Homework 1
Due Date: 9th of November EOD
Course: Stochastic Methods in Mathematical Modelling. AY 2022-2023
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Problem 1 (2 points)
A person can obviously try to generate a sequence of random numbers just by writing numbers in a way they subjectively look random. In this problem you are supposed to analyse how good you are as a random number generator.

1. Write/type down a sequence of 100 (or even more) random integers in the [1, 20] range trying to manually sample a uniform distribution.
2. Use numpy random generator to sample same number of integers from the Uniform[1, 20].
3. Plot distributions of integers for numpy versus manually generated sequences.
4. Compare mean, standard deviation, skewness and kurtosis of these sequences with their true values. For Uniform[1, 20] true values can be calculated analytically.
5. Apply autocorrelation function for these two sequences.

Can you spot any differences between your sequence and the one generated with numpy.random?

Problem 2 Direct Sampling, 2 points
In this problem we are generally interested in sampling from an arbitrary probability distribution function (PDF). Assume that $P(x)$ is a PDF and we are interested in sampling random variables from it, thus, it should be normalised. Answer the following questions:

1. What is the method of inverse transform sampling from $P(x)$?
2. Get the formula for sampling of random variables for the following distribution
   $P(x) \propto \frac{e^{-\lambda x}}{1-\frac{1}{2}e^{-\lambda x}}$ where $x \in [0, \infty)$ and $\lambda > 0$.

   Hint: First you need to normalize $P(x)$.
3. Write a code and create a sufficient number of random variables from the distribution above. (Set the parameters to $\lambda = 1$ where needed).
4. Show that your procedure works properly by plotting the histogram of your produced random variables beside the original function $P(x)$.

Problem 3 Moment generating function, 1 point
The value of a piece of factory equipment after 3 years of use is $100(0.5)^X$ where X is a random variable having the moment generating function

$$M_X(t) = \frac{1}{1-2t},\ t < 1/2.$$

Calculate the expected value of this piece after 3 years of use.

Problem 4 Transformation of Probability Density, 2 points
The unit disk is given by polar coordinates $R \in [0, 1]$ and $\theta \in [0, 2\pi)$. Write a procedure that uniformly samples a random point on the disk using two independent random variables ξ_1 and ξ_2 chosen uniformly from the interval [0, 1].

Problem 5 Correlated Random Variables, 3 points

Lets assume we have PDFs $f(x_1) = \frac{1}{2\pi}$ and $g(x_2) = \frac{1}{2\pi}$, where both x_1 and x_2 are in $[0, 2\pi]$. In a specific problem, we are interested in two sets of dependent random variables from $f(x_1)$ and $g(x_2)$. Imagine set1 as the price of a product, sampled from $f(x_1)$ and set2 as the price of the constituent materials of a product, sampled from $g(x_2)$. Obviously when the price of a constituent material is high, we expect that the final price of that specific product will be high as well.

1. Design a sampling procedure that provides two set of maximally correlated variables.
2. How we can reduce the correlation in these two sets from maximum $r = 1$ to minimum $r = -1$?
3. Write a code to apply the procedure you designed and plot x_1 v.s. x_2 for three specific cases $r = 1, 0, -1$.