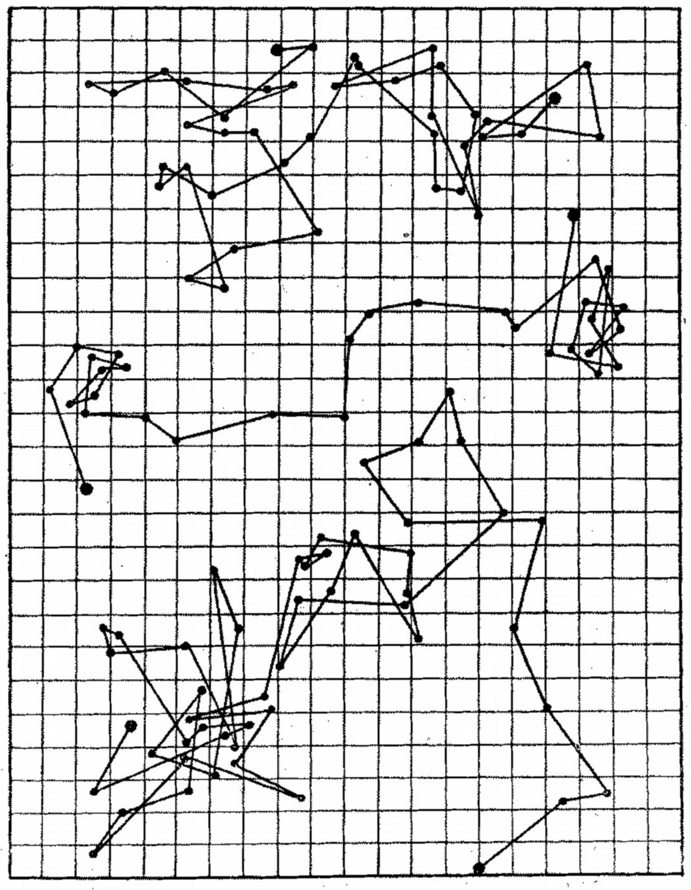


Figure 6. Survival probability for BUND futures from September 1997. The Mittag-Leffler function (full line) is compared with a stretched exponential (dashed-dotted line) and a power law (dashed line) (Mainardi *et al* 2000).

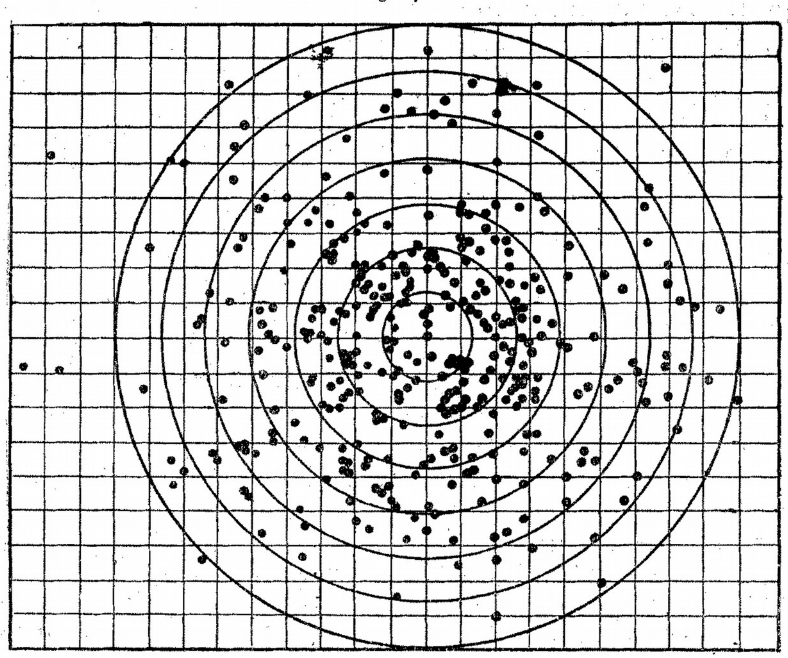
Perrin experiment. Brownian motion

Fig. 6.



$\Delta t = 30$ sec

Fig. 7.



$$P(r, \Delta t) = \frac{1}{(4\pi D \Delta t)^{d/2}} \exp\left(-\frac{r^2}{4 D \Delta t}\right)$$

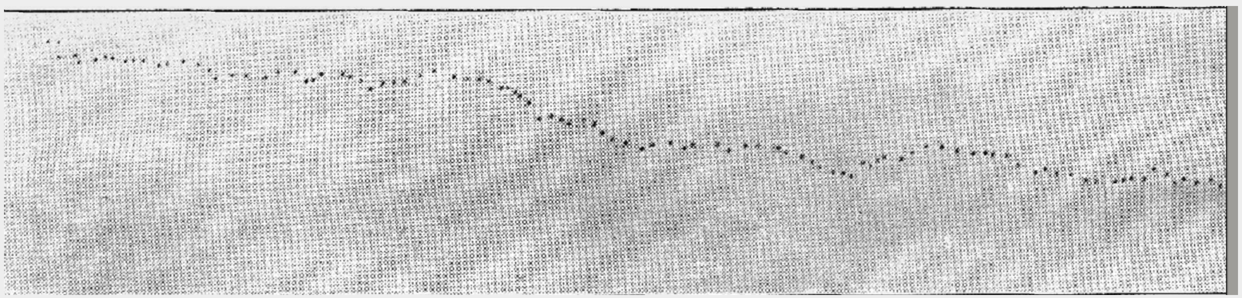
Einstein-(Smoluchowski) relation

$$D = \frac{k_B T}{m\eta} = \frac{R/N_A T}{m\eta}$$

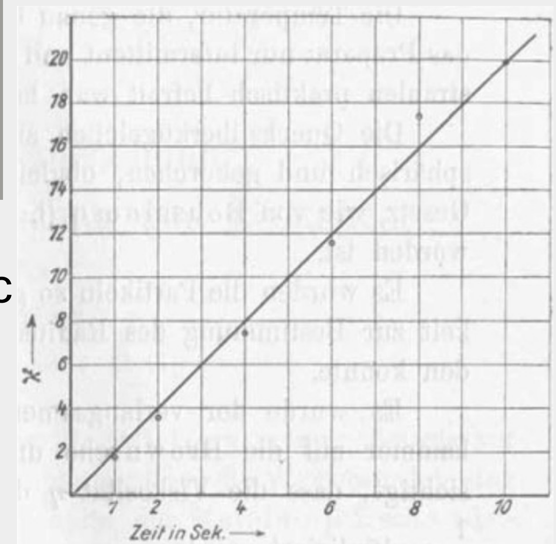
$$N_A = 7.05 \times 10^{23}$$

J Perrin, Comptes Rendus (Paris) 146 (1908) 967

Brownian motion II:
single trajectory time series analysis



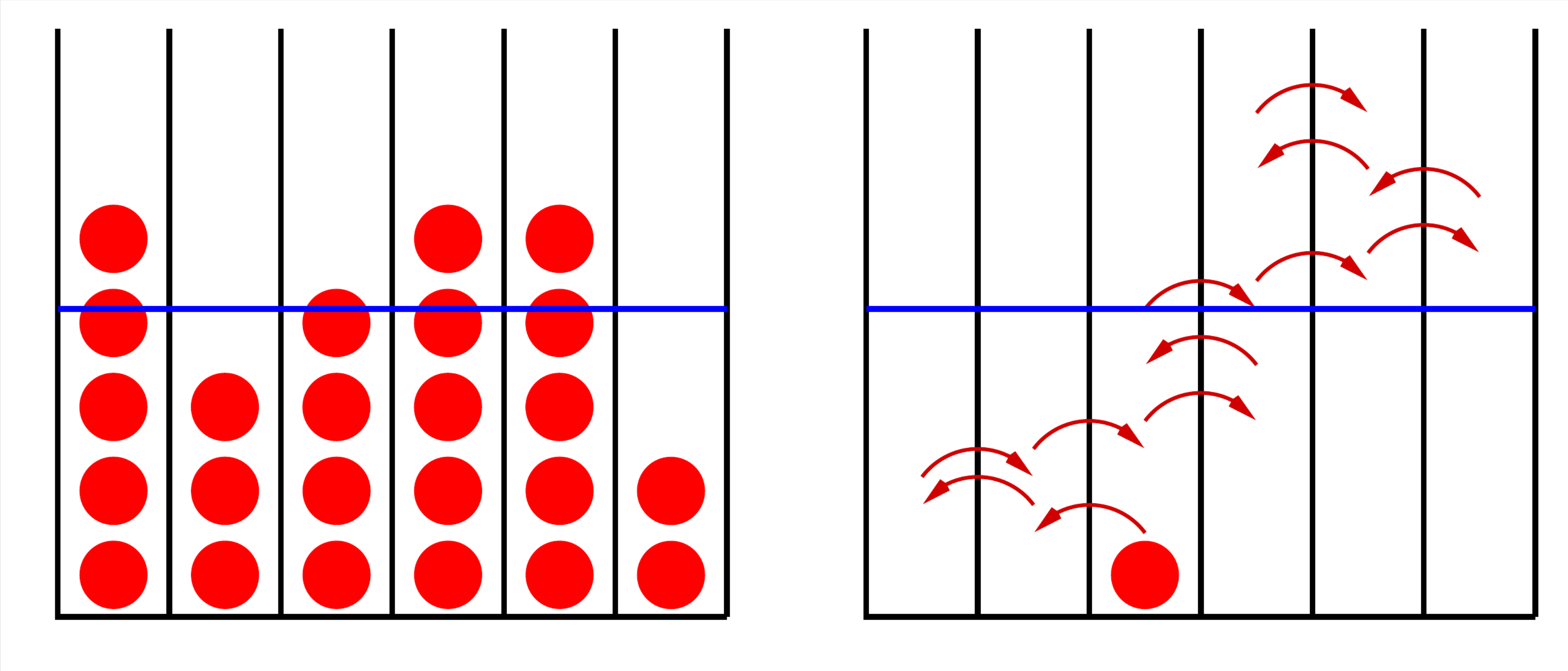
The stochastic, Brownian motion around the deterministic
sedimentation with constant velocity
can be clearly distinguished



example for the time averaged
mean squared displacement versus time
(in seconds) from a single recorded
falling mercury droplet

I Nordlund, Z Physik (1914): $N_A = 5.91 \times 10^{23}$

Time versus ensemble averages: ergodic hypothesis



Ergodicity. Ensemble average = Time average:

$$\langle p_i \rangle = \lim_{N \to \infty} \frac{N_i}{N} \equiv \bar{p}_i = \lim_{t \to \infty} \frac{t_i}{t}$$

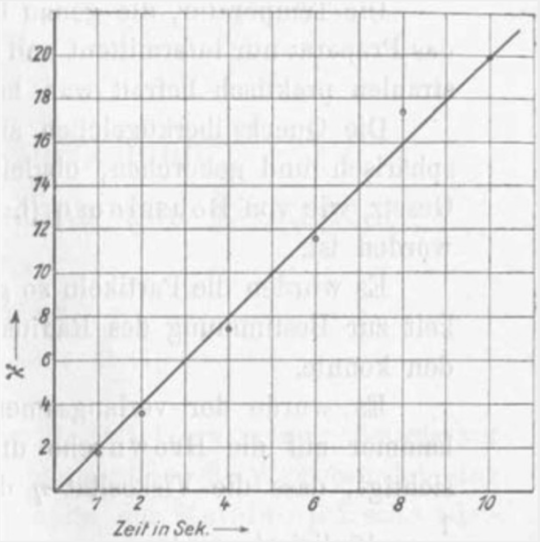
Extracting information from single trajectories

Ensemble average for normal diffusion

$$\langle x^2(t)\rangle = \int x^2 P(x,t)\,dx = 2D_1 t$$

Single particle trajectory time series x(t)

$$\overline{\delta^2(\Delta,T)} = \frac{\int_0^{T-\Delta}[x(t'+\Delta)-x(t')]^2\,dt'}{T-\Delta}$$



Normal diffusion: On average the number of jumps is proportional to the elapsed time

$$\langle [x(t+\Delta)-x(t)]^2\rangle = \langle \delta x^2\rangle n(t+\Delta,t) \simeq \frac{\langle \delta x^2\rangle}{\tau}\Delta$$

Single trajectory information equals ensemble information

$$\overline{\delta^2(\Delta,T)} \sim 2D_1\Delta = \lim_{T\to\infty}\overline{\delta^2}(\Delta,T),\ where\ D_1 = \frac{\langle \delta x^2\rangle}{2\tau}$$

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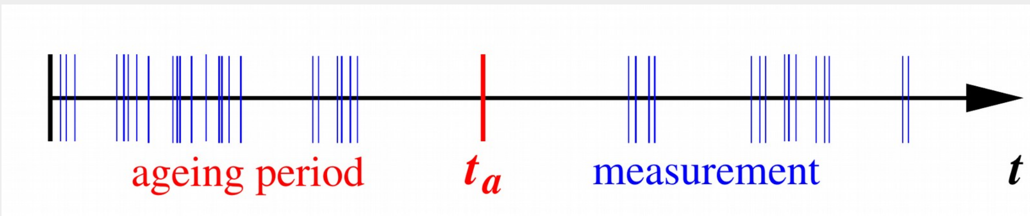
Ergodicity breaking and ageing in long rests CTRW

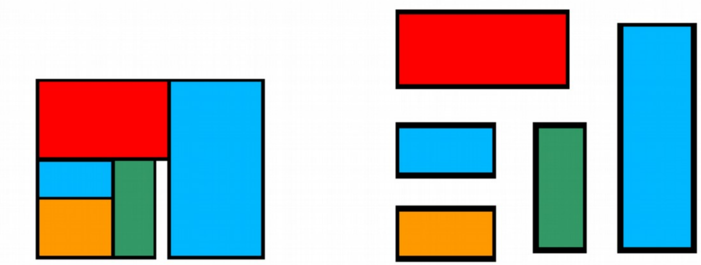
Ageing



$$\overline{\delta^2(\Delta, T)} = \int_{t_a}^{T-\Delta} \frac{\left[x(t+\Delta) - x(t)\right]^2}{T - t_a - \Delta} dt = \frac{\Lambda_\alpha(t_a/T)}{\Gamma(1+\alpha)} \frac{g(\Delta)}{T^{1-\alpha}}$$

In external compartment

$$g(\Delta) \simeq \frac{1}{T^{1-\alpha}} \begin{cases} K_\alpha \Delta, & \Delta \ll (K_\alpha \lambda_1)^{-1} \\ \Delta^{1-\alpha}, & (K_\alpha \lambda_1)^{-1} \ll \Delta \ll T \end{cases}$$



System is decomposed $\to$ strong ergodicity breaking.

System's space explored $\to$ weak ergodicity breaking.

