SNE3002 Linear Algebra – 2025 Spring

Vector Dot Cross Products & Inverse Matrix

March 12, 2025

INHA UNIVERSITY

Unit Vector

Def. Length of a Vector

Let $\boldsymbol{u} = [u_1, u_2, \dots, u_n]^\top$ be a vector in $\mathbb{R}^n$.
The length of $\boldsymbol{u}$ is given by

$$\|\boldsymbol{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}.$$

The length of a vector is also sometimes called its **magnitude** or its **norm**.

- If $\|\boldsymbol{u}\| = 1$, vector $\|\boldsymbol{u}\|$ is called a unit vector.

- Normalization: obtain a unit vector $\boldsymbol{u}$ that points in the same direction as $\boldsymbol{v}$.

$$\boldsymbol{u} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$$

Vector Dot Product

Def. Dot Product

Let $\boldsymbol{u} = [u_1, u_2, \dots, u_n]^\top$, $\boldsymbol{v} = [v_1, v_2, \dots, v_n]^\top$ be two vectors in $\mathbb{R}^n$.
We define their dot product as

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

> Q. $\boldsymbol{u} = [1, 2, 0, -1]^\top$, $\boldsymbol{v} = [0, 1, 2, 3]^\top$, compute $\boldsymbol{u} \cdot \boldsymbol{v}$

Vector Dot Product

Remark. Properties of the Dot Product

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors and k, l are scalars.

1) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
2) $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$, and $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ if and only if $\boldsymbol{u} = 0$
3) $(k\boldsymbol{u} + l\boldsymbol{v}) \cdot \boldsymbol{w} = k(\boldsymbol{u} \cdot \boldsymbol{w}) + l(\boldsymbol{v} \cdot \boldsymbol{w})$
4) $\boldsymbol{u} \cdot (k\boldsymbol{v} + l\boldsymbol{w}) = k(\boldsymbol{u} \cdot \boldsymbol{v}) + l(\boldsymbol{u} \cdot \boldsymbol{w})$
5) $\boldsymbol{u} \cdot \boldsymbol{u} = \|\boldsymbol{u}\|^{2}$

Vector Dot Product

- Cauchy-Schwarz inequality
  - The dot product satisfies the inequality

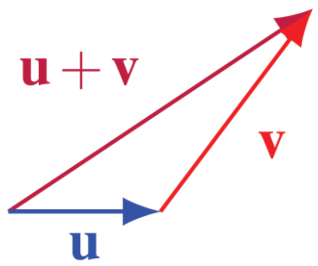
$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

- Proof.

Vector Dot Product

- Triangle inequality

$$\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$$

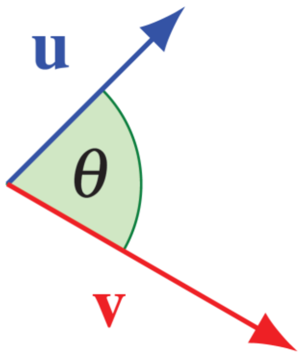


- Proof.

Vector Dot Product

> The included angle of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is the angle θ

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos(\theta)$$



> If $\boldsymbol{u} \cdot \boldsymbol{v} = 0$, then, $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

> Q. Find the angle between $\boldsymbol{u} = [2,2]^\top$ and $\boldsymbol{v} = [0,3]^\top$

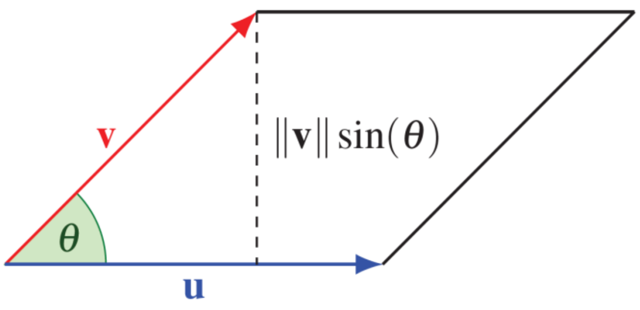
> Q. Find the angle between $\boldsymbol{u} = [4,8]^\top$ and $\boldsymbol{v} = [-10, 5]^\top$

