SNE3002 Linear Algebra – 2025 Spring

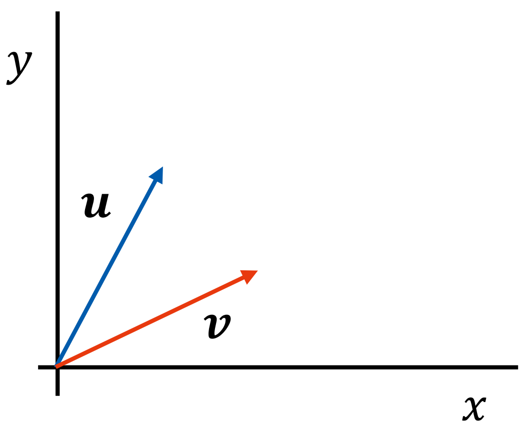
Spans, Linear Independence, and Bases

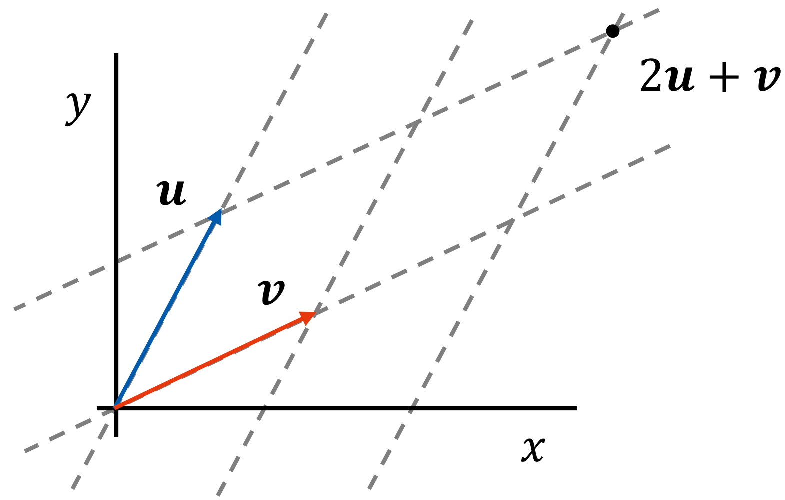
March 19, 2025

INHA UNIVERSITY

Spans

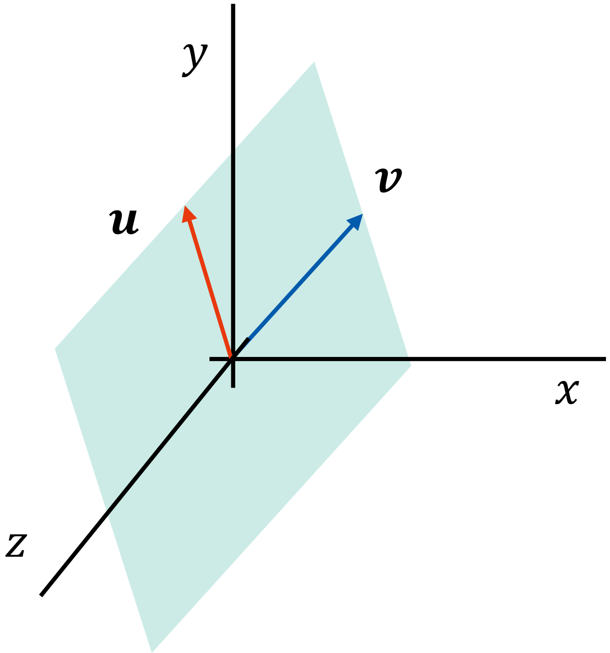
> Imagine 2 vectors in 2D space
  - Any point can be expressed by their combination unless the two vectors are aligned





Spans

- Imagine 2 vectors in 3D space
  - Any point can be expressed by their combination unless the two vectors are aligned
  - They are constructing a plane in $\mathbb{R}^n$



