SNE3002 Linear Algebra – 2025 Spring

Eigenvalues and Eigenvectors

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INHA UNIVERSITY

Eigenvalues and Eigenvectors

Def. Eigenvalues and Eigenvectors

Let $A \in \mathbb{R}^{n \times n}$. Suppose that $\boldsymbol{v} \in \mathbb{R}^n$ is a non-zero vector such that $A\boldsymbol{v}$ is a scalar multiple of $\boldsymbol{v}$. i.e., there exists a scalar λ

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

Then, $\boldsymbol{v}$ is an eigenvector of A, and λ is the corresponding eigenvalue.

Eigenvalues

Remark. Finding the Eigenvalues

Let $A \in \mathbb{R}^{n \times n}$, λ be a scalar. Then λ is an eigenvalue of A if and only if

$$\det(\lambda I - A) = 0.$$

- Note. Why?

Def. Characteristic polynomial

Let $A \in \mathbb{R}^{n \times n}$. The expression

$$p(\lambda) = \det(\lambda I - A)$$

is called the characteristic polynomial of A.

Eigenvalues

- Q. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -4 & 4 \\ 2 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvectors

Remark. Finding the Eigenvectors

For each eigenvalues λ, find a basis for the eigenvectors by solving

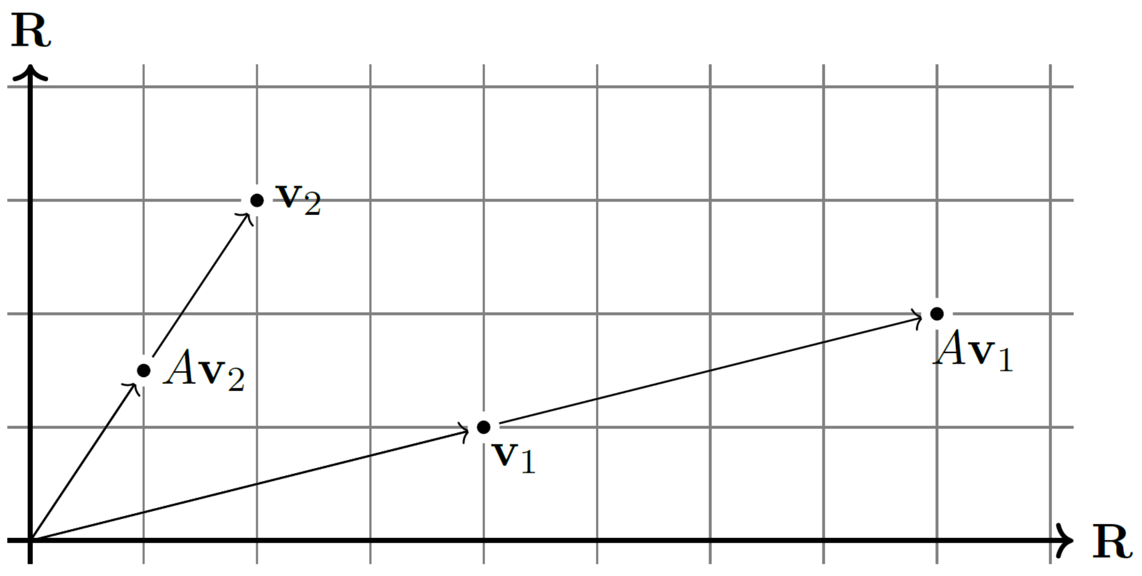
$$(\lambda I - A)\boldsymbol{v} = \mathbf{0}.$$

> Q. Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & -4 & 4 \\ 2 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

