SNE3002 Linear Algebra – 2025 Spring

Singular Value Decomposition

April 2, 2025

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Spectral Theorem for Symmetric Matrices

Remark. Spectral Theorem

All eigenvalues of a real symmetric matrix are real.

Spectral Theorem for Symmetric Matrices

Remark. Spectral Theorem

Eigenvectors of a real symmetric matrix associated with distinct eigenvalues are orthogonal.

> Proof.

Spectral Theorem for Symmetric Matrices

Remark. Spectral Theorem

Suppose $S \in \mathbb{R}^{n \times n}$ is symmetric, $S^\top = S$. Then there exist n (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and corresponding unit-length eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ such that

$$S\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i,$$

the eigenvectors from an orthonormal basis for $\mathbb{R}^n$,

$$\mathbb{R}^n = span\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\},$$

And $\boldsymbol{v}_i^\top j = 0$ when $i \neq j$, and $\boldsymbol{v}_i^\top \boldsymbol{v}_i = \|\boldsymbol{v}_i\|^2 = 1$

Spectral Theorem for Symmetric Matrices

- The Spectral Theorem leads to two convenient ways to compose the matrix.

- $$\begin{bmatrix} | & | & & | \\ S\boldsymbol{v}_1 & S\boldsymbol{v}_2 & \cdots & S\boldsymbol{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda\boldsymbol{v}_1 & \lambda\boldsymbol{v}_2 & \cdots & \lambda\boldsymbol{v}_n \\ | & | & & | \end{bmatrix}$$

$$S\begin{bmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

$$SV = V\Lambda$$

Spectral Theorem for Symmetric Matrices

- $S = V\Lambda \mathrm{V}^{-1}$

  Note. $V^\top V = I$

  $S = V\Lambda \mathrm{V}^\top$ -> Diagonalization of the symmetric matrix

- In other form,
- $S = \Sigma_{i=1}^{n} \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^\top$

Spectral Theorem for Symmetric Matrices

- Q. Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

Quadratic Form

Def. Quadratic Form

A quadratic form is a polynomial in n variables in which each term is of degree 2.

$$f(x_1, \dots, x_n)$$
$$= q_1 x_1^2 + \cdots + q_n x_n^2 + q_{12} x_1 x_2 + \cdots + q_{ij} x_i x_j + \cdots + q_{n-1,n} x_{n-1} x_n$$

> Matrix from of a quadratic form

$$f(x_1, \dots, x_n) = \boldsymbol{v}^\top A \boldsymbol{v}$$

Quadratic Form

- Q. write the quadratic form

$$f(x_1, \dots, x_n) = 5x^2 - y^2 + z^2 + 2xy - 4xz + 3yz$$

Quadratic Form

> With a symmetric matrix S

- $\boldsymbol{x}^\top S\boldsymbol{x} = \boldsymbol{x}^\top\left(\Sigma_{i=1}^n \lambda_i\left(\boldsymbol{v}_i^\top \boldsymbol{x}\right)\boldsymbol{v}_i\right) = \Sigma_{i=1}^n \lambda_i \boldsymbol{x}^\top\left(\boldsymbol{v}_i^\top \boldsymbol{x}\right)\boldsymbol{v}_i = \Sigma_{i=1}^n \lambda_i\left(\boldsymbol{v}_i^\top \boldsymbol{x}\right)(\boldsymbol{x}^\top \boldsymbol{v}_i)$

  $= \Sigma_{i=1}^n \lambda_i\left(\boldsymbol{v}_i^\top \boldsymbol{x}\right)^2$

> Suppose the eigenvalues are labeled in decreasing order

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

> $\Sigma_{i=1}^n\left(\boldsymbol{v}_i^\top \boldsymbol{x}\right)^2 = \|V^\top \boldsymbol{x}\|^2 = (V^\top \boldsymbol{x})^\top(V^\top \boldsymbol{x}) = \boldsymbol{x}^\top V V^\top \boldsymbol{x} = \boldsymbol{x}^\top \boldsymbol{x} = \|\boldsymbol{x}\|^2$

> $\boldsymbol{x}^\top S\boldsymbol{x} \leq \lambda_1 \Sigma_{i=1}^n\left(\boldsymbol{v}_i^\top \boldsymbol{x}\right)^2 = \lambda_1\|\boldsymbol{x}\|^2$, $\quad \max_{\|x\|=1} \boldsymbol{x}^\top S\boldsymbol{x} \leq \lambda_1$

Likewise, $\min_{\|x\|=1} \boldsymbol{x}^\top S\boldsymbol{x} \geq \lambda_n$

Positive Definite

Def. Positive Semidefinite and Positive Definite Matrices

Let A be a symmetric $n \times n$-matrix over the real numbers.

- A is called positive semidefinite if for all $\boldsymbol{v} \in \mathbb{R}^n$, we have $\boldsymbol{v}^\top A \boldsymbol{v} \geq 0$
- A is called positive definite if for all non−zero $\boldsymbol{v} \in \mathbb{R}^n$, we have $\boldsymbol{v}^\top A \boldsymbol{v} > 0$

Positive Definite

> Q. Which of the following matrices are positive (semi)definite?

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

Singular Value Decomposition

- For a general rectangular matrix, $A = U\Sigma V^\top$ (rank r)
  - U and V are orthogonal matrix
  - Σ is $diag(\sigma_1, \dots, \sigma_r, 0, \dots, 0)$ where $\sigma_i > 0$
- Derivation

Singular Value Decomposition

Singular Value Decomposition

Singular Value Decomposition

- Eigenvalue? Singular value?

Singular Value Decomposition

- General form

Singular Value Decomposition

- Solving $Ax = b$

Singular Value Decomposition

> Matrix (Data) approximation (MATLAB) / Noise reduction