SNE3002 Linear Algebra – 2025 Spring

Orthogonality, Projection, Gram-Schmidt

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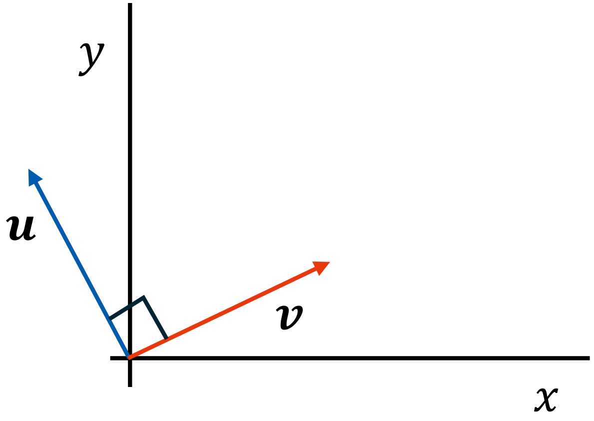
INHA UNIVERSITY

Orthogonality

Def. Orthogonality

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors. We say that $\boldsymbol{u}$ and $\boldsymbol{v}$ are **orthogonal** $\boldsymbol{u} \perp \boldsymbol{v}$ if

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{0}$$



If $\boldsymbol{u} = \begin{bmatrix} -6 \\ 3.7 \end{bmatrix}$, then $\boldsymbol{v} = \begin{bmatrix} 3.7/\alpha \\ 6/\alpha \end{bmatrix}$

> Orthogonal means ‘at right angle’

Orthogonality

Remark. Pythagorean Theorem in $\mathbb{R}^n, n > 2$

Suppose $\boldsymbol{u} \perp \boldsymbol{v}$. Then,

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$$

> Proof.

Orthogonality

- If $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ is an orthogonal set of vectors, then they are linearly independent
  - Proof.

Orthogonality

- We call orthogonal vectors that are normalized **orthonormal**.
- Given a set of orthogonal vectors $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$,
  a set of orthonormal vectors $\{\widetilde{\boldsymbol{u}}_1, \dots, \widetilde{\boldsymbol{u}}_k\}$ is obtained by

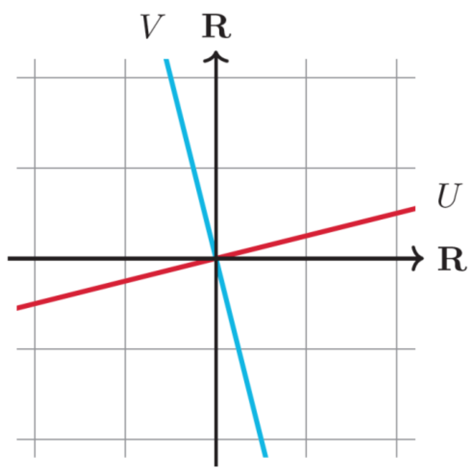
$$\widetilde{\boldsymbol{u}}_i = \frac{\boldsymbol{u}_1}{||\boldsymbol{u}_1||}$$

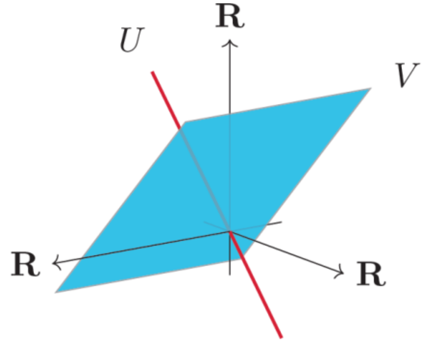
- $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ is an orthogonal basis for W if it spans W

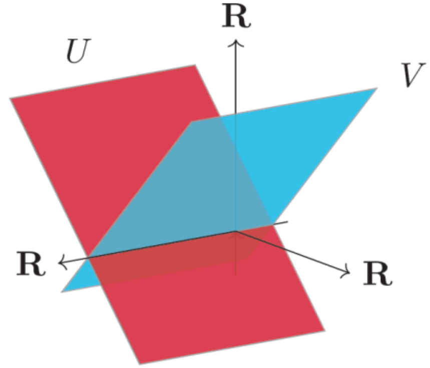
Orthogonality

> Subspaces can be orthogonal too!