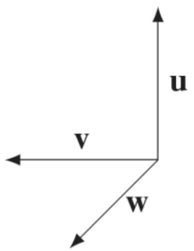
Vector Cross Product

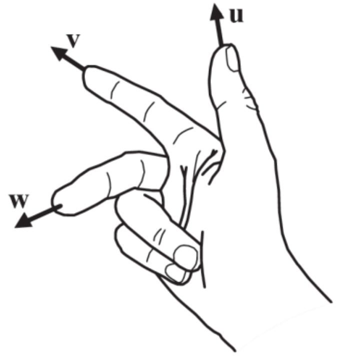
Def. Cross Product

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in $\mathbb{R}^3$. Their cross product $\boldsymbol{u} \times \boldsymbol{v}$ is

1) Its length is $\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin(\theta)$
2) It is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$
3) The vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u} \times \boldsymbol{v}$ form a right-handed system.





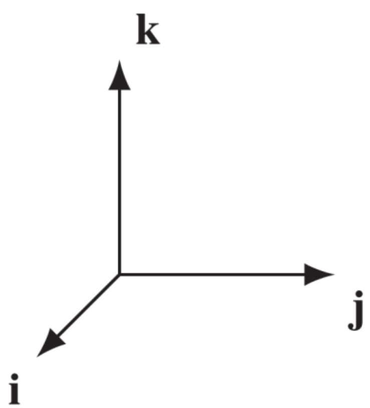


Vector Cross Product

Remark. Properties of the Cross Product

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors in $\mathbb{R}^3$, k a scalar.

1) $\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$
2) $\boldsymbol{u} \times \boldsymbol{u} = 0$
3) $(k\boldsymbol{u}) \times \boldsymbol{v} = k(\boldsymbol{u} \times \boldsymbol{w}) = \boldsymbol{u} \times (k\boldsymbol{v})$
4) $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$
5) $(\boldsymbol{v} + \boldsymbol{w}) \times \boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{u} + \boldsymbol{w} \times \boldsymbol{u}$



$$\begin{array}{lll}
\mathbf{i} \times \mathbf{j} = \mathbf{k}, & \mathbf{j} \times \mathbf{i} = -\mathbf{k}, & \mathbf{i} \times \mathbf{i} = \mathbf{0}, \\
\mathbf{k} \times \mathbf{i} = \mathbf{j}, & \mathbf{i} \times \mathbf{k} = -\mathbf{j}, & \mathbf{j} \times \mathbf{j} = \mathbf{0}, \\
\mathbf{j} \times \mathbf{k} = \mathbf{i}, & \mathbf{k} \times \mathbf{j} = -\mathbf{i}, & \mathbf{k} \times \mathbf{k} = \mathbf{0}.
\end{array}$$

Vector Cross Product

> Cross product can be computed by

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

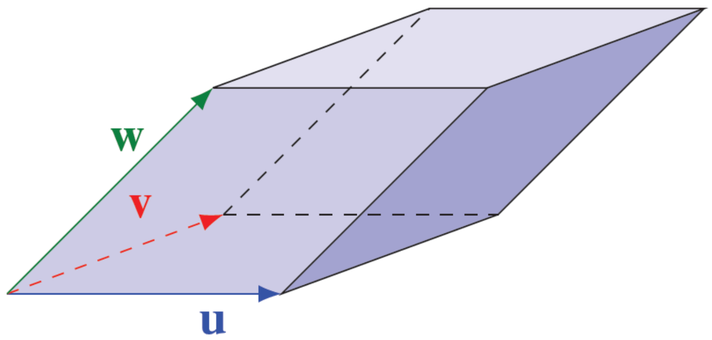
> Q. $\boldsymbol{u} = [1, -1, 2]^{\top}$, $\boldsymbol{v} = [3, -2, 1]^{\top}$, compute $\boldsymbol{u} \times \boldsymbol{v}$

> Q. Find the area of the triangle determined by the points (1, 2, 3), (0, 2, 5), and (5, 1, 2).