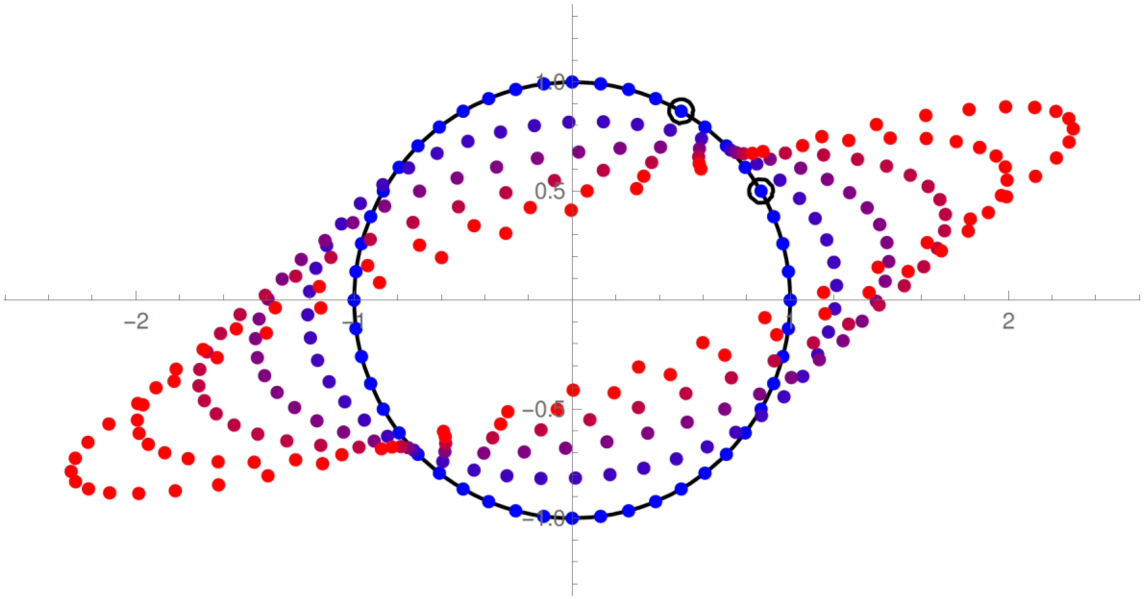
Eigenvalues and Eigenvectors

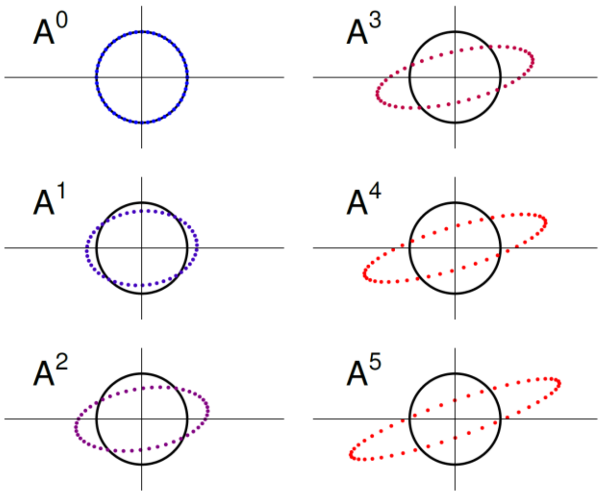
- Eigenvectors describe the direction in which a matrix changes $\mathbb{R}^n$
- Eigenvalues describe the stretching that is done in that direction.
- Suppose $A = \begin{bmatrix} 23/10 & -6/5 \\ 9/20 & 1/5 \end{bmatrix}$, which has $\boldsymbol{v}_1 = [4,\ 1]^\top$ with $\lambda_1 = 2$, and $\boldsymbol{v}_2 = [2,\ 3]^\top$ with $\lambda_2 = 0.5$.



Eigenvalues and Eigenvectors

> The vector $\boldsymbol{v}_1$ gets longer and $\boldsymbol{v}_2$ gets shorter as A is applied more times.





Eigenvalues and Eigenvectors

> The eigenvalues and eigenvectors is enough to know the effect of matrix A on any vector.

> For example, we have $A = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$,

$\boldsymbol{v}_1 = [3,\ 1]^\top$ $\lambda_1 = 3$, and $\boldsymbol{v}_2 = [1,\ 1]^\top$ with $\lambda_2 = 5$.

> Q. $A \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

Eigenvalues and Eigenvectors

Remark. Properties of Eigenvalues and Eigenvectors

1) The eigenvalues of A and A^{T} are the same.
2) If A is upper or lower triangular, its eigenvalues are on its diagonal.
3) If the rank of A is less than n, then A has an eigenvalue 0.
4) If A has an eigenpair $(\boldsymbol{v}, \lambda)$, then A^n has an eigenpair $(\boldsymbol{v}, \lambda^n)$
5) With distinct eigenvalues $\lambda_1, \dots, \lambda_k$, corresponding eigenvectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ are linearly independent

> Proof.

Eigenvalues and Eigenvectors

- Let $A \in \mathbb{R}^{n\times n}$, and their eigenvalues and eigenvectors $\boldsymbol{v}_i, \lambda_i$.
- For a general vector $\boldsymbol{x} \in \mathbb{R}^n$,

$$A\boldsymbol{x} = A(\Sigma_{i=1}^{n} c_i \boldsymbol{v}_i) = \Sigma_{i=1}^{n} c_i \lambda_i \boldsymbol{v}_i$$

$$A^{\boldsymbol{n}}\boldsymbol{x} = A^n(\Sigma_{i=1}^{n} c_i \boldsymbol{v}_i) = \Sigma_{i=1}^{n} c_i \lambda_i^n \boldsymbol{v}_i$$

