ENV 710: Lecture 4

continuous probability distributions

Duke | NICHOLAS SCHOOL OF THE ENVIRONMENT

continuous probability distributions

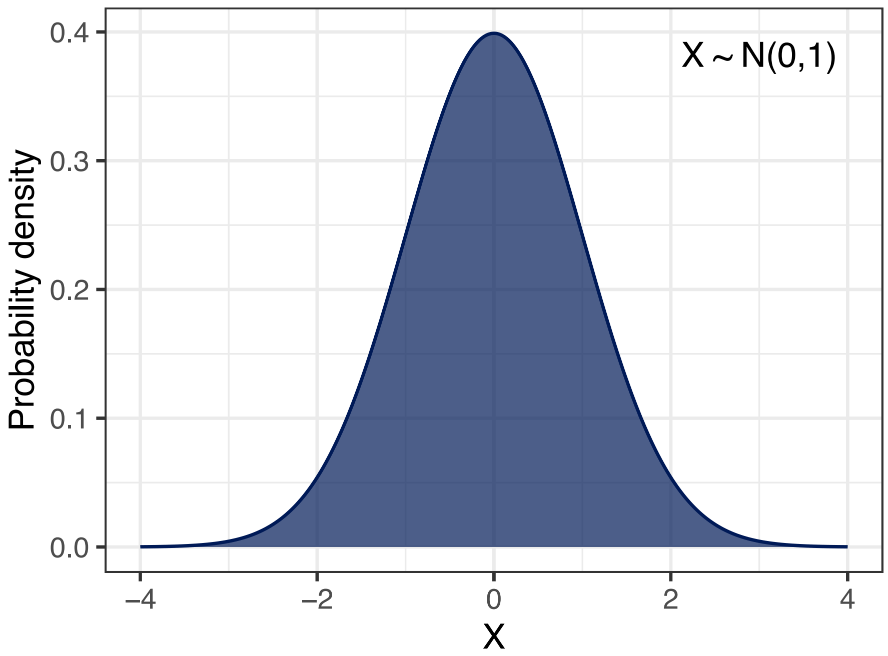
pdf and cdf and uniform distribution

learning goals

- what is the difference between a discrete and continuous probability distribution?
- what is a probability density function (pdf)?
- what is a cumulative density function (cdf)?
- how do you calculate probabilities, mean and variance of the uniform distribution?