Spans

Def. Span of a Set of Vectors

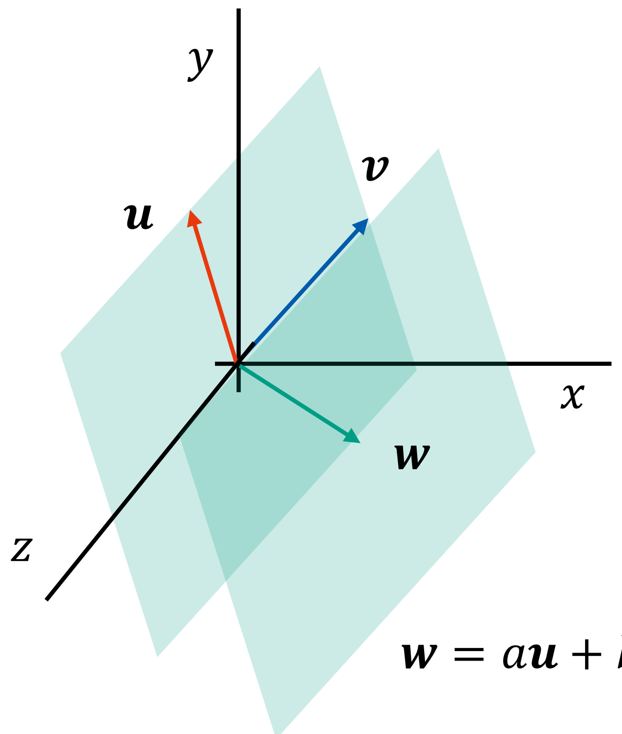
The set of all linear combinations of the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k \in \mathbb{R}^n$ is known as the span of these vectors

$$\operatorname{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\} = \{a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k | a_1, \dots, a_k \in \mathbb{R}\}$$

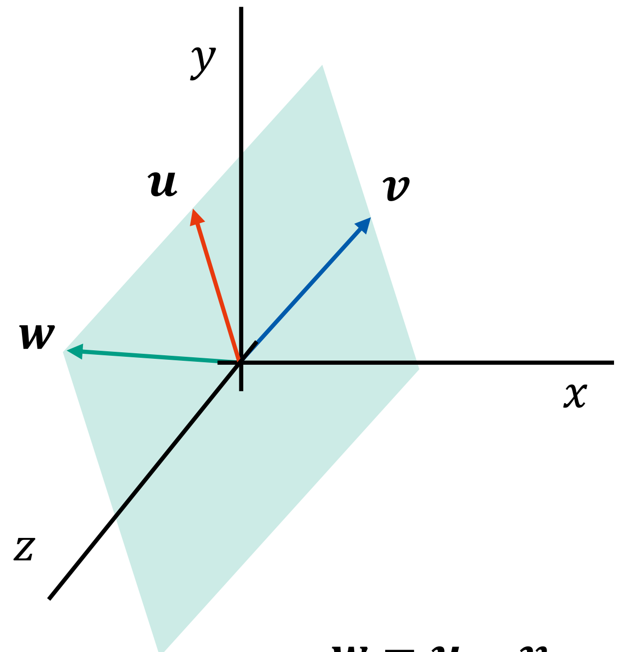
> Q. Describe the span of the vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Spans

> Imagine 3 vectors in 3D space
  - Any point can be expressed by their combination unless the three vectors do not lie in the same plane (i.e., the third vector is not contained within the plane formed by the first two vectors)



$\boldsymbol{w} = a\boldsymbol{u} + b\boldsymbol{v}$ x



$\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$

Linear Independence

Def. Redundant Vectors and Linear Independence

Consider a sequence of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in \mathbb{R}^n$. We say the vector $\boldsymbol{u}_j$ is **redundant** if it can be written as a linear combination of earlier vectors,

$$\boldsymbol{u}_j = a_1\boldsymbol{u}_1 + \cdots + a_{j-1}\boldsymbol{u}_{j-1}$$

We say that sequence of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ is **linearly dependent** if it contains one or more redundant vectors. Otherwise, the vectors are **linearly independent**.

