SNE3002 Linear Algebra – 2025 Spring

Transpose, Elementary Matrices and Linear Transformations

March 12, 2025

INHA UNIVERSITY

Transpose

Def. Transpose of a Matrix

Let $A \in \mathbb{R}^{m \times n}$, then the transpose $A^\top$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \qquad A^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Remark. Properties of the Transpose

Let A and B are matrices and r a scalar.

1) $(A^\top)^\top = A$
2) $(A + B)^\top = A^\top + B^\top$
3) $(\mathrm{r}A)^\top = rA^\top$
4) $(AB)^\top = B^\top A^\top$
5) $(A^{-1})^\top = (A^\top)^{-1}$, if A is invertible
6) $\det(A^\top) = \det(A)$, if A is square

Transpose

- Q. Show $(A^{-1})^\top = (A^\top)^{-1}$, if A is invertible --- (5)

- Q. Show $\det(A^\top) = \det(A)$, if A is square --- (6)

  recall: $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

Transpose

- $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^\top \boldsymbol{w}$

- A is said to be symmetric if $A^\top = A$
- A is said to be antisymmetric (skew symmetric) if $A^\top = -A$

- Let $A \in \mathbb{R}^{n \times m}$, then $A^\top A$ and $\mathrm{A}A^\top$ are symmetric

Transpose

> Q. Find a matrix that is both symmetric and antisymmetric

> Q. Show any square matrix can be represented as a sum of a symmetric and antisymmetric matrix.

> Q. Show that if A is an invertible square matrix, then so is $A^{\top}$

> Q. Show that A is invertible and symmetric, then so is A^{-1}

Elementary Matrices

> Recall elementary row operations
  - Switch two rows
  - Multiply a row by a non-zero number
  - Add a multiple of one row to another row

Def. Elementary Matrices and Row Operations

Let $E \in \mathbb{R}^{n\times n}$, then E is an elementary matrix if it is the result of applying one elementary row operation to the $n \times n$ identity matrix

Ex.

$$E_{switch} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_{multiply} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{add} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

Elementary Matrices

- Every elementary matrix is invertible, and its inverse is also an elementary matrix
  - If E is obtained by switching rows i and j, then E^{-1} is also obtained by switching rows i and j
  - If E is obtained by multiplying rows i and j, then E^{-1} is also obtained by switching rows i and j
  - If E is obtained by adding k times row i to row j, then E^{-1} is obtained by subtracting k times row i from row j

- Ex.

$$E_{switch} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_{multiply} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{add} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

$$E_{switch}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_{multiply}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{add}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -k & 1 \end{bmatrix}$$

Elementary Matrices

> Let $A \in \mathbb{R}^{m\times n}$ and R be its reduced echelon form. There exists an invertible $U \in \mathbb{R}^{n\times n}$ s.t.

$$R = UA$$

where U can be computed as the produce of elementary matrices

$$U = E_k E_{k-1} \cdots E_1$$

> Let $A \in \mathbb{R}^{m\times n}$, then A is invertible if and only if it can be written as a product of elementary matrices.

Proof. If A is an invertible $n \times n$-matrix, then its reduced echelon form is the $n \times n$ identity matrix I. By Theorem 4.57, we can write $I = UA$, where $U = E_k \cdots E_2 E_1$ is a product of elementary matrices. Then

$$A = U^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

By Theorem 4.55, if E_i is an elementary matrix, then so is E_i^{-1}. Therefore, A has been written as a product of elementary matrices. Conversely, if A can be written as a product of elementary matrices, then A is clearly invertible, because each elementary matrix is invertible. ♠

Elementary Matrices

> Q. $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. Write A as a product of elementary matrices

Linear Transformations

Def. Linear Function

A function $f: \mathbb{R}^p \to \mathbb{R}^q$ is called linear if

1) $f(\boldsymbol{x} + \boldsymbol{y}) = f(\boldsymbol{x}) + f(\boldsymbol{y})$, for all vectors $\forall x, y \in \mathbb{R}^p$
2) $f(a\boldsymbol{x}) = af(\boldsymbol{x})$, for all scalars $\forall a \in \mathbb{R}$ and all vectors $\forall x \in \mathbb{R}^p$

> The linear function f is given by $q \times p$ matrix. $f(\boldsymbol{x}) = A\boldsymbol{x}$

> What about $f(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$?
  - Ex. $f(x) = 2x + 5$, is it a linear function?

