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Stochastic methods in Mathematical Modelling

Lecture 12. Random Walks

A concept of random walk

294 NATURE [JULY 27, 1905

The Problem of the Random Walk.

CAN any of your readers refer me to a work wherein I should find a solution of the following problem, or failing the knowledge of any existing solution provide me with an original one? I should be extremely grateful for aid in the matter.

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+\delta r$ from his starting point, O.

The problem is one of considerable interest, but I have only succeeded in obtaining an integrated solution for *two* stretches. I think, however, that a solution ought to be found, if only in the form of a series in powers of $1/n$, when n is large.

KARL PEARSON.
The Gables, East Ilsley, Berks.

A concept of random walk

The Problem of the Random Walk.

THIS problem, proposed by Prof. Karl Pearson in the current number of NATURE, is the same as that of the composition of n iso-periodic vibrations of unit amplitude and of phases distributed at random, considered in *Phil. Mag.*, x., p. 73, 1880; xlvii., p. 246, 1899; ("Scientific Papers," i., p. 491, iv., p. 370). If n be very great, the probability sought is

$$\frac{2}{n} e^{-r^2/n} r dr.$$

Probably methods similar to those employed in the papers referred to would avail for the development of an approximate expression applicable when n is only moderately great.

RAYLEIGH.

Terling Place, July 29.

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The Problem of the Random Walk.

I HAVE to thank several correspondents for assistance in this matter. Mr. G. J. Bennett finds that my case of $n=3$ can really be solved by elliptic integrals, and, of course, Lord Rayleigh's solution for n very large is most valuable, and may very probably suffice for the purposes I have immediately in view. I ought to have known it, but my reading of late years has drifted into other channels, and one does not expect to find the first stage in a biometric problem provided in a memoir on sound. From the purely mathematical standpoint, it would still be very interesting to have a solution for n comparatively small. The sections through the axis of Lord Rayleigh's frequency surface for n large are simply the "cocked hat" or normal curve of errors type; for $n=2$ or 3 they do not resemble this form at all. For $n=2$, for example, the sections are of the form of a double U, thus UU, the whole being symmetrical about the centre vertical corresponding to $r=0$, but each U itself being asymmetrical. The system has three vertical asymptotes. It would be interesting to see how the multiplicity of types for n small passes over into the normal curve of errors when n is made large.

The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point! KARL PEARSON.

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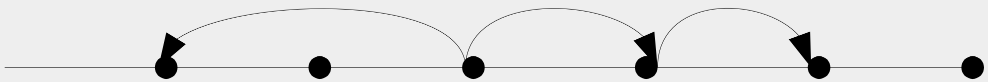
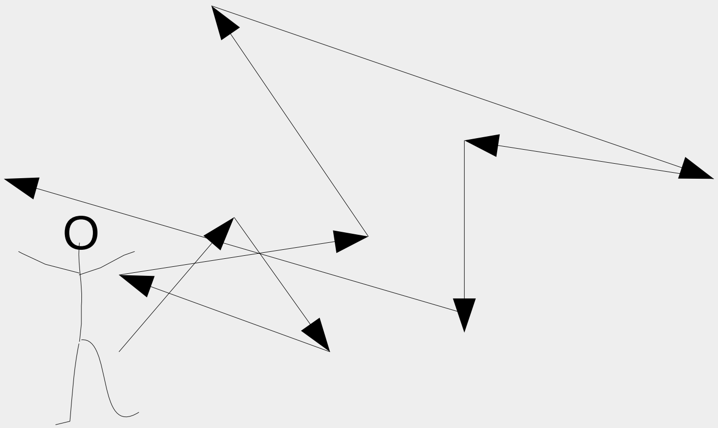
A concept of random walk