J. Bouchaud J. Phys. I France (1992).

Bel and Barkai, PRL, 2005

Ergodicity Breaking in CTRW (long rests)

EAMSD

$$\langle x^2(t)\rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)}t^\alpha, \quad 0<\alpha<1$$

$$\langle \overline{\delta^2(\Delta,T)}\rangle = \frac{1}{N}\sum_{i=1}^{N}\overline{\delta_i^2(\Delta,T)}$$

For Gaussian process

$$\langle \overline{\delta^2(\Delta,T)}\rangle = \frac{1}{T-\Delta}\int_0^{T-\Delta}\langle (x(t+\Delta)-x(t))^2\rangle\, dt$$

$$\langle n(t+\Delta)-n(t)\rangle = \langle n(t+\Delta)\rangle - \langle n(t)\rangle = \frac{t+\Delta}{\tau} - \frac{t}{\tau} = \frac{\Delta}{\tau}$$

time-averaged mean-squared displacement

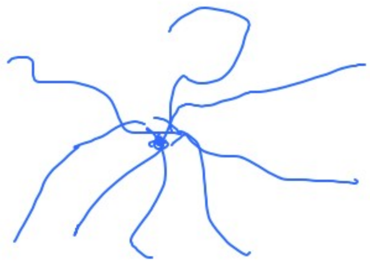
TAMSD

duration of a timestep

$$\langle \overline{\delta^2(\Delta,T)}\rangle = \frac{1}{T-\Delta}\int_0^{T-\Delta}\frac{\langle \delta x^2\rangle}{\tau}\Delta\, dt = 2K_1\Delta, \quad K_1 = \frac{\langle \delta x^2\rangle}{2\tau_1}$$

$$\langle x^2(t)\rangle = 2K_1 t$$

Ensemble averaging



$\rightarrow \quad P(\bar{r}, t)$

0 $\quad \Delta \rightarrow \quad$ T

Time averaging

$t_2 - t_1 = \Delta$

Δ

$x(t+\Delta) - x(t)$

t_1 $\quad t_2$ $\quad X(0)$ $\quad X(\Delta)$

$P(\bar{r}, \Delta)$

CTRW

ϕ ψ

$\underline{n}: \frac{\Delta}{\tau}$ for Gaussian case

Prob. of making n steps within time t

$w(t)$ is pdf of waiting times, $w(s) = \int_0^{\infty} e^{-st} w(t)\, dt = \langle e^{-st} \rangle$

Prob. of not moving $\psi(t) = 1 - \int_0^t w(t')\, dt'$

$$\psi(s) = \frac{1}{s} - \frac{w(s)}{s}$$

$t = t_1 + t_2 + \ldots + t_n$, n steps

pdf of making n steps

$$w_n(s) = \langle e^{-s(t_1 + \ldots + t_n)} \rangle = \langle e^{-st_1} \rangle \ldots \langle e^{-st_n} \rangle = w^n(s)$$

Prob. that by time t we made n steps:

$$\underline{Q_n(t)} = \int_0^t w_n(t')\,\psi(t-t')\,dt'$$

$$Q_n(s) = w_n(s)\,\psi(s) = \frac{1-w(s)}{s}\cdot w^n(s)$$

$$\langle n(s)\rangle = \sum_{i=0}^{\infty} i\,Q_i(s) = \frac{1-w(s)}{s}\,\frac{1/w(s)}{\left(1-\frac{1}{w(s)}\right)^2} = \frac{w(s)}{s(1-w(s))}$$

$s\to 0$

$t \gg \tau$ $\quad w(s) \sim 1-(\tau s)^\alpha$

$$\langle n(s)\rangle \sim \frac{1-(s\tau)^\alpha}{s(s\tau)^\alpha} \sim \frac{1}{\tau^\alpha s^{\alpha+1}} \xrightarrow{\text{Tauberian theorem}} \langle n(t)\rangle \sim \frac{(t/\tau)^\alpha}{\Gamma(1+\alpha)}$$

EAMSD $\langle x^2(t)\rangle = \langle \delta x^2\rangle\langle n(t)\rangle = \dfrac{2K_\alpha t^\alpha}{\Gamma(\alpha+1)}$, $\quad K_\alpha = \dfrac{\langle \delta x^2\rangle}{\tau^\alpha}$

TAMSD: $\langle \overline{\delta}(\Delta,T)\rangle = \dfrac{\langle \delta x^2\rangle}{(T-\Delta)\tau^\alpha}\displaystyle\int_0^{T-\Delta}\frac{(t+\Delta)^\alpha - t^\alpha}{\Gamma(1+\alpha)}\,dt \simeq \frac{2K_\alpha}{\Gamma(1+\alpha)}\,\frac{T^{1+\alpha}-\Delta^{1+\alpha}-(T-\Delta)^{1+\alpha}}{(1+\alpha)(T-\Delta)} =$

$$\langle \overline{\delta^2(\Delta, t)} \rangle \underset{T \gg \Delta}{=} \frac{2K_\alpha}{\Gamma(1+\alpha)} \frac{T^{1+\alpha} - \Delta^{1+\alpha} - T^{1+\alpha}\left(1-(1-\alpha)\frac{\Delta}{T}\right)}{(1+\alpha)(T-\Delta)} \sim \frac{2K_\alpha}{\Gamma(1+\alpha)} \cdot \frac{\Delta}{T^{1-\alpha}}$$

$$\langle \overline{\delta^2(\Delta, t)} \rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)} \frac{\Delta}{T^{1-\alpha}} \neq \langle x^2(\Delta) \rangle$$

$$T \gg \Delta$$

weak ergodicity breaking

Ageing phenomenon

↑ the start of the process

t_a – ageing time – the start of observation $\sim t^{\alpha}$

$$\langle x^2(\tau)\rangle_a = \langle x^2(t_a+\tau)\rangle - \langle x^2(t_a)\rangle \simeq \langle \delta x^2\rangle\left(\langle n(t_a+\tau)\rangle - \langle n(t_a)\rangle\right)$$

$$= \langle \delta x^2\rangle\left((t_a+\tau)^{\alpha} - t_a^{\alpha}\right)\frac{1}{\tau^{\alpha}\Gamma(1+\alpha)} = \frac{2K_\alpha}{\Gamma(1+\alpha)}\left((t_a+\tau)^{\alpha} - t_a^{\alpha}\right)$$

$\alpha = 1$ (Brownian motion)

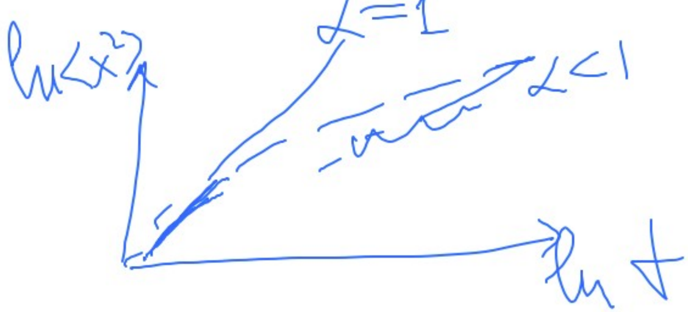
$$\langle x^2(\tau)\rangle_a = \frac{2K_1}{\underbrace{\Gamma(1+1)}_{=1}}\left(t_a+\tau-t_a\right) = 2K_1\tau$$

CTRW ageing

$$\langle x^2(\tau)\rangle_a = \frac{2K_\alpha}{\Gamma(1+\alpha)}\left((t_a+\tau)^\alpha - t_a^\alpha\right)$$

$\tau \gg t_a$ $$\langle x^2(\tau)\rangle_a = \frac{2K_\alpha \tau^\alpha}{\Gamma(1+\alpha)} = \langle x^2(\tau)\rangle$$

$\tau \ll t_a$ $$\langle x^2(\tau)\rangle_a \sim \frac{2K_\alpha \tau}{\Gamma(1+\alpha)\, t_a^{1-\alpha}}$$



ageing produces observation of normal diffusion

PDF of the first jump to occur after (ageing) for t_a $\quad \psi_1(t, t_a)$

$$\psi_n(t) = \int_0^t w_n(t') w(t_a - t' + t) dt'$$

prob. to make n steps and that we wait for $(n+1)$st step

$$\psi_1(t, t_a) = \sum_{n=0}^{\infty} \psi_n(t) = \int_0^t \left(\sum_{n=0}^{\infty} w_n(t') w(t_a - t' + t) dt' \right)$$

↓ double Laplace transform in t & t_a

$$\psi_1(s, u) = \frac{1}{1 - w(u)} \frac{w(u) - w(s)}{s - u} \sim \frac{s^\alpha - u^\alpha}{(s-u) u^\alpha}$$

$w(u) \sim 1 - (u\tau)^\alpha$

↓ Back transform →

$$\psi_1(t, t_a) = \frac{\sin(\pi\alpha)}{\pi} \frac{t_a^{\alpha}}{t^{\alpha}(t + t_a)}, \quad 0 < \alpha < 1$$

if t_a is long

$$\psi_1(t, t_a) \sim \frac{t_a^{\alpha-1}}{t^{\alpha}}$$

if t_a is small

$$\psi_1(t, t_a) \sim t_a^{\alpha} t^{-1-\alpha}$$

Aging in CTRW

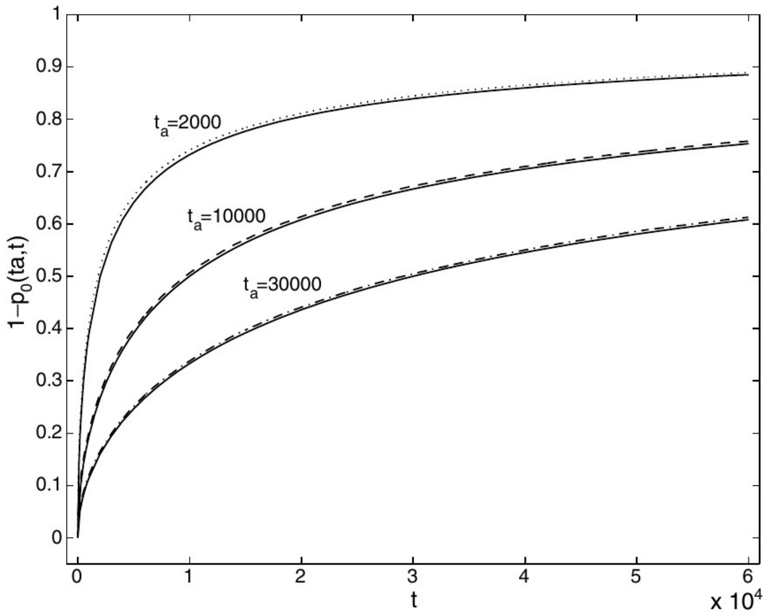


FIG. 2. The probability of making at least one step in a time interval $(0, t)$ for different aging times specified in the figure. The solid curve is the theoretical prediction Eq. (9); the dotted, dashed, and dot-dashed curves are obtained from a numerical solution of the map with $z = 3$.

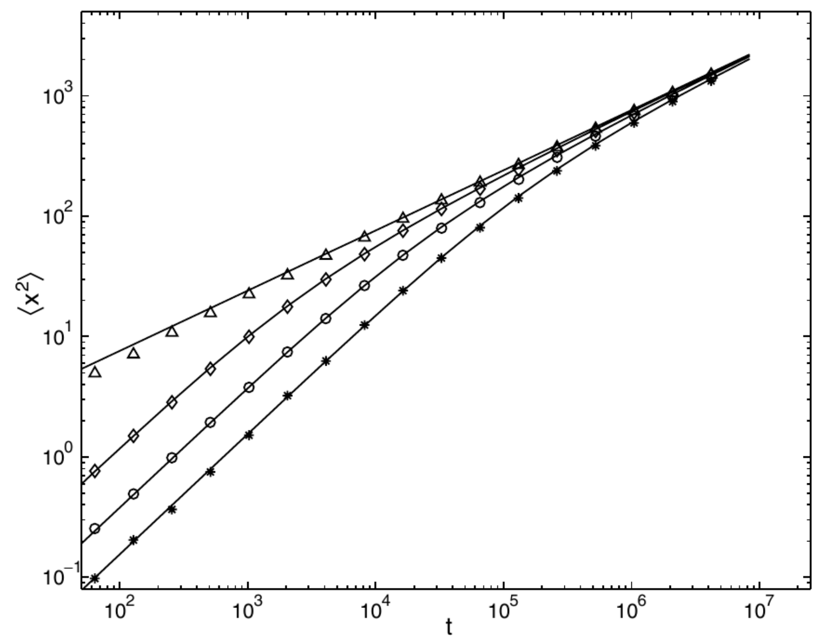


FIG. 3. The mean square displacement versus the forward time t for different aging times t_a; $t_a = 0$, triangle; $t_a = 1000$, diamond; $t_a = 10\,000$, circle; and $t_a = 60\,000$, star. The solid curves are the theoretical prediction Eq. (14). Here $z = 3$.

$$\langle x^2(t_a, t)\rangle \sim \frac{1}{A\Gamma(1+\alpha)}\left[(t+t_a)^\alpha - t_a^\alpha\right].$$

E. Barkai, Phys. Rev. Lett. 90, 104101 (2003)

Long Jumps/relocations: Fractional Diffusion Equation

Space-fractional Fokker Planck equation. Describes superdiffusion

$$\boxed{\frac{\partial P(x,t)}{\partial t} = {}_{-\infty}D_x^{\mu} K_{\mu} P(x,t)}$$

Mode relaxation

$$P(k,t) = \exp(-K^{\mu} t |k|^{\mu})$$

Solution

$$P(x,t) = \frac{1}{\mu|x|} H_{2,2}^{1,1}\left[\frac{|x|}{(K^{\mu}t)^{1/\mu}}\middle|\begin{matrix}(1,1/\mu),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right]$$

$$P(x,t) \sim \frac{K^{\mu} t}{|x|^{1+\mu}}, \quad \mu < 2 \longrightarrow \langle x^2(t)\rangle \to \infty$$

Long jumps CTRW

$$\lambda(x) = \int_{0}^{\infty} dt\, \varphi(x,t)$$

$$P(k,s) = \frac{1-w(s)}{s} \cdot \frac{P_0(k)}{1 - \varphi(k,s)}$$

$$\Sigma^2 \to \infty \qquad w(t) = \tau^{-1} e^{-t/\tau} \qquad w(s) \sim 1 - s\tau$$

jumps: $\lambda(x) \sim \frac{\sigma^\mu}{|x|^{1+\mu}}, \mu < 2$

$$\lambda(k) \sim 1 - (\sigma k)^\mu + o(k^\mu)$$

$$P(k,s) = \frac{P_0(k)}{s + \frac{\sigma^\mu}{\tau} k^\mu}$$

$$\frac{\partial P}{\partial t} = \cancel{k_\mu} \, k_\mu \, {}_{-\infty}D_x^\mu P(x,t)$$

$$s P(k,s) - P_0(k) = -\frac{\sigma^\mu}{\tau} \underline{k^\mu P(k,s)}$$

Long Jumps/relocations: Fractional Diffusion Equation

Space-fractional Fokker Planck equation

$$\boxed{\frac{\partial P(x,t)}{\partial t} = {}_{-\infty}D_x^{\mu} K_{\mu} P(x,t)}$$

$$\langle x^2(t) \rangle \to \infty$$

Let's consider

$$\langle |x|^{\delta} \rangle \propto t^{\delta/\mu} \qquad 0 < \delta < \mu$$

$$\langle |x|^{\delta} \rangle = \frac{2}{\mu} \int_0^{\infty} \mathrm{d}x\, x^{\delta-1} H_{2,2}^{1,1}\left[\frac{|x|}{(K^{\mu}t)^{1/\mu}} \middle| \begin{matrix} (1, 1/\mu), (1, 1/2) \\ (1, 1), (1, 1/2) \end{matrix} \right] \equiv \frac{2}{\mu} \mathscr{M} \left\{ H_{2,2}^{1,1} \left(\frac{|x|}{(K^{\mu}t)^{1/\mu}} \right) \right\}$$

Where *M* is a Mellin transform,

$$\mathscr{M}\{f(t); s\} = \int_0^{\infty} \mathrm{d}t\, t^{s-1} f(t)$$

Long Jumps/relocations: Fractional Diffusion Equation

Space-fractional Fokker Planck equation

$$\boxed{\frac{\partial P(x,t)}{\partial t} = {}_{-\infty}D_t^{\mu} K_{\mu} P(x,t)}$$

Let's consider

$$\langle |x|^{\delta} \rangle \propto t^{\delta/\mu} \qquad 0 < \delta < \mu$$

$$\int_0^{\infty} \mathrm{d}x\, x^{s-1} H_{p,q}^{m,n}\left[ax \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] = a^{-s}\chi(-s)$$

$$\chi(s) = \frac{\prod_1^m \Gamma(b_j - B_j s) \prod_1^n \Gamma(1 - a_j + A_j s)}{\prod_{m+1}^q \Gamma(1 - b_j + B_j s) \prod_{n+1}^p \Gamma(a_j - A_j s)}$$

Long Jumps/relocations: Fractional Diffusion Equation

Space-fractional Fokker Planck equation

$$\boxed{\frac{\partial P(x,t)}{\partial t} = {}_{-\infty}D_x^{\mu} K_{\mu} P(x,t)}$$

Let's consider

$$\langle |x|^{\delta} \rangle \propto t^{\delta/\mu} \qquad 0 < \delta < \mu$$

$$\int_0^{\infty} \mathrm{d}x\, x^{s-1} H_{p,q}^{m,n}\left[ax \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] = a^{-s} \chi(-s)$$

$$\chi(s) = \frac{\prod_1^m \Gamma(b_j - B_j s) \prod_1^n \Gamma(1 - a_j + A_j s)}{\prod_{m+1}^q \Gamma(1 - b_j + B_j s) \prod_{n+1}^p \Gamma(a_j - A_j s)}$$