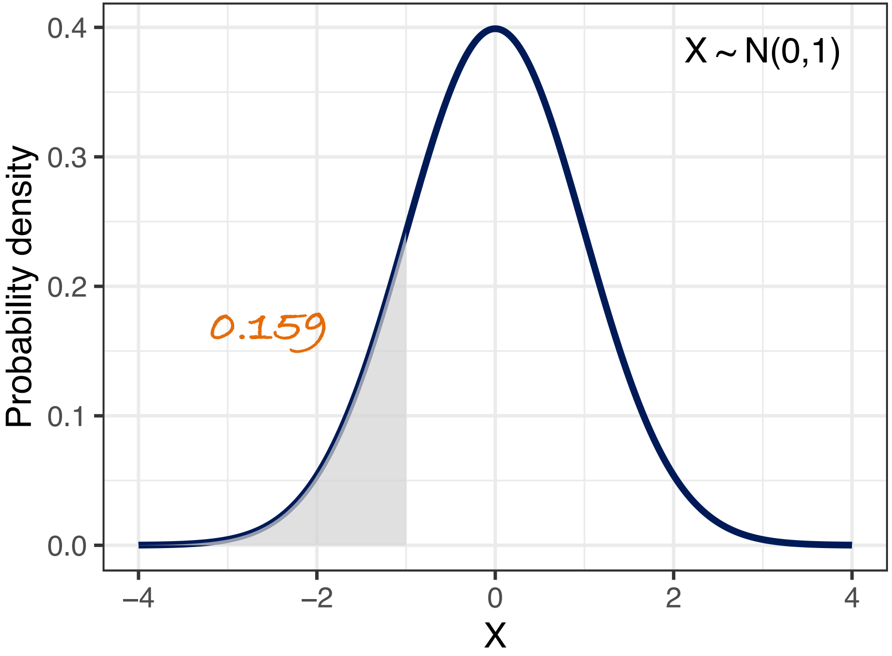
stuff you should know

continuous probability distribution



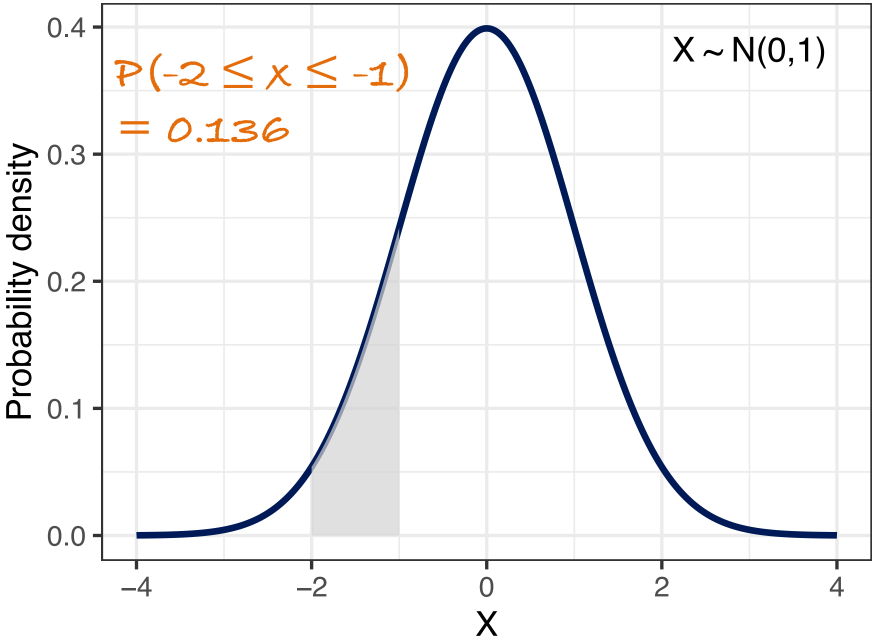
defined by a function that gives the density of probability rather than the probability mass

continuous probability distribution



defined by a function that gives the density of probability rather than the probability mass

continuous probability distribution



defined by a function that gives the density of probability rather than the probability mass

probability density function (pdf)

$$P(a < x < b) = \int_a^b f(x)dx$$

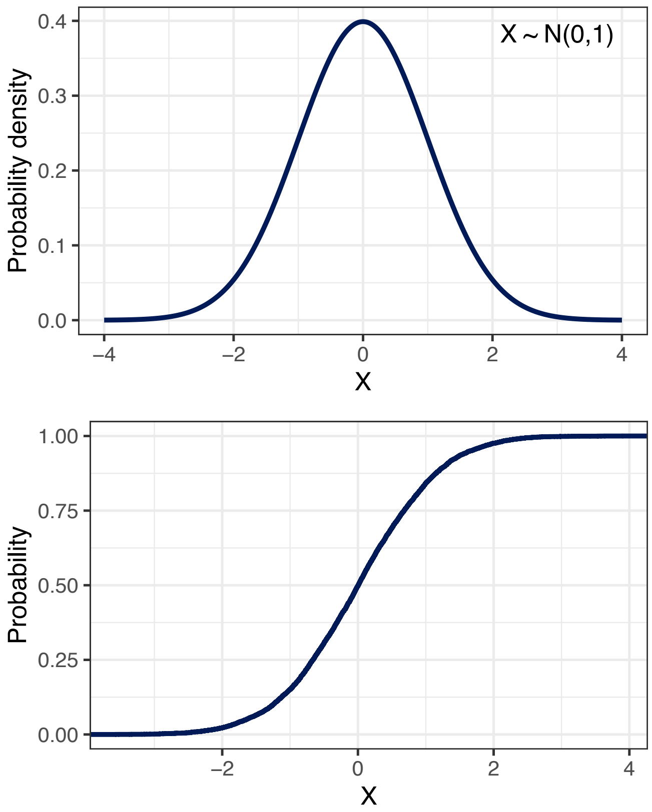
$$\int_{\infty}^{-\infty} f(x)d(x) = 1$$

