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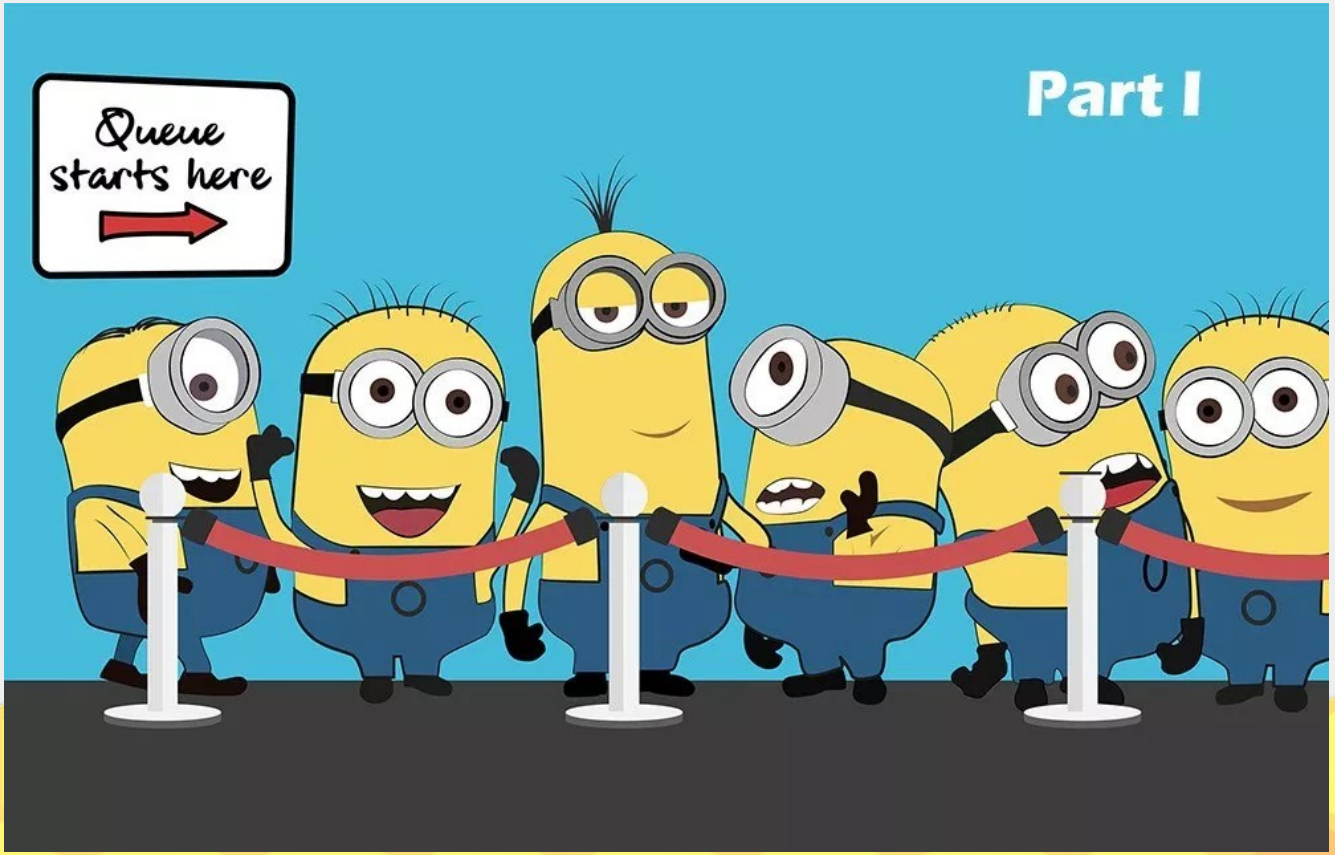
Stochastic methods in Mathematical Modelling

Lecture 8. Queueing Systems.



https://michaelscodingspot.com/c-job-queues/

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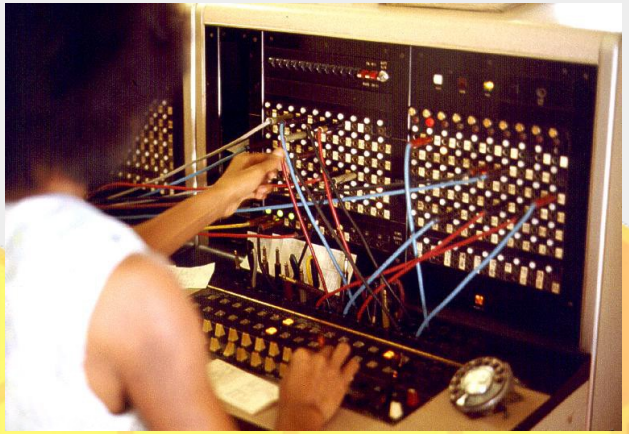
Queueing Systems

Agner Krarup Erlang published first paper in queueing theory in 1909. He modelled a number of phone calls arriving at an exchange by a Poisson process, thus, founding a queueing theory and telephone networks analysis.