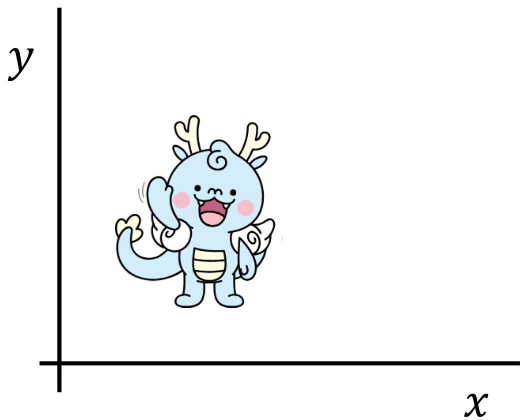
Linear Transformations

- Linear maps can be categorized geometrically,
  - Scaling
  - Rotation
  - Reflection
  - Shear
  - …

- Linear maps
  - Origin remains fixed
  - Straight lines remain straight
  - Parallelism is preserved

Linear Transformations

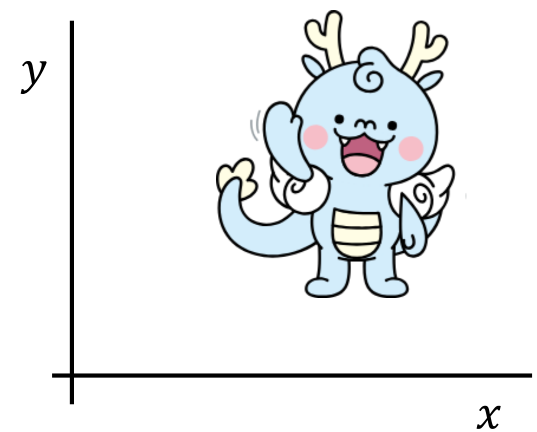
- Image sitting in $\mathbb{R}^2$



Scaling

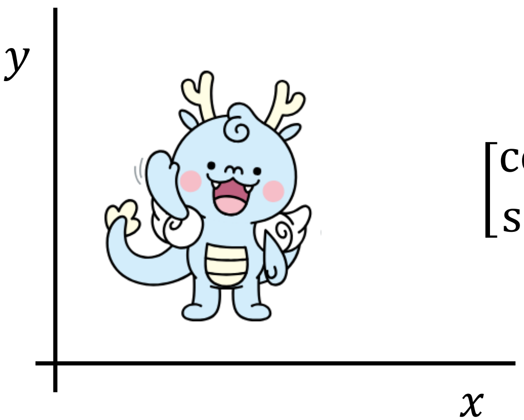
$$\boldsymbol{x}' = A\boldsymbol{x}$$

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$



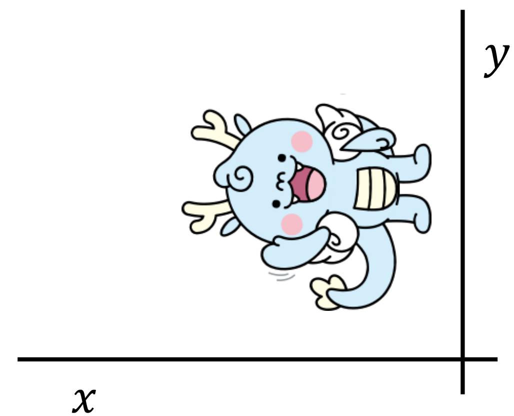
Linear Transformations

- Image sitting in $\mathbb{R}^2$



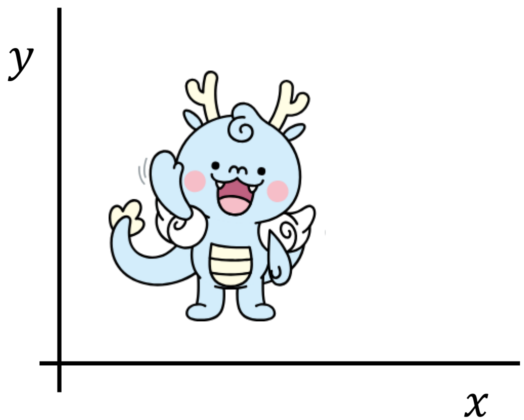
Rotation

$$\begin{bmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{bmatrix}$$



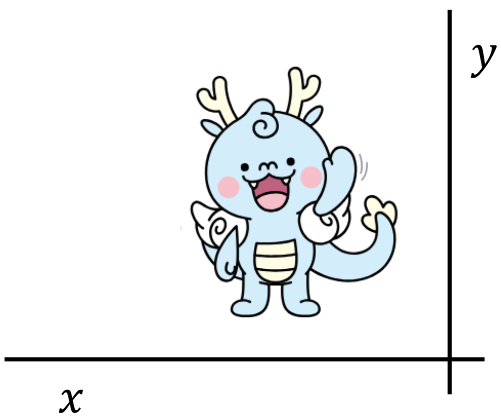
Linear Transformations

- Image sitting in $\mathbb{R}^2$



Reflection

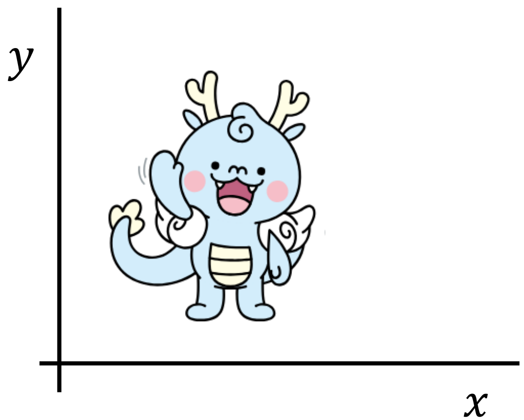
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



x

Linear Transformations

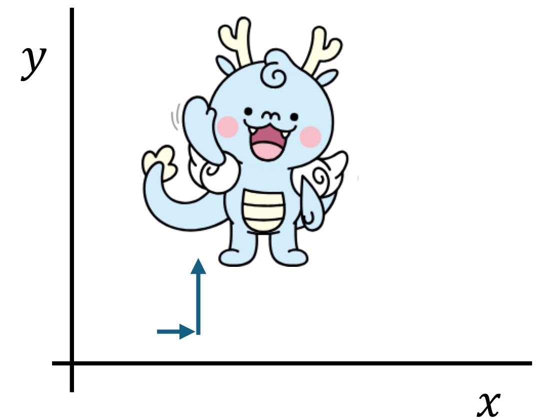
> Image sitting in $\mathbb{R}^2$



Translation

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{b}$$