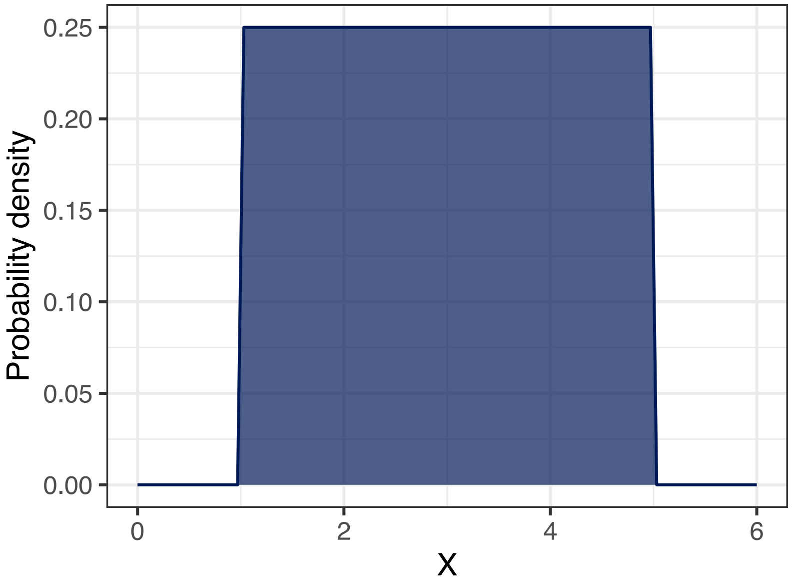
a function of a continuous random variable who's integral across an interval gives the probability that the value of the variable lies within the same interval

cumulative distribution function (cdf)

"area in so far function"

cdf is the probability that a random variable X will be found at a value less than or equal to X




0.25
0.20
0.15
0.10
0.05
0.00
Probability density
0
2
4
6
X

uniform probability distribution

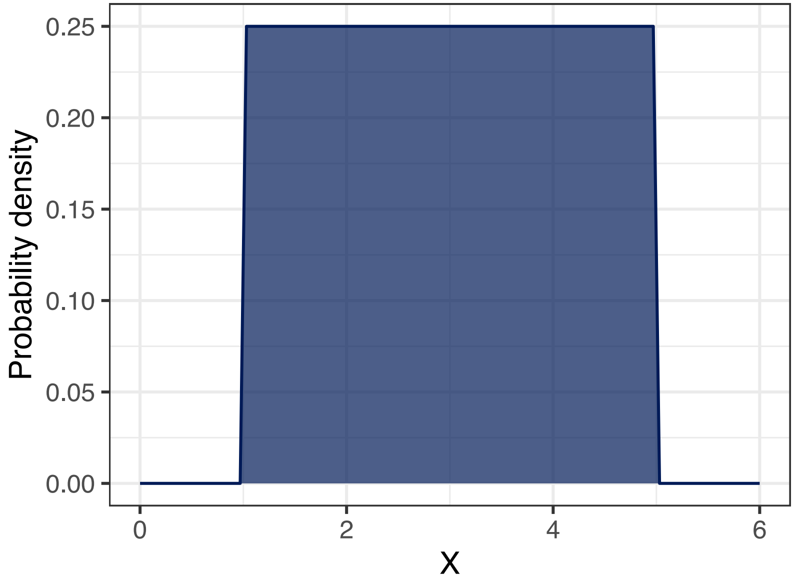
- all intervals of the same length along the distribution have the same probability
- any number selected from the interval $[a, b]$ has an equal chance of being selected

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b. \\ 0 & \text{for } x < a \text{ or } x > b. \end{cases}$$

probability density function (pdf)

uniform probability distribution

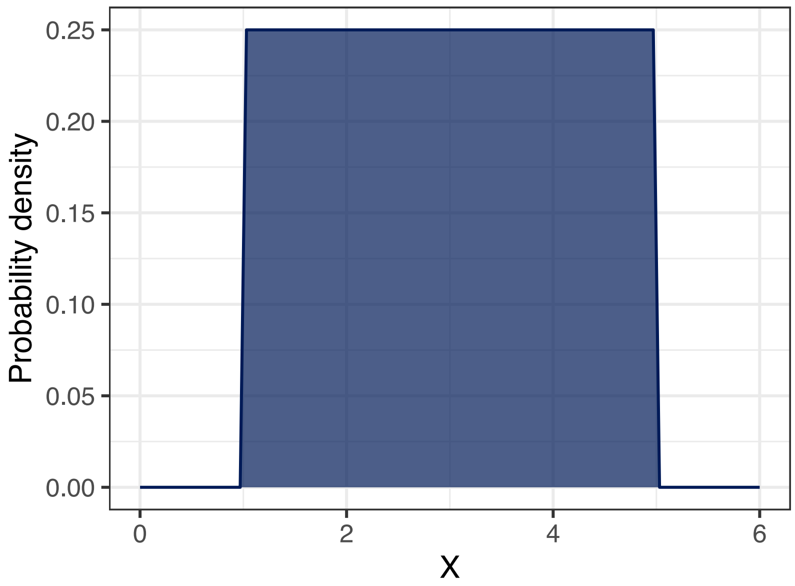
$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b. \\ 0 & \text{for } x < a \text{ or } x > b. \end{cases}$$



uniform probability distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b. \\ 0 & \text{for } x < a \text{ or } x > b. \end{cases}$$