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Queuing theory. Applications

Any kind of systems where arrivals (generally, coming units) and processing fits the framework

Call centres

Manufacturing

Supermarket check out

Airport border controls

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Poisson processes

Continuous version of Bernoulli process or, alternatively, random time intervals between successes

Assumptions

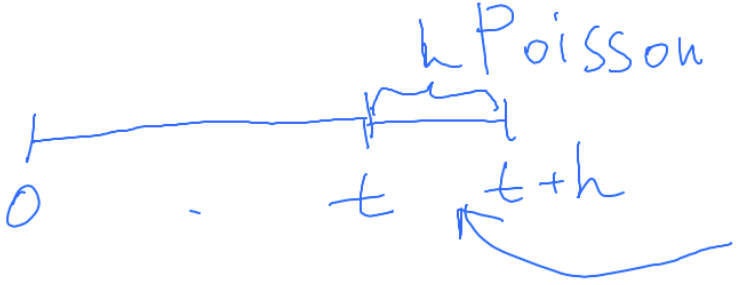
- Numbers of arrivals in disjoint intervals are independent
- For a small *h* the probability of *k* events P(*k*;*h*)

$$1-P_0(h)=\lambda h+o(h),$$
$$P_1(h)=\lambda h+o(h),$$
$$P_{k\geq 2}(h)=o(h).$$

- λ is the arrival rate of the process

For *k* successes within time interval *t*

$$P_k(t)=\frac{\lambda t}{k!}e^{-\lambda t}$$



$$\begin{cases} P_1(h) = \lambda h + o(h) \\ 1 - P_0(h) = \lambda h + o(h) \\ P_{h\geq 2}(h) = o(h) \end{cases}$$

4. $P_n(t+h)$

1. 0 successes $(t, t+h]$
   n events in $[0, t]$
2. 1 event in $[t, t+h]$
   $n-1$ events in $[0, t]$
3. $k \geq 2$ events in $[t, t+h]$
   $n-k$ events in $[0, t]$

$$P_n(t+h) = P_n(t)(1-\lambda h) + P_{n-1}(t)\lambda h + o(h) \rightarrow$$

$$\frac{P_n(t+h) - P_n(t)}{h} = -\lambda P_n + \lambda P_{n-1} + \frac{o(h)}{h}$$

$$\downarrow h \to 0$$

$$\frac{\partial P_n}{\partial t} = -\lambda P_n + \lambda P_{n-1}$$

$$\begin{cases} P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), & n \geq 1 \\ P_0'(t) = -\lambda P_0(t), & n = 0 \end{cases}$$

initial condition

$$P_0(0) = 1$$

$$P_0(t) = e^{-\lambda t}$$

$$P_1(t) = \lambda t e^{-\lambda t}$$

...

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

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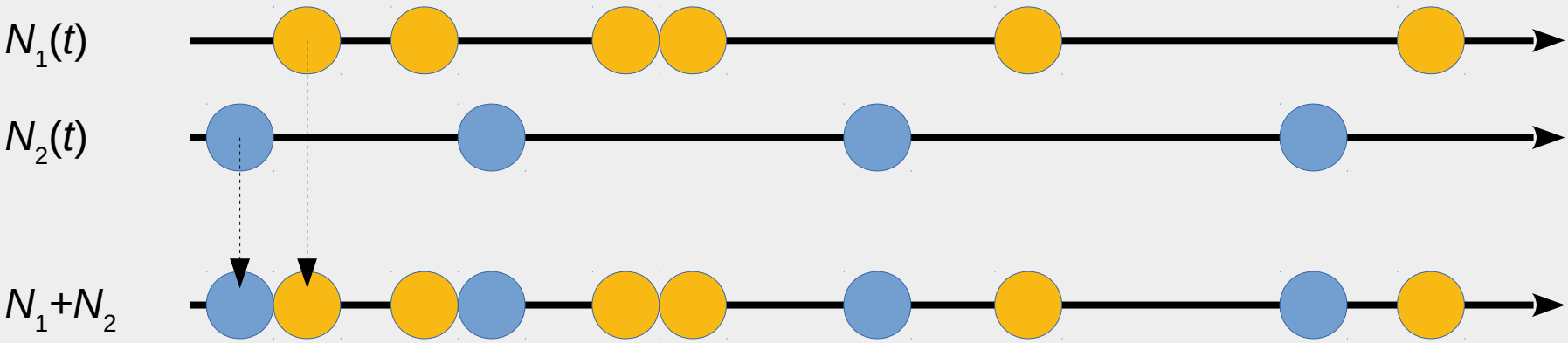
Poisson processes. Merging