Hence

$$\langle |x|^{\delta} \rangle = \frac{2}{\mu} (K^{\mu} t)^{\delta/\mu} \chi(-\delta) = \frac{2}{\mu} (K^{\mu} t)^{\delta/\mu} \frac{\Gamma(-\delta/\mu)\Gamma(1+\delta)}{\Gamma(-\delta/2)\Gamma(1+\delta/2)}$$

Long Jumps/relocations: Fractional Diffusion Equation

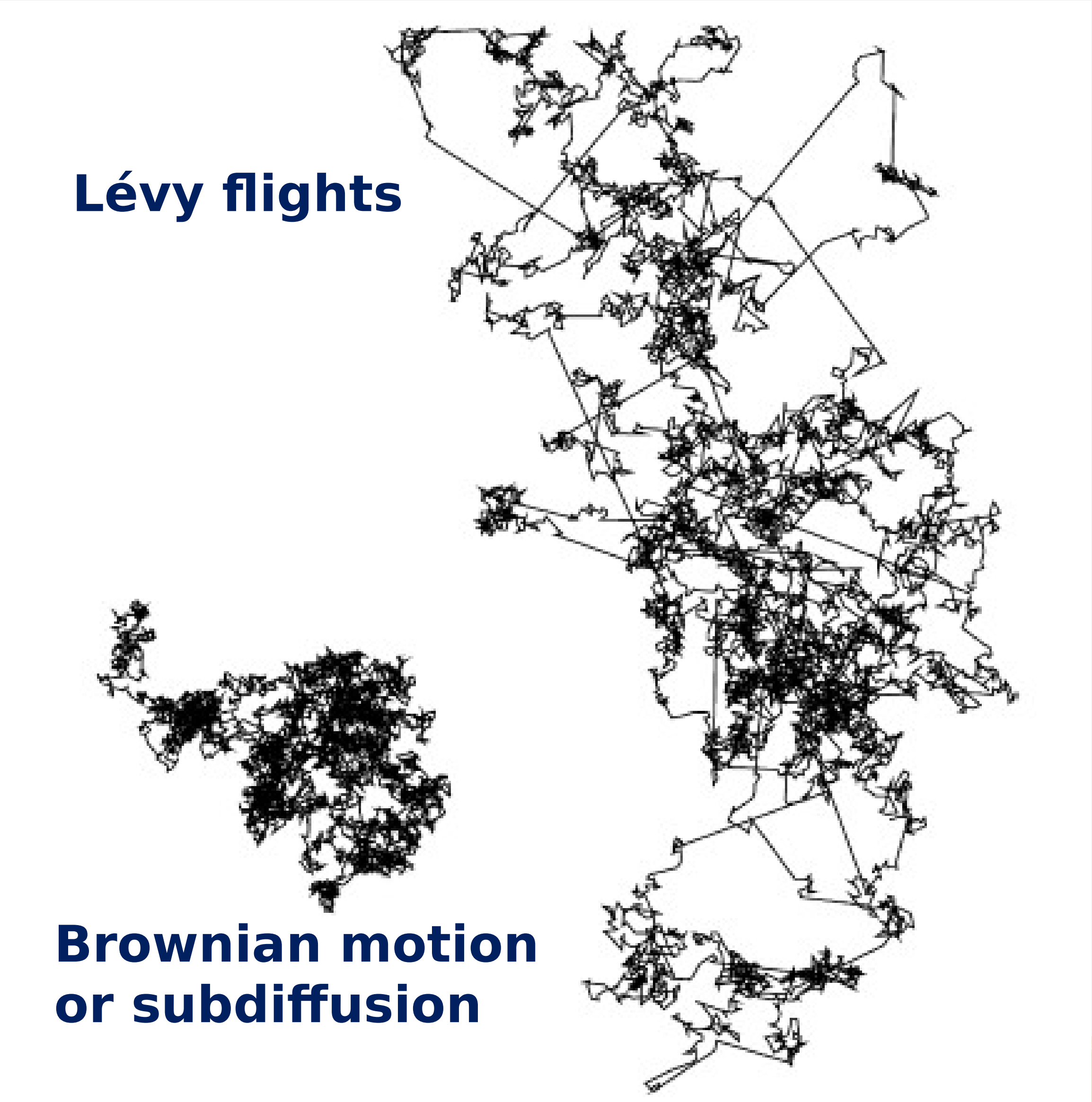
Space-fractional Fokker Planck equation. Lévy flights

$$\langle|x|^{\delta}\rangle = \frac{2}{\mu}(K^{\mu}t)^{\delta/\mu}\chi(-\delta) = \frac{2}{\mu}(K^{\mu}t)^{\delta/\mu}\frac{\Gamma(-\delta/\mu)\Gamma(1+\delta)}{\Gamma(-\delta/2)\Gamma(1+\delta/2)} \quad 0<\delta<\mu$$

$$\langle|x|^{\delta}\rangle^{2/\delta} \sim t^{2/\mu}, \mu<2$$

Hence, we proved that space-fractional Fokker-Planck equation and CTRW with diverging second moment of λ(x) produce superdiffusion

Lévy flights. Superdiffusion



Lévy flights. Superdiffusion

Markovian

CTRW superdiffusion (Lévy flights)
CTRW normal diffusion (Brownian motion)

Non-Markovian

CTRW subdiffusion

$$X(t) = X_0 + \sum_{i=1}^{N(t)} \Delta X_i$$

Trajectories of CTRW normal diffusion and CTRW subdiffusion are identical.
Whence comes the concept of *subordination*:

$$P(x,t) = \int_0^\infty T(t,\tau)\, G(x,\tau)\, d\tau$$

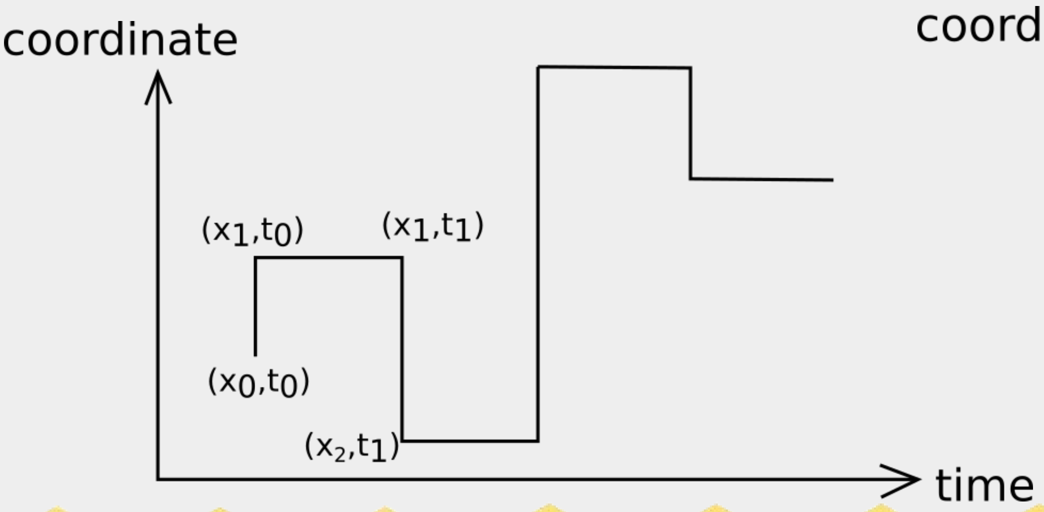
The probability for the walker to arrive at position *x* at time *t* equals the probability of being at *τ* on the path at time *t*, multiplied by the probability of being at position *x* for this path length τ, summed over all path lengths.

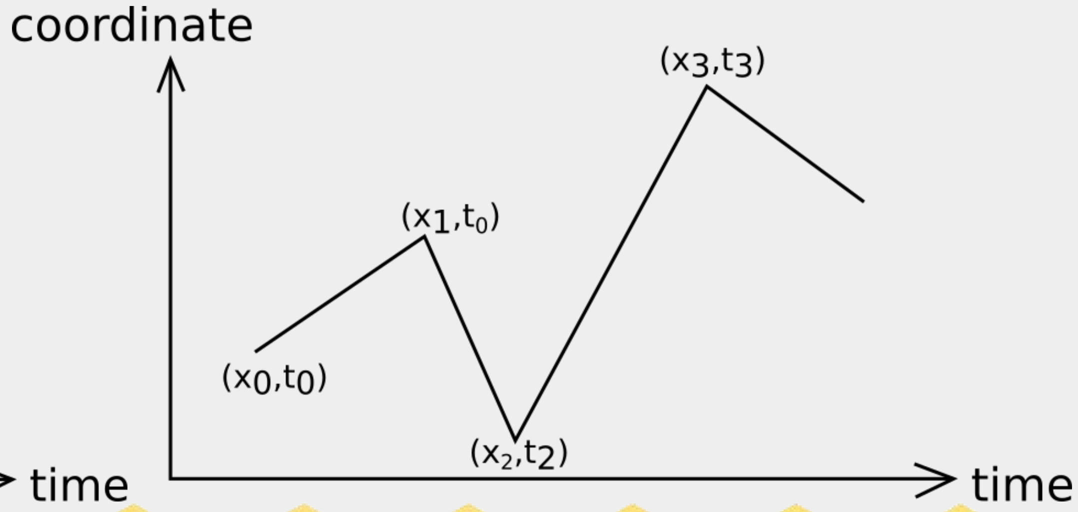
Subdiffusive CTRW processes are directly subordinated to their analogous Gaussian process

Lévy walks:
an example of a process with a coupled jump lengths/ waiting times distribution

$$\psi(x,t) = \frac{1}{2}\,\delta(|x| - vt)\,w(t)$$

Lévy flights **Lévy walks**





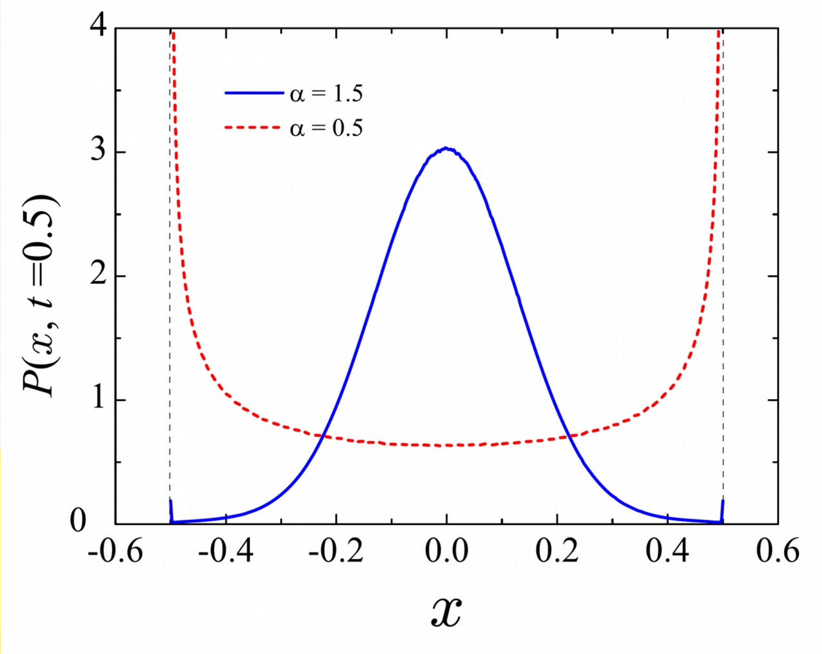
Lévy Walks

an example of the process with a coupled jump lengths/
waiting times distribution. Can be obtained through subordination from Lévy Flights

Joint PDF (classical realisation)

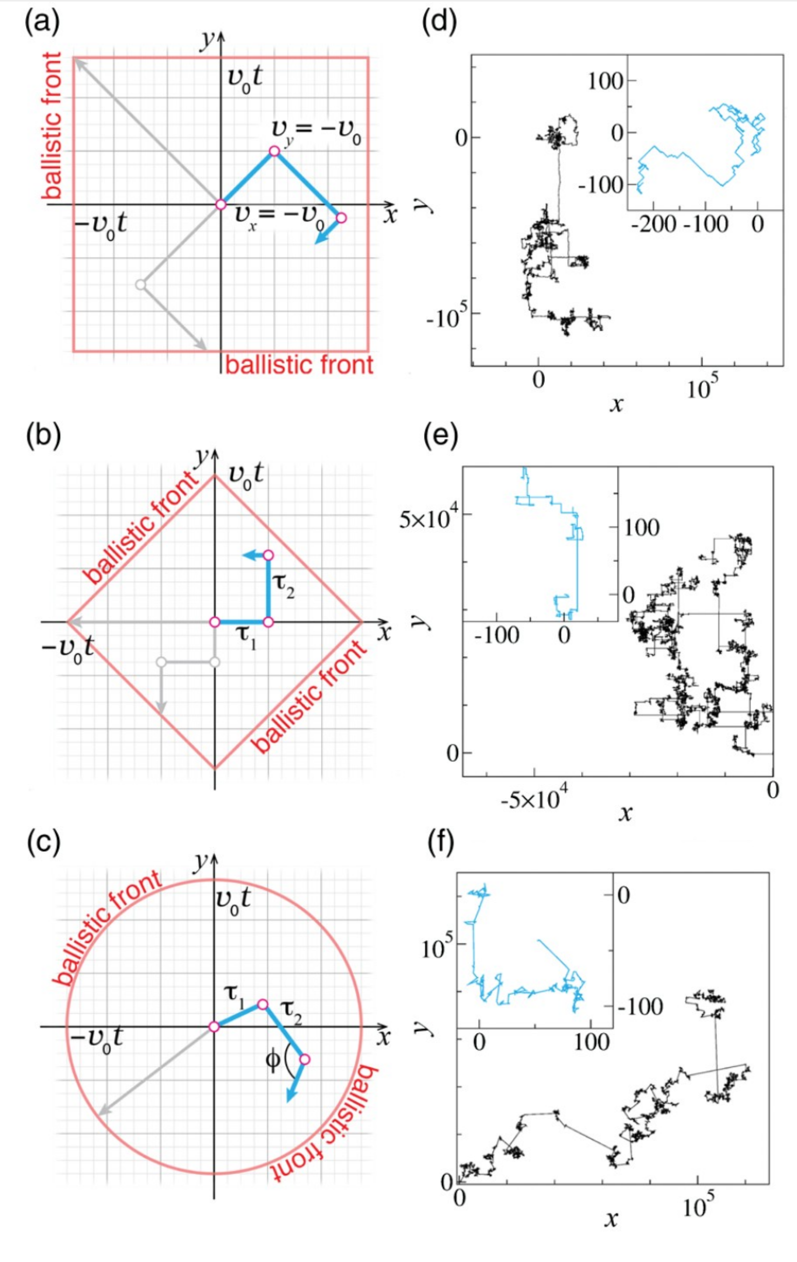
$$\psi(x,t)=\frac{1}{2}\delta(|x|-vt)w(t)$$

$$P(k,s)=\frac{[\Psi(s+ikv)+\Psi(s-ikv)]P_0(k)}{2-[\psi(s+ikv)+\psi(s-ikv)]}$$



VVP et al, NJP, 2019

Lévy Walks on the plane



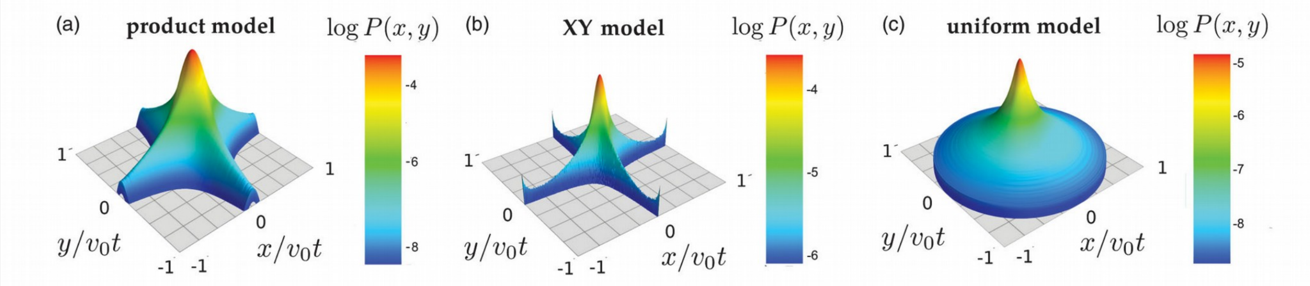


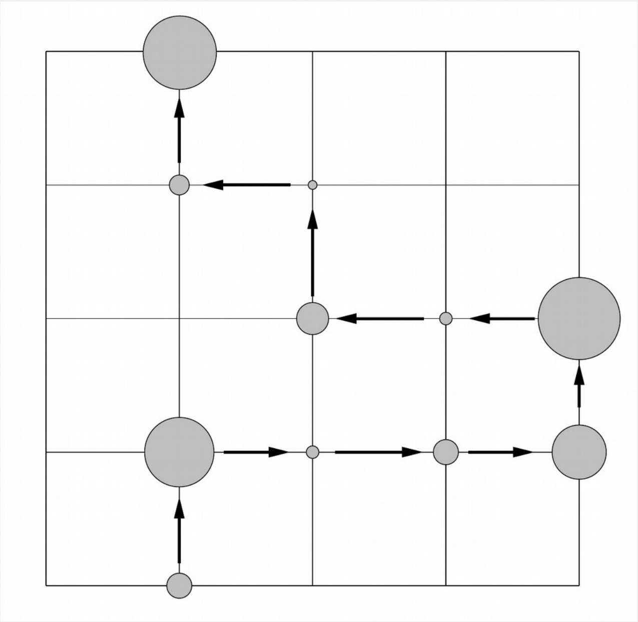
FIG. 2. Probability density functions of the three models in the superdiffusive regime. The distributions are plotted on a log scale for the time $t/\tau_0 = 10^4$. The PDF for the product model (a) was obtained by multiplying PDFs of two identical one-dimensional LW processes. The PDFs for the XY (b) and uniform (c) models were obtained by sampling over 10^{14} realizations. The parameters are $\gamma = 3/2$, $v_0 = 1$, and $\tau_0 = 1$.