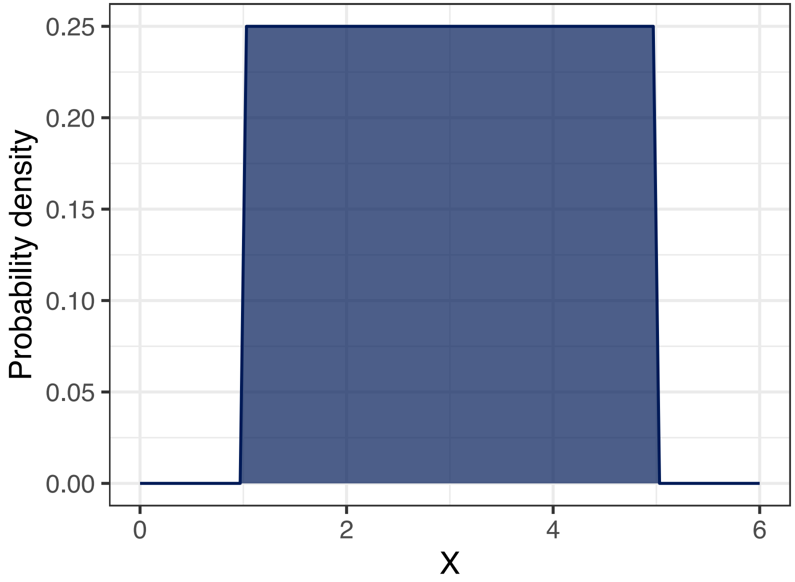
```
dunif(x = x, min = a, max = b)
```



uniform probability distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b. \\ 0 & \text{for } x < a \text{ or } x > b. \end{cases}$$

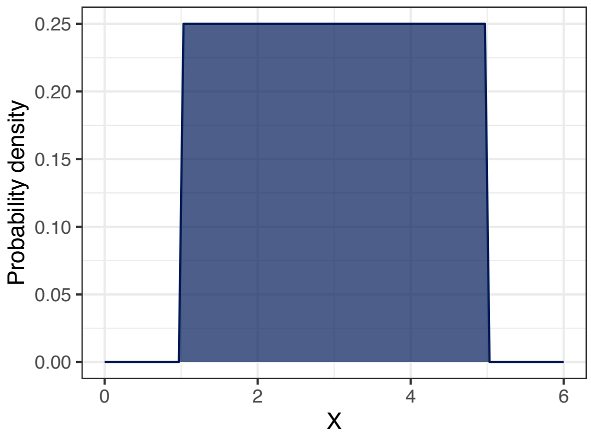
```
dunif(x = 2, min = 1, max = 5)
```



uniform probability distribution

$$E(X) = (b + a)/2$$

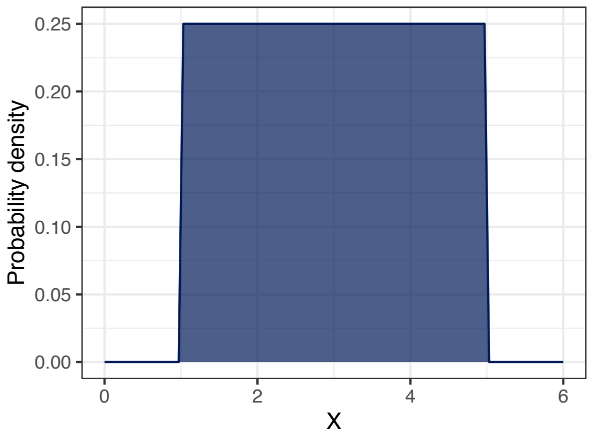
$$\sigma^2(X) = \frac{(b-a)^2}{12}$$



uniform probability distribution

$$E(X) = (b + a)/2$$

$$\sigma^2(X) = \frac{(b - a)^2}{12}$$




```
> (1+5)/2
[1] 3

> ((5-1)^2)/12
[1] 1.333333
```

example

what is the probability of obtaining any number from a uniform distribution with a minimum of 5.7 and a maximum of 8.9?

- what is the mean of the distribution?
- what is the standard deviation of the distribution?