Merging: Assume that we have two independent Poisson processes $N_1(t)$ and $N_2(t)$ with rates λ_1 and λ_2. Then $N(t)=N_1(t)+N_2(t)$ is also a Poisson process with the rate $\lambda = \lambda_1 + \lambda_2$



$$N(0)=N_1(0)+N_2(0)$$

$$1-P_0(h)=\lambda_1 h+\lambda_2 h+o(h),$$

$$P_1(h)=\lambda_1 h+\lambda_2 h,$$

$$P_2(h)=o(h).$$

For k successes within time interval t

Hence $$P_k(t)=\frac{(\lambda_1+\lambda_2)t}{k!}e^{-(\lambda_1+\lambda_2)t}$$

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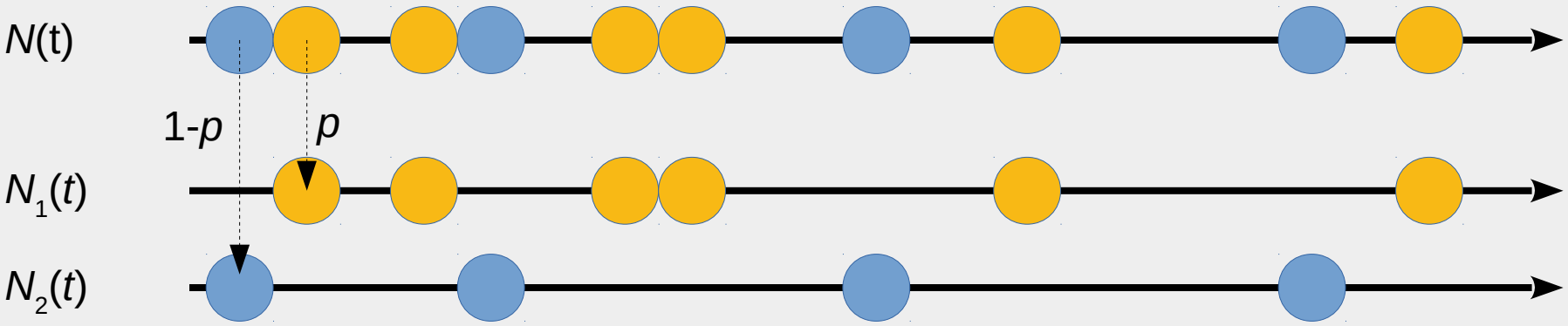
Poisson processes. Splitting

Splitting: Let $N(t)$ will be a Poisson process with the rate λ. We split it into two Poisson processes $N_1(t)$ and $N_2(t)$ and the splitting is decided by the Bernoulli process with the probability p, i.e. with probabilities p and $1-p$ the arrival is classified as N_1 or N_2.

Then:

- $N_1(t)$ is a Poisson process with the rate $\lambda_1 = p\lambda$
- $N_2(t)$ is a Poisson process with the rate $\lambda_2 = (1-p)\lambda$
- $N_1(t)$ and $N_2(t)$ are independent



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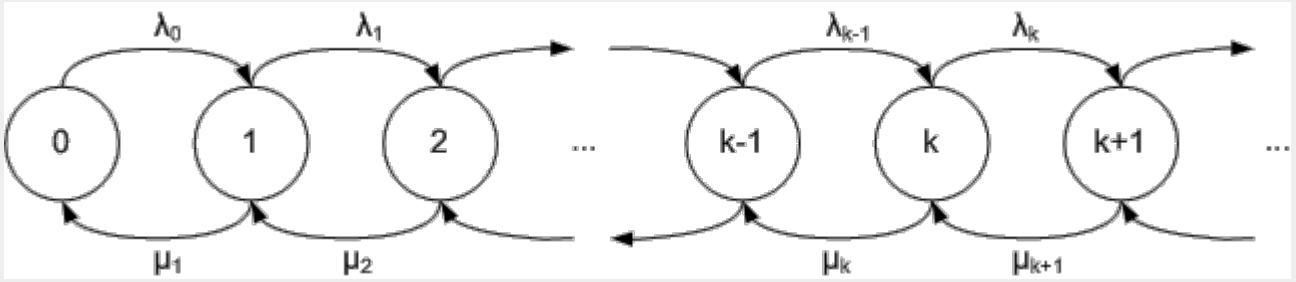
Birth-death processes



$$\frac{dp_0(t)}{dt} = \mu_1 p_1(t) - \lambda_0 p_0(t)$$

$$\frac{dp_k(t)}{dt} = \lambda_{k-1} p_{k-1}(t) + \mu_{k+1} p_{k+1}(t) - (\lambda_k + \mu_k) p_k(t), k = 1,2,\ldots$$

The graph is also a Markov chain!!

