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Efficient Monte Carlo and Quasi–Monte Carlo Option Pricing Under the Variance Gamma Model

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We develop and study efficient Monte Carlo algorithms for pricing path-dependent options with the variance gamma model. The key ingredient is difference-of-gamma bridge sampling, based on the representation of a variance gamma process as the difference of two increasing gamma processes. For typical payoffs, we obtain a pair of estimators (named low and high) with expectations that (1) are monotone along any such bridge sampler, and (2) contain the continuous-time price. These estimators provide pathwise bounds on unbiased estimators that would be more expensive (infinitely expensive in some situations) to compute. By using these bounds with extrapolation techniques, we obtain significant efficiency improvements by work reduction. We then combine the gamma bridge sampling with randomized quasi–Monte Carlo to reduce the variance and thus further improve the efficiency. We illustrate the large efficiency improvements on numerical examples for Asian, lookback, and barrier options.

Key words: option pricing; efficiency improvement; extrapolation; quasi–Monte Carlo; variance gamma
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1. Introduction

Madan and Seneta (1990) and Madan et al. (1998) introduced the variance gamma (VG) model with application to modeling asset returns and option pricing. The VG process is a Brownian motion with a random time change that follows a stationary gamma process. This model permits more flexibility in modeling skewness and kurtosis relative to Brownian motion. We developed closed-form solutions for European option pricing and provided empirical evidence that this model gives a better fit to market option prices than the classical Black-Scholes model. One approach to option pricing under VG is based on characteristic function representations and employs efficient techniques such as the fast Fourier transform (Carr and Madan 1999); however, it does not appear to be sufficiently well developed for path-dependent options. Monte Carlo (MC) methods offer a simple way of estimating option values in that context (Glasserman 2004). However, these methods can be very time consuming if a precise estimator is needed.

Our aim in this paper is to develop and study strategies that improve the efficiency of MC estimators in the context of pricing path-dependent options under the VG model. Our approach exploits the key property that a VG process can be written as the difference of two (increasing) gamma processes. By simulating a path of these two gamma processes over a coarse time discretization, we can compute bounds on the "more detailed" (or continuous-time) sample path and approximate the corresponding payoff by extrapolation, with explicit bounds on the approximation error. We simulate the process paths over a given time interval by successive refinements via bridge sampling truncated after a small number of steps and combine this with extrapolation and randomized quasi–Monte Carlo sampling.

Quasi–Monte Carlo (QMC) methods have attracted a lot of interest in finance over the past 10 years or so (Caflisch and Moskowitz 1995, Moskowitz and Caflisch 1996, Acworth et al. 1997, Åkesson and Lehoczy 2000, Owen 1998, Glasserman 2004, L'Ecuyer 2004a). Randomized QMC (RQMC) methods replace the vectors of independent uniform random numbers that drive the simulation by sets or sequences of points that cover the space more uniformly than random points, so the variance is reduced, and yet the points are randomized in a way that provides an unbiased estimator of the integral together with a confidence interval.

In applications where RQMC really improves on ordinary MC, the integral has typically low effective dimension in some sense (L'Ecuyer and Lemieux 2002). Roughly, this means that the variance of the integrand depends (almost) only on a small number of uniforms, or is a sum of functions that depends (almost) only on a small number of uniforms. Caflisch and Moskowitz (1995) and Moskowitz and Caflisch (1996) introduced a bridge sampling method for generating the path of a Brownian motion at discrete points over a finite time interval: First generate the value at the end, then the value in the middle conditional on that at the end, then the values at one quarter and at three quarters conditional on the previous ones, and so on, halving each interval recursively. This method rewrites the integrand so as to concentrate the variance to the first few coordinates, and this makes QMC and RQMC much more effective.

Path-dependent option pricing under the VG model via QMC with bridge sampling was recently proposed and studied empirically by Ribeiro and Webber (2004) and Avramidis et al. (2003). The algorithms proposed there are similar in spirit and structure to the bridge sampling algorithm mentioned for the Brownian motion. Both are based on the premise that the bridge sampling methodology improves the effectiveness of QMC by concentrating the variance on the first few random numbers (i.e., by reducing the effective dimension of the integrand).

Our contributions are a theoretical and empirical analysis of a particular bridge sampling technique named *difference-of-gammas bridge sampling* (DGBS) (Avramidis et al. 2003). We exploit this in two ways: (i) to approximate continuous-time prices, or prices of options whose payoffs are based on a large number of observation times, by simulating the process only over a limited number of observation times, and (ii) to improve the effectiveness of RQMC.

The representation of a VG process that underlies DGBS yields a pair of estimators (named low and high) whose expectations are monotone along any sequence of paths with increasing resolution. Under monotonicity assumptions on the payoff that hold for many frequently traded options, if bridge sampling is stopped at a coarser resolution than the one on which the payoff is based, then the low and high estimators have negative and positive bias, respectively. By extrapolating estimators based on different levels of resolution, one can reduce the bias of these inexpensive estimators and improve their efficiency relative to estimators that compute the exact payoff. (As usual, we define *efficiency* as one over the product of the mean square error by the expected computing time of the estimator.)

We focus on prototypical Asian, lookback, and barrier options. For Asian options, we obtain the convergence rate of the expected gap between the low and high estimators and develop an estimator, based on Richardson's extrapolation, whose bias has a better convergence rate. For the lookback case, we exhibit the low and high estimators and provide empirical evidence that the extrapolation method can improve both convergence of the bias and efficiency. For the barrier case, we exhibit low and high estimators and develop bridge sampling that continues until the gap between the two estimators becomes zero; this procedure yields an unbiased estimator whose expected work requirement is often considerably smaller than full-dimensional path sampling. Ribeiro and Webber (2003) also develop and study empirically a method for "correcting for simulation bias" for lookback and barrier options when estimating continuous-time prices for models driven by Lévy processes; this method heuristic does not yield error bounds and risks increasing the bias.

We compare empirically the variance and efficiency of MC and RQMC for three option pricing examples on a single asset, namely an arithmetic average Asian option, a lookback option, and a barrier option. We experiment with different types of QMC point sets from those used by Ribeiro and Webber (2004) and randomize our QMC point sets to obtain unbiased estimators of both the mean and variance (which Ribeiro and Webber do not have). Our bridge sampling algorithms combined with RQMC improve simulation efficiency by large factors in these examples.

The remainder of the paper is organized as follows. Section 2, reviews the VG model and option pricing under this model. In §3, we introduce sampling algorithms for the gamma and VG processes. In §4, we develop and study truncated DGBS, examine special cases, and suggest a few estimators for each case. In §5, we compare the estimators (in terms of variance and efficiency) on a numerical example, with and without truncation and extrapolation, for MC and RQMC. In §6, we briefly review Lévy processes, generalize the approach to Lévy processes of finite variation, and state the practical difficulties that arise beyond the VG case.

2. Background and Notation

2.1. The Variance Gamma Process and Option Pricing

We provide the relevant background for the VG model developed by Madan and Seneta (1990), Madan and Milne (1991), and Madan et al. (1998), with emphasis on pricing path-dependent options via MC sampling methods. The notation $X \sim \mathcal{N}(\mu, \sigma^2)$ means that X has the normal distribution with mean μ and variance σ^2. $\text{Gamma}(\alpha, \beta)$ denotes the gamma distribution with mean $\alpha\beta$ and variance $\alpha\beta^2$.

Let $B = \{B(t) = B(t; \theta, \sigma),\ t \geq 0\}$ be a Brownian motion with drift parameter θ and variance parameter σ. That is, $B(0) = 0$, the process B has independent increments, and $B(t+\delta) - B(t) \sim \mathcal{N}(\theta\delta, \sigma^2\delta)$, for all $t \geq 0$ and $\delta > 0$. Let $G = \{G(t) = (t; \mu, \nu),\ t \geq 0\}$ be a gamma process independent of B, with drift $\mu > 0$ and volatility $\nu > 0$. That is, $G(0) = 0$, the process G has independent increments, and $G(t+\delta) - G(t) \sim \text{Gamma}(\delta\mu^2/\nu, \nu/\mu)$ for $t \geq 0$ and $\delta > 0$.

A VG process with parameters (θ, σ, ν) is defined by

$$X = \{X(t) = X(t; \theta, \sigma, \nu) = B(G(t; 1, \nu), \theta, \sigma),\ t \geq 0\}. \quad (1)$$

In simple terms, the VG process is obtained by subjecting the Brownian motion to a random time change obeying a gamma process with parameter $\mu = 1$. The gamma process has increasing paths, and this ensures that the time change makes sense. The Brownian motion, the gamma process, and the VG process are Lévy processes (Bertoin 1998); see §6 for a brief overview of this class of processes.

Madan et al. (1998) developed risk-neutral valuation formulas based on a general equilibrium argument that has the effect of specifying a market price of risk to relate a statistical VG process to an equilibrium (or risk-neutral) VG process (see p. 85 and their reference to Madan and Milne 1991). In this paper, we work exclusively with risk-neutral parameters and processes. In practice, such parameters are estimated to fit market prices of simple options (e.g., European).

The (risk-neutral) asset price process $\{S(t)\colon t \geq 0\}$ is defined by

$$S(t) = S(0) \exp\{(\omega + r - q)t + X(t)\}, \quad (2)$$

where $X(t) = X(t; \theta, \sigma, \nu)$, r is the (constant) risk-free interest rate, q is the asset's continuously compounded dividend yield, and the constant $\omega = \log(1 - \theta\nu - \sigma^2\nu/2)/\nu$ is chosen so that the discounted value of a portfolio invested in the asset is a martingale. In particular,

$$\mathrm{E}[S(t)] = S(0) \exp[(r - q)t]. \quad (3)$$

We therefore require

$$(\theta + \sigma^2/2)\nu < 1, \quad (4)$$

which ensures $\mathrm{E}[S(t)] < \infty$ for all $t > 0$. We will assume (4) throughout the paper.

We consider the pricing of a general path-dependent option under the VG model. The payoff may depend on the value of the process at a finite number of observation times, say $0 = t_0 < t_1 < \ldots < t_d = T$, where T is the expiration epoch. These observation times are often built explicitly into the option contract. The equilibrium value of such a path-dependent option, with payment occurring at time T, can be expressed as $c = \mathrm{E}[C]$, where the random variable

$$C = e^{-rT} f(S(t_1), \ldots, S(t_d)) \quad (5)$$

is the discounted payoff and $f\colon \mathbb{R}^d \to \mathbb{R}$ is the payoff function. In typical applications, the time discretization is often composed of equal-length intervals over (a subset of) $[0, T]$. In the case of a continuous-time option, where the payoff depends on the entire path $\{S(t),\ 0 \leq t \leq T\}$, the discounted payoff C is simply redefined as

$$C = e^{-rT} f(\{S(t),\ 0 \leq t \leq T\}). \quad (6)$$

3. Bridge Sampling for Gamma and Variance Gamma Processes

3.1. Sampling the Brownian Motion

A Brownian motion B with parameters (θ, σ) can be sampled at arbitrary discrete times $0 = t_0 < t_1 < \cdots < t_m$ via either sequential sampling or bridge sampling. In sequential sampling, one defines $B(0) = 0$ and $B(t_i) = B(t_{i-1}) + (t_i - t_{i-1})\theta + \sqrt{t_i - t_{i-1}}\sigma Z_i$, for $i = 1, \ldots, m$, where the Z_i are independent and identically distributed (i.i.d.) $\mathcal{N}(0, 1)$ random variables.