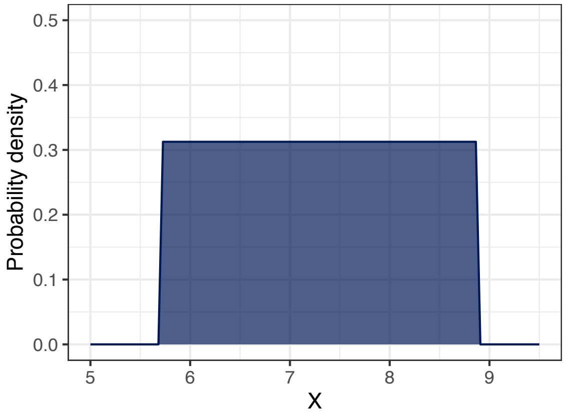
example

what is the probability of obtaining any number from a uniform distribution with a minimum of 5.7 and a maximum of 8.9?

- what is the mean of the distribution?
- what is the standard deviation of the distribution?

$$E(x) = \frac{8.9 + 5.7}{2} = 7.3$$

$$\sigma(x) = \sqrt{\frac{(8.9 - 5.7)^2}{12}} = 0.85$$



example

what is the probability of obtaining any number from a uniform distribution with a minimum of 5.7 and a maximum of 8.9?

- what is the probability of obtaining an x of 6 for this distribution?

example

what is the probability of obtaining any number from a uniform distribution with a minimum of 5.7 and a maximum of 8.9?

- what is the probability of obtaining an x of 6 for this distribution?

$$f(x) = \frac{1}{8.9 - 5.7} = 0.3125$$

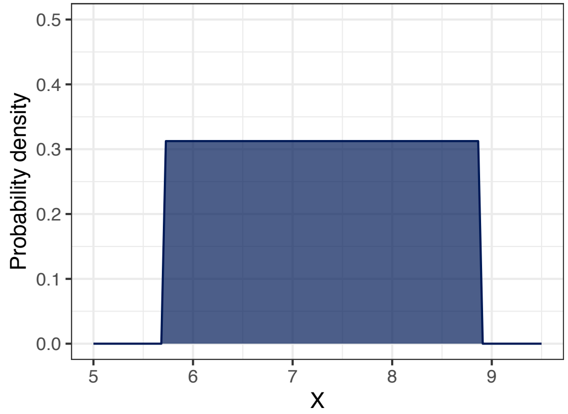
```
>dunif(x = 6, min = 5.7, max = 8.9)
[1] 0.3125
```

calculating probability of an interval

- probability can be derived for area under the curve
- use punif()which calculates cumulative probability

```
dunif(x = 6.1, min = 5.7, max = 8.9)* (6.1-5.7)
```

```
punif(q, min, max)
punif(q = 6.1, min = 5.7, max = 8.9)
```



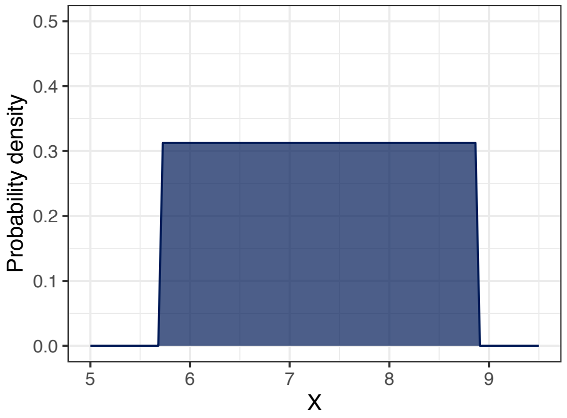
calculating probability of an interval

- probability can be derived for area under the curve
- use punif()which calculates cumulative probability

$P(X \leq 8.9) = ???$

$P(X \geq 8) = ???$

$P(6 \leq X \leq 7) = ???$



calculating probability of an interval

- probability can be derived for area under the curve
- use punif()which calculates cumulative probability

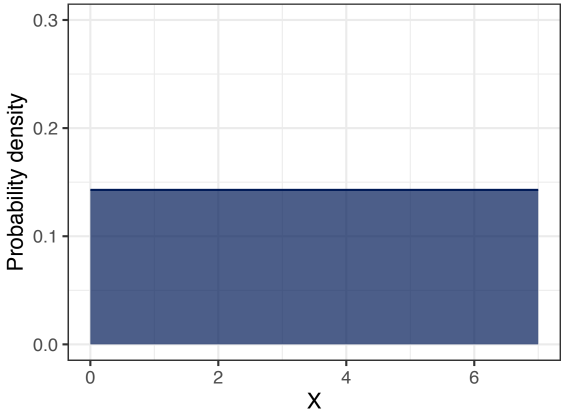
$P(X \leq 8.9) = ???$ `punif(q = 8.9, min = 5.7, max = 8.9)`