- Two subspaces $U, V \subseteq \mathbb{R}^n$ are orthogonal if every pair of vectors $\boldsymbol{u} \in U, \boldsymbol{v} \in V$ is orthogonal







- If two subspaces $U, V \subseteq \mathbb{R}^n$ are orthogonal and $\dim(U) + \dim(V) = n$, then each is the orthogonal complement $U = V^{\perp}$

Orthogonality

- An $n \times m$ rectangular matrix Q is **orthonormal**:
  - If $n > m$, its columns are orthonormal vectors, which is equivalent to $Q^\top Q = I_m$
  - If $n > m$, its columns are orthonormal vectors, which is equivalent to $QQ^\top = I_n$

Remark. Orthogonal Matrix

A square $n \times m$ matrix is **orthogonal** if $Q^\top Q = QQ^\top = I_n$ and hence, $Q^{-1} = \mathrm{Q}^\top$

From $Q^\top Q = I_n$, $[\det(\mathrm{Q})^2] = 1$, and $\det(Q) = \pm 1$

$$Q \text{ orthogonal } \Rightarrow \det(Q) = \pm 1$$

Orthogonality

> $$Q = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & \cdots & | \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_2 & - \\ \vdots & \vdots & \vdots \\ - & \vec{v}_k & - \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \cdots & \vec{v}_1 \cdot \vec{v}_k \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \cdots & \vec{v}_2 \cdot \vec{v}_k \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_k \cdot \vec{v}_1 & \vec{v}_k \cdot \vec{v}_2 & \cdots & \vec{v}_k \cdot \vec{v}_k \end{bmatrix}$$

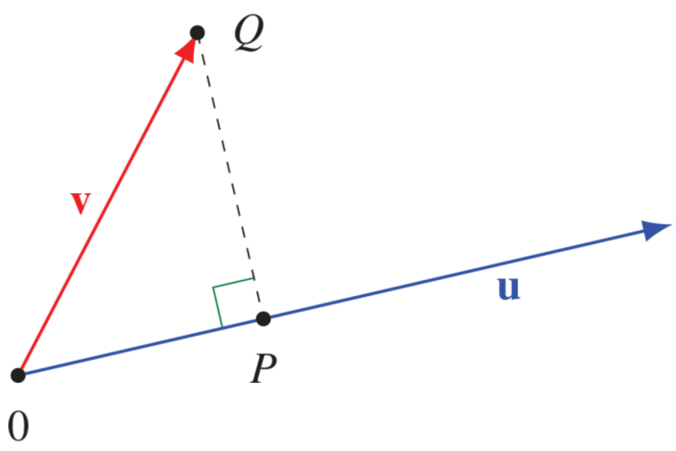
> This is $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ if and only if the vectors are orthonormal

Projection

Def. Vector Projection

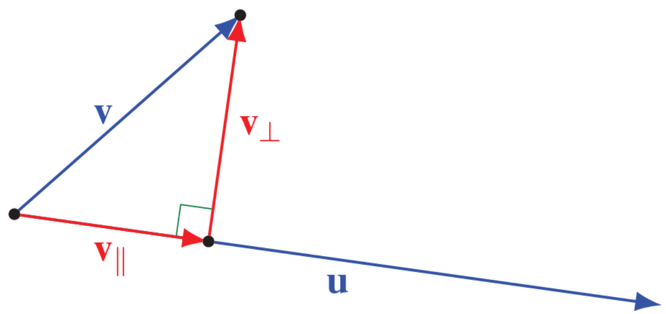
Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. The projection of $\boldsymbol{v}$ onto $\boldsymbol{u}$ is defined to be the vector

$$\mathrm{proj}_{\mathbf{u}}(\boldsymbol{v}) = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}} \boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|^2} \boldsymbol{u}$$

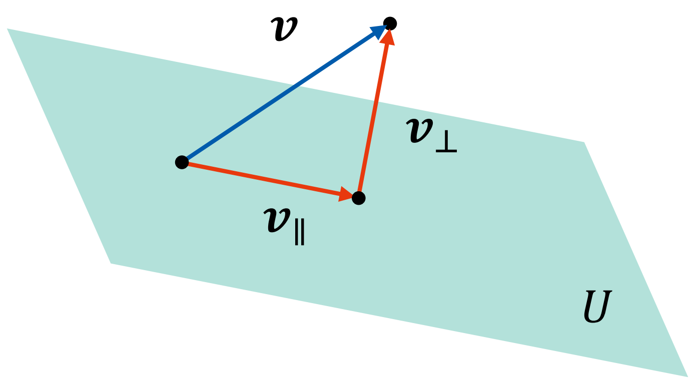


Projection

- Decomposition into components
  - $\boldsymbol{v} = \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp}$