Bridge sampling exploits the property of Brownian motion that, for arbitrary times $0 \leq \tau_1 < t < \tau_2$, the conditional distribution of $B(t)$ given $B(\tau_1)$ and $B(\tau_2)$ is $\mathcal{N}(aB(\tau_1) + (1-a)B(\tau_2),\ a(t-\tau_1)\sigma^2)$, where $a = (\tau_2 - t)/(\tau_2 - \tau_1)$. If we assume that m is a power of 2, say $m = 2^k$, the bridge sampling algorithm operates as follows: Let $B(0) = 0$, generate first $B(t_m) \sim \mathcal{N}(t_m\theta, t_m\sigma^2)$, then $B(t_{m/2})$ conditional on $(B(0), B(t_m))$, then $B(t_{m/4})$ conditional on $(B(0), B(t_{m/2}))$, then $B(t_{3m/4})$ conditional on $(B(t_{m/2}), B(t_m))$, and so on. With this method, the first values that are generated already determine a rough sketch of the process path. For that reason, for typical payoff functions, most of the variance depends only on the first few random numbers, and the integrand is much more QMC-friendly than with sequential sampling (Moskowitz and Caflisch 1996, Glasserman 2004).

3.2. Sampling the Gamma Process

A gamma process G with parameters (μ, ν) can be sampled at discrete times $0 = t_0 < t_1 < \cdots < t_m$ in ways similar to sampling a Brownian motion, via either sequential sampling or bridge sampling.

For gamma sequential sampling (GSS), let $G(0) = 0$ and $G(t_i) = G(t_{i-1}) + \Delta_i$, where

$$\Delta_i \sim \mathrm{Gamma}((t_i - t_{i-1})\mu^2/\nu,\ \nu/\mu),$$

and the Δ_is are independent, for $i = 1, \ldots, m$.

Gamma bridge sampling (GBS) relies on the observation that for arbitrary times $0 \leq \tau_1 < t < \tau_2$, the conditional distribution of $G(t)$ given $G(\tau_1)$ and $G(\tau_2)$ is the same as $G(\tau_1) + (G(\tau_2) - G(\tau_1))Y$, where $Y \sim \mathrm{Beta}((t-\tau_1)\mu^2/\nu,\ (\tau_2 - t)\mu^2/\nu)$, and $\mathrm{Beta}(\alpha, \beta)$ denotes the beta distribution with parameters (α, β) over the interval $(0, 1)$, whose density is $f(x) = x^{\alpha-1}(1-x)^{\beta-1}/\mathrm{B}(\alpha, \beta)$, where $\mathrm{B}(\alpha, \beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\,dy$ is the beta function. Moreover, because of the independent-increments property of G, additionally conditioning on any portion of the path before τ_1 and/or after τ_2 does not change the conditional distribution of $G(t)$. When $\tau_2 - \tau_1$ becomes small, the parameters of the beta distribution decrease, and its (bimodal) density concentrates near the extreme values 0 and 1. In the limit as $\tau_2 - \tau_1 \to 0$, the probability that there is more than one jump in the interval goes to zero (see §6), and the beta distribution converges to a Bernoulli. The GBS algorithm operates in essentially the same way as bridge sampling for the Brownian motion: Let $G(0)=0$, generate $G(t_m)\sim \text{Gamma}(t_m\mu^2/\nu, \nu/\mu)$, then $Y\sim \text{Beta}(t_{m/2}\mu^2/\nu, (t_m-t_{m/2})\mu^2/\nu)$, and set $G(t_{m/2}) = G(t_0)+[G(t_m)-G(t_0)]Y$, and so on.

Madan et al. (1998, p. 82) say in their discussion of the gamma process, "the dynamics of the continuous-time gamma process is best explained by describing a simulation of the process," and proceed to describe the standard, general-purpose, but only approximate method for generating paths of a Lévy process with infinite Lévy measure, namely truncation of the Lévy measure near zero (i.e., ignoring jumps below a certain small threshold) and simulation from the appropriate compound Poisson processes. In the case where the process value needs to be observed only at fixed discrete points in time, this approximate approach is unnecessary: The GSS and GBS algorithms simulate the process exactly via gamma or beta random or both variate generators.

3.3. Bridge Sampling of the Variance Gamma Process

We now describe two bridge sampling algorithms for the VG process and then generalize the second one. In analogy with Brownian bridge sampling and gamma bridge sampling, these algorithms sample the VG process at a time partition that becomes increasingly fine. To simulate a path of the asset price dynamics, it suffices to generate a path of the VG process X and transform it to a path of the process S via the exponential transformation (2).

For the first algorithm, named Brownian-gamma bridge sampling (BGBS), we observe that, because of the independence of the two processes G and B, conditional on any collection of increments of G, the increments of B are independent normal random variables. Thus, we can first sample the increments of G via GBS and then sample the increments of $B(G(\cdot))$ by Brownian bridge sampling, conditional on the corresponding increments of G. This can be done in two ways: (a) sampling first all increments of G and then all increments of B, or (b) sampling them in pairs. The pseudocode given in Figure 1 uses the second approach. This method is equivalent to the one sketched in §4 of Ribeiro and Webber (2004).

Our second algorithm, named *difference-of-gammas bridge sampling* (DGBS), depends crucially on an alternative representation of the VG process as the difference between two independent gamma processes as follows (Madan et al. 1998):

$$X(t)=\Gamma^+(t)-\Gamma^-(t), \tag{7}$$

where Γ^+ and Γ^- are independent gamma processes (defined on a common probability space) with

Figure 1 **Brownian-Gamma Bridge Sampling of a VG Process X with Parameters (μ,ν,θ,σ) at Arbitrary Times $0=t_0<t_1<\cdots<t_m$, Where $m=2^k$ (All Variates Are Independent)**

$G(0)\leftarrow 0$; $X(0)\leftarrow 0$
Generate $G(t_m)\sim \text{Gamma}(t_m\mu^2/\nu, \nu/\mu)$
Generate $X(t_m)\sim \mathcal{N}(\theta G(t_m), \sigma^2 G(t_m))$
For $\ell=1$ to k {
  $n\leftarrow 2^{k-\ell}$
  For $j=1$ to $2^{\ell-1}$ {
    $i\leftarrow(2j-1)n$
    $\delta_1\leftarrow(t_i-t_{i-n})\mu^2/\nu$
    $\delta_2\leftarrow(t_{i+n}-t_i)\mu^2/\nu$
    Generate $Y\sim\text{Beta}(\delta_1,\delta_2)$
    $G(t_i)\leftarrow G(t_{i-n})+[G(t_{i+n})-G(t_{i-n})]Y$
    Generate $Z\sim\mathcal{N}(0,[G(t_{i+n})-G(t_i)]\sigma^2 Y)$
    $X(t_i)\leftarrow YX(t_{i+n})+(1-Y)X(t_{i-n})+Z$
  }
}

parameters $(\mu_{\text{p}},\nu_{\text{p}})$ and $(\mu_{\text{n}},\nu_{\text{n}})$, respectively, with $\mu_{\text{p}}=(\sqrt{\theta^2+2\sigma^2/\nu}+\theta)/2$, $\mu_{\text{n}}=(\sqrt{\theta^2+2\sigma^2/\nu}-\theta)/2$, $\nu_{\text{p}}=\mu_{\text{p}}^2\nu$, and $\nu_{\text{n}}=\mu_{\text{n}}^2\nu$. To concentrate the sampling of macro-effects of paths of X (and of S) to the first sampling coordinates, one can simulate X by applying GBS to both Γ^+ and Γ^- simultaneously.

The general version of the DGBS algorithm works as follows. We consider a finite time interval $[0,T]$ and an infinite sequence of distinct real numbers $y_0=0$, $y_1=T$, and $y_2,y_3,\ldots,$ dense in $(0,T)$, but otherwise arbitrary. This is the sequence of times at which the two gamma processes are generated, in order (i.e., first at y_1, then at y_2 conditional on their values at y_1, then at y_3 conditional on their values at y_1 and y_2, and so on). For each positive integer m, let $0=t_{m,0}<t_{m,1}<\cdots<t_{m,m}=T$ denote the values of $y_0,y_1,\ldots,y_m$ sorted by increasing order, and let $\iota(m)$ be the index i such that $t_{m,i}=y_m$. That is, $t_{m,\iota(m)}$ is the new observation time added at step m. Figure 2 outlines the DGBS algorithm with an infinite loop. In an actual implementation, the algorithm can be stopped after any number of steps. A special case of particular interest involves the dyadic partition $t_i=Ti/m$, $i=1,2,\ldots,m$, where m is a power of 2; by choosing $t_{m,\iota(m)}$ as the midpoint of $(t_{m,\iota(m)-1},t_{m,\iota(m)+1})$ for all m, all beta variates are symmetric (the density is symmetric with respect to 1/2). In the numerical results in §5, we work exclusively with dyadic partitions and use a fast beta random-variate generator that exploits the symmetry.

4. Truncated Difference-of-Gammas Bridge Sampling

The representation of a VG process as a difference of two gamma processes has the nice property that, given the values of the two gamma processes at a finite set of observation times, one can compute bounds on the path of the VG process everywhere between these observation times, as we shall explain

Figure 2 Difference-of-Gammas Bridge Sampling of a VG Process X with Parameters $(1, \nu, \theta, \sigma)$ at an Infinite Sequence of Times $y_0 = 0$, $y_1 = T$, and $y_2, y_3, \ldots$ in $(0, T)$ (All Variates Are Independent)

$t_{1,0} \leftarrow 0$; $t_{1,1} \leftarrow T$
$\Gamma^+(0) \leftarrow 0$; $\Gamma^-(0) \leftarrow 0$
Generate $\Gamma^+(T) \sim \text{Gamma}(T/\nu, \nu_p/\mu_p)$
Generate $\Gamma^-(T) \sim \text{Gamma}(T/\nu, \nu_n/\mu_n)$
For $m = 2$ to ∞ {
  $i \leftarrow \iota(m)$
  $\delta_1 \leftarrow (y_m - t_{m,i-1})/\nu$
  $\delta_2 \leftarrow (t_{m,i+1} - y_m)/\nu$
  Generate $Y^+ \sim \text{Beta}(\delta_1, \delta_2)$
  $\Gamma^+(y_m) \leftarrow \Gamma^+(t_{m,i-1}) + [\Gamma^+(t_{m,i+1}) - \Gamma^+(t_{m,i-1})]Y^+$
  Generate $Y^- \sim \text{Beta}(\delta_1, \delta_2)$
  $\Gamma^-(y_m) \leftarrow \Gamma^-(t_{m,i-1}) + [\Gamma^-(t_{m,i+1}) - \Gamma^-(t_{m,i-1})]Y^-$
  $X(y_m) \leftarrow \Gamma^+(y_m) - \Gamma^-(y_m)$
}

in this section. This implies in particular that with DGBS, one can stop (truncate) the sampling process at any intermediate step and compute bounds on the value of the VG process at all other times that are involved in the option payoff. From that, one can obtain bounds on the final payoff without having to simulate all the process values that determine the payoff, and some intermediate value between these bounds (instead of the exact payoff) can be taken as an estimator of the expected payoff. This truncated DGBS procedure can save work in exchange for some bias. In many practical settings, the saving can be significant, and the additional bias contributes little to the mean square error. Truncating early can therefore improve the overall estimator's efficiency by an important factor, especially when the computing budget is small (so the squared bias can be dominated by the variance). For certain types of options, the bias can be completely eliminated. In other cases, the bias can be reduced by computing the truncated DGBS estimators at different truncation points and extrapolating. This methodology is also very convenient for approximating the values of options based on continuous-time observation of the process.