Vector Cross Product

Remark. The Box Product

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors in $\mathbb{R}^3$ that define a parallelepiped. The box product $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ is equal to the volume of the parallelepiped, if vectors form a right-handed system.



Note that,

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = (\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{u} = (\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v}$$

Inverse Matrix

Def. Inverse of a Matrix

Let A and B be $\mathrm{n} \times n$-matrices. We say that B is an inverse of A if

$$BA = I \quad \text{and } \mathrm{A}B = I$$

In this case, $B = A^{-1}$

Remark. Uniqueness

Both B and C are inverses of A, then $B = C$

$$B = BI = B(AC) = (BA)C = IC = C$$

Inverse Matrix

- How to find the inverse of a matrix
  - From $[\mathrm{A} \mid \mathrm{I}]$, do row operations until you get $[\mathrm{I} \mid B]$

- Q. Find A^{-1}, $\mathrm{A} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$

- Check $\mathrm{AA}^{-1} = \mathrm{A}^{-1}\mathrm{A} = \mathrm{I}$

Inverse Matrix

Remark. Inverse of Matrix Product

The inverse of ‘put on your socks, then put on your shoes’ is ‘first take off your shoes, then take off your socks’.

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$AB(B^{-1}A^{-1}) = A(BB^{-1}A^{-1}) = AIA^{-1} = I$$

Inverse Matrix

Remark. Properties of the Inverse

Let A and B be $\mathrm{n} \times n$-matrices, I the identity matrix.

1) I is invertible and $I^{-1} = I$
2) If A and B are invertible then AB is invertible
3) If A is invertible then so is A^{-1}, and $(A^{-1})^{-1} = A$
4) If A is invertible then so is A^k, and $\left(A^{-k}\right)^{-1} = (A^{-1})^k$
5) If A is invertible and p is a non-zero scalar, then pA is invertible and $(pA)^{-1} = \frac{1}{p}A^{-1}$

Inverse Matrix

- Fact: $\det(AB) = \det(A) \cdot \det(B)$

- $1 = \det(\mathrm{I_n}) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$

  $\det(A^{-1}) = \frac{1}{\det(A)}$

- If and only if $\det(A) \neq 0$, A is invertible

Remark. Invertibility

An $\mathrm{n} \times n$ matrix A is invertible if and only if $\det(A) \neq 0$

We call a matrix that is not invertible a **singular matrix**

Inverse Matrix

- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$,

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in \mathbb{R}^{2\times 2}$,

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$where\, \det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Inverse Matrix

Remark. Inverse Matrix using the Determinant

The inverse of a matrix A can be expressed using its determinant

$$A^{-1} = \frac{1}{\det(A)} \cdot adj(A)$$

where $\mathrm{adj}(A)$ is the transpose of the cofactor matrix $adj(A) = C^{\top}$

Inverse Matrix

- Inverse of a matrix is useful to find the solution of a system of linear equations.

- When we solve $A\mathbf{x} = \mathbf{b}$, we can easily get $\mathbf{x} = A^{-1}\mathbf{b}$

Inverse Matrix

> Q. Wanda the witch owns three types of pets – cats, ravens, and snakes. She has 10 pets in total, with a total of 16 legs and 4 wings. How many of each type of pet does she own?

$$x = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 16 \\ 4 \end{bmatrix}$$

Inverse Matrix

- Note.
  - It is much better to solve $A\mathbf{x} = \mathbf{b}$ using LU decomposition or QR decomposition, which we will study later
  - Explicitly computing A^{-1} can lead to numerical instability

  - If A has any special structure such as sparsity, A^{-1} in general will not preserve it.