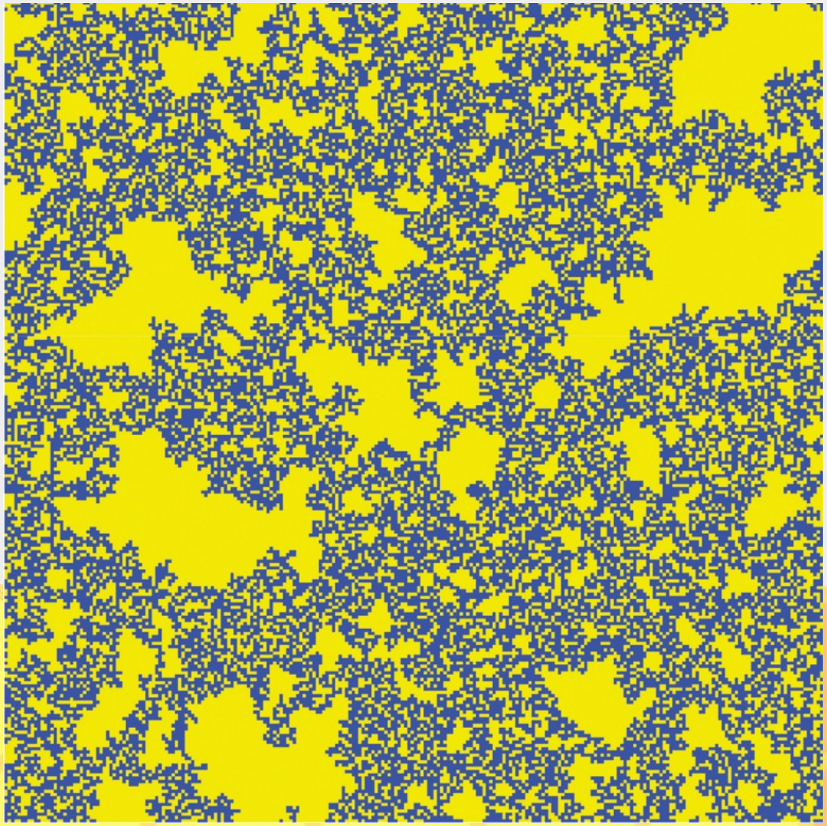
1D walk on a lattice



2D walk on a plane



Random walk on a percolation cluster



Elementary arguments

We assume that displacements are independent.
$\boldsymbol{R}_n$ is the position after *n* steps

$$\boldsymbol{R}_n = \sum_{j=1}^{n} \boldsymbol{Y}_j$$

Displacement at *j*-th step

Mean displacement at *j*-th step

$$\langle \boldsymbol{R}_n \rangle = \sum_{j=1}^{n} \boldsymbol{m}_j$$

If $\sigma_j^2 = \langle |\boldsymbol{Y}_j - \boldsymbol{m}_j|^2 \rangle$

$$Var(\boldsymbol{R}_n) = \langle |\boldsymbol{R}_n - \langle \boldsymbol{R}_n \rangle|^2 \rangle = \sum_{j=1}^{n} \boldsymbol{\sigma}_j^2$$

If all means and all the variances are the same $\boldsymbol{m}_j = \boldsymbol{m}$, $\boldsymbol{\sigma}_j = \boldsymbol{\sigma}$

$$\langle \boldsymbol{R}_n \rangle = n\boldsymbol{m}, Var(\boldsymbol{R}_n) = n\sigma^2$$

$$fluctuations \sim \sqrt{Var(\boldsymbol{R}_n)} = \sqrt{n}$$

Formal solution

General equation for the evolution of the walk

$$P_{n+1}(\mathbf{r}) = \int p_{n+1}(\mathbf{r} - \mathbf{r}')P_n(\mathbf{r}')d^d\mathbf{r}'$$

In Fourier space

$$\tilde{P}_{n+1}(\mathbf{q}) = \tilde{p}_{n+1}(\mathbf{q})\tilde{P}_n(\mathbf{q})$$

With $P_0(\mathbf{r})$ being the PDF of the initial position and $P_0(\mathbf{q})$ its Fourier transform

$$\tilde{P}_n(\mathbf{q}) = \tilde{P}_0(\mathbf{q})\prod_{j=1}^{n}\tilde{p}_j(\mathbf{q})$$