4.1. Bounds on the VG Process

Define $\zeta = \omega + r - q$, $\zeta^+ = \max(\zeta, 0)$, $\zeta^- = \max(-\zeta, 0)$, and

$$S(t) = S(0)\exp[\zeta t + X(t)] = S(0)\exp[\zeta t + \Gamma^+(t) - \Gamma^-(t)] \quad \text{for } t \geq 0. \tag{8}$$

We now develop bounds on the path of S from the values of the two gamma processes at a finite set of observation times in the setting of the DGBS algorithm of Figure 2.

Define the two processes L_m and U_m over $[0, T]$ by

$$\begin{aligned} L_m(t) &= S(t_{m,i-1})\exp[\zeta(t - t_{m,i-1}) - \Gamma^-(t_{m,i}) + \Gamma^-(t_{m,i-1})] \\ &= S(0)\exp[\zeta t - \Gamma^-(t_{m,i}) + \Gamma^+(t_{m,i-1})] \end{aligned}$$

and

$$\begin{aligned} U_m(t) &= S(t_{m,i-1})\exp[\zeta(t - t_{m,i-1}) + \Gamma^+(t_{m,i}) - \Gamma^+(t_{m,i-1})] \\ &= S(0)\exp[\zeta t + \Gamma^+(t_{m,i}) - \Gamma^-(t_{m,i-1})] \end{aligned}$$

for $t_{m,i-1} < t < t_{m,i}$, and $L_m(t_{m,i}) = U_m(t_{m,i}) = S(t_{m,i})$, for $i = 0, \ldots, m$. These two processes are both left and right discontinuous at the observation times $t_{m,i}$ (where they match). The following proposition states that the process S is squeezed between these two bounding processes and that these bounds are narrowing monotonously when m increases.

Proposition 1. *For every sample path of S, any integer $m > 0$, and all $t \in [0, T]$, we have*

$$L_m(t) \leq L_{m+1}(t) \leq S(t) \leq U_{m+1}(t) \leq U_m(t).$$

Proof. At $t = t_{m,i}$, for $0 \leq i \leq m$, we have $L_m(t) = U_m(t) = S(t)$ by definition. The increase of the process $X = \Gamma^+ - \Gamma^-$ in any subinterval of $(t_{m,i-1}, t_{m,i})$ cannot exceed the increase of the process Γ^+ during that interval, that is, cannot exceed $\Gamma^+(t_{m,i}) - \Gamma^+(t_{m,i-1})$. Similarly, its decrease cannot exceed the increase of Γ^- during that interval, that is, cannot exceed $\Gamma^-(t_{m,i}) - \Gamma^-(t_{m,i-1})$. Combining these observations with (8), we obtain that $L_m(t) \leq S(t) \leq U_m(t)$.

To show that $U_{m+1}(t) \leq U_m(t)$, let $(t_{m,j-1}, t_{m,j})$ be the interval that contains the point y_{m+1}. Outside this interval $(t_{m,j-1}, t_{m,j})$, we have $U_{m+1}(t) = U_m(t)$. We also have $t_{m+1,j-1} = t_{m,j-1}$, $t_{m+1,j} = y_{m+1}$, and $t_{m+1,j+1} = t_{m,j}$, so this interval splits into the two intervals $(t_{m+1,j-1}, t_{m+1,j})$ and $(t_{m+1,j}, t_{m+1,j+1})$, when m increases to $m+1$. For $t \in (t_{m+1,j-1}, t_{m+1,j})$, we have

$$\begin{aligned} U_{m+1}(t) &= S(t_{m+1,j-1})\exp[\zeta(t - t_{m+1,j-1}) \\ &\qquad + \Gamma^+(t_{m+1,j}) - \Gamma^+(t_{m+1,j-1})] \\ &= S(t_{m,j-1})\exp[\zeta(t - t_{m,j-1}) \\ &\qquad + \Gamma^+(t_{m+1,j}) - \Gamma^+(t_{m,j-1})] \\ &\leq S(t_{m,j-1})\exp[\zeta(t - t_{m,j-1}) \\ &\qquad + \Gamma^+(t_{m+1,j+1}) - \Gamma^+(t_{m,j-1})] \\ &= U_m(t) \end{aligned}$$

because $t_{m+1,j+1} = t_{m,j}$. For $t \in [t_{m+1,j}, t_{m+1,j+1})$, we have

$$\begin{aligned} U_{m+1}(t) &= S(t_{m+1,j})\exp[\zeta(t - t_{m+1,j}) + \Gamma^+(t_{m+1,j+1}) \\ &\qquad - \Gamma^+(t_{m+1,j})] \\ &= S(t_{m+1,j-1})\exp[\zeta(t - t_{m+1,j-1}) + \Gamma^+(t_{m+1,j}) \\ &\qquad - \Gamma^-(t_{m+1,j}) - \Gamma^+(t_{m+1,j-1}) + \Gamma^-(t_{m+1,j-1}) \\ &\qquad + \Gamma^+(t_{m+1,j+1}) - \Gamma^+(t_{m+1,j})] \end{aligned}$$

$$\begin{aligned}
&\le S(t_{m+1,\,j-1})\exp[\zeta(t-t_{m+1,\,j-1})+\Gamma^+(t_{m+1,\,j+1})\\
&\qquad -\Gamma^+(t_{m+1,\,j-1})]\\
&= S(t_{m+1,\,j-1})\exp[\zeta(t-t_{m+1,\,j-1})+\Gamma^+(t_{m,\,j})\\
&\qquad -\Gamma^+(t_{m,\,j-1})]\\
&= U_m(t).
\end{aligned}$$

The inequality $L_m(t) \le L_{m+1}(t)$ can be proved by a symmetrical argument. □

The following simplified (piecewise constant) version of the bounds of Proposition 1 will be convenient. Define

$$\begin{aligned}
L_{m,\,i} &= \inf_{t_{m,\,i-1}<t<t_{m,\,i}} L_m(t)\\
&= S(t_{m,\,i-1})\exp[-\zeta^-(t_{m,\,i}-t_{m,\,i-1})-\Gamma^-(t_{m,\,i})\\
&\qquad +\Gamma^-(t_{m,\,i-1})],\\
U_{m,\,i} &= \sup_{t_{m,\,i-1}<t<t_{m,\,i}} U_m(t)\\
&= S(t_{m,\,i-1})\exp[\zeta^+(t_{m,\,i}-t_{m,\,i-1})+\Gamma^+(t_{m,\,i})\\
&\qquad -\Gamma^+(t_{m,\,i-1})],
\end{aligned}$$

$L_m^*(t) = L_{m,\,i}$ and $U_m^*(t) = U_{m,\,i}$ for $t_{m,\,i-1} < t < t_{m,\,i}$, and $L_m^*(t_{m,\,i}) = U_m^*(t_{m,\,i}) = S(t_{m,\,i})$, for $i = 1, \ldots, m$.

Corollary 1. *For every sample path of S, any integer $m > 0$, and all $t \in [0, T]$, we have*

$$L_m^*(t) \le L_{m+1}^*(t) \le S(t) \le U_{m+1}^*(t) \le U_m^*(t).$$

4.2. Estimators Based on Truncated DGBS and Extrapolation

Under monotonicity conditions on the option's payoff as a function of the path of S, conditional on the values observed so far, these bounds on S translate into bounds on the conditional payoff. Any value lying between these bounds can be taken as a (generally biased) estimator of the expected payoff. When the two bounds on the payoff coincide (which can happen for barrier options, for example), there is no bias. To state this more formally, we define

$$\begin{aligned}
\mathcal{F}_m = (&t_{m,1}, \Gamma^+(t_{m,1}), \Gamma^-(t_{m,1}), \ldots,\\
&t_{m,m}, \Gamma^+(t_{m,m}), \Gamma^-(t_{m,m})),
\end{aligned} \tag{9}$$

which contains the values of the two gamma processes at the first m observation times. Denote by $C_{L,m}$, $C_{U,m}$, $C^*_{L,m}$, and $C^*_{U,m}$ the discounted payoffs (that correspond to C) when S is replaced in (5) and (6) by L_m, U_m, L_m^*, and U_m^*, respectively.

Corollary 2. *Suppose that conditional on $\mathcal{F}_m$, C is a monotone nondecreasing function of $S(t)$ for all values of t not in $\{t_{m,0}, \ldots, t_{m,m}\}$. Then,*

$$C^*_{L,m} \le C_{L,m} \le C \le C_{U,m} \le C^*_{U,m}.$$

If C is (conditionally) nonincreasing instead, the reverse inequality holds. In both cases, these bounds are narrowing when m increases.

Proof. This follows from Proposition 1 and straightforward monotonicity arguments. □

Recall that d is the number of times at which the option contract requires the underlying asset to be observed. When d is large (possibly infinite), we suggest the following sampling strategy for estimating $c = \mathrm{E}[C]$: Generate the process X by DGBS for some integer k such that $m = 2^k < d$ (for simplicity, we may assume that the sampling points are $y_1 = T$, $y_2 = T/2$, $y_3 = T/4$, $y_4 = 3T/4, \ldots$) and compute the values of $C_{L,m}$ and $C_{U,m}$, which readily provide bounds on the value of C. In this context, we are interested in an estimator C_m of c based only on the information in $\mathcal{F}_m$. There are of course several possibilities for C_m. In general, these estimators are biased.

A simple one is the average of the payoff bounds, $C_{A,m} = (C_{L,m} + C_{U,m})/2$. It may not be the best choice if the bounds are highly asymmetric with respect to the exact payoff C. Another possibility is to take C_m as the discounted payoff obtained when the process S is replaced by the arithmetic average $\bar{S}_m = (L_m + U_m)/2$, or by the geometric average $\bar{S}_{G,m}$ defined by

$$\begin{aligned}
\bar{S}_{G,m}(t) &= (L_m(t)U_m(t))^{1/2}\\
&= S(0)\exp[\zeta t + (X(t_{m,\,i}) + X(t_{m,\,i-1}))/2]
\end{aligned}$$

for $t_{m,\,i-1} \le t \le t_{m,\,i}$, or by the process defined by replacing X in (8) by its conditional expectation given $\mathcal{F}_m$, that is,

$$\begin{aligned}
&\bar{S}_{X,m}(t)\\
&\quad = S(0)\exp[\zeta t + \mathrm{E}[X(t) \mid \mathcal{F}_m]]\\
&\quad = S(0)\exp[\zeta t + \alpha_i(t)X(t_{m,\,i}) + (1-\alpha_i(t))X(t_{m,\,i-1})]
\end{aligned}$$

for $t_{m,\,i-1} \le t \le t_{m,\,i}$, where $\alpha_i(t) = (t - t_{m,\,i-1})/(t_{m,\,i} - t_{m,\,i-1})$. We shall denote by $C_m^{(G)}$ and $C_m^{(X)}$ the values of C_m when S is replaced by $\bar{S}_{G,m}$ and $\bar{S}_{X,m}$, respectively. Yet another approach is to let C_m be the payoff of the corresponding discrete-time option with observation times $t_{m,1}, \ldots, t_{m,m}$, which we denote by $C_m^{(D)}$.