$P(X \geq 8) = ???$ `1-punif(q = 8, min = 5.7, max = 8.9)`

$P(6 \leq X \leq 7) = ???$ `punif(q = 7, min = 5.7, max = 8.9) - punif(q = 6, min = 5.7, max = 8.9)`

example

After a rain, it takes 5 hours for water to reach all leaves of a plant. If we assume water restoration rate is uniformly distributed over the interval 0 to 7 hours, what is the probability that it will take more than 3 hours for water to reach all leaves?



example

After a rain, it takes 5 hours for water to reach all leaves of a plant. If we assume water restoration rate is uniformly distributed over the interval 0 to 7 hours, what is the probability that it will take more than 3 hours for water to reach all leaves?

```
1 - punif(q = 3, min = 0, max = 7)
```

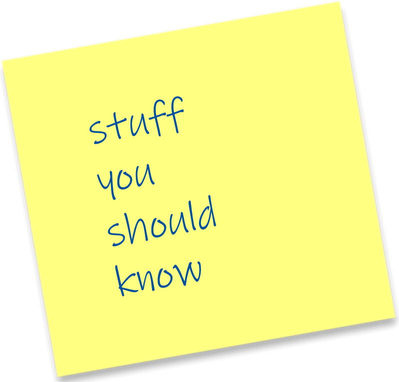
$P(X > 3) = 0.57$

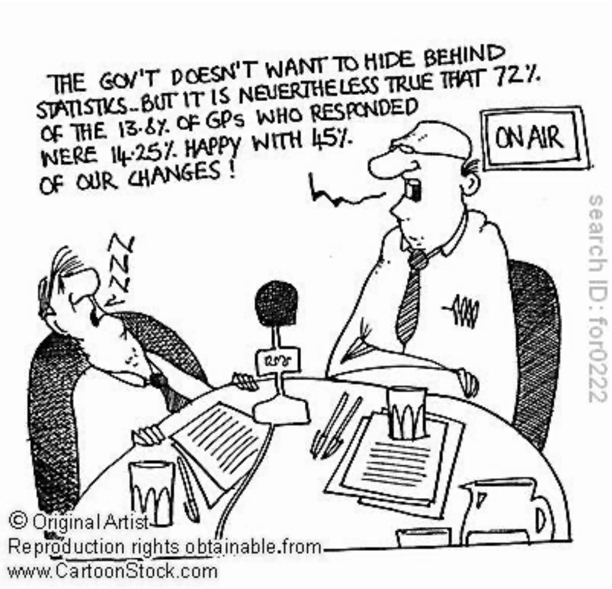
continuous probability distributions

normal distribution

learning goals

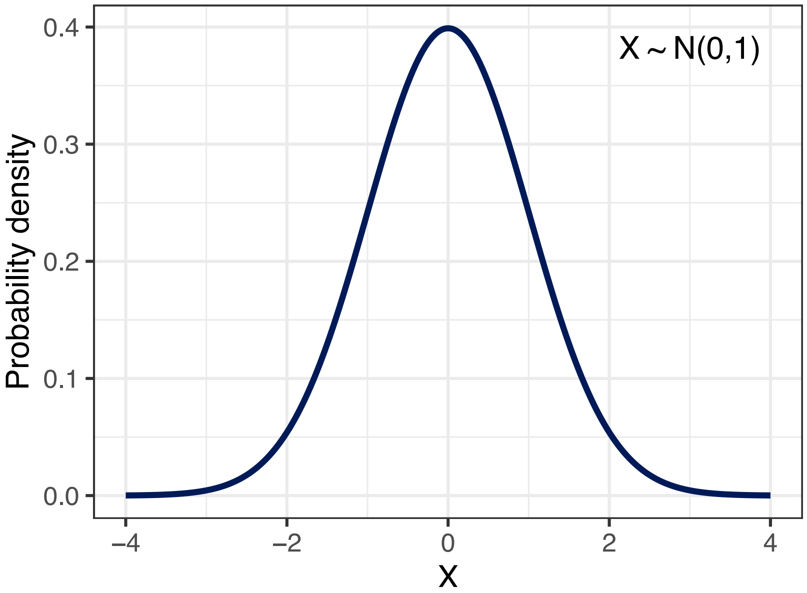
- describe the normal distribution
- differentiate between the standard normal distribution and other normal distributions
- use the probability density function to calculate probabilities and cumulative probabilities



