SNE3002 Linear Algebra – 2025 Spring

Determinants

March 26, 2025

INHA UNIVERSITY

Recall

> The determinant is a function that maps a square matrix to a real number.

- $\det([a]) = \mathrm{a}$

- $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathrm{ad} - \mathrm{bc}$

- $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \mathrm{a} \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - \mathrm{b} \cdot \det(\begin{bmatrix} d & f \\ g & i \end{bmatrix}) + \mathrm{c} \cdot \det(\begin{bmatrix} d & e \\ g & h \end{bmatrix})$

  $= \mathrm{a(ei - hf) - b(di - fg) + c(dh - eg)}$

Recall

> For higher dimensional matrix, $\det(A)$ can be calculated by picking a row or column.

> $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

or $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

> C_{ij} is a **cofactor** defined by **minor** M_{ij}: $C_{ij} = (-1)^{i+j}M_{ij}$

> Minor M_{ij} is the determinant of the $(\mathrm{n}-1) \times (n-1)$-matrix that is obtained by deleting the i^th row and the j^th column of A.

M_{12}

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Determinant

- Let $A \in \mathbb{R}^{n\times n}$, according to row operations
  - If B is obtained from A by switching two rows,

$$\det(B) = -\det(A)$$

  - If B is obtained from A by multiplying one row by a non-zero scalar k,

$$\det(B) = k\det(A)$$

  - *If* B is obtained from A by adding a multiple of one row to another row,

$$\det(B) = \det(A)$$

Determinant

- Let $A \in \mathbb{R}^{n\times n}$, according to row operations
  - If $B = E_{switch}A$,

$$\det(B) = -\det(A) = \det(\mathrm{E}_{\mathrm{switch}})\det(A)$$

  - If $B = E_{multiply}A$,

$$\det(B) = \mathrm{k}\det(A) = \det\left(E_{multiply}\right)\det(A)$$

  - *If* $B = E_{add}A$,

$$\det(B) = \det(A) = \det(E_{add})\det(A)$$

Examples

$$E_{switch} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_{multiply} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{add} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

Determinant of a Product

> Recall

- $1 = \det(\mathrm{I_n}) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$
  $\det(A^{-1}) = \frac{1}{\det(A)}$

- $A = LU$
  $\det(\mathrm{A}) = \det(\mathrm{L}) \det(\mathrm{U}) = u_{11} u_{22} \cdots \mathrm{u_{nn}}$

Remark. Determinant of a Product

Let $A, B \in \mathbb{R}^{n \times n}$

$$\det(\mathrm{A}B) = \det(\mathrm{A}) \det(\mathrm{B})$$

Determinant of a Product

- Proof

Determinant

Remark. Determinants and Invertible Matrices

$A \in \mathbb{R}^{n\times n}$ is invertible if and only if $\det(A) \neq 0$

> Recall: $A \in \mathbb{R}^{m\times n}$, then A is invertible if and only if it can be written as a product of elementary matrices. (Lecture 4-8)

Proof. We know that every matrix A can be converted to echelon form by elementary row operations. We also know from Theorem 7.16 that no elementary row operation changes whether the determinant is zero or not. Let R be an echelon form of A. Because R is an echelon form, it is also an upper triangular matrix. Case 1: A is invertible. In that case, the rank of R is n, and every diagonal entry of R is a pivot entry (therefore non-zero). It follows that $\det(R) \neq 0$, which implies $\det(A) \neq 0$. Case 2: A is not invertible. In that case, the triangular matrix R contains a row of zeros. It follows that $\det(R) = 0$, and therefore $\det(A) = 0$. ♠

$$
\begin{aligned}
1 &= \det(\mathrm{I_n}) = \det(A \cdot A^{-1}) \\
&= \det(A) \cdot \det(A^{-1})
\end{aligned}
$$

Determinant

Remark. Properties of Determinants

$A, B \in \mathbb{R}^{n\times n}$

1) $\det(AB) = \det(A)\det(B)$
2) $\det(I) = 1$
3) $\det(\mathrm{A}^{-1}) = \frac{1}{\det(\mathrm{A})}$
4) $\det(kA) = k^n \det(A)$
5) $\det(A^\top) = \det(A)$

Determinant

- Q. Is it true $\det(A + B) = \det(A) + \det(B)$

- Q. A is orthogonal if $A^\top A = I$, what is $\det(\mathrm{A})$

- Q. We say A is similar to B, if there exists $A = P^{-1}BP$.
  Show $\det(A) = \det(B)$

Cramer’s Rule

Remark. Cramer’s Rule

Suppose $A \in \mathbb{R}^{n \times n}$ is invertible and we wish to solve $A\boldsymbol{x} = \boldsymbol{b}$

$$x_i = \frac{\det(A_i)}{\det(A)}$$

Where A_i is the matrix obtained by replacing the i^{th} column of A with $\boldsymbol{b}$

Cramer’s Rule

> Proof.