- Projection onto subspace



Projection

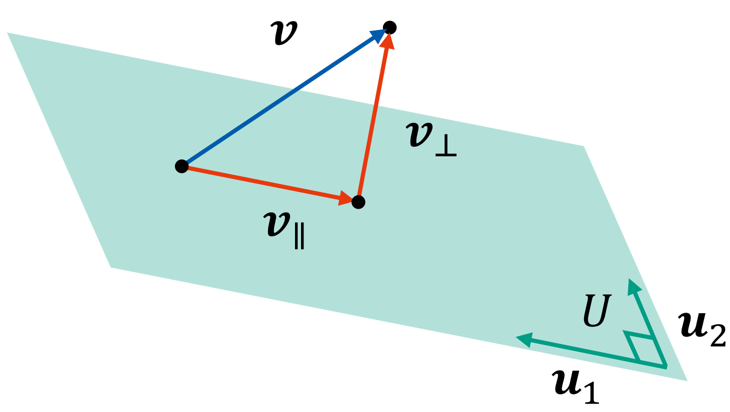
- Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be an orthonormal basis of U
- Since $\boldsymbol{v}_\parallel$ is in U, we can write

$$\boldsymbol{v}_\parallel = a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k$$

$$\boldsymbol{u}_{\boldsymbol{i}} \cdot \boldsymbol{v} = \boldsymbol{u}_{\boldsymbol{i}} \cdot (\boldsymbol{v}_\parallel + \boldsymbol{v}_\perp) = \boldsymbol{u}_{\boldsymbol{i}} \cdot \boldsymbol{v}_\parallel = \boldsymbol{u}_{\boldsymbol{i}} \cdot (a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k) = a_{\boldsymbol{i}}$$

$$\boldsymbol{v}_\parallel = (\boldsymbol{u}_1 \cdot \boldsymbol{v})\boldsymbol{u}_1 + \cdots + (\boldsymbol{u}_k \cdot \boldsymbol{v})\boldsymbol{u}_k$$

$$\boldsymbol{v}_\perp = \boldsymbol{v} - \boldsymbol{v}_\parallel$$



Projection

$$\begin{aligned} \boldsymbol{v}_{\parallel} &= (\boldsymbol{u}_1 \cdot \boldsymbol{v})\boldsymbol{u}_1 + \cdots + (\boldsymbol{u}_k \cdot \boldsymbol{v})\boldsymbol{u}_k \\ &= (\boldsymbol{u}_1^\top \boldsymbol{v})\boldsymbol{u}_1 + \cdots + \left(\boldsymbol{u}_k^\top \boldsymbol{v}\right)\boldsymbol{u}_k \\ &= \boldsymbol{u}_1(\boldsymbol{u}_1^\top \boldsymbol{v}) + \cdots + \mathbf{u}_\mathrm{k}(\mathbf{u}_\mathrm{k}^\top \boldsymbol{v}) \\ &= QQ^\top \boldsymbol{v} \end{aligned}$$

$$Q = [\boldsymbol{u}_1 \quad \ldots \quad \boldsymbol{u}_k]$$

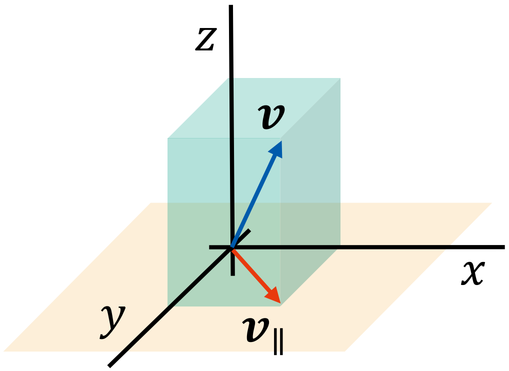
- So, if Q is an orthonormal basis matrix for a subspace U, then $QQ^\top$ is the orthogonal projection operator onto U

- Note, if Q is not orthonormal, then projection would be

$$\boldsymbol{v}_{\parallel} = Q(Q^\top Q)^{-1} Q^\top \boldsymbol{v}$$

Projection

> Q. Find a projection of $\boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ onto $x - y$ plane.



> Q. Find a projection of $\boldsymbol{v}$ onto $U = \text{span}\left\{\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}\right\}$

Gram-Schmidt Algorithm

- Find an orthogonal basis of a subspace
- Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$ be a basis for some subspace W

$$\boldsymbol{u}_1 = \boldsymbol{v}_1$$

$$\boldsymbol{u}_2 = \boldsymbol{v}_2 - \frac{\boldsymbol{u}_1 \cdot \boldsymbol{v}_2}{\boldsymbol{u}_1 \cdot \boldsymbol{u}_1} \boldsymbol{u}_1$$

$$\boldsymbol{u}_3 = \boldsymbol{v}_3 - \frac{\boldsymbol{u}_1 \cdot \boldsymbol{v}_3}{\boldsymbol{u}_1 \cdot \boldsymbol{u}_1} \boldsymbol{u}_1 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_3}{\boldsymbol{u}_2 \cdot \boldsymbol{u}_2} \boldsymbol{u}_2$$

…

$$\boldsymbol{u}_k = \boldsymbol{v}_k - \frac{\boldsymbol{u}_1 \cdot \boldsymbol{v}_k}{\boldsymbol{u}_1 \cdot \boldsymbol{u}_1} \boldsymbol{u}_1 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_k}{\boldsymbol{u}_2 \cdot \boldsymbol{u}_2} \boldsymbol{u}_2 - \cdots - \frac{\boldsymbol{u}_{k-1} \cdot \boldsymbol{v}_k}{\boldsymbol{u}_{k-1} \cdot \boldsymbol{u}_{k-1}} \boldsymbol{u}_{k-1}$$

Normalize!

