SNE3002 Linear Algebra – 2025 Spring

Triangular Matrices

March 26, 2025

INHA UNIVERSITY

Diagonal Matrices

Def. Diagonal Matrices

A square matrix A is diagonal if $A_{ij} = 0$ for $i \neq j$

> $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

> Computing the solution of the set of diagonal equations is trivial
  - $Ax = b \ \rightarrow \ x_i = b_i / a_{ii}$

Diagonal Matrices

- $\det(A) = a_{11}a_{22}\cdots a_{nn}$
  - Recall $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

Lower Triangular

- $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$
- $\det(A) = a_{11}a_{22}\cdots a_{nn}$
- Q. Solve $Ax = b$ $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}$

Upper Triangular

- $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

- $\det(A) = a_{11}a_{22}\cdots a_{nn}$

- Q. Solve $Ax = b$ $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}$

Triangular Matrices

Remark. Inverse of Triangular Matrices

Inverse of a triangular matrix is also a triangular matrix
Diagonal elements of inverse matrix is reciprocal of the original's

> Assume a lower triangular matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$A^{-1}A = I$$

Triangular Matrices

- Inverse of triangular matrices can be easily calculated
- To find L^{-1}, solve $L\,\boldsymbol{x}_j = \boldsymbol{e}_j$

- Q. $L = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$, L^{-1} ?

LU Factorization

> What if we multiply a lower triangular matrix by an upper triangular matrix?

> Form the matrix product of $A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} 5 & 2 \\ 0 & -1 \end{bmatrix}$

LU Factorization

- Consider a square matrix $A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 11 & 17 \\ 6 & 18 & 58 \end{bmatrix}$
- We want to find a column vector C_1 and a row vector R_1

s.t. $A - C_1 R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$

LU Factorization

- Consider a square matrix $A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 11 & 17 \\ 6 & 18 & 58 \end{bmatrix}$
- We want to find a column vector C_1 and a row vector R_1

s.t. $A - C_1 R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$

LU Factorization

- $A - C_1 R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 6 & 43 \end{bmatrix}$
- We want to find a column vector C_2 and a row vector R_2

  s.t. $A - C_1 R_1 - C_2 R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}$

LU Factorization

- $A - C_1R_1 - C_2R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 29 \end{bmatrix}$
- We want to find a column vector C_3 and a row vector R_3

  s.t. $A - C_1R_1 - C_2R_2 - C_3R_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

LU Factorization

- $A = C_1R_1 + C_2R_2 + C_3R_3$
- $L = [\mathrm{C}_1 \quad \mathrm{C}_2 \quad \mathrm{C}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ $\quad U = [\mathrm{R}_1 \quad \mathrm{R}_2 \quad \mathrm{R}_3]^\top = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 3 & 7 \\ 0 & 0 & 29 \end{bmatrix}$
- $A = LU$, the product of a lower triangular matrix and an upper triangular matrix

LU Factorization

- We wish to solve the system of linear equations $Ax = b$

  $\rightarrow \quad LUx = b$

  $\rightarrow \quad Ly = b, Ux = y$

LU Factorization

- Can we always perform LU factorization ?
- $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

LU Factorization

- Use row permutation (row switching elementary matrix)
- $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad PA = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$
- $PA = LU$
- $Ax = b$

  $\rightarrow PAx = Pb$

  $\rightarrow Ly = Pb, Ux = y$

LU Factorization

Remark. Determinant from LU Factorization

Suppose $A = LU$

$$\det(\mathrm{A}) = \det(\mathrm{L})\det(\mathrm{U}) = u_{11}u_{22}\cdots \mathrm{u}_{\mathrm{nn}}$$

If $\mathrm{P}A = LU$

$$\mathrm{A} = \mathrm{P}^{\top}\mathrm{LU}$$

$$\det(\mathrm{A}) = \det(P^{\top})\det(L)\det(U) = \pm\det(\mathrm{L})\det(U)$$

LU Factorization

- Advantages
  - Numerical stability
  - Faster determinant calculation
  - Efficient inverse computation
  - Useful in sparse matrix computations
  - Reusability