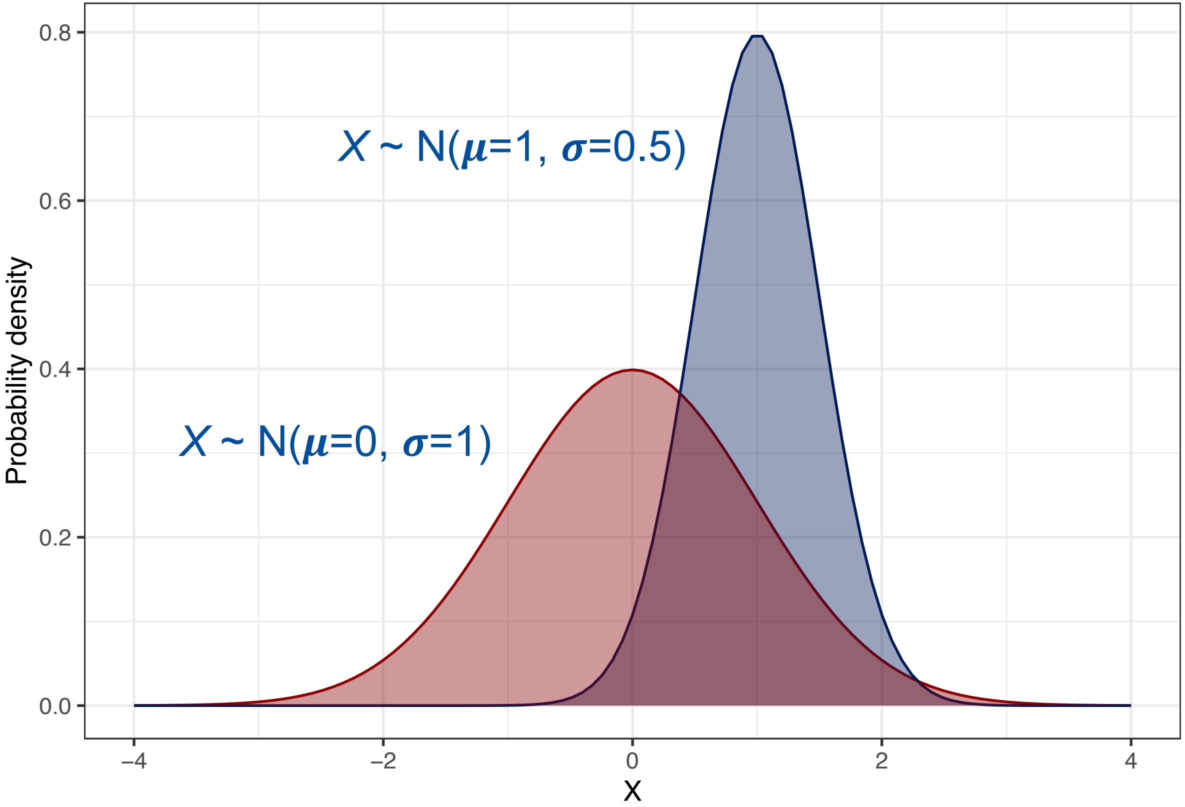


normal probability distribution

- normal distribution is symmetrical about the mean
- total area under the curve is equal to 1