Zaburdaev, Fouxon, Denisov, Barkai, PRL, 2016

CTRW summary

$$X(t) = X_0 + \sum_{i=1}^{N(t)} \Delta X_i$$

$$\boxed{P(k,s) = \frac{1-w(s)}{s}\,\frac{P_0(k)}{1-\varphi(k,s)}}$$



$$\varphi(x,t)$$

Superdiffusion

Subdiffusion

Normal

Levy flights

Levy walks

Same trajectory.
Connected through subordination

Same trajectory.
Connected through subordination

Non-Markovian

Markovian

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FFPE with potential

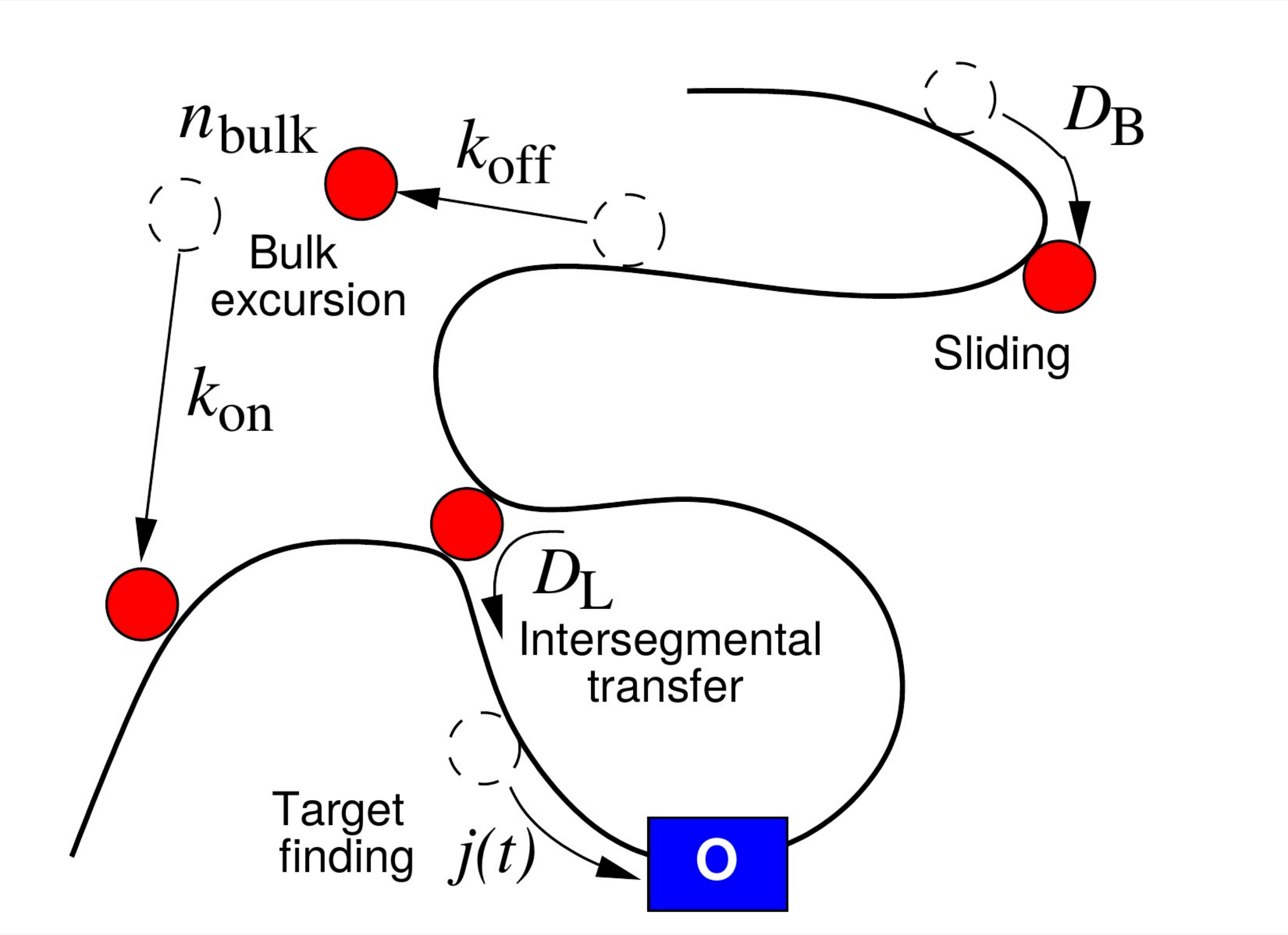
For superdiffusion (Markovian)

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial (U'(x)P(x,t))}{\partial x} + {}_{-\infty}D_t^{\mu} K_{\mu} P(x,t)$$

For subdiffusion (Non-Markovian)

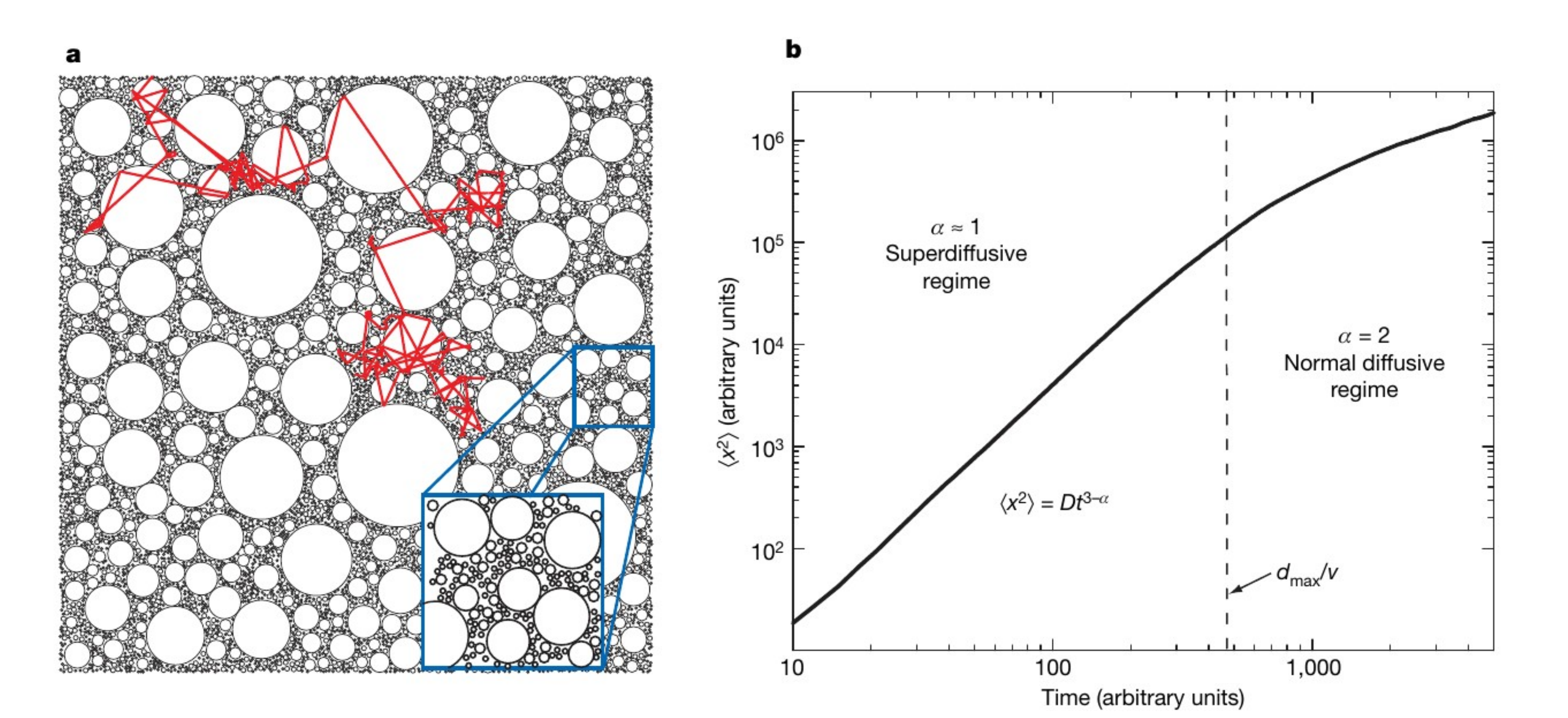
$$\frac{\partial P(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left(-\frac{\partial (U'(x)P(x,t))}{\partial x} + K_{\alpha}\frac{\partial^2}{\partial x^2}P(x,t)\right)$$

Lévy Flight in a chemical space. Protein on DNA search



M. Lomholt, T. Ambjörnsson & R. Metzler, PRL (2005)

Light as a Lévy Flight



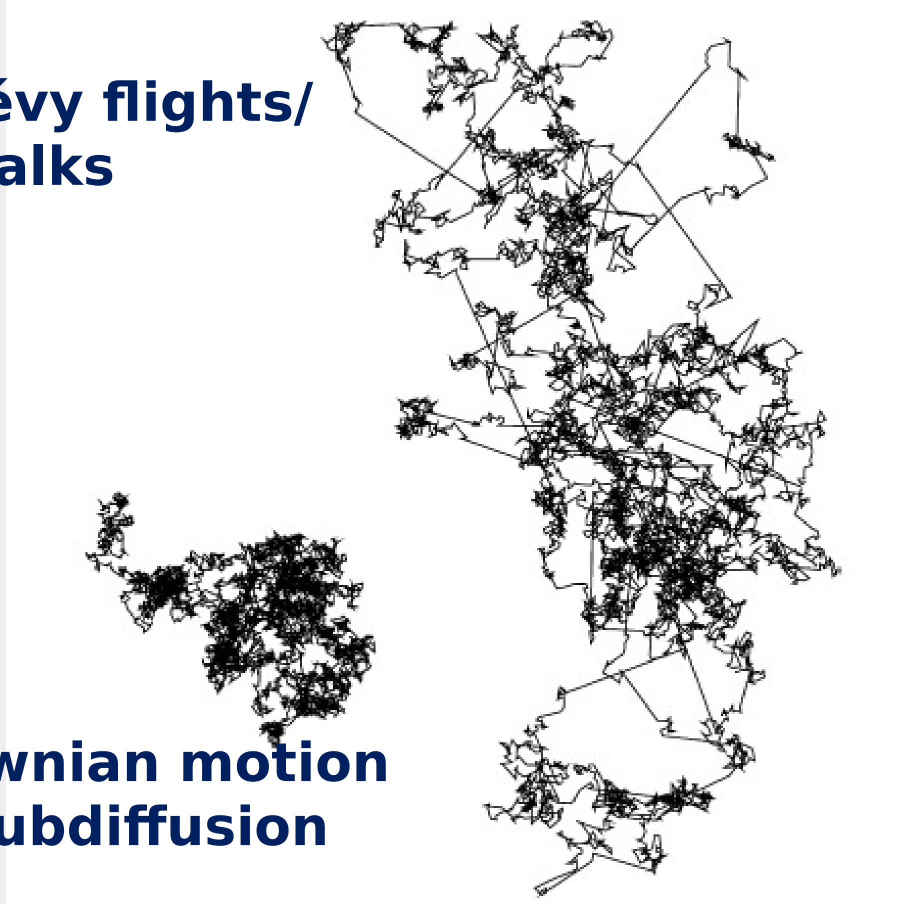
high-refractive-index scattering particles (of titanium dioxide) in a glass matrix

Pierre Barthelemy et al., Nature, volume 453, pp 495–498(2008)

Foraging and search

Are Lévy flights more efficient?

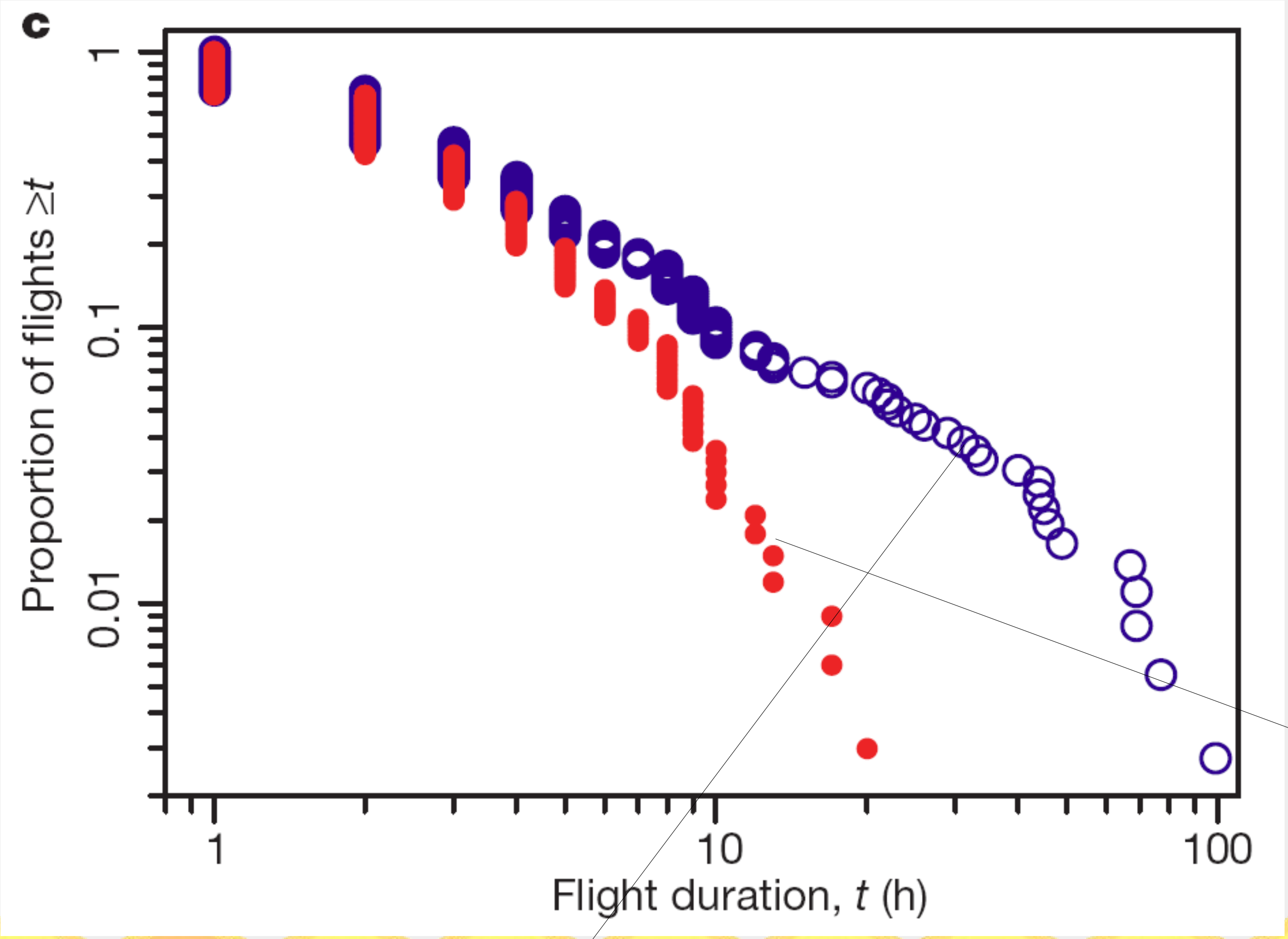
Search by Lévy flights/walks does not lead to oversampling in 1D and 2D which is typical for Brownian search



Universality?