Taking the inverse Fourier transform and for the case $P_0(\mathbf{r})=\delta(\mathbf{r})$ and $P_0(\mathbf{q})=1$

$$P_n(\mathbf{r}) = \frac{1}{(2\pi)^d}\int e^{-i\,\mathbf{qr}}\tilde{p}(\mathbf{q})^n d^d\mathbf{q}$$

Example: Pearson's walk in the plane

2D isotropic walk

$\langle a_j \rangle = a < \infty$

$$p_j(\vec{r}) = \frac{1}{2\pi|\vec{r}|}\,\delta(|\vec{r}| - a_j)$$

$$P_n(\vec{r}) = \frac{1}{2\pi}\int_0^\infty u\, J_0(u|\vec{r}|)\prod_{\ell=1}^{n} J_0(u a_\ell)\, du$$

$n \to \infty$

$\langle a_j \rangle = a$

$$P_n(r) \sim \frac{1}{\pi n a^2}\exp\left(-\frac{r^2}{n a^2}\right)$$

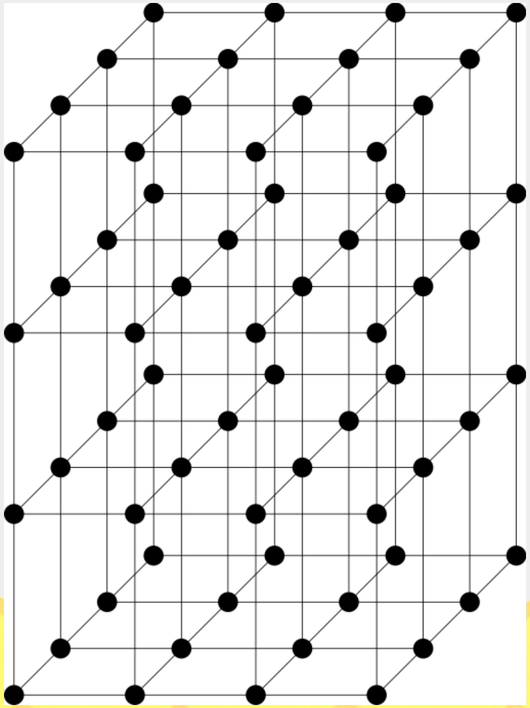
Rayleigh (1919) $a_j = \text{const} = a$

$$P_n(r) = \frac{1}{\pi n a^2}\exp\left(-\frac{r^2}{n a^2}\right)\left(1 - \frac{1}{4n}\left(2 - \frac{4r^2}{na^2} + \frac{r^4}{n^2a^4}\right) + \dots\right)$$

Recurrency and transience of random walks. Return probability. Pólya theorem

The random walk is called *recurrent* if the eventual return to the starting cite is certain. Otherwise the walk is called *transient*.

Q: Imagine a walker performing simple random walk
on a n-dimensional periodic lattice. Is the walk recurrent in 1D?
What about 2D? 3D? 4D?



Recurrency and transience of random walks. Pólya theorem

The random walk is called *recurrent* if the eventual return to the starting cite is certain. Otherwise the walk is called *transient*.

Pólya theorem

The nearest-neighbour unbiased walk on a lattice is *recurrent* in 1D and 2D but *transient* for $d \geq 3$

Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz.

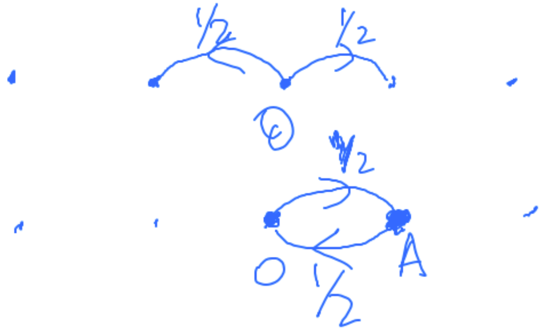
Von

Georg Pólya in Zürich.

