According to row operations,

1) $B \leftarrow$ switch rows of A

$$\det(B) = -\det(A)$$

2) $B \leftarrow$ multiply one row of A by k

$$\det(B) = k\det(A)$$

3) $B \leftarrow$ add a multiplication of one row to other row

$$\det(B) = \det(A)$$

Proof.

1) For $n \times n$ matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & \cdots & a_{in} \\ a_{j1} & \cdots & a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{j1} & \cdots & a_{jn} \\ a_{i1} & \cdots & a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

For 2×2 matrix.

$$\det\left(\begin{vmatrix} a & b \\ c & d \end{vmatrix}\right) = ad - bc = -(bc - ad) = -\det\left(\begin{vmatrix} c & d \\ a & b \end{vmatrix}\right)$$

$$\det(A) = -\det(B)$$

Assume for $k \times k$ matrix, $\det(A) = -\det(B)$

Investigate $k+1 \times k+1$ matrix

$$\det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$$
$$= a_{i1}(-1)^{i+1}M_{i1} + \cdots a_{in}(-1)^{i+n}M_{in}$$

$$\det(B) = b_{j1}\hat{C}_{j1} + \cdots + b_{jn}\hat{C}_{jn} = a_{i1}\hat{C}_{j1} + \cdots + a_{in}\hat{C}_{j1}$$
$$= a_{i1}(-1)^{j+1}\hat{M}_{j1} + \cdots + a_{in}(-1)^{j+1}\hat{M}_{j1}$$

$M_{ij} \Rightarrow$ A matrix 에서 a_{ij} 가 포함된 행, 열을 제외한 $k \times k$ matrix의 determinant

$\hat{M}_{ij} \Rightarrow$ B matrix 에서 b_{ij}가 포함된 행, 열을 제외한 $k \times k$ matrix의

determinant

i와 j가 홀수번 떨어져 있다면 $\hat{M}_{ij}$는 M_{ij}에서 row switch를

$|i-j-1|$번 적용하면 만들수 있는 matrix 이므로, $M_{ij} = \hat{M}_{ij}$

i와 j가 짝수번 떨어져 있다면, $M_{ij} = -\hat{M}_{ij}$

ex) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $2 \leftrightarrow 3$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$

⇒ 이해돕는 예시

$M_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$ $\hat{M}_{31} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$

1번 떨어져 있으면 동일.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $1 \leftrightarrow 3$ $B = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$ $\hat{M}_{31} = \begin{vmatrix} 8 & 9 \\ 5 & 6 \end{vmatrix}$

$= -\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$

$= -M_{11}$

$\therefore$ det(B) ⇒ if $i-j$가 짝수

$\det(B) = a_{i1}(-1)^{j+1}\hat{M}_{i1} + \cdots + a_{in}(-1)^{j+1}\hat{M}_{in}$

$= a_{i1}(-1)^{j+1+i-j}(-M_{i1}) + \cdots + a_{in}(-1)^{j+1+i-j}(-M_{in})$

$i+j$ 더해줘도 값변화 x 홀수번 row switch

$= a_{i1}(-1)^{i+1}(-M_{i1}) + \cdots + a_{in}(-1)^{i+1}(-M_{in})$

$= -\{a_{i1}(-1)^{i+1}M_{i1} + \cdots + a_{in}(-1)^{i+1}M_{in}\}$

$= -\det(A)$

else if $i-j$가 홀수

$\det(B) = a_{i1}(-1)^{j+1}\hat{M}_{i1} + \cdots + a_{in}(-1)^{j+1}\hat{M}_{in}$

$= -\{a_{i1}(-1)^{i+1}M_{i1} + \cdots + a_{in}(-1)^{j+1}M_{in}\}$

$= -\det(A)$ //

3) $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ $B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{i1} + r a_{i1} & \cdots & a_{in} + r a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

$$\det(B) = (a_{i1} + r a_{k1}) C_{i1} + \cdots + (a_{in} + r a_{kn}) C_{in}$$

$$= a_{i1}C_{i1} + \cdots + a_{in}C_{in} + \underbrace{r a_{k1} C_{i1} + \cdots + r a_{kn} C_{in}}_{= Y}$$

$$Y = r(a_{k1} C_{i1} + \cdots + r a_{kn} C_{in})$$

새로운 matrix $\hat{A} = \begin{array}{cl} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{k1} & \cdots & a_{kn} \\ a_{k1} & \cdots & a_{kn} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{l} \leftarrow 1st \\ \leftarrow kth \\ \leftarrow ith \\ \leftarrow nth \end{array} \end{array}$

($\hat{A}$는 i번째 row를 k번째 row로 대체함.)

Y는 $\hat{A}$의 determinant에 r을 곱한 것.

$\hat{A}$는 같은 row vector가 있으므로, not linearly independant

$\Rightarrow$ not invertible $\Rightarrow \det(\hat{A}) = 0$

$\therefore \det(A) = \det(B)$