Linear Independence

> Q. Find the redundant vectors in the following sequence of vectors.

$$\boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix},\ \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\ \boldsymbol{u}_4 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix},\ \boldsymbol{u}_5 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix},\ \boldsymbol{u}_6 = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

Linear Independence

> How to find out redundant vectors easily?
  - Casting-out algorithm: reduce to (reduced) echelon form. Every non-pivot column, if any, corresponds to a redundant vector.

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 3 \\ 0 & 2 & 1 & 3 & 1 & 3 \\ 0 & 2 & 1 & 3 & 2 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 \end{bmatrix} \simeq \ldots \simeq \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear Independence

Remark. Characterization of Linear Independence

Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k \in \mathbb{R}^n$. Then, they are linearly independent if and only if the homogeneous equation

$$a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k = \boldsymbol{0}$$

has only the trivial solution $a_1 = a_2 = \cdots = a_k = 0$

Linear Independence

- Why linear independence is important?
  - We wanted to solve $A\boldsymbol{x} = \boldsymbol{b}$, and found a solution $\overline{\boldsymbol{x}}$
  - Suppose there exists a solution $\boldsymbol{\alpha} \neq 0$ for $A\boldsymbol{\alpha} = 0$
  - $A\overline{\boldsymbol{x}} = \boldsymbol{b}$
    $A\overline{\boldsymbol{x}} + A\boldsymbol{\alpha} = \boldsymbol{b}$
    $A(\overline{\boldsymbol{x}} + \boldsymbol{\alpha}) = b$
  - Then, $\overline{\boldsymbol{x}} + \boldsymbol{\alpha}$ must be a solution too!

- In a linearly dependent system, $A\boldsymbol{x} = \boldsymbol{b}$ will not have a unique solution.

Linear Independence

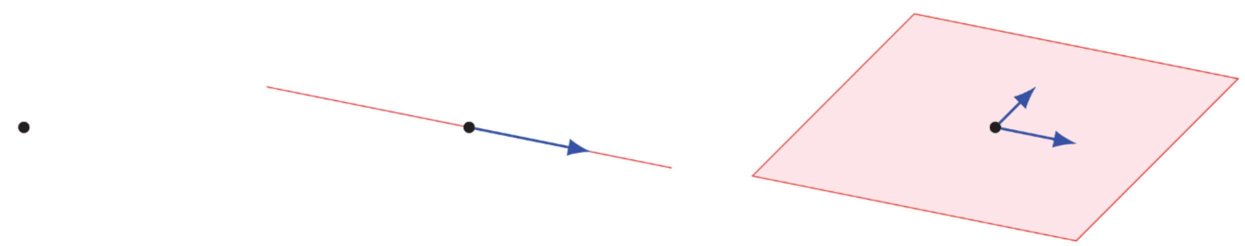
Remark. Properties of Linear Independence

1) If a sequence of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ is linearly independent, then so is any reordering of the sequence.
2) If $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ is linearly independent, then so are $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_j$ for any $j < k$
3) Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in \mathbb{R}^n$. If $k > n$, then the vectors are linearly dependent.
4) If $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ are linearly independent, then every vector $\mathrm{v} \in \mathrm{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ can be written as a linear combination of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ in a unique way

> Proof of 4)

Subspace

> A subset V of $\mathbb{R}^n$ is called a subspace if
  - V contains the zero vector, $\mathbf{0} \in V$
  - V is closed under addition
  - V is closed under scalar multiplication



Span of 0 vectors: a point　　Span of one vector: a line　　Span of two vectors: a plane

> $\text{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ is a subspace of $\mathbb{R}^n$

$\Longleftrightarrow$

If V is a subspace of $\mathbb{R}^n$, then there exist linearly independent vectors $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ s.t. $V = \text{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$

Bases

Def. Basis of a Subspace

Let V be a subspace of $\mathbb{R}^n$. Then $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ is a basis for V if

1) $\operatorname{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\} = V$
2) $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ are linearly independent