Ja, wenn $d = 1$ oder $d = 2$, *nein*, wenn $d \geqq 3$.

In continuum the recurrency stops when the fractal dimension of a walk reaches the dimension of space, i.e. Brownian motion (Gaussian continuous process) is recurrent when dimensions are less the d = 2. On the plane we can get as close to any given point as possible, but not hit the value exactly. For alpha-stable Levy processes with $\alpha \leq 1$ the probability to hit the point on a line is 0.

R is return prob.



Pólya theorem

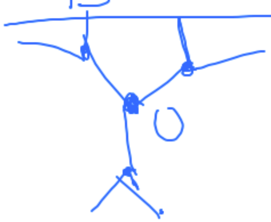
$p = q = 1/2$

$$R = \frac{1}{2} + \frac{1}{2}\left(\frac{R}{2} + \frac{R}{2}\left(\frac{R}{2} + \dots\right)\right) = \frac{1}{2} + \frac{R}{4} + \frac{R^2}{8} + \dots + \frac{R^m}{2^{m+1}} + \dots$$

$$= \frac{1}{2-R}$$

$$R^2 - 2R + 1 = 0 \Rightarrow R = 1$$

Bethe lattice



Cayley tree

z

$$R = \frac{1}{z-1}$$

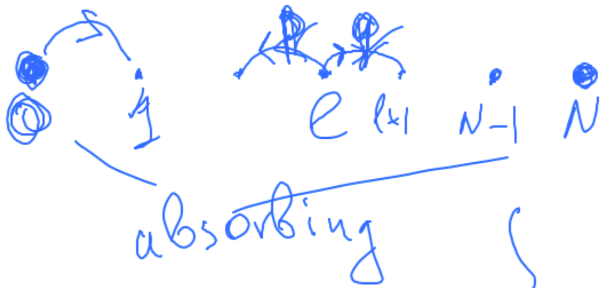
Biased walk

$p \neq q$ $\quad p^2 + q^2 + 2pq = 1$

$$R = 1 - \sqrt{1-4pq} =$$

$$= 1 - |2p-1| = 1 - |p-q| < 1$$

Lifetimes of Pólya walks



$\underline{T(\ell)}$?

$$\times n \left\{ \begin{array}{l} 1)\ \Pr\{T(\ell)=n\} = p\, P\{T(\ell+1)=n-1\} + q\, P\{T(\ell-1)=n-1\} \\ 2)\ \Pr\{T(1)=n\} = p \Pr\{T(2)=n-1\} + q\, \delta_{n,1} \\ 3)\ \Pr\{T(N-1)=n\} = q \Pr\{T(N-2)=n-1\} + p\, \delta_{n,1} \end{array} \right.$$

$$\langle T(\ell)\rangle = \sum_{n=0}^{\infty} n \Pr\{T(\ell)=n\}$$

after multiplication & summation:

$$\left\{ \begin{array}{l} \langle T(\ell)\rangle = p\langle T(\ell+1)\rangle + q\langle T(\ell-1)\rangle + 1 \qquad 1 \le \ell \le N-1 \\ \langle T(0)\rangle = \langle T(N)\rangle = 0 \end{array} \right.$$

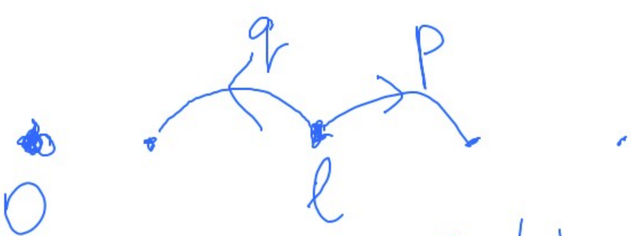
$$\langle T(\ell)\rangle = \frac{\ell}{q-p} + A + B\left(\frac{q}{p}\right)^{\ell}$$

$$\boxed{\langle T(\ell)\rangle = \alpha^{\ell}}$$

$$\langle T(\ell)\rangle = \begin{cases} \ell(N-\ell) & p=q=1/2 \\ \dfrac{1}{q-p}\left\{\ell - N\dfrac{(q/p)^{\ell}-1}{(q/p)^{N}-1}\right\}, & p\neq q \end{cases}$$