Shlesinger and Klafter 1986: LFs visit more locations and return less than BM

Albatross story and Lévy Flight Foraging Hypothesis

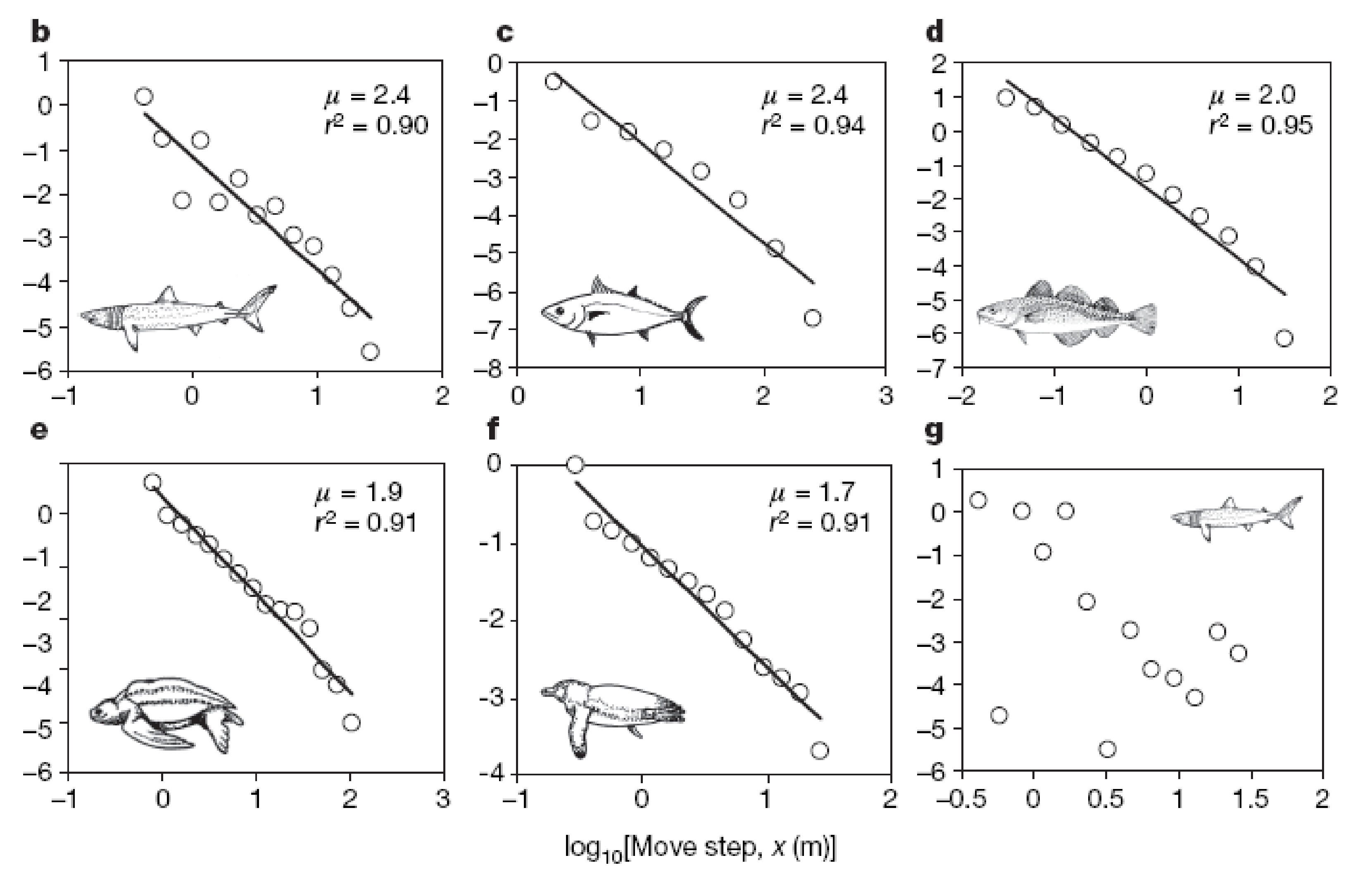


recalculated in Edwards AM, et al., Nature, 2007

Viswanathan GM, et al. (1999) Optimizing the success of random searches. Nature 401: 911–914

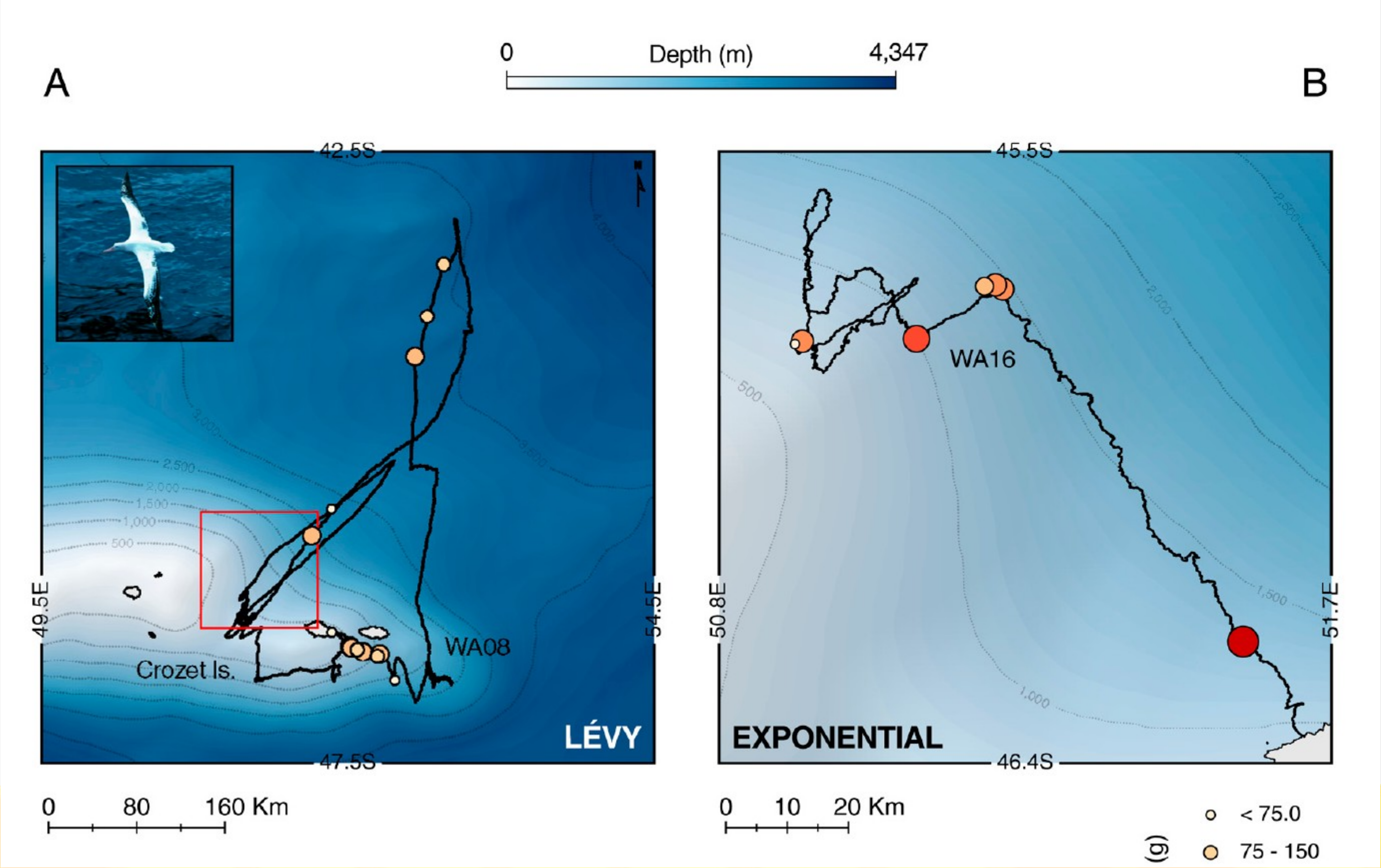
Viswanathan GM, et al. (1996) Lévy flight search patterns of wandering albatrosses. Nature 381: 413–415

Albatross story and Lévy Flight Foraging Hypothesis



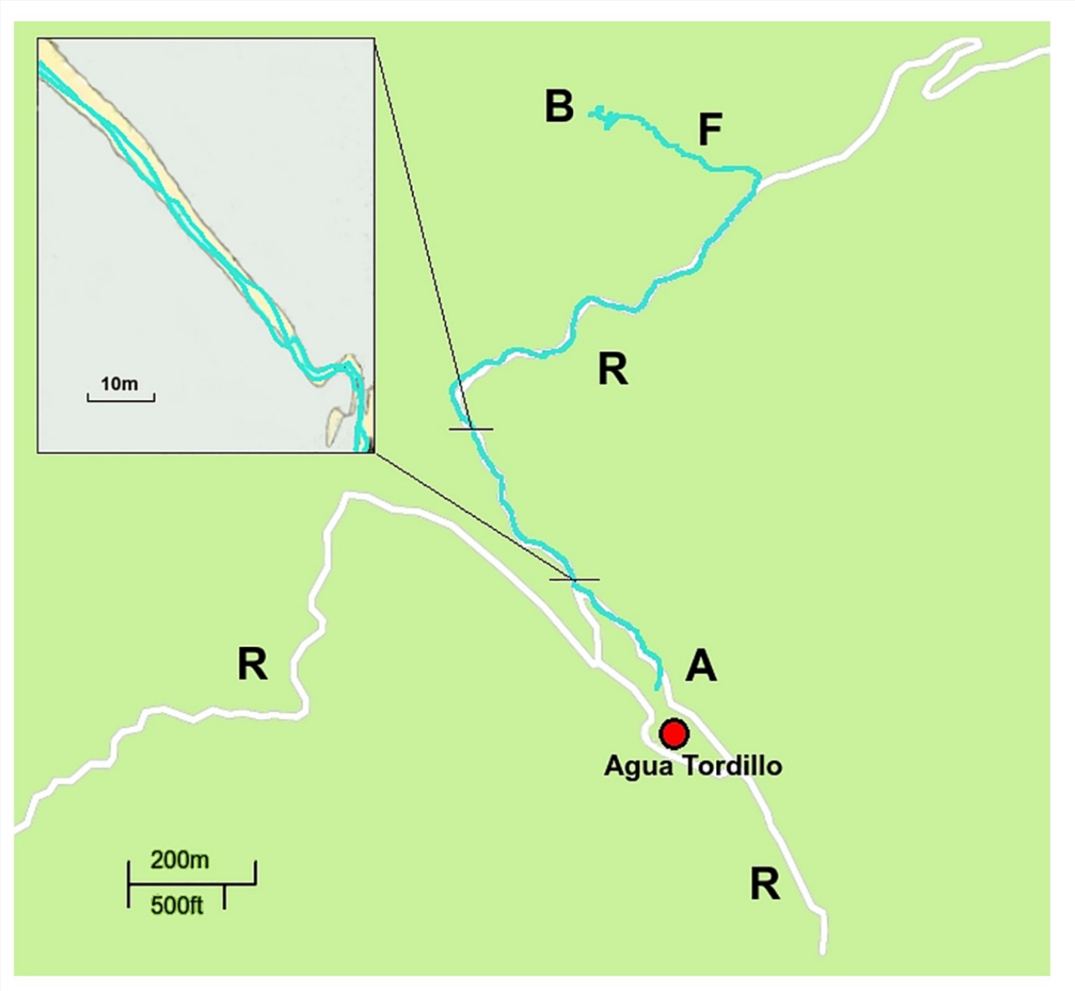
D.W. Sims et al, Nature, Vol 451, 1059 (2008)

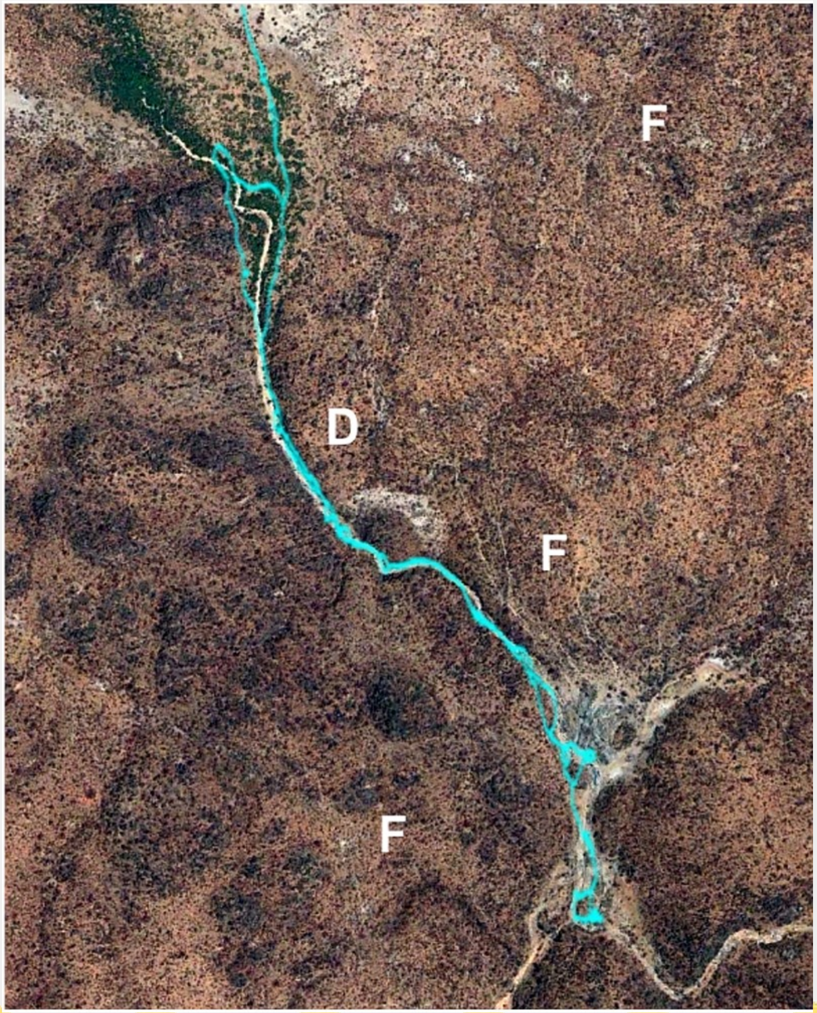
Albatross story and Lévy Flight Foraging Hypothesis



Humphries et al, PNAS, 2012

Lévy foraging patterns of rural humans





Reynolds A, Ceccon E, Baldauf C, Karin Medeiros T, Miramontes O (2018) . PLoS ONE 13(6): e0199099. https://doi.org/10.1371/journal.pone.0199099