Note plural of basis is bases

> Standard basis of $\mathbb{R}^n$

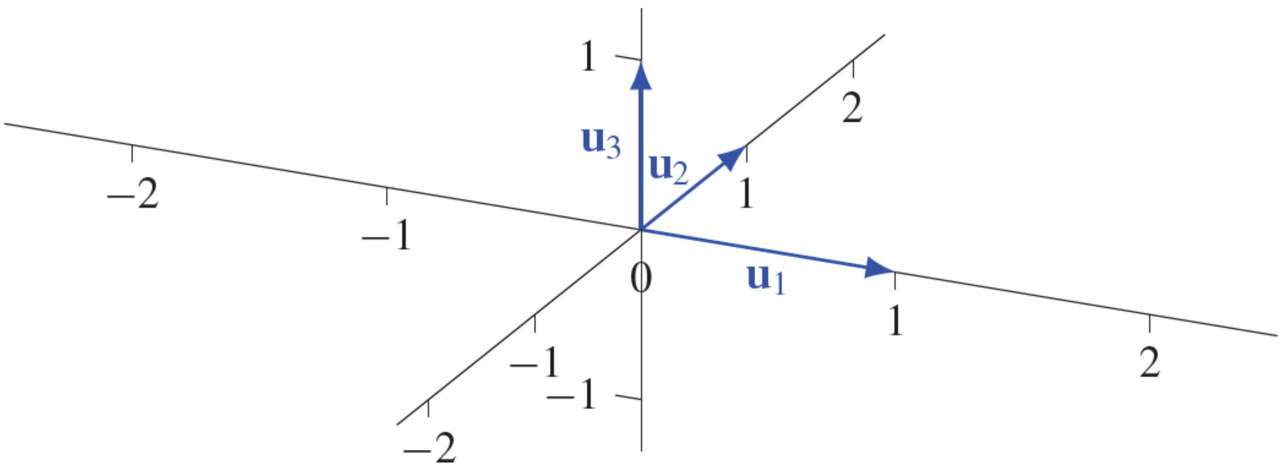
$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \dots, \qquad \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

> Let A be an $\mathrm{n} \times n$-matrix. Then the columns of A form a basis of $\mathbb{R}^n$ if and only A is invertible

Bases

- A basis of V is essentially equivalent to a coordinate system for V
- We say $a_1, \dots, a_k$ are the coordinates of $\boldsymbol{v}$ with respect to the basis B

$$[\boldsymbol{v}]_{\mathrm{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$



Basis as a coordinate system

Bases

> The dimension of V, $\dim(V)$, is the number of vectors in a basis.

> Let V be a k-dimensional subspace of $\mathbb{R}^n$

Linearly independent elements of V can extend to a basis of V

- Every linearly independent set of vectors has at most k vectors
- If we have k linearly independent vectors, then they form a basis

A set of vectors spanning V can shrink to obtain a basis of V

- Every spanning set of vectors has at least k vectors
- If we have k vectors spanning V, then they form a basis

> The **basis** of a vector space is a set of **linearly independent** vectors that **span** the full space

Bases

> Q. Find a basis and determine the dimension

1) $V_1 = \mathrm{span}\left\{\begin{bmatrix}2\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\\-1\end{bmatrix}, \begin{bmatrix}5\\2\\3\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\-2\\-2\end{bmatrix}\right\}$

2) $S_1 = \left\{\begin{bmatrix}2u+6v+7w\\-3u-9v-12w\\2u+6v+6w\\u+3v+3w\end{bmatrix}\middle|\, u,v,w \in \mathbb{R}\right\}$

Matrix Subspaces

Def. Column Space, Row Space, Null Space

Let $A \in \mathbb{R}^{n \times n}$.
The column space of A, $\mathrm{col}(A)$ is the span of the columns
The row space of A, $\mathrm{row}(A)$ is the span of the rows
The null space of A, $\mathrm{null}(A)$ is the set

$$\mathrm{null}(A) = \{\boldsymbol{x} | A\boldsymbol{x} = \boldsymbol{0}\}$$

- Let A and B be row equivalent. $\mathrm{row}(A) = \mathrm{row}(B)$, $\mathrm{null}(A) = \mathrm{null}(B)$
  - Row equivalent if one can be obtained from the other by performing elementary row operations