$$\mathrm{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



Linear Transformations

Def. Linear Function

A function $f: \mathbb{R}^p \rightarrow \mathbb{R}^q$ is called **affine** or **affine linear** if there exists a constant vector $\boldsymbol{b} \in \mathbb{R}^q$ s.t.

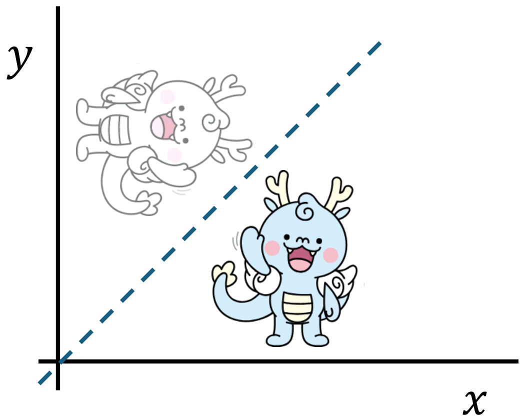
$$f(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$$

- There is a trick to represent the affine map as a linear map in an augmented space
  - Extend the vector $\tilde{\boldsymbol{x}} = [\boldsymbol{x}^\top, 1]^\top$
  - Augment the transformation matrix $\tilde{A} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$
  - New linear map $\boldsymbol{x}' = \tilde{A}\tilde{\boldsymbol{x}}$

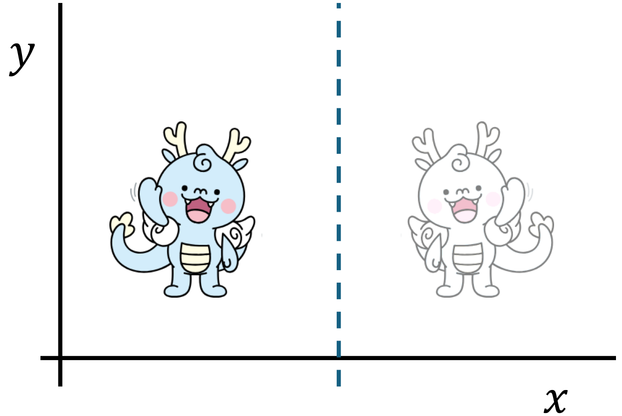
- We call this homogeneous coordinates

Linear Transformations

> Q. Find the reflection matrix A for the line $\mathrm{y} = \mathrm{x}$



> Q. Find the reflection matrix A for the line $\mathrm{x} = 4$



Linear Transformations

- Check
  - Affine Transformations · Arcane Algorithm Archive
  - Quick Understanding of Homogeneous Coordinates for Computer Graphics