Suppose that $d = \infty$ and that the bias $\beta(m) = \mathrm{E}[C_m] - c$ has the asymptotic form

$$\beta(m) = \beta_0 m^{-\gamma_1} + O(g(m)) \tag{10}$$

for some constants β_0 and $\gamma_1 > 0$ and a function g such that $g(m) = o(m^{-\gamma_1})$. One frequently encounters $\gamma_1 = 1$ (see the next subsections). Assume $\beta_0 \ne 0$ and replace the estimator C_m by

$$\tilde{C}_m = \frac{2^{\gamma_1} C_m - C_{m/2}}{2^{\gamma_1} - 1}, \tag{11}$$

where m is assumed to be a power of 2 (for simplicity). We have

$$\begin{aligned}\mathrm{E}[\tilde{C}_m] &= [2^{\gamma_1}(c+\beta(m)) - (c+\beta(m/2))]/(2^{\gamma_1}-1)\\ &= [2^{\gamma_1}c + 2^{\gamma_1}\beta_0 m^{-\gamma_1} + O(g(m)) - c - \beta_0(m/2)^{-\gamma_1}\\ &\quad - O(g(m/2))]/(2^{\gamma_1}-1)\\ &= c + O(g(m)).\end{aligned}$$

That is, $\tilde{C}_m$ has reduced bias order compared with C_m. In case $\beta_0 = 0$, the bias order remains $O(g(m))$, and the dominant (i.e., nonzero, slowest-decaying) bias term has a (generally) different constant. As an example, if $\beta(m) = \beta_2 m^{-\gamma_2}$, where $\beta_2 \neq 0$ and $\gamma_2 > \gamma_1 > 0$, then it is easy to see that $\tilde{C}_m$ has bias $c_2 m^{-\gamma_2}$, where $c_2 = \beta_2(2^{\gamma_1} - 2^{\gamma_2})/(2^{\gamma_1}-1)$; the bias remains in order $m^{-\gamma_2}$, but the constant in front has opposite sign and larger absolute value than β_2. Further, a simple calculation shows that the absolute bias may also increase if the dominant term in (10) alternates sign as a function of m.

This method is an application of a general class of techniques called *extrapolation to the limit* or *Richardson's extrapolation* (Joyce 1971, Conte and de Boor 1972), also proposed by Boyle et al. (1997) in a simulation-based security-pricing context. Our numerical experiments in §5 show that the bias reduction can be significant, typically without a significant increase of the variance. This bias reduction permits one to obtain an estimator with a given (target) mean square error using a smaller value of m (i.e., with a smaller amount of work).

Although many traded options meet the monotonicity assumption of Corollary 2, there exist actively traded types of options that fail the assumption, for example, if the payoff is the realized volatility (sample-path standard deviation), or a monotone function of it, or the first (in time) large drop or increase of the underlying; in all cases, the conditional payoff has "interactions" between the not-yet-sampled path values, destroying monotonicity. For such options, truncated DGBS would yield biased estimates, and the error bounds of Corollary 2 would not apply.

In the remainder of this section, we specialize these bounds to specific option types, namely Asian options in §4.3, lookback options in §4.4, and barrier options in §4.5. For an Asian option with continuous-time observation, we also prove that (10) holds with $\gamma_1 = 1$ (where β_0 could be zero) for an equal-length partition (i.e., $t_{m,i} = iT/m$) for both $C_{\mathrm{L},m}$ and $C_{\mathrm{U},m}$.

4.3. Asian Options

Suppose we want to estimate the continuous-time Asian call price $c = c_{\mathrm{A}}(\infty) = \mathrm{E}[C_{\mathrm{A}}(\infty)]$, where

$$C = C_{\mathrm{A}}(\infty) = e^{-rT}\left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)^+,$$

by a discrete-time, generally biased, Monte Carlo estimator. We consider the DGBS algorithm of Figure 2. Here, conditional on $\mathcal{F}_m$, the discounted payoff C is clearly a monotone nondecreasing function of $S(t)$ for all t. Therefore,

$$C_{\mathrm{L},m-1} \le C_{\mathrm{L},m} \le C \le C_{\mathrm{U},m} \le C_{\mathrm{U},m-1}$$

for all $m > 1$ and all sample paths, from Corollary 2. This implies that $C_{\mathrm{L},m}$ and $C_{\mathrm{U},m}$ are negatively and positively biased estimators of $c_{\mathrm{A}}(\infty)$, respectively. A better estimator could be $\tilde{C}_m$ defined in (11) for some C_m between these bounds, if we know the correct value of γ_1.

The upper bound $C_{\mathrm{U},m}$ here can be written as

$$C_{\mathrm{U},m} = e^{-rT}\left(\frac{1}{T}\int_0^T U_m(t)\,dt - K\right)^+,$$

where

$$\begin{aligned}&\frac{1}{T}\int_0^T U_m(t)\,dt\\ &= \frac{1}{T}\sum_{i=1}^m \int_{t_{m,i-1}}^{t_{m,i}} S(0)\exp[\zeta t + \Gamma^+(t_{m,i}) - \Gamma^-(t_{m,i-1})]\,dt\\ &= \frac{1}{T}\sum_{i=1}^m S(0)\exp[\Gamma^+(t_{m,i}) - \Gamma^-(t_{m,i-1})]\int_{t_{m,i-1}}^{t_{m,i}} \exp[\zeta t]\,dt\\ &= \begin{cases} \dfrac{S(0)}{\zeta T}\displaystyle\sum_{i=1}^m \exp[\Gamma^+(t_{m,i}) - \Gamma^-(t_{m,i-1})]\\ \quad\cdot[\exp(\zeta t_{m,i}) - \exp(\zeta t_{m,i-1})], & \zeta \neq 0\\ \dfrac{S(0)}{T}\displaystyle\sum_{i=1}^m \exp[\Gamma^+(t_{m,i}) - \Gamma^-(t_{m,i-1})]\\ \quad\cdot[t_{m,i} - t_{m,i-1}], & \zeta = 0.\end{cases}\end{aligned}$$

Similarly,

$$C_{\mathrm{L},m} = e^{-rT}\left(\frac{1}{T}\int_0^T L_m(t)\,dt - K\right)^+,$$

where

$$\frac{1}{T}\int_0^T L_m(t)\,dt = \begin{cases} \dfrac{S(0)}{\zeta T}\displaystyle\sum_{i=1}^m \exp[-\Gamma^-(t_{m,i}) + \Gamma^+(t_{m,i-1})]\\ \quad\cdot[\exp(\zeta t_{m,i}) - \exp(\zeta t_{m,i-1})], & \zeta \neq 0\\ \dfrac{S(0)}{T}\displaystyle\sum_{i=1}^m \exp[-\Gamma^-(t_{m,i}) + \Gamma^+(t_{m,i-1})]\\ \quad\cdot[t_{m,i} - t_{m,i-1}], & \zeta = 0.\end{cases}$$

Besides $C_{\mathrm{L},m}$ and $C_{\mathrm{U},m}$, natural choices of C_m are $C_{\mathrm{A},m}$, $C_m^{(\mathrm{D})}$, $C_m^{(\mathrm{G})}$, and $C_m^{(\mathrm{X})}$ defined in §4.2, and the symmetrical version of $C_m^{(\mathrm{D})}$ defined by

$$C_m^{(\mathrm{S})} = e^{-rT}\left(\frac{1}{m}[(S(0)+S(T))/2 + S(t_1) + \cdots + S(t_{m-1})] - K\right)^+.$$

For the Asian option, $C_m^{(G)}$ and $C_m^{(X)}$ can be computed via the following expressions (we leave out the details of the derivations):

$$\begin{aligned} C_m^{(G)} &= e^{-rT}\left(\frac{1}{T}\int_0^T \bar{S}_{G,m}(t)\,dt - K\right)^+ \\ &= e^{-rT}\left(\frac{1}{T}\int_0^T S(0)\exp[\zeta t + (X(t_{m,i}) \right.\\ &\qquad\left. + X(t_{m,i-1}))/2]\,dt - K\right)^+ \\ &= e^{-rT}\left(\frac{S(0)}{\zeta T}\sum_{i=1}^m \exp[(X(t_{m,i}) - X(t_{m,i-1}))/2]\right.\\ &\qquad\left. \cdot[\exp(\zeta t_{m,i}) - \exp(\zeta t_{m,i-1})] - K\right)^+ \end{aligned}$$

and

$$\begin{aligned} C_m^{(X)} &= e^{-rT}\left(\frac{1}{T}\int_0^T \bar{S}_{X,m}(t)\,dt - K\right)^+ \\ &= e^{-rT}\left(\frac{S(0)}{T}\sum_{i=1}^m \exp\left[\frac{t_{m,i}X(t_{m,i-1}) - t_{m,i-1}X(t_{m,i})}{t_{m,i} - t_{m,i-1}}\right]\right.\\ &\qquad\left. \cdot\left[\frac{\exp(D_{m,i}t_{m,i}) - \exp(D_{m,i}t_{m,i-1})}{D_{m,i}}\right] - K\right)^+, \end{aligned}$$

where $D_{m,i} = [X(t_{m,i}) - X(t_{m,i-1})]/[t_{m,i} - t_{m,i-1}]$ and we assumed $\zeta \neq 0$; obvious modifications apply when $\zeta = 0$.

Proposition 2 establishes that for $C_{L,m} \le C_m \le C_{U,m}$ and equal-length time discretizations of the form $t_{m,i} = iT/m$, the bias converges to zero as $O(m^{-1})$ when $m \to \infty$ and as $o(m^{-1})$ when extrapolating with $\gamma_1 = 1$.

Proposition 2. *Assume* (4). *Consider equal-length discretizations of the form $t_{m,i} = iT/m$, where $m = 1, 2, \ldots$. We have*

$$m\mathrm{E}[C_{U,m} - C_{L,m}] \le \left\{\begin{array}{l} S(0)(Q_+^{1/m} - Q_-^{1/m})\dfrac{e^{-rT} - e^{-qT}}{1 - e^{-(r-q)T/m}} \\ \qquad \text{if } r \neq q \\ S(0)(Q_+^{1/m} - Q_-^{1/m})e^{-rT}m \\ \qquad \text{if } r = q \end{array}\right\} \to \kappa_0 \quad (12)$$

as $m \to \infty$, where

$$\kappa_0 = \begin{cases} S(0)\dfrac{e^{-rT} - e^{-qT}}{(r-q)T}\log(Q_+/Q_-) & \text{if } r \neq q \\ S(0)e^{-rT}\log(Q_+/Q_-) & \text{if } r = q, \end{cases}$$

$$Q_+ = e^{T\zeta^+}\left(1 - \frac{\nu_p}{\mu_p}\right)^{-T/\nu}, \text{ and } Q_- = e^{-T\zeta^-}\left(1 + \frac{\nu_n}{\mu_n}\right)^{-T/\nu}.$$

This implies that whenever $C_{L,m} \le C_m \le C_{U,m}$ for all m,

$$|\mathrm{E}[C_m] - c_A(\infty)| = \tilde{\beta}/m + o(1/m) \quad (13)$$

as $m \to \infty$, where $0 \le \tilde{\beta} \le \kappa_0$.