Matrix Subspaces

> Q. Find a basis for column space, row space, and null space

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{bmatrix}$$

Matrix Subspaces

- Let $A \in \mathbb{R}^{n \times n}$. The **rank** of a matrix is equal to the number of pivot entries of its reduced echelon form.
  - $\dim\big(\mathrm{col}(A)\big) = \mathrm{rank}(A)$
  - $\dim\big(\mathrm{row}(A)\big) = \mathrm{rank}(A)$
  - $\dim\big(\mathrm{null}(A)\big) = \mathrm{n} - \mathrm{rank}(A)$
  - $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$

Matrix Subspaces

Remark. Useful Theorems about Matrices

The following are equivalent for an $m \times n$ matrix A

1) $\operatorname{rank}(A) = n$
2) $\mathrm{r}ow(A) = \mathbb{R}^n$, i.e., the rows of A span $\mathbb{R}^n$
3) The columns of A are linearly independent in $\mathbb{R}^m$
4) The $\mathrm{n} \times n$ matrix $A^\top A$ is invertible ($\det(A^\top A) \neq 0$)
5) The system $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution

And

1) $\operatorname{rank}(A) = m$
2) $\operatorname{col}(A) = \mathbb{R}^m$, i.e., the columns of A span $\mathbb{R}^m$
3) The rows of A are linearly independent in $\mathbb{R}^n$
4) The $m \times m$ matrix $AA^\top$ is invertible ($\det(AA^\top) \neq 0$)
5) The system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b} \in \mathbb{R}^n$

*consistent = there exists at least one solution

Matrix Subspaces

> Q. Proof of 4)

To determine linear independence or dependence, we are looking to exclude (or find) non-trivial solutions to $A\alpha = 0$, where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix},$$

and $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}$, the matrix formed from our set of vectors $\{v_1, v_2, \cdots, v_m\}$. Motivated by this, we let

$$y = A\alpha.$$

We then note the following chain of implications[3]

$$(A\alpha = 0) \implies (A^\top \cdot A\alpha = 0) \implies (\alpha^\top A^\top \cdot A\alpha = 0) \implies \left((A\alpha)^\top \cdot (A\alpha) = 0\right) \implies (A\alpha = 0),$$

where the last implication follows from $y = 0_{n\times 1} \iff y^\top y = 0$.

From logic, we know that when we have

$$(a) \implies (b) \implies (c) \implies (d) \implies (a),$$

a chain of implications that begins and ends with the same proposition, then we deduce that

$$(a) \iff (b) \iff (c) \iff (d).$$

In our case, we are only interested in $(a) \iff (b)$, that is,

$$\boxed{A\alpha = 0 \iff \left(A^\top A\right)\alpha = 0.} \tag{7.31}$$

We next note that the matrix $A^\top \cdot A$ is $m \times m$, because it is the product of $m \times n$ and $n \times m$ matrices, $A^\top$ and A, respectively. Hence, the equation

$$\left(A^\top \cdot A\right)\alpha = 0 \tag{7.32}$$

has a unique solution if, and only if, $\det(A^\top \cdot A) \neq 0$.

Now, why are we done? If $\alpha = 0_{m\times 1}$ is the ONLY solution to (7.32), then it is also the only solution to $A\bar{\alpha} = 0$, and we deduce that the columns of A are linearly independent. If $\alpha = 0_{m\times 1}$ is not a unique solution to (7.32), then there exists a non-zero vector $\bar{\alpha} \in \mathbb{R}^m$ that is also a solution to (7.32), meaning that $\left(A^\top A\right)\bar{\alpha} = 0$. But we know from (7.31) that this also means that $\bar{\alpha} \neq 0$ is a solution of $A\bar{\alpha} = 0$, and hence the columns of A are linearly dependent. ■

Matrix Subspaces

- Q. True or False?
  - In a linearly dependent system, $A\boldsymbol{x} = \boldsymbol{b}$ will have infinitely many solutions if $\boldsymbol{b}$ lies in the column space of A
  - In a linearly independent system, $A\boldsymbol{x} = \boldsymbol{b}$ will have unique solution