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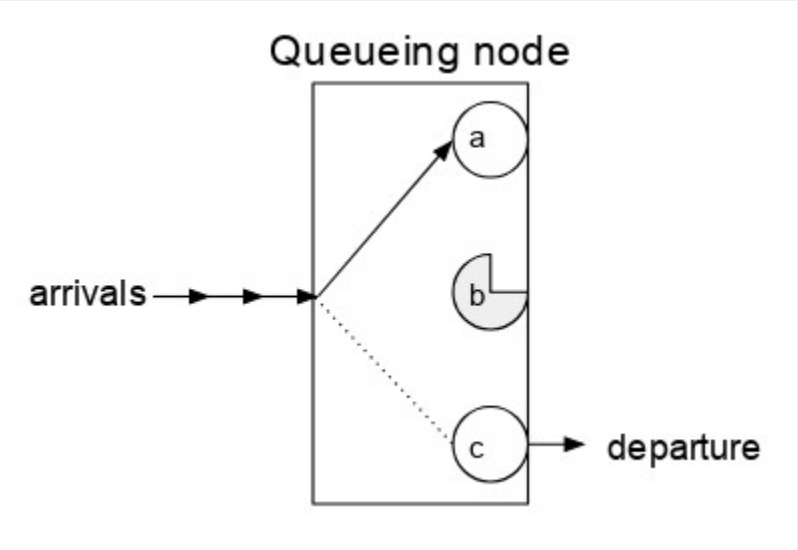
Queueing Systems

Kendall's notation (extended) A/S/c/K/N/D



A denotes the time distribution between arrivals to the queue

S is the service time distribution

c is the number of service channels open at the node

K is the capacity of the queue

N is the size of the population of jobs to be served

D is the queueing discipline

When the final three parameters are not specified (e.g. M/M/1 queue), it is assumed K = ∞, N = ∞ and D = “First in First out”

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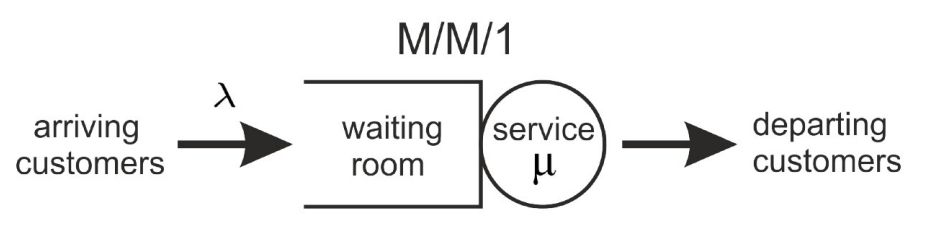
Queueing Systems

Kendall's notation (examples)

M/M/1 queue: A queue with 1 server, arrival rate λ and departure rate μ.



M/G/1 queue
M/D/1/∞ queue
M/D/k/∞ queue

- **M** stands for *Markov* or *memoryless* and it means that arrivals occur according to a Poisson process. Arrivals may be also deterministic, **D**.
- **D** stands for *deterministic* and means that the jobs arriving at the queue require a fixed=deterministic amount of service/processing. Processing can also be stochastic, Markovian (or non-Markovian, in which case it is custom to denote it as **G** - generic service; arrival can also be **G**=generic).
- **k** describes the *number of servers* at the queueing node **k** = 1, 2, ... If there are more jobs at the node than the servers then jobs will queue and wait for service