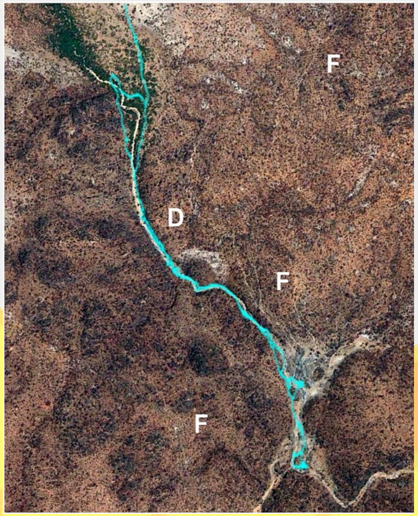
Lévy Flight Paradigm

Lévy search hypothesis (*internal reason*)

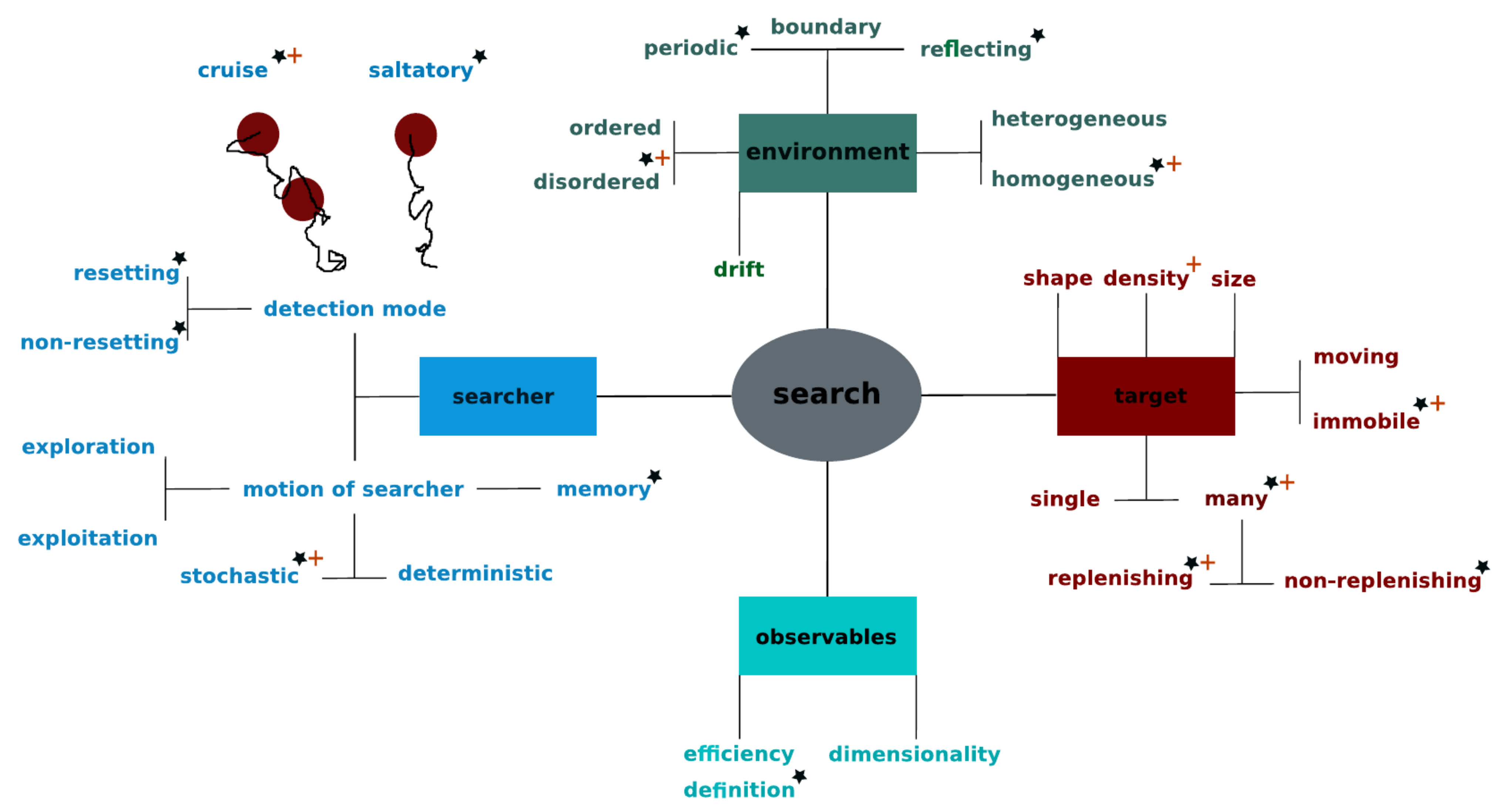
- Minimal cues
- Evolutionary view
- Lab experiments
- Internal interaction complexity

Lévy environmental hypothesis (*external reason*)

- Environmental complexity
- Maximal cues
- Ecological view
- Field experiments



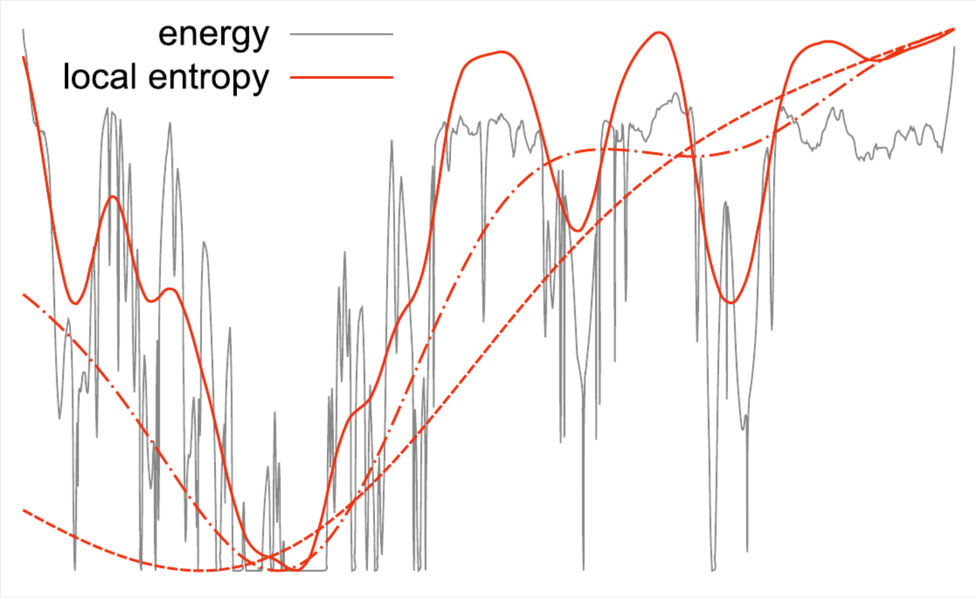
Complexity of the search



S. Mohsen J. Khadem, Sabine H. L. Klapp, and Rainer Klages
Phys. Rev. Research 3, 023169 (2021)

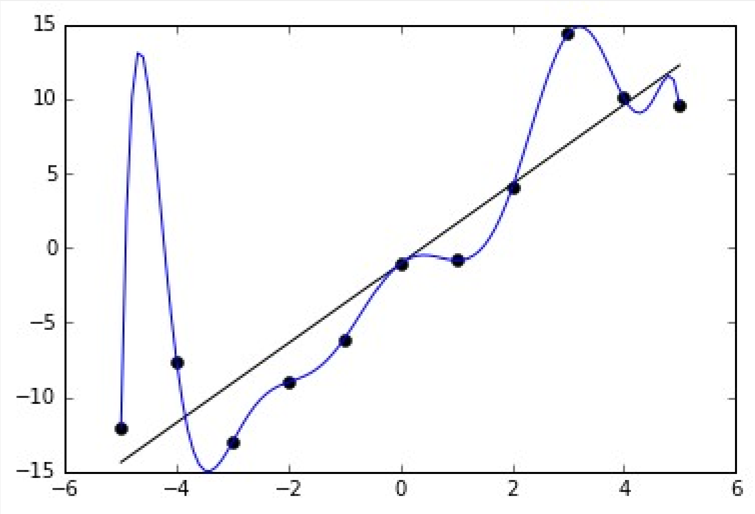
Other kinds of search/exploration settings

Neural networks propagation as a search in an energy landscape

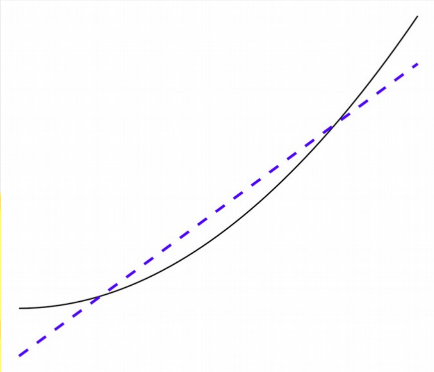


Baldassi et al., PNAS 2016

Overfitting



Underfitiing



By Ghiles - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=47471056

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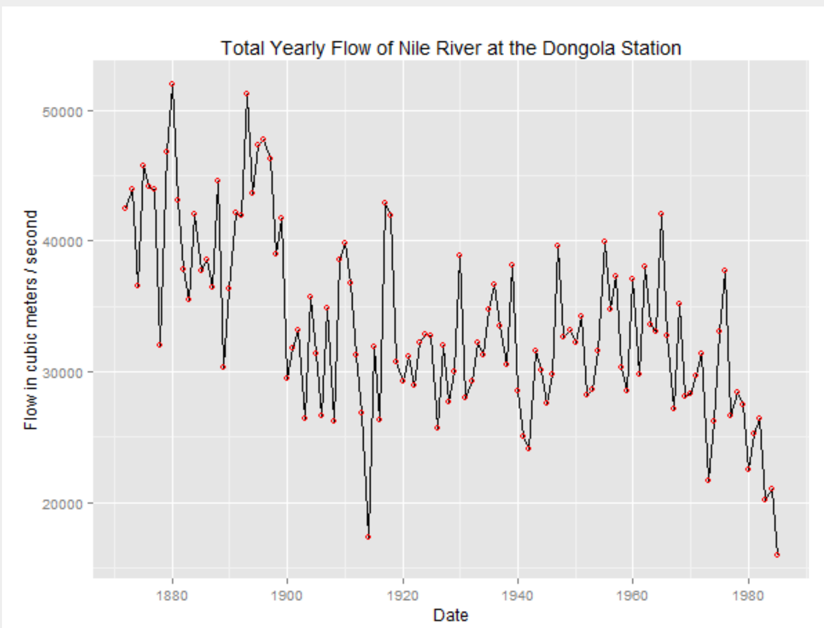
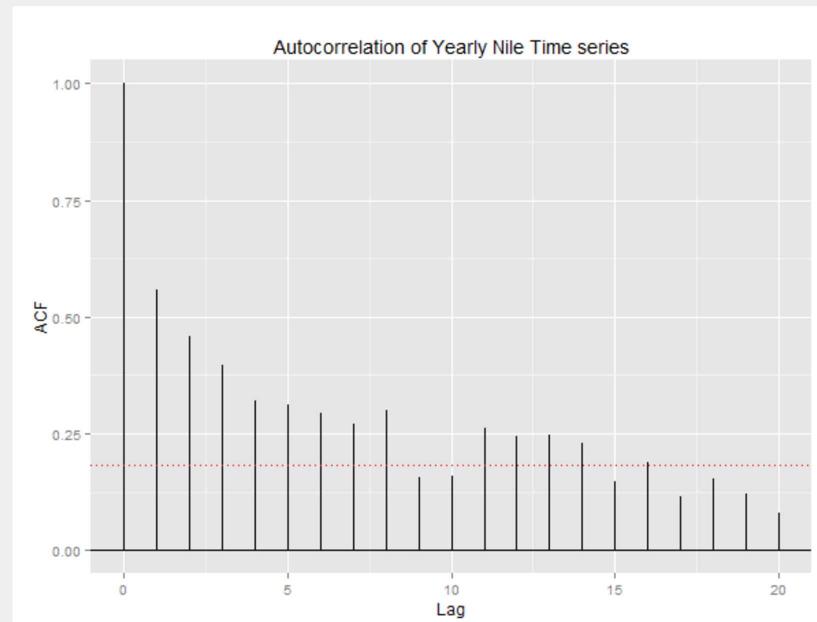
Fractional Brownian Motion. Intro

Harold Edwin Hurst studied Egyptian 875-year record of the Nile River's overflows and observed that flood occurrences are correlated, i.e. heavier floods were accompanied by above average flood occurrences, while below average occurrences were followed by minor floods





https://blog.revolutionanalytics.com/2014/09/intro-to-long-memory-herodotus-hurst-and-h.html

FBM applications: Hydrology, Seismology, Turbulence, Finance etc etc

Fractional Brownian Motion

increments are correlated

Kolmogorov 1940, Mandelbrot and van Ness 1968

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \xi_{\mathrm{fGn}}(t)$$

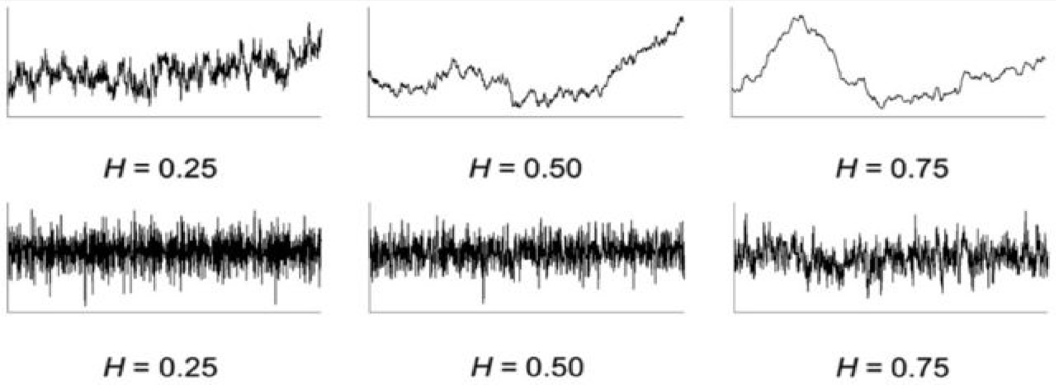
$$\langle \xi_{fGn}(t) \rangle = 0$$

$$\langle \xi_{\mathrm{fGn}}(t_1)\xi_{\mathrm{fGn}}(t_2)\rangle = \alpha(\alpha - 1)K_\alpha{}^{*}|t_1 - t_2|^{\alpha-2}$$

Note, in FBM literature the so-called Hurst exponent is used, H = α/2

H<1/2 is antipersistent motion

H>1/2 is a persistent motion



Fractional Brownian motion

$$\frac{dx(t)}{dt} = \xi_{fGn}(t)$$

$$P(\xi) = \frac{1}{4\pi}\exp\left(-\frac{\xi^2}{4}\right)$$ Gaussian noise

$$\langle \xi(t_1)\,\xi(t_2)\rangle \sim \alpha \cdot \underbrace{K_\alpha}_{\text{diff. coefficient}} \cdot (\alpha-1)\cdot |t_1 - t_2|^{\alpha-2}$$

$$X(t) = \int_0^t \xi(t')\,dt'$$

$$\langle x(t_1)\,x(t_2)\rangle = \int_0^{t_1} dt' \int_0^{t_2} dt'' \langle \xi(t')\,\xi(t'')\rangle = \int_0^{t_1} dt' \int_0^{t_2} \alpha \cdot K_\alpha (\alpha-1)\,|t'-t''|^{\alpha-2}\,dt'' =$$

$$= \alpha K_\alpha(\alpha-1)\int_0^{t_1} dt' \left\{ \int_0^{t'} (t'-t'')^{\alpha-2}\,dt'' + \int_{t'}^{t_2} (t''-t')^{\alpha-2}\,dt'' \right\} =$$

$$= \alpha K_\alpha(\alpha-1)\int_0^{t_1} dt' \left\{ \frac{(t')^{\alpha-1}}{\alpha-1} + \frac{|t_2-t'|^{\alpha-1}}{\alpha-1} \right\} = K_\alpha\left[t_1^\alpha + t_2^\alpha + |t_1-t_2|^\alpha\right]$$

$$\text{MSD: } \langle x^2(t)\rangle = 2K_\alpha t^\alpha$$

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Fractional Brownian Motion

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \xi_{\mathrm{fGn}}(t)$$

$$\langle \xi_{\mathrm{fGn}}(t_1)\xi_{\mathrm{fGn}}(t_2)\rangle = \alpha(\alpha - 1)K_\alpha{}^{*}|t_1 - t_2|^{\alpha-2}$$

$$P(x,t) = \frac{1}{\sqrt{4\pi K_\alpha{}^{*} t^\alpha}}\exp\left(-\frac{x^2}{4K_\alpha{}^{*} t^\alpha}\right)$$

$$\langle x(t_1)x(t_2)\rangle = K_\alpha{}^{*}(t_1^\alpha + t_2^\alpha - |t_1 - t_2|^\alpha)$$

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle = 2K_\alpha{}^{*}\Delta^\alpha = \langle x^2(\Delta)\rangle$$

Anomalous process but the PDF is Gaussian!!