Gram-Schmidt Algorithm

> Proof.

Proof. First, it is clear that $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ and $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ span the same subspace, as each $\mathbf{v}_i$ is a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_i$ and conversely, each $\mathbf{u}_i$ is a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_i$. So the only thing we must check is that $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is an orthogonal set. In other words, we must show that $\langle \mathbf{u}_j,\mathbf{u}_i\rangle = 0$ for

all $j < i$. We prove this by induction on i, i.e., we assume it is already true for all pairs of indices smaller than i. To show $\langle \mathbf{u}_j,\mathbf{u}_i\rangle = 0$, we calculate:

$$\begin{aligned}
\langle \mathbf{u}_j,\mathbf{u}_i\rangle &= \langle \mathbf{u}_j, \mathbf{v}_i - \tfrac{\langle \mathbf{u}_1,\mathbf{v}_i\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 - \ldots - \tfrac{\langle \mathbf{u}_j,\mathbf{v}_i\rangle}{\langle \mathbf{u}_j,\mathbf{u}_j\rangle}\mathbf{u}_j - \ldots - \tfrac{\langle \mathbf{u}_{i-1},\mathbf{v}_i\rangle}{\langle \mathbf{u}_{i-1},\mathbf{u}_{i-1}\rangle}\mathbf{u}_{i-1}\rangle \\
&= \langle \mathbf{u}_j,\mathbf{v}_i\rangle - \tfrac{\langle \mathbf{u}_1,\mathbf{v}_i\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\langle \mathbf{u}_j,\mathbf{u}_1\rangle - \ldots - \tfrac{\langle \mathbf{u}_j,\mathbf{v}_i\rangle}{\langle \mathbf{u}_j,\mathbf{u}_j\rangle}\langle \mathbf{u}_j,\mathbf{u}_j\rangle - \ldots - \tfrac{\langle \mathbf{u}_{i-1},\mathbf{v}_i\rangle}{\langle \mathbf{u}_{i-1},\mathbf{u}_{i-1}\rangle}\langle \mathbf{u}_j,\mathbf{u}_{i-1}\rangle \\
&= \langle \mathbf{u}_j,\mathbf{v}_i\rangle - 0 - \ldots - \tfrac{\langle \mathbf{u}_j,\mathbf{v}_i\rangle}{\langle \mathbf{u}_j,\mathbf{u}_j\rangle}\langle \mathbf{u}_j,\mathbf{u}_j\rangle - \ldots - 0 \\
&= \langle \mathbf{u}_j,\mathbf{v}_i\rangle - \langle \mathbf{u}_j,\mathbf{v}_i\rangle \\
&= 0.
\end{aligned}$$

It follows that the set $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is orthogonal, as desired. ♠

Gram-Schmidt Algorithm

> Q. Find an orthogonal basis for

$$\text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right\}$$

Gram-Schmidt Algorithm

- Note. Modified Gram-Schmidt

$$
\begin{aligned}
&\boldsymbol{u}_1 = \boldsymbol{v}_1 \\
&\boldsymbol{u}_1 \leftarrow \boldsymbol{u}_1/\|\boldsymbol{u}_1\| \\
&\boldsymbol{u}_2 = \boldsymbol{v}_2 - (\boldsymbol{u}_1 \cdot \boldsymbol{v}_2)\boldsymbol{u}_1 \\
&\boldsymbol{u}_2 \leftarrow \boldsymbol{u}_2/\|\boldsymbol{u}_2\| \\
&\boldsymbol{u}_3 = \boldsymbol{v}_3 - (\boldsymbol{u}_1 \cdot \boldsymbol{v}_3)\boldsymbol{u}_1 - (\boldsymbol{u}_2 \cdot \boldsymbol{v}_3)\boldsymbol{u}_2 \\
&\boldsymbol{u}_3 \leftarrow \boldsymbol{u}_3/\|\boldsymbol{u}_3\| \\
&\dots \\
&\boldsymbol{u}_k = \boldsymbol{v}_k - (\boldsymbol{u}_1 \cdot \boldsymbol{v}_k)\boldsymbol{u}_1 - (\boldsymbol{u}_2 \cdot \boldsymbol{v}_k)\boldsymbol{u}_2 - \cdots - (\boldsymbol{u}_{k-1} \cdot \boldsymbol{v}_k)\boldsymbol{u}_{k-1} \\
&\boldsymbol{u}_k \leftarrow \boldsymbol{u}_k/\|\boldsymbol{u}_k\|
\end{aligned}
$$