SNE3002 Linear Algebra – 2025 Spring

Scalars, Vectors, Matrices

INHA UNIVERSITY

Scalars and Arrays

- Scalars: numbers

  Ex. 25.773, $\sqrt{175}$, 0, -4.8, π^2, 2.99999$\cdots$, ∞

- Arrays: scalars that have been organized somehow into lists

25.773	$\sqrt{101}$	e
25.774	$\sqrt{103}$	π
25.802	$\sqrt{107}$	i
25.103	$\sqrt{109}$	e^{3i}

Vectors

Def. Vectors and $\mathbb{R}^n$

A n-dimensional **column vector**, often simply called a vector, is an ordered list of n real numbers, written as a column within square brackets:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Likewise, row vector is an ordered list of n real numbers arranged in a single row:

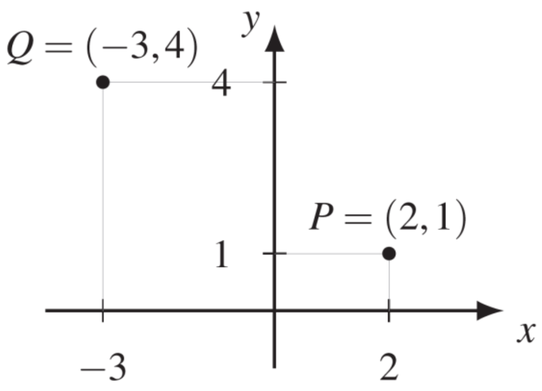
$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

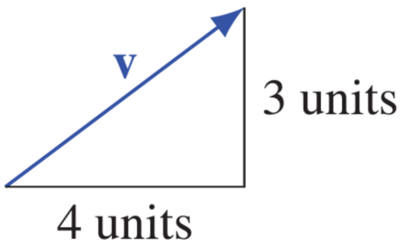
We write $\mathbb{R}^n$ for the set of all n-dimensional column vectors. It is also known as **n-dimensional Euclidean space**.

Vectors

Remark. Points vs Vectors

Algebraically, they seem to be almost the same thing because they are both an ordered pair of real numbers. Geometrically, a point is a location in space, and has neither a length nor direction, whereas a vector has length and direction, but is not fixed at any location.





Vectors

- Zero vector: a vector with all elements equal to zero
  - $0_n = \underbrace{[0 \quad \cdots \quad 0]}_{n}^\top$

- Unit vector: a vector with all elements equal to zero, except on element which is equal to one.
  - $e_1 = [1 \quad 0 \quad 0]^\top,\ e_2 = [0 \quad 1 \quad 0]^\top,\ e_3 = [0 \quad 0 \quad 1]^\top$

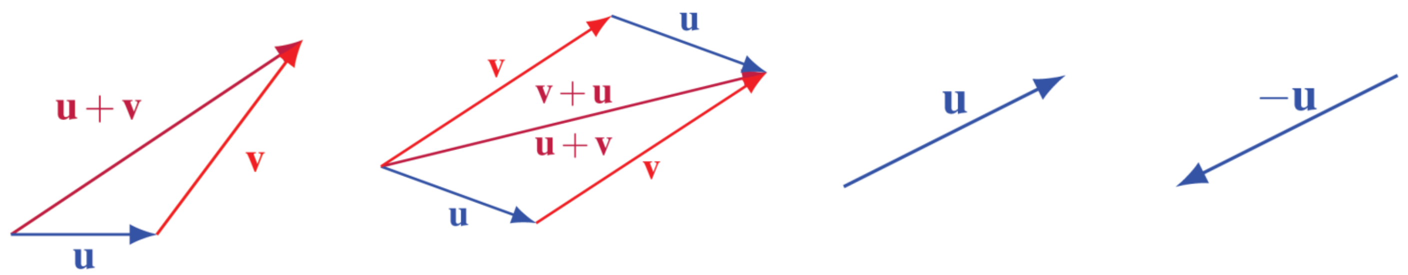
- Ones vector: a vector with all elements equal to one
  - $1_n = \underbrace{[1 \quad \cdots \quad 1]}_{n}^\top$

Vectors

Def. Addition of Vectors in $\mathbb{R}^n$

For vectors $\mathbf{u} = [u_1 \quad \cdots \quad u_n]^\top, \mathbf{v} = [v_1 \quad \cdots \quad v_n]^\top \in \mathbb{R}^n$,
the sum $\mathbf{u} + \mathbf{v}$ is defined by

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1 \quad \cdots \quad u_n + v_n]^\top.$$



Remark. Properties of Vector Addition

1) The commutative law of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2) The associative law of addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3) The existence of an additive unit: $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4) The existence of an additive inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Vectors

Def. Scalar Multiplication of Vectors in $\mathbb{R}^n$

If $\mathrm{k} \in \mathbb{R}$ is a scalar and $\boldsymbol{u} \in \mathbb{R}^n$ is a vector, then their scalar multiplication $k\mathbf{u} \in \mathbb{R}$ is defined by

$$k\mathbf{u} = [ku_1 \quad \cdots \quad ku_n]^\top.$$

Remark. Properties of Scalar Multiplication

1) The distributive law over vector addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{v} + k\mathbf{u}$
2) The distributive law over scalar addition: $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
3) The associative law for scalar multiplication: $k(l\mathbf{u}) = (kl)\mathbf{u}$
4) The rule for multiplication by 1: $1\mathbf{u} = \mathbf{u}$

Matrices

Def. Matrix

A **matrix** is a rectangular array of numbers

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix},$$

where the a_{ij} are scalars, called the entries or components of A. The size or dimension of a matrix is defined as $m \times n$, where m and n are the number of rows and columns, respectively.