Fractional Langevin Equation (subdiffusion)

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\gamma^*\int_0^t (t-t')^{\alpha-2}\left(\frac{\mathrm{d}x(t')}{\mathrm{d}t'}\right)\mathrm{d}t' + \eta^*\xi_{\mathrm{fGn}}(t)$$

$1<\alpha<2$

Overdamped FLE

$$\gamma^*\int_0^t (t-t')^{\alpha-2}\left(\frac{\mathrm{d}x(t')}{\mathrm{d}t'}\right)\mathrm{d}t' = \eta^*\xi_{\mathrm{fGn}}(t)$$

Caputo fractional derivative

$$\frac{\mathrm{d}^{2-\alpha}x(t)}{\mathrm{d}t^{2-\alpha}} = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-t')^{\alpha-2}\left(\frac{\mathrm{d}x(t')}{\mathrm{d}t'}\right)\mathrm{d}t'$$

Fractional Langevin equation

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\gamma^*\Gamma(\alpha-1)\frac{\mathrm{d}^{2-\alpha}x(t)}{\mathrm{d}t^{2-\alpha}} + \eta^*\xi_{\mathrm{fGn}}(t)$$

MSD of FLE

$$\langle x^2(\Delta)\rangle = \lim_{t\to\infty}\overline{\delta^2(\Delta)} = \frac{2k_{\mathrm{B}}T\Delta^2}{m}E_{\alpha,3}\left(-\Gamma(\alpha-1)\frac{\gamma^*}{m}\Delta^\alpha\right)$$

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Fractional Langevin Equation

$$m\frac{d^2x(t)}{dt^2} = -\gamma^* \Gamma(\alpha - 1)\frac{d^{2-\alpha}x(t)}{dt^{2-\alpha}} + \eta^* \xi_{\mathrm{fGn}}(t)$$

$$\langle x^2(\Delta)\rangle = \lim_{t\to\infty} \overline{\delta^2(\Delta)} = \frac{2k_B T \Delta^2}{m} E_{\alpha,3}\left(-\Gamma(\alpha-1)\frac{\gamma^*}{m}\Delta^\alpha\right)$$

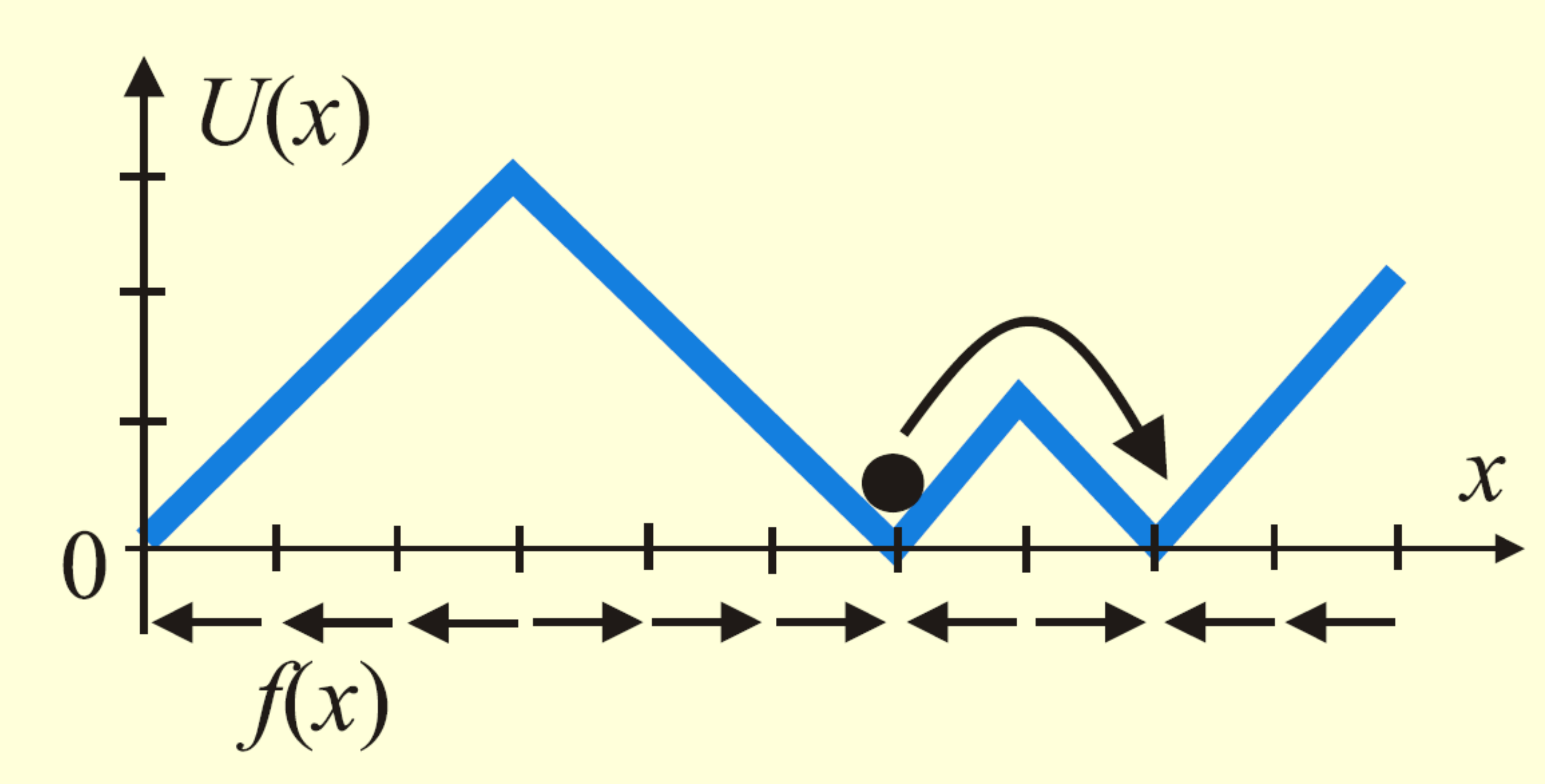
$$\langle x^2(t)\rangle \sim \begin{cases} k_B T t^2/m, & t \ll [m/\gamma^*]^{1/\alpha} \\ 2k_B T(\Gamma(\alpha-1)\gamma^*)^{-1} t^{2-\alpha}, & t \gg [m/\gamma^*]^{1/\alpha} \end{cases}$$

Occurs naturally in viscoelastic environment. Was used to describe a motion of a monomer in a polymer chain, internal dynamics of proteins, hydrodynamic interactions etc.

Sinai diffusion

A simple random walk in a potential landscape which is a simple random walk itself

$$\langle x^2(t)\rangle \sim \log^4 t$$



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Sinai diffusion

A simple random walk in a potential landscape
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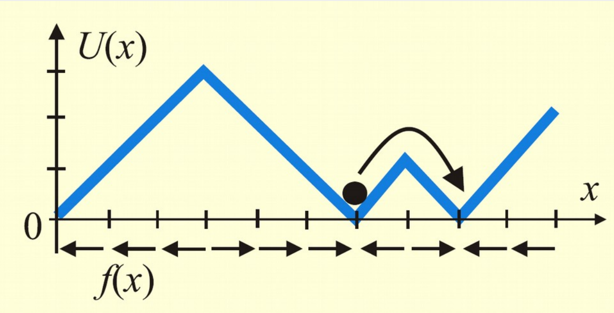

Scaling argument

$$U_{max}(|\Delta x|) \sim \sqrt{\Delta x}$$

$$t(|\Delta x|) \sim \tau_0 e^{\frac{U_{max}(|\Delta x|)}{k_B T}} \sim \tau_0 e^{const \sqrt{\Delta x}}$$

$$\sqrt{\Delta x} \sim \log t \qquad \Rightarrow \qquad \langle \Delta x^2 \rangle \sim \log^4 t$$

CTRW description

$$\psi(t) \sim \frac{\tau}{t \log^{\kappa+1}(t/\tau)}$$

$$P(x,t) \sim \exp\left(-A \frac{|x|}{\log^{\kappa/2} t}\right)$$

Sinai $\kappa = 4$

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Ultra-slow diffusion example: crack propagation

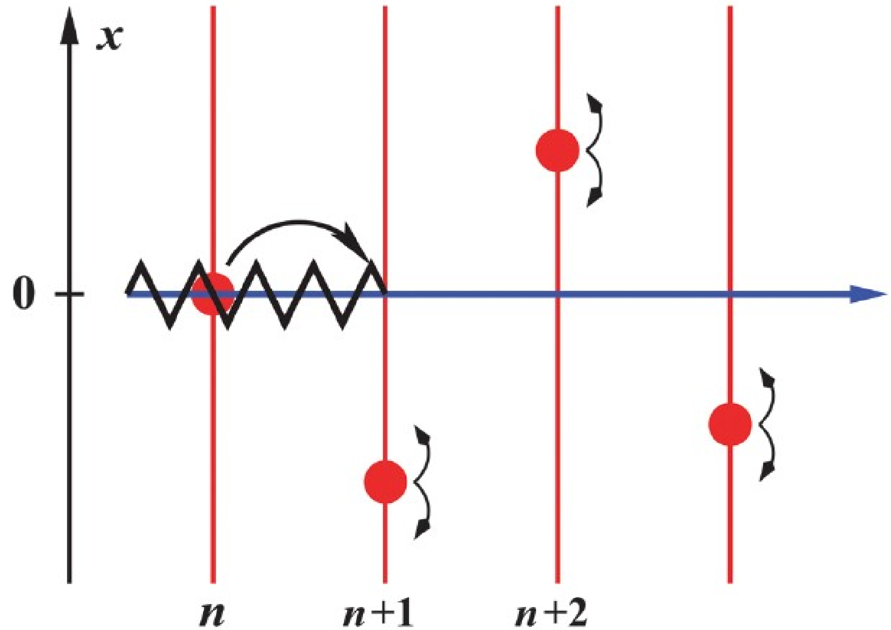

Fig. 14 Sketch of the crack propagation model discussed in the text. The tip of the crack (black zig-zag line) propagates from site n to $n + 1$ when the vacancy represented by the red circle diffuses to the origin ($x = 0$) at point n.

$$\langle n(t) \rangle \sim \frac{\ln(t/\tau_0)}{\mu}$$

Figure from R Metzler, JH Jeon, AG Cherstvy, E Barkai, Physical Chemistry Chemical Physics 16, 24128-24164 (2014)

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Time-dependent diffusivity: Scaled Brownian motion

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sqrt{2K(t)} \times \xi(t)$$

$$\langle \xi(t)\xi(t') \rangle = \delta(t-t')$$

If we choose

$$K(t) = \alpha K_\alpha t^{\alpha-1}$$

Then

$$\langle x^2(t) \rangle \sim K_\alpha t^\alpha$$

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle = \frac{2K_\alpha^\star t^{1+\alpha}}{(\alpha+1)(t-\Delta)} \left[1 - \left(\frac{\Delta}{t}\right)^{1+\alpha} - \left(1 - \frac{\Delta}{t}\right)^{1+\alpha} \right]$$

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \sim 2K_\alpha^\star \frac{\Delta}{t^{1-\alpha}}$$

Heterogeneous diffusion

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sqrt{2K(x)} \times \xi(t)$$

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}\left(\sqrt{K(x)}\frac{\partial}{\partial x}\left[\sqrt{K(x)}P(x,t)\right]\right)$$

$$K(x) = K_0|x|^{\beta}$$

$$\left\langle x^2(t)\right\rangle = \frac{\Gamma(\alpha+1/2)}{\pi^{1/2}}\left(\frac{2}{\alpha}\right)^{2\alpha}(K_0 t)^{\alpha}$$

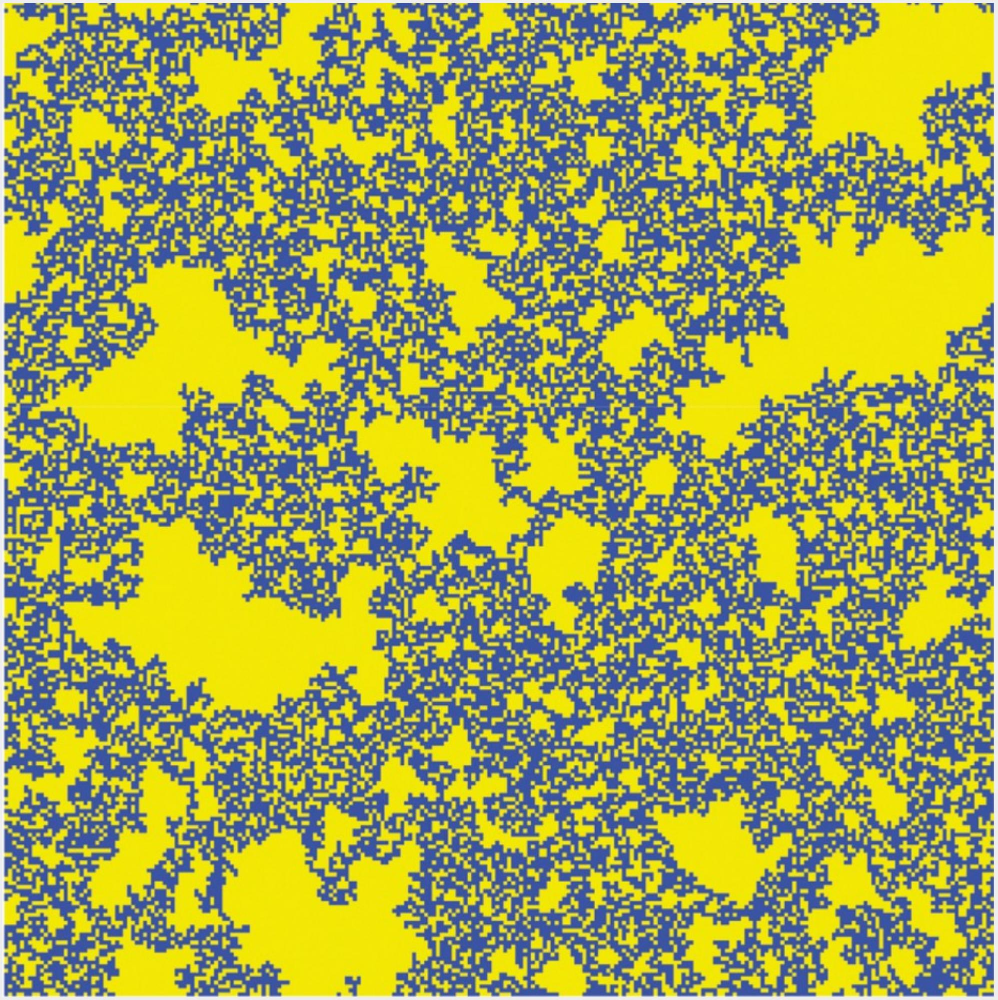
$$\alpha = \frac{2}{2-\beta}$$

$$P(x,t) = \frac{|x|^{-\beta/2}}{\sqrt{4\pi K_0 t}}\exp\left(-\frac{|x|^{2-\beta}}{(2-\beta)^2 K_0 t}\right)$$

$$\left\langle \overline{\delta^2(\Delta)}\right\rangle = \left(\frac{\Delta}{t}\right)^{1-\alpha}\left\langle x^2(\Delta)\right\rangle = \frac{\Gamma(\alpha+1/2)}{\pi^{1/2}}\left(\frac{2}{\alpha}\right)^{2\alpha}K_0^{\alpha}\frac{\Delta}{t^{1-\alpha}}$$

A. G. Cherstvy, A. V. Chechkin and R. Metzler, New J. Phys., 2013, 15, 083039

Diffusion on a percolation cluster (fractal)



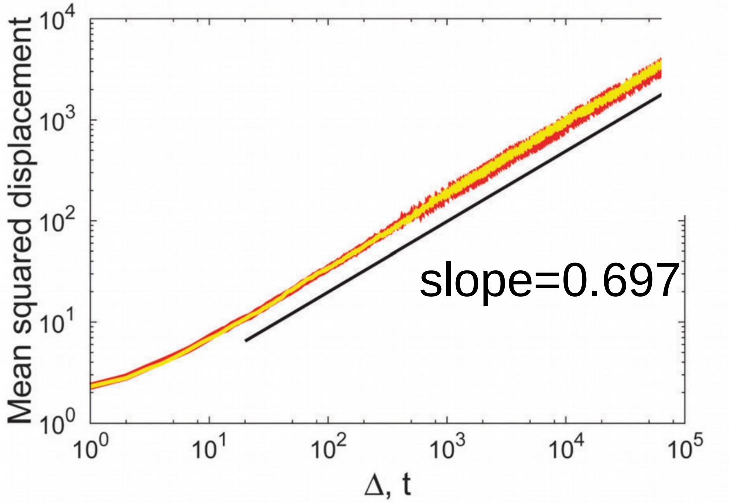


Fig. 26 MSD (thicker red curve) and time averaged MSD (thinner yellow curve) of a random walk on the infinite critical percolation cluster shown in Fig. 25. Both MSD and time averaged MSD perfectly overlap, *i.e.*, diffusion on a fractal is stationary and ergodic. The straight black line shows the expected slope α = 0.697 to guide the eye. Data provided by Y. Meroz, corresponding to those used in ref. 218.