Proof. To prove the inequality in (12), define $\Delta\Gamma_{m,i}^+ = \Gamma^+(iT/m) - \Gamma^+((i-1)T/m)$, $\Delta\Gamma_{m,i}^- = \Gamma^-(iT/m) - \Gamma^-((i-1)T/m)$, and observe that

$$\begin{aligned} 0 &\le C_{U,m} - C_{L,m} \\ &\le e^{-rT}\frac{1}{m}\sum_{i=1}^m [U_{m,i} - L_{m,i}] \\ &\le e^{-rT}\frac{1}{m}\sum_{i=1}^m S((i-1)T/m)\left[\exp\left(\frac{\zeta^+T}{m} + \Delta\Gamma_{m,i}^+\right)\right.\\ &\qquad\left. - \exp\left(-\frac{\zeta^-T}{m} - \Delta\Gamma_{m,i}^-\right)\right]. \end{aligned}$$

We take expectations, observing that $S((i-1)T/m)$, $\Delta\Gamma_{m,i}^+$, and $\Delta\Gamma_{m,i}^-$ are independent, with $\mathrm{E}[S(iT/m)] = S(0)\exp[(r-q)iT/m]$ from (3); $\mathrm{E}[\exp(\Delta\Gamma_{m,i}^+)] = (1 - \nu_p/\mu_p)^{-T/\nu m}$ (the finiteness of this moment follows from (4), which implies $\nu_p/\mu_p < 1$); and $\mathrm{E}[\exp(-\Delta\Gamma_{m,i}^-)] = (1 + \nu_p/\mu_p)^{-T/\nu m}$. We obtain

$$\begin{aligned} &\mathrm{E}[C_{U,m} - C_{L,m}] \\ &\quad\le e^{-rT}S(0)\frac{1}{m}(Q_+^{1/m} - Q_-^{1/m})\sum_{i=0}^{m-1}\exp[(r-q)iT/m] \end{aligned}$$

and summing the geometric series proves the inequality in (12). The convergence part of (12) follows by L'Hôpital's rule.

In view of (12) and the fact that $\mathrm{E}[C_{L,m}] \le c_A(\infty) \le \mathrm{E}[C_{U,m}]$ from Corollary 2, we have that $|\mathrm{E}[C_m] - c_A(\infty)| = O(1/m)$. It is either $o(1/m)$, in which case (13) holds trivially with $\tilde{\beta} = 0$, or of the form (13) for $\tilde{\beta} > 0$. □

For the bias model (10), a numerical experiment with one example suggests that for estimators $C_{L,m}$, $C_{U,m}$, $C_m^{(D)}$, $C_m^{(G)}$, and $C_m^{(X)}$, we have $\gamma_1 = 1$ and $g(m) = m^{-2}$. For $C_m^{(S)}$ and $C_{A,m}$, we observed $\gamma_1 \approx 2$.

Consider now an Asian option based on discrete observation times $0 = t_0 < t_1 < \cdots < t_d = T$, whose value is $c = c_A(d) = \mathrm{E}[C_A(d)]$, where

$$C = C_A(d) = e^{-rT}\left(\frac{1}{d}\sum_{i=1}^d S(t_i) - K\right)^+.$$

Because $C_{L,d} \le C_A(d) \le C_{U,d}$, the bounds $C_{L,m} \le C_A(d) \le C_{U,m}$ also hold in this case whenever $m \le d$ and $\{t_{m,1}, \ldots, t_{m,m}\} \subseteq \{t_1, \ldots, t_d\}$. If d is a large or moderate power of 2 and if $t_i = iT/d$, for example, we may consider replacing the estimator $C_A(d)$ by a less expensive extrapolated one.

4.4. Lookback Options

We now examine a prototypical lookback option, the *floating-strike call*. The approach extends easily to other related payoff types; for these and further information, see Hull (2000). The continuous-time price of this option is $c = c_F(\infty) = \mathrm{E}[C_F(\infty)]$ with payoff

$$C_F(\infty) = e^{-rT}\Big[S(T) - \inf_{0\le t\le T} S(t)\Big]. \tag{14}$$

Conditional on $S(T)$, this payoff is a nonincreasing function of $S(t)$ for each t, which implies from the second part of Corollary 2 that $C_{U,m} \le C_F(\infty) \le C_{L,m}$. These bounds are narrowing when m increases. The low and high estimators are

$$\begin{aligned} C_{U,m} &= e^{-rT}\Big[S(T) - \inf_{0\le t\le T} U_m(t)\Big] \\ &= e^{-rT}\Big[S(T) - \min_{0\le i\le m} S(t_{m,i})\Big] \end{aligned} \tag{15}$$

and

$$\begin{aligned} C_{L,m} &= e^{-rT}\Big[S(T) - \inf_{0\le t\le T} L_m(t)\Big] \\ &= e^{-rT}\Big[S(T) - \min_{0\le i\le m} L_{m,i}\Big], \end{aligned} \tag{16}$$

respectively.

A numerical experiment with one example indicates that for an equal-length discretization where $t_{m,i} = iT/m$, when C_m is taken as either $C_{U,m}$ or $C_{L,m}$, $\beta(m)$ has approximately the form $\beta(m) = \beta_0 m^{-1} + O(m^{-5/3})$. Thus, the extrapolation from either of these high and low estimators reduces the bias (empirically) from $O(m^{-1})$ to $O(m^{-5/3})$ approximately.

The discretely observed version of this option has discounted payoff

$$C_F(d) = e^{-rT}\Big[S(T) - \min_{0\le i\le d} S(t_i)\Big]. \tag{17}$$

We have $C_{U,d} \le C_F(d) \le C_{L,d}$, and therefore $C_{U,m} \le C_F(d) \le C_{L,m}$ whenever $\{t_{m,1}, \dots, t_{m,m}\} \subseteq \{t_1, \dots, t_d\}$.

4.5. Barrier Options

Barrier options appear to pose greater computational challenges relative to Asians and lookbacks. Ribeiro and Webber (2004), pricing with the VG model and using BGBS sampling, achieved substantial efficiency improvement for Asian and lookback options but very little efficiency improvement for barrier options. A barrier option can be classified as either knock-in or knock-out. A *knock-in* option has zero payoff unless the underlying asset price reaches a barrier, whereas a *knock-out* option has zero payoff whenever the underlying asset price reaches a barrier. For further information, see Hull (2000).

As a prototypical barrier option, we consider an *up-and-in call*, whose continuous-time payoff is

$$C_B(\infty) = e^{-rT}(S(T)-K)^+ I\Big\{\sup_{0\le t\le T} S(t) > b\Big\}, \tag{18}$$

where $b > S(0)$ is the barrier, K is the strike price, and I denotes the indicator function. Conditional on $\mathcal{F}_m$ for $m > 0$, this payoff is nondecreasing in $S(t)$ for each t, so the first part of Corollary 2 applies. The low and high estimators are

$$\begin{aligned} C_{L,m} &= e^{-rT}(S(T)-K)^+ I\Big\{\sup_{0\le t\le T} L_m(t) > b\Big\} \\ &= e^{-rT}(S(T)-K)^+ I\Big\{\max_{1\le i\le m} S(t_{m,i}) > b\Big\} \end{aligned}$$

and

$$\begin{aligned} C_{U,m} &= e^{-rT}(S(T)-K)^+ I\Big\{\sup_{0\le t\le T} U_m(t) > b\Big\} \\ &= e^{-rT}(S(T)-K)^+ I\Big\{\max_{1\le i\le m} U_{m,i} > b\Big\}. \end{aligned}$$

The gap between the low and high estimators vanishes if $S(T) \le K$ and also whenever the indicator function takes the same value in both cases, that is, whenever

$$\max_{1\le i\le m} U_{m,i} \le b \quad \text{or} \quad \max_{1\le i\le m} S(t_{m,i}) > b.$$

One approach to estimating the continuous-time price is to increase m in DGBS until this gap is closed. The smallest m at which the gap is closed is a random variable, say M, and the resulting estimator $C_{L,M} = C_{U,M} = C_M$ is unbiased. This unbiased estimator may require considerably less expected computational time than a (generally biased) estimator based on a large but fixed value of m. To avoid the possibility of excessively large values of m, one may select some upper bound m^* and increase m only up to $\min(M, m^*)$. For the cases where m reaches m^*, one can use an extrapolation-based estimator as in the previous examples. These cases would happen when the path gets very close to the barrier, without crossing it at an observation time $t_{m,i}$.

Our approach extends easily to other related payoff types. For example, for a *down-and-in call* option, we have $b < S(0)$, and we replace "$\sup_{0\le t\le T} S(t) > b$" in the indicator function of (18) by "$\inf_{0\le t\le T} S(t) < b$," with the corresponding replacements in the low and high estimators. The cases up-and-out call, down-and-out call, and the put versions can be handled in a similar way.

The computationally most challenging setting for an up-and-in call option occurs when the barrier is far from the initial asset price (i.e., $b \gg S(0)$), making the barrier crossing an event of small probability.

In such instances, truncating the DGBS algorithm at the random stopping time M typically yields a significant computational saving, because $\mathrm{E}[M]$ tends to be small relative to a reasonable preselected value of m. Empirical evidence is provided in §5.4.

The saving can also be important for the discretely observed version of this option, with discounted payoff

$$C_{\mathrm{B}}(d) = e^{-rT}(S(T)-K)^+ I\Big\{\max_{1\le i\le d} S(t_i) > b\Big\}.$$

We have $C_{\mathrm{L},m} \le C_{\mathrm{B}}(d) \le C_{\mathrm{U},m}$ whenever $\{t_{m,1}, \ldots, t_{m,m}\} \subseteq \{t_1, \ldots, t_d\}$, so we can use the same truncated DGBS procedure, stopped at $m = \min(M, d)$.

5. Numerical Results

We illustrate the efficiency improvement provided by truncated DGBS with MC and RQMC. We work with dyadic partitions (§3.3) in all examples.

5.1. Efficiency

The efficiency of our MC estimators depends on the truncation parameter m and on the number of independent replications n. Suppose that the truncated estimator C_m has expected computing cost (e.g., CPU time) $c(m)$, bias $\beta(m) = \mathrm{E}[C_m] - \mathrm{E}[C]$, and variance $\mathrm{Var}[C_m] = \sigma^2(m)$. Let $\bar{C}_{m,n}$ be the average of n i.i.d. copies of C_m. The efficiency of $\bar{C}_{m,n}$ is

$$\begin{aligned}\mathrm{Eff}[\bar{C}_{m,n}] &= \frac{1}{nc(m)[\beta^2(m) + \sigma^2(m)/n]}\\ &= \frac{1}{c(m)[n\beta^2(m) + \sigma^2(m)]}. \end{aligned} \tag{19}$$

This expression goes to zero if m is fixed and $n \to \infty$, reflecting the fact that increasing n eventually becomes a waste of computing resources if m is fixed, because the bias eventually dominates the mean square error. This means that as the available computing budget increases, both m and n should increase simultaneously, in a way that depends on the functions c, β, and σ^2.