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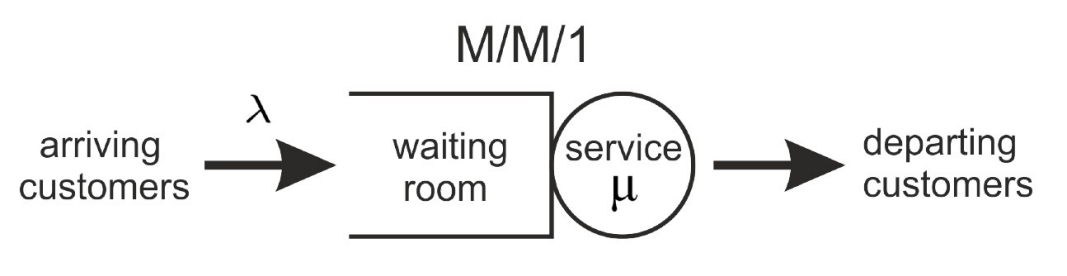
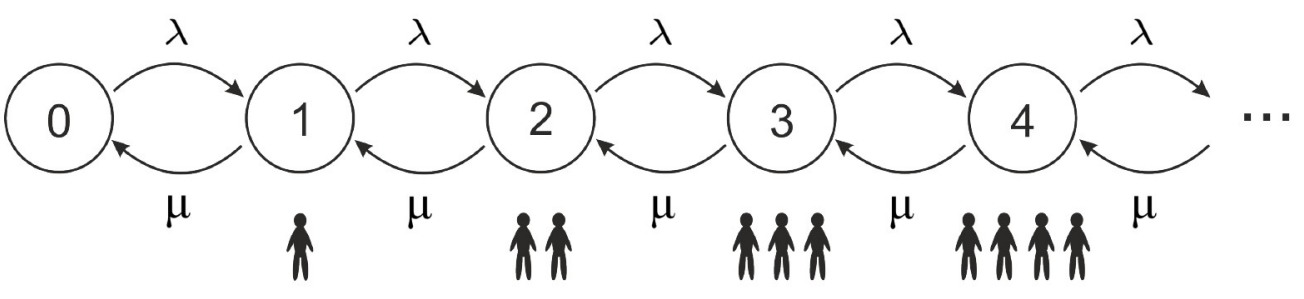
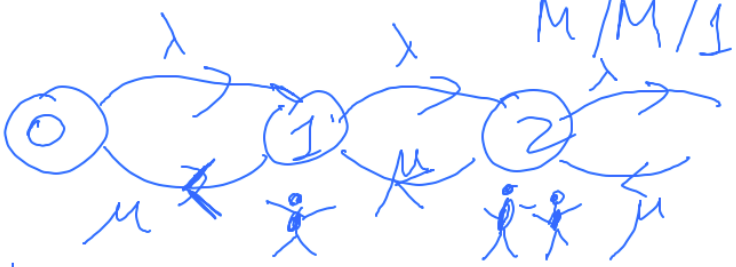
M/M/1 queue



Equivalent to the following birth-death process



M/M/1 Queue



$$\begin{cases} \dfrac{dp_0}{dt} = -\lambda p_0 + \mu p_1 \\ \dfrac{dp_n}{dt} = \lambda p_{n-1} - (\lambda+\mu) p_n + \mu p_{n+1}, \quad n \geq 1 \end{cases}$$

Steady-state analysis

$$\begin{cases} \mu p_1 = \lambda p_0 & 0 \text{ customers} \\ p_0 + \mu p_2 = (\lambda+\mu) p_1 & 1 \text{ customer} \\ \lambda p_{n-1} + \mu p_{n+1} = (\lambda+\mu) p_n & n \text{ customers} \end{cases} \Rightarrow$$

$$\frac{dp_i}{dt} = 0 \qquad \boxed{\rho \equiv \frac{\lambda}{\mu}}$$

$$p_1 = \frac{\lambda}{\mu} p_0$$

$$p_2 = \frac{1}{\mu}\left[(\lambda+\mu)\frac{\lambda}{\mu} p_0 - \lambda p_0\right] = \frac{\lambda^2}{\mu^2} p_0$$

$$\dots$$

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0$$

$$1 = \sum_{n=0}^{\infty} p_n = p_0 \sum_{n=0}^{\infty} \rho^n = \frac{p_0}{1-\rho} \Leftarrow$$

$P_0 = 1-\rho$

$P_n = (1-\rho)\rho^n$

$\rho = \frac{\lambda}{\mu}$

$$\langle N \rangle = \sum_{n=0}^{\infty} n P_n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$$

$\lambda \to \mu$, $\langle N \rangle \to \infty$

if we start with m customers

$$P_n(t) = e^{-(\lambda+\mu)t}\left[\rho^{\frac{n-m}{2}} I_{n-m}(at) + \rho^{\frac{n-m-1}{2}} I_{n+1+m}(at) + \right.$$

$$\left. + (1-\rho)\rho^n \sum_{j=n+m+2} \rho^{-\frac{j}{2}} I_j(at)\right], \quad a = 2\sqrt{\lambda\mu}$$