Cramer’s Rule

- Q. Use Cramer’s rule to solve the system of equations

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

Cramer's Rule

> Q. Use Cramer's rule to solve the system of equations

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & t & t^2 \\ 1 & s & s^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ s \\ 1 \end{bmatrix}$$

Geometry of Determinant

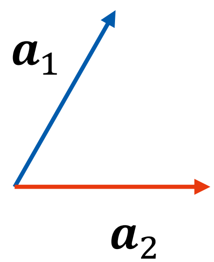
- Let $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots \boldsymbol{a}_{n-1}, \boldsymbol{x}], B = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{n-1}, \boldsymbol{y}]$,

$$\det([\boldsymbol{a}_1, \boldsymbol{a}_2, \dots \boldsymbol{a}_{n-1}, \boldsymbol{x} + \boldsymbol{y}]) = \det(A) + \det(B)$$

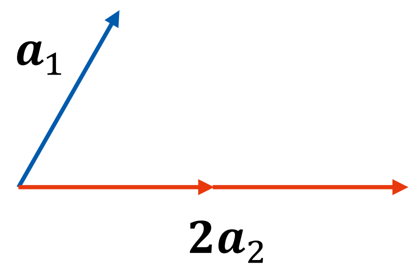
- Proof.

Geometry of Determinant

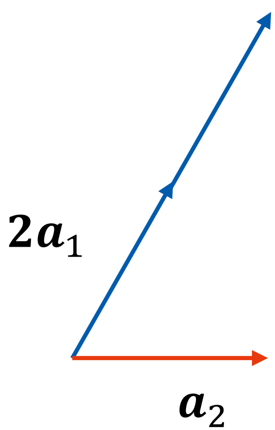
> What does it mean?



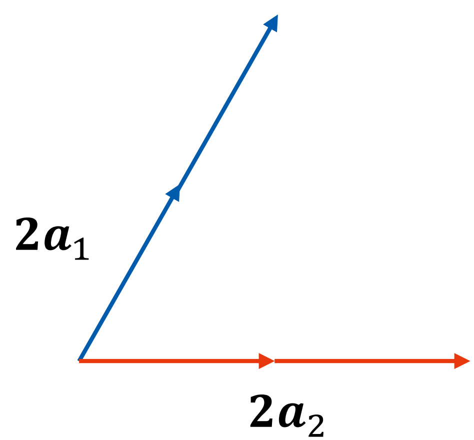
$\det([\boldsymbol{a}_1, \boldsymbol{a}_2]) = \alpha$



$\det([\boldsymbol{a}_1, \boldsymbol{2a}_2]) = 2\alpha$



$\det([\boldsymbol{2a}_1, \boldsymbol{a}_2]) = 2\alpha$

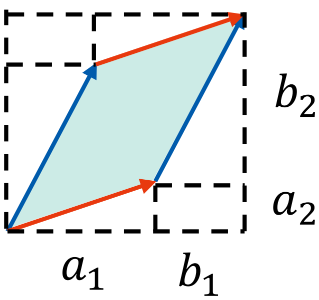


$\det([\boldsymbol{2a}_1, \boldsymbol{2a}_2]) = 4\alpha$

Geometry of Determinant

> Let $A = [\boldsymbol{a}, \boldsymbol{b}] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

> $\det(A) = a_1 b_2 - a_2 b_1 =$ (area of a parallelogram)



Geometry of Determinant

- Determinant represents the volume in n-dim space
  - Multilinearity

$$f(v_1, \dots, av_i + bw_i, \dots, v_n) \\ = a\, f(v_1, \dots, v_i, \dots, v_n) + b\, f(v_1, \dots, w_i, \dots, v_n)$$

  - Alternating property

$$f(v_1, \dots, v_i, \dots, v_i, \dots, v_n) = 0$$

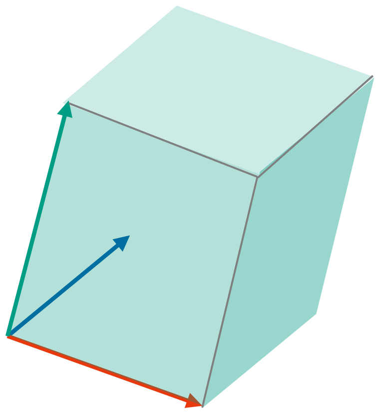
  - Normalization

$$f(I) = f(e_1, e_2, \dots, e_n) = 1$$

Geometry of Determinant

- Let $A = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

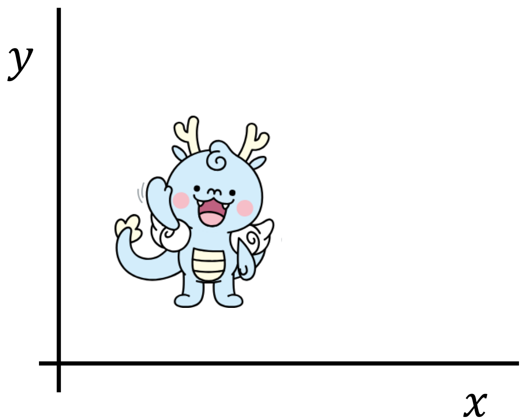
- $|\det(A)| = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} =$ (volume of a parallelepiped)



Geometry of Determinant

> For any shape X at all,

$$\mathrm{Vol}(AX) = |\det(A)|\mathrm{Vol}(X)$$



Scaling

$$\boldsymbol{x}' = A\boldsymbol{x}$$

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$