Suppose that $c(m) = c_0 + c_1 m$, $\beta(m) = \beta_0 m^{-\gamma}$, and $\sigma^2(m) = \sigma_0^2$ for some positive constants c_0, c_1, β_0, γ, and σ_0. This is often a good approximation to reality. Then

$$\mathrm{Eff}[\bar{C}_{m,n}] = \frac{1}{(c_0 + c_1 m)[n\beta_0^2 m^{-2\gamma} + \sigma_0^2]}. \tag{20}$$

For a fixed computing budget $b = (c_0 + c_1 m)n$, this expression is maximized by taking

$$m = m^* = \left(\frac{2\gamma\beta_0^2 b}{c_1\sigma_0^2}\right)^{1/(2\gamma+1)}. \tag{21}$$

That is, m must increase proportionally to $b^{1/(2\gamma+1)}$. To quantify the efficiency improvement of one estimator over another, in our examples, we estimate the constants involved in (20).

The constants c_0 and c_1 depend on the sampling algorithm and on the method used for generating the normal, gamma, and beta random variables. In our experiments, all random variables were generated by inversion (Law and Kelton 2000, Hörmann et al. 2004) to make them compatible with RQMC. To approximate the normal and gamma inverse distribution functions, we used the rational Chebyshev approximation of Blair et al. (1976), which gives 16 decimal digits of accuracy, and the algorithm of Moshier (2000), respectively. Because we worked exclusively with dyadic partitions, all required beta random variables have density that is symmetric with respect to 1/2, and we developed a special method to approximate the inverse distribution function for such symmetric beta random variables (?). This special method is much faster than all available methods for inverting the beta distribution with general parameters. The ratio-of-gammas method for generating beta random variables should not be used in this context because it becomes unstable when the density of the beta distribution concentrates near 0 and 1.

With these inversion methods, the normal random variables are faster to generate than the symmetric betas (roughly by a factor of 20), and the gamma random variables are the slowest to generate (roughly by a factor of 10 compared with the symmetric betas, and this factor increases when the shape parameter α of the gamma distribution becomes smaller). Therefore, GBS is significantly faster than GSS (typically by a factor of 10 or more) for sampling the gamma processes when using these variate-generation methods. In what follows, efficiency comparisons are for the DGBS algorithm only. Our simulations were performed with the Java simulation library SSJ (L'Ecuyer 2004b).

For comparison, we ran MC experiments where all random variates were generated by the fastest noninversion methods available in SSJ (an acceptance/complement ratio method for the normal, a rejection method with log-logistic transform for the beta, and an acceptance/rejection method for the gamma). The MC simulation ran faster with these methods than with inversion when m is small but became slower when m is large. Roughly, in our examples, they cut the CPU time in half for $m = 16$ and take about the same time for $m = 512$. The explanation is that the performance of these beta and gamma generators depends highly on the distribution's parameters and, as m increases, we get into the parameter range where the methods are less efficient.

For RQMC, we tried various types of methods implemented in SSJ. The results reported here are only for the F2wLFSR point sets taken from Panneton

(2004), randomized by a linear matrix scrambling plus a random digital shift (Owen 2003). One of the alternatives are the well-known Sobol′ point sets with the same randomization, but with the available implementation of these point sets we were limited to $m \leq 128$, due to an upper bound on their dimension. The variance reduction with the Sobol′ point sets was actually better than for the selected point sets for some small values of m. To estimate the variance and efficiency with RQMC, we performed $N = 100$ independent randomizations of the $2m$-dimensional n-point F2wLFSR point set, for several values of n equal to powers of 2. In this context, if $\bar{C}_{m,n,N}$ denotes the general average, we get

$$\text{Eff}[\bar{C}_{m,n,N}] = \frac{1}{Nn(c_0 + c_1 m)[\beta_0^2 m^{-2\gamma} + \sigma^2(m,n)/N]}, \tag{22}$$

in which c_0 and c_1 are easily estimated, whereas $\sigma^2(m,n)$ depends on both m and n in a complicated way. We could not easily fit an expression to this $\sigma^2(m,n)$, so in our efficiency computations, we simply replaced it by its empirical value for each m and n. Expression (22) always reaches its maximum for $N = 1$, but we need to take $N > 1$ to be able to estimate the variance.

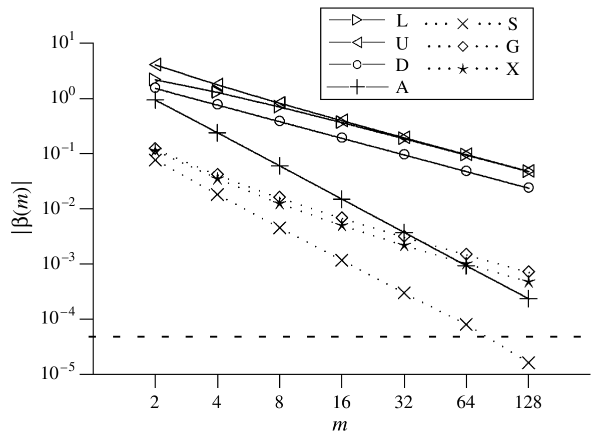
5.2. An Asian Option

We borrow VG model parameters from an unpublished draft of Hirsa and Madan (2004), where the VG model parameters were calibrated against options on the S&P 500 index using data for June 30, 1999. In our example, $T = 0.40504$, $\sigma = 0.1927$, $\nu = 0.2505$, $\theta = -0.2859$, $r = 0.0548$, and $q = 0$. These parameter values can be considered representative for the case of intermediate-maturity options for a nondividend-paying asset. Our first example is an Asian option with parameters $S(0) = 100$ and $K = 100$. Our aim is to estimate the continuous-time price $c_A(\infty)$.

To estimate the bias as a function of m for the estimators C_m, we computed very precise RQMC estimates of $E[C_m]$ for $m = 2^k$, $k = 1, \ldots, 11$. For that, we used 100 copies of a $2m$-dimensional F2wLFSR point set with $n = 2^{18}$ points, randomized independently by an affine linear scramble. Figures 3 and 4 show the estimated absolute bias as a function of m (in logarithmic scale), without and with Richardson's extrapolation, respectively. To estimate this bias, we used an estimated value of $c_A(\infty)$ obtained by extrapolating the values for $k = 10$ and 11. This estimated exact value is 3.68538 ± 0.000048 with 95% confidence.

In this example, (10) holds well for all estimators, and we estimated γ_1 and β_0 by fitting the linear regression model $\log \beta(m) = \log \beta_0 - \gamma_1 \log m$ to the data points for which the estimated bias differed significantly from zero (i.e., was more than twice the half-width of the confidence interval on $c_A(\infty)$). The rate estimates were $\hat{\gamma}_1 \approx 2$ for $C_{A,m}$ and $C_m^{(S)}$ and $\hat{\gamma}_1 \approx 1$ for the other estimators (the slopes in Figure 3 seem to converge to -2 or -1 when $m \to \infty$). These estimates may not apply generally. Figure 4 summarizes the bias of all extrapolated estimators based on the associated estimated rate and of the extrapolated estimators $\tilde{C}_{A,m}$ and $\tilde{C}_m^{(S)}$ with $\gamma_1 = 1$. The empirical bias of all estimators extrapolated with the rate 1 is also well modeled by (10) and converges as $O(m^{-2})$. Of these, the best performer is $\tilde{C}_m^{(D)}$, followed by $\tilde{C}_m^{(S)}$, $\tilde{C}_m^{(G)}$, and $\tilde{C}_m^{(X)}$, which are almost indistinguishable; last come $\tilde{C}_{L,m}$, $\tilde{C}_{U,m}$, and $\tilde{C}_{A,m}$, which are almost indistinguishable. The best estimators in terms of bias are $\tilde{C}_{A,m}$ and $\tilde{C}_m^{(S)}$ using the rate 2; the bias is below our confidence interval half-width for $m \geq 16$. The estimators $C_m^{(S)}$, $C_m^{(G)}$, and $C_m^{(X)}$ without extrapolation are

Figure 3 Estimated Absolute Bias $|\beta(m)|$ of Various Estimators without Extrapolation, as a Function of m, for the Asian Option Example



Note. In the legend, "L" refers to $C_{L,m}$, "U" refers to $C_{U,m}$, and so on. The dashed line indicates the half-width of the confidence interval on the bias. Bias estimates that fall below that line are too noisy to be reliable.

Figure 4 Estimated Absolute Bias $|\beta(m)|$ of Various Estimators with Extrapolation (with $\gamma_1 = 1$, Unless Otherwise Indicated) as a Function of m, for the Asian Option Example

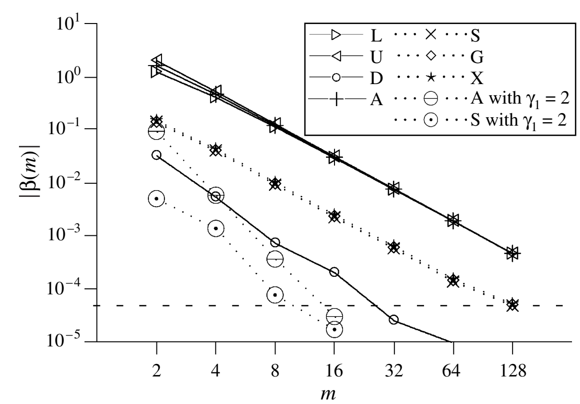
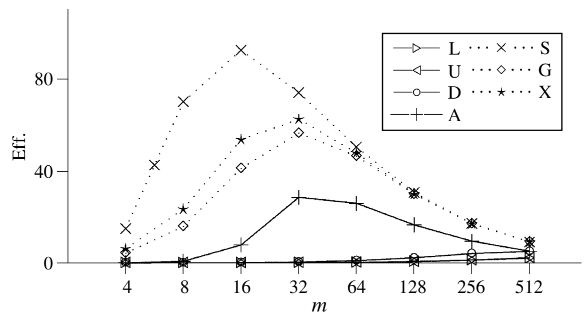


Figure 5 Estimated MC Efficiency for the Asian Option Example, without Extrapolation



doing practically as well in terms of bias as the extrapolated $\tilde{C}_{\mathrm{L},m}$, $\tilde{C}_{\mathrm{U},m}$, and $\tilde{C}_{\mathrm{A},m}$ with $\gamma_1 = 1$.