- $\lambda_{max} > 1, \lambda_{max} < 1$

Diagonalization

- Recall, diagonal matrix $D \in \mathbb{R}^{n\times n}$

$$\mathrm{D} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

- $A \in \mathbb{R}^{n\times n}$ is diagonalizable if there exists an invertible matrix P and a diagonal matrix A such that

$$P^{-1}AP = D.$$

- Then, $A = PDP^{-1}$. $A^n = PDP^{-1}PDP^{-1} \cdots PDP^{-1} = PD^nP^{-1}$

Diagonalization

Remark. Diagonalization and eigenvectors

$A \in \mathbb{R}^{n\times n}$ is diagonalizable if and only if A has n linearly independent eigenvectors.

In this case, columns of P are n linearly independent eigenvectors of A and diagonal entries of D are corresponding eigenvalues when $P^{-1}AP = D$

- Proof

Diagonalization

- Q. Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

Diagonalization

> Q. Find A^{50} and $\sqrt{A}$ when $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$.

Matrix Exponential

> Power series of functions
  - $\mathrm{e}^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$
  - $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}\mathrm{x}^5 \pm \cdots$
  - $\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}\mathrm{x}^4 \pm \cdots$

> Power series of matrix
  - $\mathrm{e}^A = 1 + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$
  - $\sin(A) = A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 \pm \cdots$
  - $\cos(A) = 1 - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 \pm \cdots$

Matrix Exponential

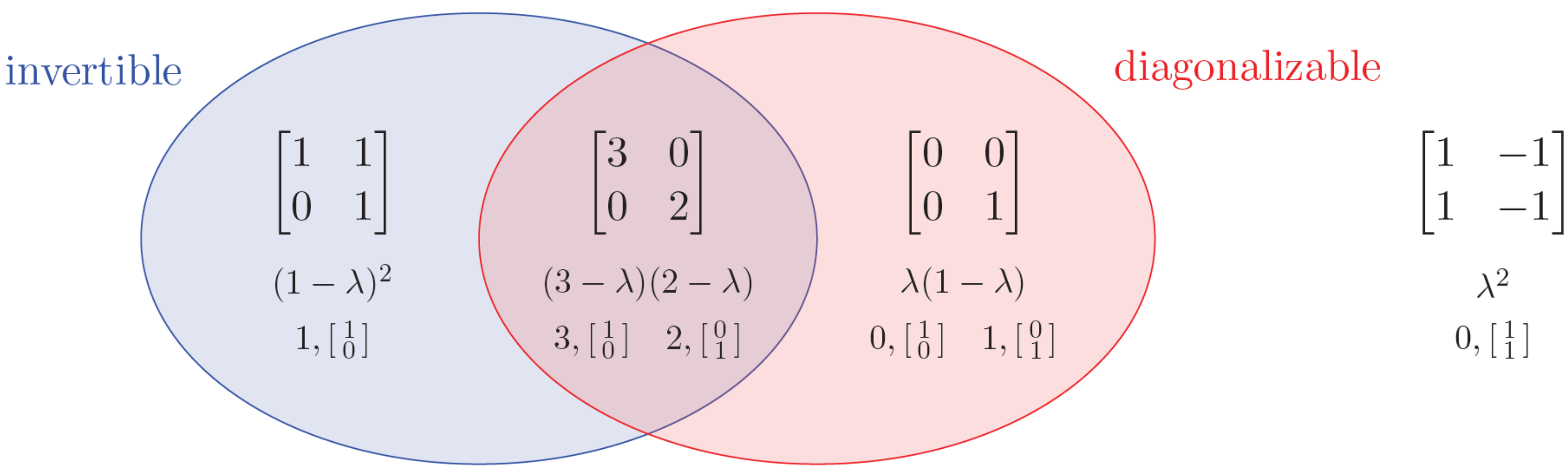
> Q. Compute e^D, where $D = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$

> Note, if $A = PDP^{-1}$, then

- $e^A = Pe^D P^{-1}$
- $\sin(A) = P\sin(D)\,P^{-1}$
- $\cos(A) = P\cos(D)\,P^{-1}$

Invertibility and Diagonalizability

- Invertibility: $\det(A) \neq 0$
- Diagonalizability: $\det(\lambda I - A) = 0$
  $\rightarrow \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ *are linearly independent*



Symmetric Matrix

Remark. Eigenvalues and Eigenvectors of a Symmetric Matrix

$A \in \mathbb{R}^{n \times n}$ is symmetric, then A has n real eigenvalues and n orthogonal eigenvectors.

> This will be linked with SVD. (Next lecture)