I is a modified Bessel function

Busy period of the server

PDF: $f(t) = \frac{1}{t\sqrt{\rho}} e^{-(\lambda+\mu)t} I_1(2t\sqrt{\lambda\mu})$

$\langle t \rangle = \int_0^\infty t f(t)\,dt$

$\langle t \rangle$: $\mathcal{L}(f(t)) = \frac{1}{2\lambda}\left(\lambda+\mu+s - \sqrt{(\lambda+\mu+s)^2 - 4\lambda\mu}\right) = \hat{f}(s)$

Laplace transform

$\langle t \rangle, \langle t^2 \rangle, \langle t^3 \rangle$ are finite: $\mu > \lambda$

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\,dt = \underbrace{\int_0^\infty f(t)\,dt}_{1} - s\underbrace{\int_0^\infty t f(t)\,dt}_{\langle t \rangle} + \ldots =$$

$$= 1 - \langle t \rangle s + \frac{\langle t^2 \rangle}{2} s^2 + \ldots$$

$$\hat{f}(s) = \frac{1}{2\lambda}\left(\lambda+\mu+s - \sqrt{\lambda^2+\mu^2+s^2+2\lambda\mu+2s(\lambda+\mu)-4\lambda\mu}\right) = \frac{1}{2\lambda}\left(\lambda+\mu+s-(\mu-\lambda)\sqrt{1+\frac{2s(\lambda+\mu)}{(\mu-\lambda)^2}}\right)$$

$$= 1 - \frac{s}{\mu-\lambda} \Rightarrow \boxed{\langle t \rangle = \frac{1}{\mu-\lambda}}$$

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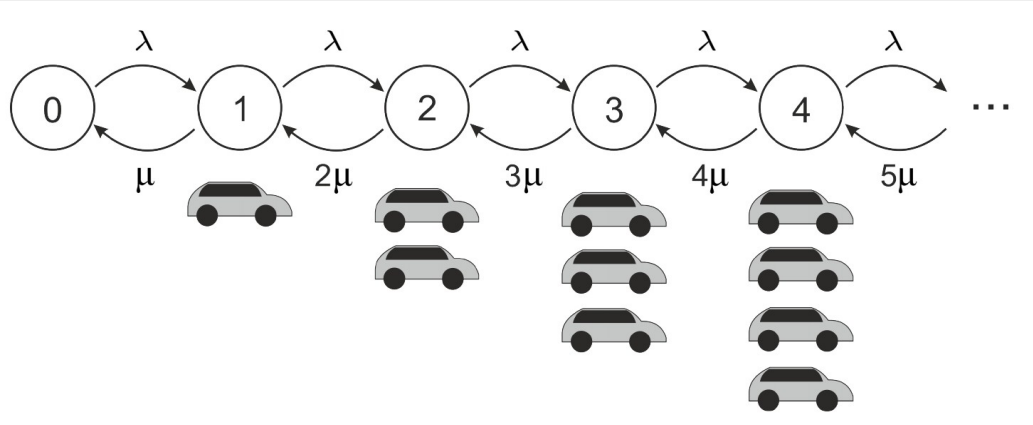
M/M/∞ queue

Parking in the area of unlimited capacity



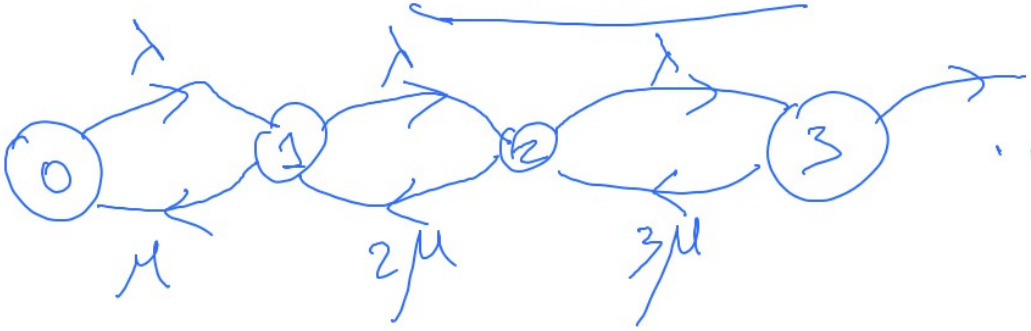
M/M/∞

Steady-state solution

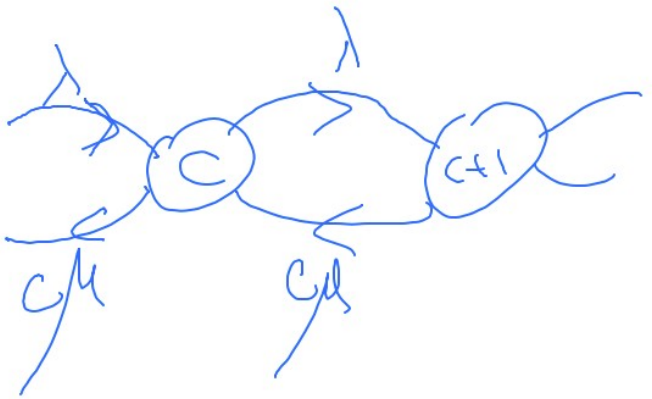
$$p_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-\lambda/\mu}$$