Figures 5 and 6 show the estimated efficiency of standard MC as a function of m, without and with the extrapolation, respectively. Without the extrapolation, $C_{\mathrm{L},m}$, $C_{\mathrm{U},m}$, and $C_m^{(\mathrm{D})}$ do very poorly for moderate m (they lie close to the horizontal axis in the figure), whereas $C_m^{(\mathrm{S})}$ with $m = 16$ is the best combination for this example. With the extrapolation, $\tilde{C}_m^{(\mathrm{S})}$ with $\gamma_1 = 2$ is the best performer, followed by $\tilde{C}_m^{(\mathrm{D})}$ and then by $\tilde{C}_m^{(\mathrm{S})}$ with $\gamma_1 = 1$. The efficiency of all extrapolated estimators decreases for $m > 64$. For $\tilde{C}_{\mathrm{A},m}$ and $\tilde{C}_m^{(\mathrm{S})}$, extrapolating with $\gamma_1 = 2$ versus $\gamma_1 = 1$ increases efficiency, and the efficiency peaks at smaller m, because of the faster decay of bias.

With RQMC, for $n = 2^{14}$, 2^{16}, and 2^{18}, the efficiency as a function of m behaves very similarly to that under MC, except that it is significantly larger. The best estimator in our MC experiments, excluding those extrapolated with $\gamma_1 = 2$, is $\tilde{C}_8^{(\mathrm{D})}$ and has (empirical) efficiency 130. For comparison, the naive estimators $C_{2048}^{(\mathrm{D})}$ and $C_8^{(\mathrm{D})}$, which correspond to estimating the continuous-time option by a discrete-time one with 2,048 and 8 observation times, respectively, have efficiencies 2.2 and 0.03. For RQMC with $n = 2^{12}$, $\tilde{C}_{16}^{(\mathrm{D})}$ has (empirical) efficiency 6,973, whereas $C_{2048}^{(\mathrm{D})}$ and $C_{16}^{(\mathrm{D})}$ have efficiencies 30.5 and 0.10, respectively. The most efficient RQMC estimator, $\tilde{C}_{16}^{(\mathrm{D})}$, is approximately, 3,000 times more efficient than the naive estimator $C_{2048}^{(\mathrm{D})}$. That is, it requires a total CPU time approximately 3,000 times smaller to estimate the option value with a given precision.

Figure 6 Estimated MC Efficiency for the Asian Option Example, with Extrapolation

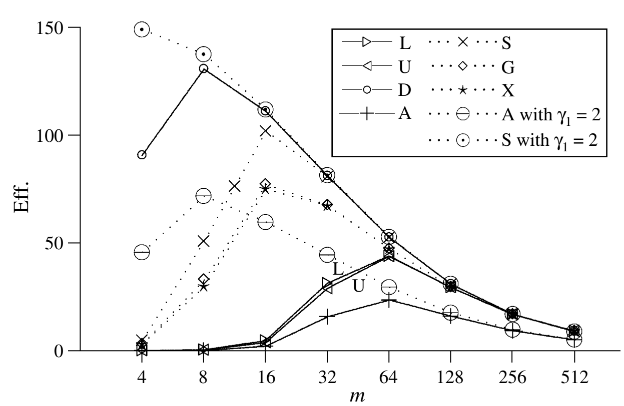
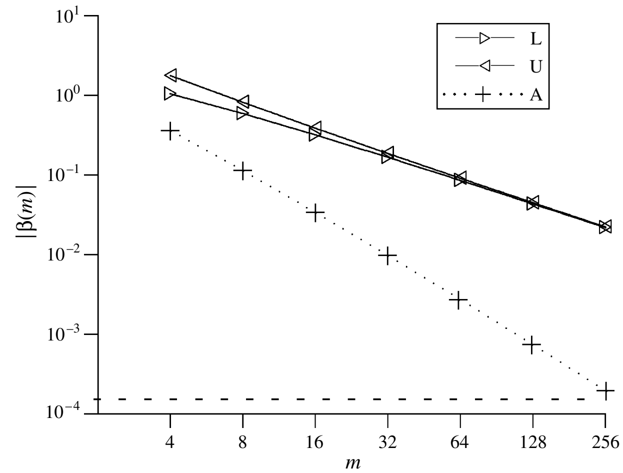


Figure 7 Estimated Absolute Bias $|\beta(m)|$ of Various Estimators without Extrapolation, as a Function of m, for the Lookback Option Example



5.3. A Lookback Option

Our second example is a lookback option, with the same parameters as for the Asian option. The bias and efficiency are estimated in the same way as in the Asian option case, for the three estimators $C_{\mathrm{L},m}$, $C_{\mathrm{U},m}$, and $C_{\mathrm{A},m}$ considered in §4.4. The estimated exact value is 9.39805 ± 0.00015 with 95% confidence. Figure 7 shows the bias without extrapolation; the points for $C_{\mathrm{L},m}$ and $C_{\mathrm{U},m}$ fall well along a straight line, and the rate estimates are $\hat{\gamma}_1 \approx 1$. The points for $C_{\mathrm{A},m}$ show mild concave nonlinearity, suggesting extrapolation may be less effective. Figure 8 shows the bias when we heuristically use $\gamma_1 = 1$ in the extrapolation for all estimators, and some concave nonlinearity is

Figure 8 Estimated Absolute Bias $|\beta(m)|$ of Various Estimators with Extrapolation, as a Function of m, for the Lookback Option Example

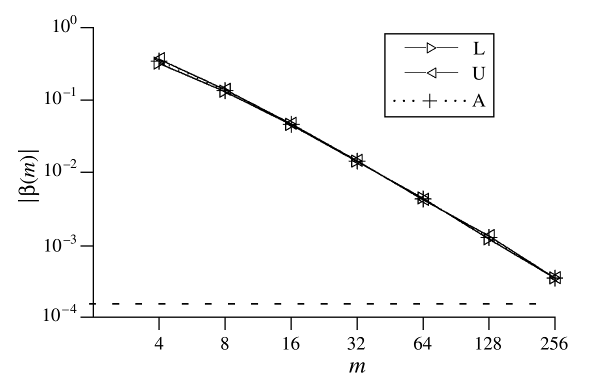


Figure 9 Estimated RQMC Efficiency for the Lookback Option Example, without Extrapolation

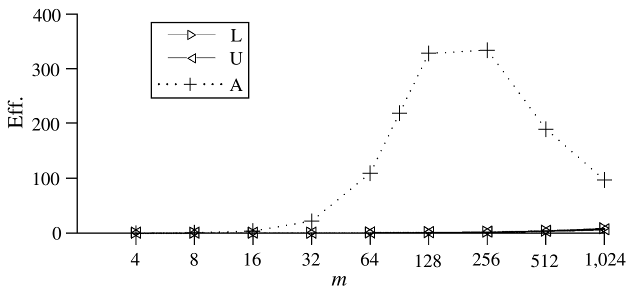
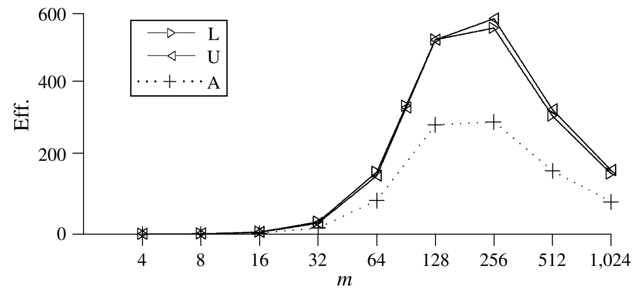


Figure 10 Estimated RQMC Efficiency for the Lookback Option Example, with Extrapolation



visible. A linear regression to the points in this figure gives slopes of about $-5/3$, that is, the bias converges approximately as $O(m^{-5/3})$. Extrapolation reduces the bias order of $C_{\mathrm{L},m}$, $C_{\mathrm{U},m}$, but results in a slight increase in the bias of $C_{\mathrm{A},m}$.

Figures 9 and 10 display the estimated efficiency for RQMC, without and with extrapolation. The MC efficiencies are smaller but have a very similar behavior.

The best estimator in our MC experiments, $\tilde{C}_{64}^{(\mathrm{U})}$, has (empirical) efficiency 46.2. For comparison, the naive estimator $C_{2048}^{(\mathrm{U})}$, which corresponds to estimating the continuous-time option by a discrete-time one with $d = 2{,}048$, has efficiency 2.03. The best RQMC estimator is $\tilde{C}_{256}^{(\mathrm{U})}$, with (empirical) efficiency 585. It is approximately 300 times more efficient than the naive MC estimator $C_{2048}^{(\mathrm{U})}$.

5.4. A Barrier Option

We report results for a barrier option with $b = 120$ and all other parameters as in the Asian option. In this case, the DGBS randomly truncated at M, as explained in §4.5, gives exactly the same payoff as a "full-dimensional" estimator that would sample the entire path at all m^* observation times. Therefore, the efficiency improvement factor can be characterized by the ratio of expected work between these two estimators.

The efficiency is estimated in the same way as in the Asian option case, for the two estimators $C_{\mathrm{L},m^*} \equiv C_{\mathrm{B}}(m^*)$ and C_{U,m^*} considered in §4.5. The estimated exact value is 2.1575 ± 0.0010 with 95% confidence. If we compare an estimator truncated at m^* with one truncated at $\min(M, m^*)$, we obtain the efficiency improvement ratio

$$q(m^*) \approx \frac{c_0 + c_1 m^*}{c_0 + c_1 \mathrm{E}[\min(M, m^*)]}$$

for some constants c_0 and c_1. This is simply the ratio of computing costs, because the estimator takes the same value in both cases. Table 1 displays values of $\mathrm{E}[\min(M, m^*)]$ and $q(m^*)$ for m^* equal to powers of 2. These values are approximately the same for both MC and RQMC because their computing costs are about the same. The table also gives the estimated values based on $n = 10^6$ replicates of C_{L,m^*} with RQMC, without extrapolation (est. value) and with extrapolation (est. value extra.). Despite the fact that $\mathrm{E}[\min(M, m^*)]$ converges rather slowly with m^*, the estimated value stabilizes rapidly (i.e., the bias quickly becomes negligible), especially when we use extrapolation. Further evidence of the efficiency of DGBS with MC, for a wider set of examples than presented here, is provided in Avramidis (2004).

The efficiency of the RQMC estimator truncated at $\min(M, m^*)$ is approximately 12 times that of the corresponding truncated MC estimator. So the efficiency improvement factor of this RQMC estimator compared with the standard MC estimator truncated at m^* is approximately $12q(m^*)$. This improvement factor is approximately 12,800 for $m^* = 2^{14} = 16{,}386$, for instance.

6. Generalization to Finite-Variation Lévy Processes

The DGBS method for option pricing under the VG model can in principle be extended to option-pricing

Table 1 Expected Truncation Value and Efficiency Improvement Ratio Under MC, and Estimated Value with RQMC with $n = 2^{20}$, as a Function of m^*, for the Barrier Option Example

m^*	4	16	64	256	1,024	4,096	16,386	65,536
$\mathrm{E}[\min(M, m^*)]$	2.204	2.554	2.894	3.221	3.598	3.893	4.159	4.527
$q(m^*)$	1.13	2.0	5.4	18.8	71	277	1,065	4,094
Est. value	1.9877	2.0980	2.1402	2.1528	2.1561	2.1569	2.1570	2.1571
Est. value extra.	2.0543	2.1428	2.1553	2.1572	2.1571	2.1571	2.1571	2.1571

models driven by any Lévy process whose paths have finite variation. (The variation of a function f over the interval $[a, b]$ is $V(f) = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$, where $a = x_0 < x_1 < \cdots < x_n = b$ determines a partition of $[a, b]$ and the supremum is taken over all such partitions and all n.) We will explain the extension beyond the VG case and point to the difficulties that arise. Exponentials of Lévy processes are increasingly used as models of asset prices in the derivative-pricing literature, for example, the CGMY model (Carr et al. 2002) that contains VG as a special case. We make a minimal introduction to Lévy processes, following Asmussen (1999).