Def. Square Matrix

A matrix of size $n \times n$ is called a square matrix.

Remark. Properties of Scalar Multiplication

A matrix of size $n \times 1$ is called a column vector.
A matrix of size $1 \times n$ is called a row vector.

Matrices

Def. Addition of Matrices in $\mathbb{R}^n$

Let $A = [a_{ij}]$, and $B = [b_{ij}]$ be two $m \times n$-matrices. Then $A + B = C$ where C is the $m \times n$-matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

Def. Scalar Multiplication of a Matrix in $\mathbb{R}^n$

If k is a scalar and $A = [a_{ij}]$ is a matrix, then $kA = [ka_{ij}]$

Remark. Concatenation of Matrices

Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times p}$,
horizontal concatenation $\quad C = [A|B] \in \mathbb{R}^{n\times(m+p)}$

Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times m}$,
vertical concatenation $\quad C = \begin{bmatrix} A \\ B \end{bmatrix} \in \mathbb{R}^{(n+p)\times m}$

Matrices

Def. Matrix Multiplication

Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B = [b_{jk}] \in \mathbb{R}^{n\times p}$. Then their product is the $m \times p$-matrix $AB = [c_{ik}]$ whose (i, k)-*entry is defined by*

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ \ldots + a_{in}b_{nk}$$

> Q. Compute $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 9 & 6 \\ 8 & 5 \\ 7 & 4 \end{bmatrix}$

Matrices

> Q. $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}$

> Q. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

> Q. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} ?$

Matrices

> Square matrix: size $n \times n$ matrix
  - The diagonal of the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \iff \mathrm{diag}(A) = \begin{bmatrix} a_{11} & a_{22} & \cdots & a_{nn} \end{bmatrix}$$

  - Other elements? Off-diagonal

> Identity matrix has ones along the diagonal, zeros everywhere else.
  - Other words, $\mathrm{I} = [\delta_{ij}]$ where δ is a Kronecker delta function.

$$\underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{\mathrm{I}_1}, \quad \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathrm{I}_2}, \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathrm{I}_3}, \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathrm{I}_4}$$

Matrices

Remark. Properties of Matrix Multiplication

1) The associative law of multiplication: $(AB)C = A(BC)$
2) The existence of multiplicative units: $I_m A = A = A I_n, \ \ A \in \mathbb{R}^{m \times n}$
3) Compatibility with scalar multiplication: $(rA)B = r(AB) = A(rB)$
4) The distributive laws of multiplication over addition

$$A(B + C) = AB + AC$$
$$(B + C)A = BA + CA$$

Matrices

> Q. $\begin{bmatrix} 4 & 2 & 6 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 7 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$

Determinant

> The determinant is a function that maps a square matrix to a real number.

- $\det([a]) = \mathrm{a}$

- $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathrm{ad} - \mathrm{bc}$

- $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \mathrm{a} \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - \mathrm{b} \cdot \det(\begin{bmatrix} d & f \\ g & i \end{bmatrix}) + \mathrm{c} \cdot \det(\begin{bmatrix} d & e \\ g & h \end{bmatrix})$

  $= \mathrm{a(ei - hf) - b(di - fg) + c(dh - eg)}$

Determinant

- Q. compute $\det(A)$, where $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

Determinant

> For higher dimensional matrix, $\det(A)$ can be calculated by picking a row or column.

> $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

or $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

> C_{ij} is a **cofactor** defined by **minor** M_{ij}: $C_{ij} = (-1)^{i+j}M_{ij}$

> Minor M_{ij} is the determinant of the $(\mathrm{n}-1) \times (n-1)$-matrix that is obtained by deleting the i^th row and the j^th column of A.

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{array}{c} M_{12} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{array}$$

Determinant

> Q. $\det\left(\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 2 & 3 \\ 0 & 3 & 0 & 5 \\ 3 & 0 & 3 & 2 \end{bmatrix}\right)$

Determinant

> Q. $\det\left(\begin{bmatrix} 15 & 1 & 6 & 36 \\ 7 & 0 & 6 & 12 \\ 0 & 0 & 0 & 3 \\ 3 & 0 & 1 & 62 \end{bmatrix}\right)$

Note

> AlphaTensor by DEEPMIND finds new Algorithms for Matrix Multiplication