$$\langle N \rangle = \frac{\lambda}{\mu}$$

M/M/C



1 < C < ∞



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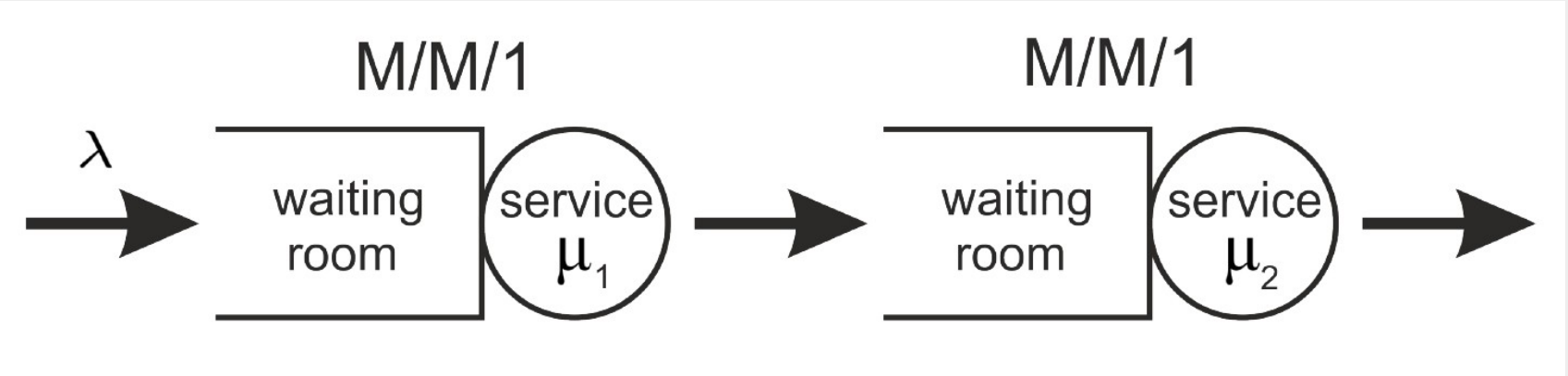
Tandem queues