R Metzler, JH Jeon, AG Cherstvy, E Barkai, Physical Chemistry Chemical Physics 16, 24128-24164 (2014)

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Diffusing diffusivity model

$$\langle x_N^2\rangle = \sum_{i=1}^{N}\langle(\Delta x_i)^2\rangle + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\langle\Delta x_i \Delta x_j\rangle$$

At a time step *i* we move according to Gaussian with diffusivity D_i:

$$P(\Delta x_i) = \frac{1}{\sqrt{4\pi D_i}}\exp\left(-\frac{\Delta x_i^2}{4D_i}\right)$$

$$\langle x_N^2\rangle = \sum_{i=1}^{N}\langle\Delta x_i^2\rangle = 2\sum_{i=1}^{N}\langle D_i\rangle = 2\langle D\rangle N$$

$$\langle x_N^4\rangle - 3\langle x_N^2\rangle^2 = 12(\langle D^2\rangle - \langle D\rangle^2)N + 24\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(\langle D_i D_j\rangle - \langle D_i\rangle\langle D_j\rangle)$$

M.V. Chubynsky and G.W. Slater, Physical Review Letters, 113, 098302 (2014)

Diffusing diffusivity model

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$$G(x;t) = \int_0^{D_{\max}} \frac{\pi(D)}{2\sqrt{\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right) dD$$

$$\pi(D) = \frac{1}{D_0} \exp(-D/D_0)$$

$$G(x;t) = \frac{1}{2\sqrt{D_0 t}} \exp\left(-\frac{|x|}{\sqrt{D_0 t}}\right), \quad t \ll \tau_D$$

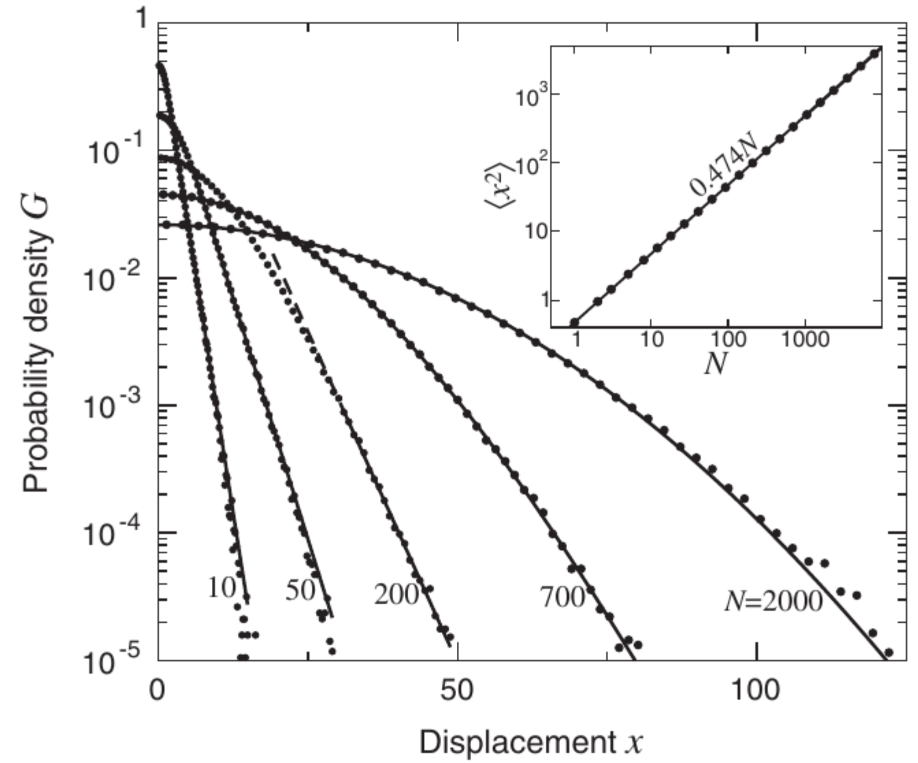

FIG. 1. The displacement distributions after different numbers of steps N for the diffusing diffusivity model [Eqs. (3), (6), and (7)] with $d = 0.0025$ and $s = 0.01$ simulated as described in the text. The solid line fits are exponential for the three smallest values of N, Gaussian for $N = 2000$, and the interpolating function $G(x) = A\exp(-B\sqrt{1 + (x/X_0)^2})$ for $N = 700$. The inset shows the MSD and a linear fit.

M.V. Chubynsky and G.W. Slater, Physical Review Letters, 113, 098302 (2014)

The zoo of anomalous models

Process	WEB	$\langle x^2(t)\rangle$	$\left\langle \overline{\delta^2(\Delta)} \right\rangle$
Correlated jump lengths	Yes	$\simeq t^3$	$\simeq \Delta^2 t$
Lévy walk, $0 < \alpha < 1$	Yes	$\simeq A(\alpha)t^2$	$\simeq \dfrac{A(\alpha)}{1-\alpha}\Delta^2$
Lévy walk, $1 < \alpha < 2$	Yes	$\simeq A^*(\alpha)t^{3-\alpha}$	$\simeq \dfrac{A^*(\alpha)}{\alpha-1}\Delta^{3-\alpha}$
Lévy flight	Yes	$=\infty[\langle\lvert x\rvert^q\rangle^{2/q} \simeq t^{2/\alpha}]$	$\simeq \Delta t^{2/\alpha-1}$
FBM $0 < \alpha < 2$	No	$\simeq t^\alpha$	$\simeq \Delta^\alpha$
Brownian motion	No	$\simeq t$	$\simeq \Delta$
FLE motion $0 < \alpha < 1$	No	$\simeq t^\alpha$	$\simeq \Delta^\alpha$
Fractal environment	No	$\simeq t^{2/d_w}$	$\simeq \Delta^{2/d_w}$
HDP $K(x) = K_0\lvert x\rvert^\beta$	Yes	$\simeq t^{2/(2-\beta)}$	$\simeq \Delta t^{2/(2-\beta)-1}$
Correlated waiting times	Yes	$\simeq t^{\gamma/(1+\gamma)}$	$\simeq \Delta t^{\gamma/(1+\gamma)-1}$
Subdiffusive CTRW	Yes	$\simeq t^\alpha$	$\simeq \Delta t^{\alpha-1}$
Confined subdiffusive CTRW	Yes	$\simeq t^0$	$\simeq (\Delta/t)^{1-\alpha}$
Quenched trap/patch models	Yes	$\simeq t^\alpha$	$\simeq \Delta t^{\alpha-1}$
Ageing CTRW	Yes	$\simeq \begin{cases} t/t_a^{1-\alpha}, & t \ll t_a, \\ t, & t \gg t_a \end{cases}$	$\simeq \Lambda_\alpha(t_a/t)\Delta t^{\alpha-1}$
Scaled Brownian motion	Yes	$\simeq t^\alpha$	$\simeq \Delta t^{\alpha-1}$
Ultraslow CTRW	Yes	$\simeq \log^\alpha(t)$	$\simeq \log^\alpha(t)\Delta/t$
Sinai (quenched)	Yes	$\simeq \log^4(t)$	$\simeq \log^4(t)\Delta/t$
CTRW in ageing environment	Yes	$\simeq \log(t)$	$\simeq \log(t)\Delta/t$
HDP $K(x) = (K_0/2)e^{-2x/x^*}$	Yes	$\simeq \log^2(t)$	$\simeq (\Delta/t)^{1/2}$

R Metzler, JH Jeon, AG Cherstvy, E Barkai, Physical Chemistry Chemical Physics 16, 24128-24164 (2014)

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Literature

1. B.D. Hughes, Random Walks and Random Environments, Vol. 1 Random Walks, Clarendon Press, Oxford, 1995.
2. R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Physics reports 339, 1-77 (2000).
3. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics R Metzler, J Klafter, Journal of Physics A: Math. and Gen. 37, R161 (2004)
4. Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking R Metzler, JH Jeon, AG Cherstvy, E Barkai, Physical Chemistry Chemical Physics 16, 24128-24164 (2014)

Literature.
Fractional derivatives and fractional differential equations

1. I. Podlubny, Fractional Differential Equations:
An Introduction to Fractional Derivatives, Fractional Differential Equations,
to Methods of Their Solution and Some of Their Applications. Elsevier, 1998.

2. Samko, S.; Kilbas, A.A.; Marichev, O. (1993). Fractional Integrals
and Derivatives: Theory and Applications. Taylor & Francis Books.

3. Publications of F. Mainardi

Fox H-functions:

1. A.M. Mathai, R.K. Saxena, H.J. Haubold, The H-Function
Theory and Applications, Springer, 2009.

Fractional derivatives

Riemann-Liouville fractional derivative

$$ {}_0D_t^q P(x,t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{P(x,t')}{(t-t')^q} dt' \quad q<1 $$

$$ {}_aD_t^p P(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_a^t P(t')(t-t')^{n-p-1} dt', n-1 \le p < n $$

Examples

$$ {}_0D_t^\nu t^\mu = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\nu)} t^{\mu-\nu} $$

$$ {}_0D_t^\nu 1 = \frac{1}{\Gamma(1-\nu)} t^{-\nu} $$

$$ {}_0D_t^\nu \mathrm{e}^t = \frac{t^{-\nu}}{\Gamma(1-\nu)} {}_1F_1(1, 1-\nu, t) $$

$$ {}_{-\infty}D_t^\nu \mathrm{e}^t = \mathrm{e}^t $$

Weyl fractional operator

Fractional derivatives

Riemann-Liouville fractional derivative

$$ {}_0D_t^q P(x,t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{P(x,t')}{(t-t')^q} dt' $$

q<1

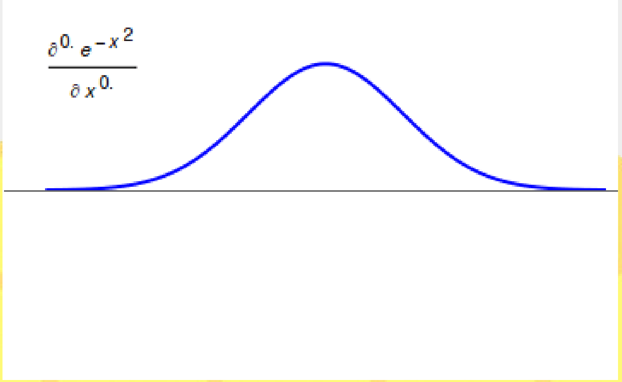
$$ \mathcal{L}\{{}_0D_t^{-q} f(t)\} = u^{-q} f(u) $$

q>0

$$ \mathcal{L}\{{}_0D_t^p f(t)\} = u^p f(u) - \sum_{j=0}^{n-1} u^j c_j $$

$$ c_j = \lim_{t\to 0} {}_0D_t^{p-1-j} f(t) $$

Example: a fractional derivative from a Gaussian changing from 0 to 1



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Fractional derivatives

Weyl fractional derivative

$$\mathscr{F}\{{}_{-\infty}D_x^{\mu} f(x)\} = (\mathrm{i}k)^{\mu} f(k)$$

q<1

Grünwald-Letnikov fractional derivative: Finite element representation

$$f'(t) = \frac{df}{dt} = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h}$$

$$f'''(t) = \frac{d^3 f}{dt^3} = \lim_{h \to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3}$$

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h \to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh)$$

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

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Fractional derivatives

Grünwald-Letnikov fractional derivative: Finite element representation

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh)$$

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

$$f_h^{(p)}(t) = \frac{1}{h^p} \sum_{r=0}^{n} (-1)^k \binom{p}{r} f(t - rh)$$

$$\boxed{{}_aD_t^p f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t - rh) = \lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(p)}(t)}$$

If we consider a class of functions f(t) having m + 1 continuous derivatives for t, then the Grünwald-Letnikov definition is equivalent to the Riemann-Liouville definition

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Fox H-functions

$$H_{pq}^{mn}(z) = H_{pq}^{mn}\left[z \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] = H_{pq}^{mn}\left[z \left| \begin{matrix} (a_1, A_1), (a_2, A_2), \ldots, (a_p, A_p) \\ (b_1, B_1), (b_2, B_2), \ldots, (b_q, B_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_L \mathrm{d}s\, \chi(s) z^s$$

$$\chi(s) = \frac{\prod_1^m \Gamma(b_j - B_j s) \prod_1^n \Gamma(1 - a_j + A_j s)}{\prod_{m+1}^q \Gamma(1 - b_j + B_j s) \prod_{n+1}^p \Gamma(a_j - A_j s)}$$

Examples:

$$e^{-z} = H_{01}^{10}\left[z \left| \begin{matrix} \overline{} \\ (0, 1) \end{matrix} \right. \right]$$

$$\frac{1}{p + k^{2-\alpha}} = \frac{1}{p} \frac{1}{2-\alpha} H_{11}^{11}\left[\frac{k}{p^{\frac{1}{2-\alpha}}} \left| \begin{matrix} \left(0, \frac{1}{2-\alpha}\right) \\ \left(0, \frac{1}{2-\alpha}\right) \end{matrix} \right. \right]$$

Way to compute is to use the expansion for z → 0 under the set of conditions

$$H_{pq}^{mn}(z) = \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{\prod_{j=1, j\neq h}^{m} \Gamma(b_j - B_j (b_h + \nu)/B_h)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j (b_h + \nu)/B_h)} \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + A_j (b_h + \nu)/B_h)}{\prod_{j=n+1}^{p} \Gamma(a_j - A_j (b_h + \nu)/B_h)} \times \frac{(-1)^\nu z^{(b_h + \nu)/B_h}}{\nu! B_h}$$

Fox H-functions

$$H_{pq}^{mn}(z) = H_{pq}^{mn}\left[z \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] = H_{pq}^{mn}\left[z \left| \begin{matrix} (a_1, A_1), (a_2, A_2), \ldots, (a_p, A_p) \\ (b_1, B_1), (b_2, B_2), \ldots, (b_q, B_q) \end{matrix} \right. \right] = \frac{1}{2\pi \mathrm{i}} \int_L \mathrm{d}s\, \chi(s) z^s$$

Pros and contras

Pros:

1. Perform integrations one cannot do otherwise
2. Analytical inverse Laplace transform
3. Always either $z \to 0$ or $z \to \infty$ expansion exists
4. Very helpful for equations with fractional derivatives

Contras

1. Not always possible to express a needed function as an *H*-function
2. Just by looking at coefficients we do not have a clue what the function is