$N \to \infty$

$$\langle T(\ell)\rangle = \begin{cases} \infty, & q \le p \\ \dfrac{\ell}{q-p}, & q > p \end{cases}$$

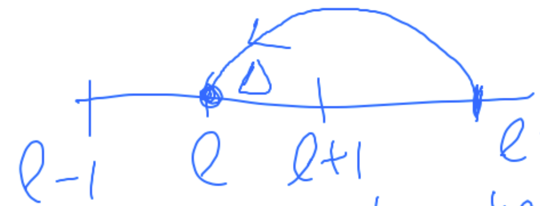


Fokker-Planck

$$T = \frac{1}{2D} x_0(1-x_0)$$

$$\langle T(\ell)\rangle = N^2 \frac{\ell}{N}\left(1-\frac{\ell}{N}\right)\tau \Rightarrow \boxed{D = \frac{1}{2N^2\tau}}$$

Derivation of Fokker-Planck equation from RW:



$$P_{n+1}(\ell) = \sum_{\ell'=-\infty}^{+\infty} p(\ell-\ell')\, P_n(\ell')$$

τ is a timestep

$x = \ell\Delta$

we would like a continuous approx:

$$P_n(\ell) = \Delta\, p(\ell\Delta, n\tau)$$

$$p(x, t+\tau) = \sum_{\ell'=-\infty}^{+\infty} p(\ell-\ell')\, \underline{p(x-\Delta(\ell-\ell'), t)}$$

$$p(x, t+\tau) = p(x,t) + \tau \frac{\partial p(x,t)}{\partial t} + o(\tau)$$

$$p(x-\Delta(\ell-\ell'), t) = p(x,t) - \underline{\Delta(\ell-\ell')} \frac{\partial p(x,t)}{\partial x} + \frac{1}{2} \underline{\Delta^2 (\ell-\ell')} \frac{\partial^2 p}{\partial x^2} + \dots$$

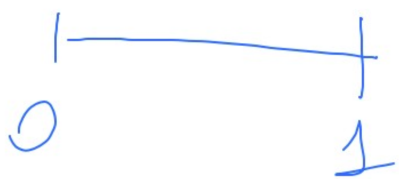
Assume that mean & 2nd moment of our steps are finite

$$m_1 = \sum_{\ell=-\infty}^{\infty} \ell\, p(\ell) < \infty \qquad m_2 = \sum_{-\infty}^{\infty} \ell^2 p(\ell) < \infty$$

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\Delta}{\tau} m_1 \frac{\partial p(x,t)}{\partial x} + \frac{\Delta^2}{2\tau} m_2 \frac{\partial^2 p}{\partial x^2}$$

$$\begin{cases} D \equiv \lim\limits_{\Delta,\tau\to 0} \dfrac{m_2 \Delta^2}{2\tau} \\[2ex] V \equiv \lim\limits_{\Delta,\tau\to 0} \dfrac{m_1 \Delta}{\tau} \end{cases}$$

$$\frac{\partial P}{\partial t} = -\vec{V} \frac{\partial P}{\partial x} + D \frac{\partial^2 P}{\partial x^2}$$



$$\Delta = \frac{1}{N} \qquad m_1 = 0$$

$$p = q \qquad m_2 = 1$$

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Literature

1. B.D. Hughes, Random Walks and Random Environments, Vol. 1 Random Walks, Clarendon Press, Oxford, 1995.
2. N.G. Van Kampen, Stochastic Processes in Physics and Chemistry, 3rd Edition, Elsevier, 2007