Joint pdf for number of customers n_1, n_2 queueing in 1, 2

$$p(n_1, n_2) = (1-\rho_1)(1-\rho_2)\rho_1^{n_1}\rho_2^{n_2}$$

$$\rho_1 = \lambda/\mu_1, \rho_2 = \lambda/\mu_2$$

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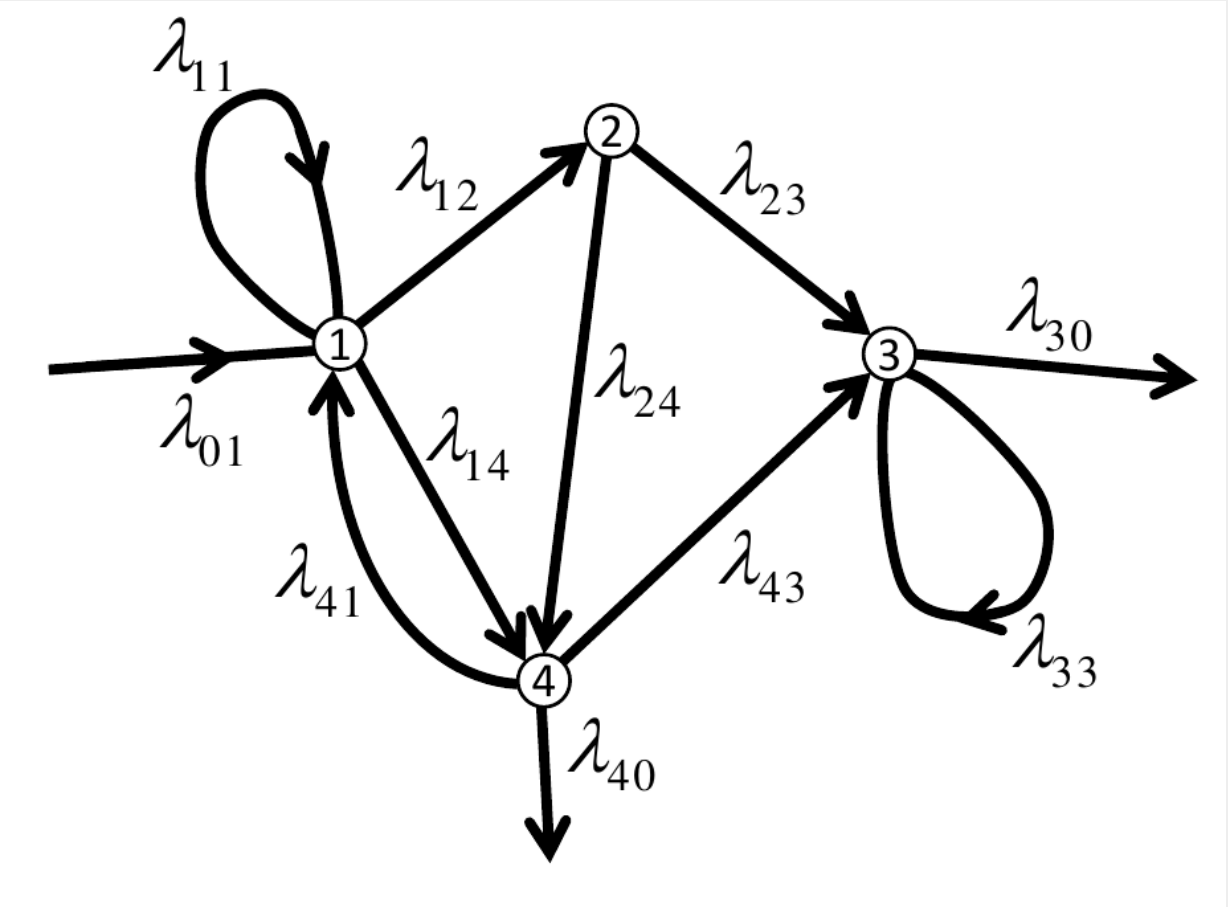
Generalisation. Jackson Networks



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Heavy traffic limit

• The number of servers is fixed and the traffic intensity (utilization) λ/μ approaches unity (from below). The queue length approximation is the so-called “reflected Brownian motion”

• Traffic intensity is fixed and the number of servers and arrival rate are increased to infinity. Here the queue length limit converges to the normal distribution

M/M/∞ queue. Heavy traffic limit

If N_t is the number of customers at time t and $\lambda/\mu \to \infty$ then the variable

$$X_t = \frac{N_t - \lambda/\mu}{\sqrt{\lambda/\mu}}$$

Behaves according to Ornstein-Uhlenbeck process

$$dX_t = -X dt + \sqrt{2}\, dW_t \longrightarrow \text{Brownian motion}$$

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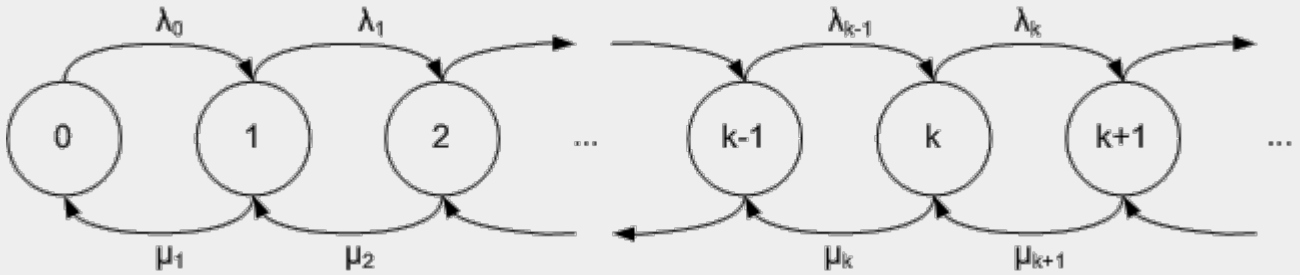
Exercise. M/M/c queue



1. Program a Markov chain for M/M/c queue for a general c (parameter)
2. Then study the case c=1 and approach and plot the steady state distribution $p_n=(1-\lambda/\mu)(\lambda/\mu)^n$ for $\lambda=0.2$, $\mu=0.6$
3. Perform the same for $c \to \infty$ to check $p_n=(1/n!)(\lambda/\mu)^n \exp(-\lambda/\mu)$

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Literature

1. F. P. Kelly. Reversibility and Stochastic Networks (Wiley, Chichester, 1979, reprinted 1987, 1994)