A Lévy process $X = \{X(t), t \geq 0\}$ is defined as a continuous-time real-valued process with stationary independent increments and $X(0) = 0$. This class of processes is in one-to-one correspondence with the class of infinitely divisible distributions via the distribution of, say, $X(1)$. A Lévy process can be written as the independent sum $X(t) = ct + \sigma B(t) + J(t)$ of a deterministic drift ct, a Brownian component $\sigma B(t)$, and a pure-jump Lévy process $\{J(t), t \geq 0\}$ characterized by its Lévy measure π, which can be any nonnegative measure on $\mathbb{R}$ satisfying $\pi(\{0\}) = 0$ and

$$\int_{-\infty}^{\infty} \min(x^2, 1)\, \pi(dx) < \infty. \tag{23}$$

A rough description of the process J is that jumps of size x occur with intensity (rate per unit time) $\pi(dx)$. The paths of $\{J(t), t \geq 0\}$ over any finite time interval are functions of finite variation if and only if

$$\int_{-\infty}^{\infty} \min(|x|, 1)\, \pi(dx) < \infty. \tag{24}$$

A subordinator is a nondecreasing Lévy process. A pure-jump Lévy process can be written as the difference of two independent subordinators defined by the restriction of π to $(0, \infty)$ and the negative of the restriction of π to $(-\infty, 0)$, respectively, if and only if its Lévy measure satisfies (24).

Generalizing the setting we have studied so far, we can replace the VG process by any pure-jump Lévy process X of finite variation, that is, with Lévy measure satisfying (24). Because of the representation $X = X^+ - X^-$ where X^+ and X^- are subordinators, the pathwise bounding of X is straightforward if one samples the subordinators via the bridge method. For the general case of bridge sampling, assuming knowledge of the increment density p_t^+ and p_t^- of $X^+(t)$ and $X^-(t)$, respectively, for $\tau_1 < t < \tau_2$, we can write the conditional density of $X^+(t)$ at y given $X^+(\tau_1) = x_1$ and $X^+(\tau_2) = x_2$ as $p^+(y; \tau_1, \tau_2, x_1, x_2) = p_{t-\tau_1}^+(y - x_1) \cdot p_{\tau_2 - t}^+(x_2 - y)/p_{\tau_2-\tau_1}^+(x_2 - x_1)$. So if we can sample from this density, we have a generalization of DGBS.

However, there is a major difficulty: Except for the VG case, p_t^+ and p_t^- do not appear to be known explicitly for other pure-jump finite-variation Lévy processes. According to Asmussen (1999, pp. 88–89 and references therein), the increment density of a pure-jump Lévy process is only known in special cases, notably the gamma, Cauchy, and inverse Gaussian. The inverse Gaussian process (Barndorff-Nielsen 1998) is not a subordinator, and a Cauchy subordinator is unlikely to be useful because the increment has an infinite mean. So at this point there is no clear practical generalization of our approach beyond the case where the subordinators are gamma processes. In particular, we do not have an algorithm to sample from the conditional density of the increment for the CGMY process.

7. Conclusions and Future Research

An insight that emerges from this paper is that in certain applications the idea of integrand structuring to concentrate the variance to a few coordinates can also be used to reduce the work by eliminating sampling of uninteresting coordinates altogether, given some assurance that the resulting estimator's bias is acceptable. This is the theme behind the estimation approach proposed in §4, where it is possible to bound the bias a priori (pre-sampling) for Asian options, or a posteriori (after-sampling) for lookback and barrier options. An extreme case is coordinate elimination, arising with barrier options, where there is an a posteriori guarantee—once the stopping condition is satisfied—that the full-dimensional and truncated estimators are equal. Special properties of the variance gamma process have been exploited to enable this structuring.

Our theoretical and numerical results suggest the attractiveness of difference-of-gammas bridge sampling combined with Richardson extrapolation when pricing path-dependent options under the variance gamma model. In our numerical illustrations, we observed that combining this approach with RQMC improved the efficiency by yet another large factor. The integrand structuring proposed here, via DGBS, not only permits one to truncate with bounds on the bias, it also boosts the effectiveness of RQMC. Our illustrations focused on a particular set of VG model parameters to study the bias, variance, work, and efficiency in some depth; we expect these measures to be fairly representative of what one may encounter in typical applications.

It would be desirable to know the asymptotic (convergence rate) of the bias of estimators for options other than the Asian type. Such knowledge would support better extrapolation, leading to bias reduction and efficiency improvement.

Our efficiency improvement techniques could certainly be combined further with other variance reduction methods such as control variates, for example.

Perhaps the most natural control variates are the powers of the asset price or the powers of the two gamma processes sampled, for example, at an equal-length partition of time; the means are directly derived from the moment-generating function of the gamma distribution (see the proof of (12)) and are only finite for powers $r < \mu_p/\nu_p$. When the number of replications is very large, there is no problem in using many control variates, and the savings could be significant.

Acknowledgments
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References

Acworth, P., M. Broadie, P. Glasserman. 1997. A comparison of some Monte Carlo and quasi-Monte Carlo techniques for option pricing. P. Hellekalek, H. Niederreiter, eds. *Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing*. Number 127 in *Lecture Notes in Statistics*. Springer-Verlag, New York, 1–18.

Åkesson, F., J. P. Lehoczy. 2000. Path generation for quasi-Monte Carlo simulation of mortgage-backed securities. *Management Sci.* **46** 1171–1187.

Asmussen, S. 1999. Stochastic simulation, with a view toward stochastic processes. MaPhySto Lecture Notes 2, University of Aarhus, Aarhus, Denmark.

Avramidis, A. N. 2004. Efficient pricing of barrier options with the variance gamma model. R. G. Ingalls, M. D. Rossetti, J. S. Smith, B. A. Peters, eds. *Proc. 2004 Winter Simulation Conf.* IEEE Press, Piscataway, NJ, 1574–1578.

Avramidis, A. N., P. L'Ecuyer, P.-A. Tremblay. 2003. Efficient simulation of gamma and variance-gamma processes. *Proc. 2003 Winter Simulation Conf.* IEEE Press, Piscataway, NJ, 319–326.

Barndorff-Nielsen, O. E. 1998. Processes of normal inverse Gaussian type. *Finance Stochastics* **2** 41–68.

Bertoin, J. 1998. *Lévy Processes*. Cambridge University Press, Cambridge, UK.

Blair, J. M., C. A. Edwards, J. H. Johnson. 1976. Rational Chebyshev approximations for the inverse of the error function. *Math. Comput.* **30** 827–830.

Boyle, P., M. Broadie, P. Glasserman. 1997. Monte Carlo methods for security pricing. *J. Econom. Dynam. Control* **21**(8–9) 1267–1321.

Caflisch, R. E., B. Moskowitz. 1995. Modified Monte Carlo methods using quasi-random sequences. H. Niederreiter, P. J.-S. Shiue, eds. *Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing*. Number 106 in *Lecture Notes in Statistics*. Springer-Verlag, New York, 1–16.

Carr, P., D. Madan. 1999. Option valuation using the fast Fourier transform. *J. Comput. Finance* **2** 61–73.

Carr, P., H. Geman, D. Madan, M. Yor. 2002. The fine structure of asset returns: An empirical investigation. *J. Bus.* **75** 305–332.

Conte, S. D., C. de Boor. 1972. *Elementary Numerical Analysis: An Algorithmic Approach*, 2nd ed. McGraw-Hill, New York.

Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering*. Springer-Verlag, New York.

Hirsa, A., D. B. Madan. 2004. Pricing American options under variance gamma. *J. Comput. Finance* **7**(2) 63–80.

Hörmann, W., J. Leydold, G. Derflinger. 2004. *Automatic Nonuniform Random Variate Generation*. Springer-Verlag, Berlin, Germany.

Hull, J. 2000. *Options, Futures, and Other Derivative Securities*, 4th ed. Prentice-Hall, Englewood-Cliff, NJ.

Joyce, D. C. 1971. Survey of extrapolation processes in numerical analysis. *SIAM Rev.* **13**(4) 435–490.

Law, A. M., W. D. Kelton. 2000. *Simulation Modeling and Analysis*, 3rd ed. McGraw-Hill, New York.

L'Ecuyer, P. 2004a. Quasi-Monte Carlo methods in finance. R. G. Ingalls, M. D. Rossetti, J. S. Smith, B. A. Peters, eds. *Proc. 2004 Winter Simulation Conf.*, IEEE Press, Piscataway, NJ, 1645–1655.

L'Ecuyer, P. 2004b. SSJ: A Java library for stochastic simulation. Software user's guide, University of Montréal, Montréal, Canada, http://www.iro.umontreal.ca/∼lecuyer.

L'Ecuyer, P., C. Lemieux. 2002. Recent advances in randomized quasi-Monte Carlo methods. M. Dror, P. L'Ecuyer, F. Szidarovszky, eds. *Modeling Uncertainty: An Examination of Stochastic Theory, Methods, and Applications*. Kluwer Academic Publishers, Boston, MA, 419–474.

L'Ecuyer, P., R. Simard. 2004. Inverting the symmetrical beta distribution. *ACM Trans. Math. Software*. Forthcoming.

Madan, D. B., F. Milne. 1991. Option pricing with V.G. martingale components. *Math. Finance* **1** 39–55.

Madan, D. B., E. Seneta. 1990. The variance gamma (V.G.) model for share market returns. *J. Bus.* **63** 511–524.

Madan, D. B., P. P. Carr, E. C. Chang. 1998. The variance gamma process and option pricing. *Eur. Finance Rev.* **2** 79–105.

Moshier, S. L. 2000. Cephes math library. Retrieved August 7, 2006, http://www.moshier.net.

Moskowitz, B., R. E. Caflisch. 1996. Smoothness and dimension reduction in quasi-Monte Carlo methods. *J. Math. Comput. Model.* **23** 37–54.

Owen, A. B. 1998. Latin supercube sampling for very high-dimensional simulations. *ACM Trans. Model. Comput. Simulation* **8**(1) 71–102.

Owen, A. B. 2003. Variance with alternative scramblings of digital nets. *ACM Trans. Model. Comput. Simulation* **13**(4) 363–378.

Panneton, F. 2004. Construction d'ensembles de points basée sur des récurrences linéaires dans un corps fini de caractéristique 2 pour la simulation Monte Carlo et l'intégration quasi-Monte Carlo. Unpublished doctoral thesis, Département d'informatique et de recherche opérationnelle, Université de Montréal, Canada.

Ribeiro, C., N. Webber. 2003. Correcting for simulation bias in Monte Carlo methods to value exotic options in models driven by Lévy processes. Working paper, Cass Business School, London, UK.

Ribeiro, C., N. Webber. 2004. Valuing path-dependent options in the variance-gamma model by Monte Carlo with a gamma bridge. *J. Computational Finance* **7**(2) 81–100.