can take on lots of shapes…

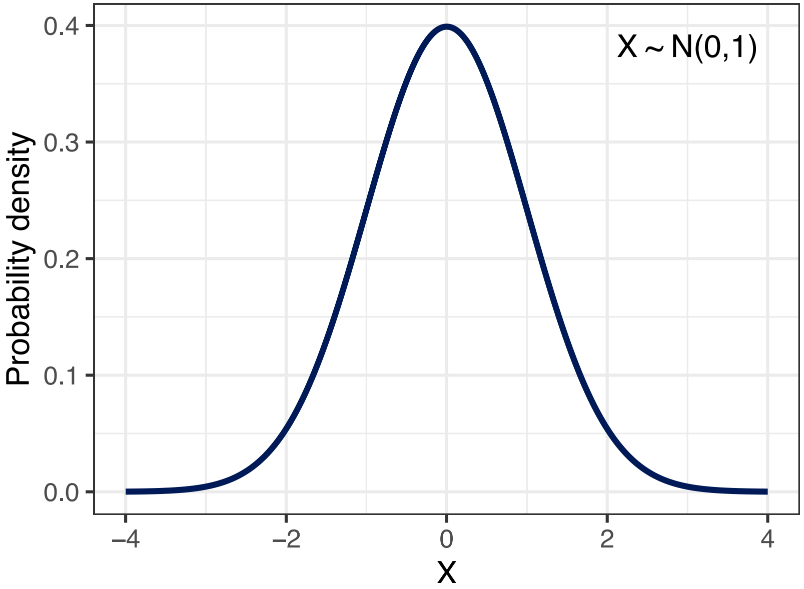


normal probability distribution

- the probability density function derives the *density* of any single X value

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

```
dnorm(x, mean, sd)
```



normal probability distribution

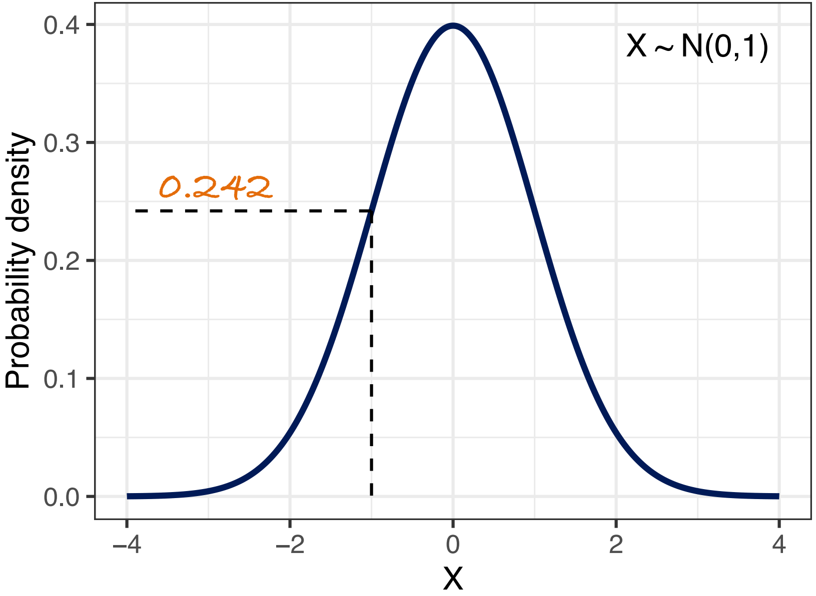
- the probability density function derives the *density* of any single X value

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$P(x=-1) = \frac{1}{1\sqrt{2\pi}} e^{-\frac{(-1-0)^2}{(2\cdot 1^2)}} = 0.242$$

```
dnorm(x, mean, sd)

dnorm(x = -1, mean = 0, sd = 1)
```



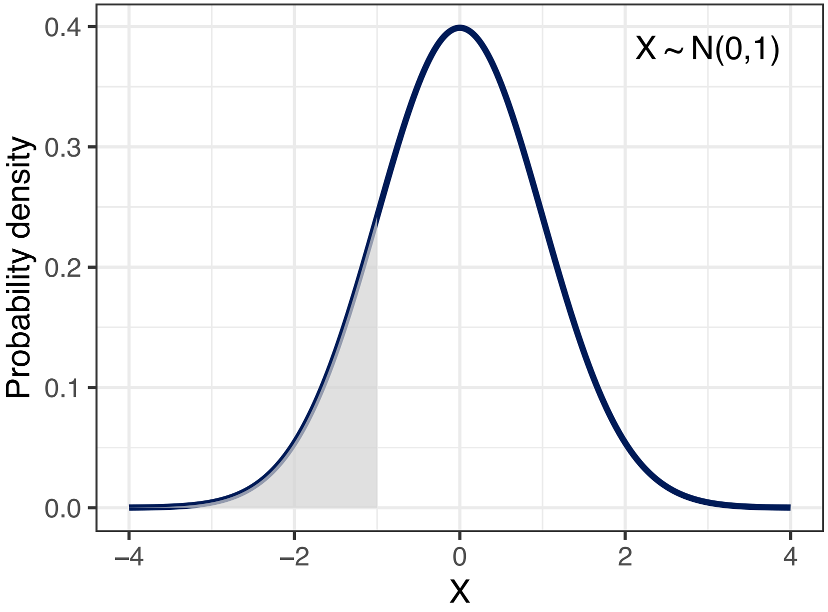
normal probability distribution

- *probability* can be derived for area under the curve
- use `pnorm()` which calculates cumulative probability

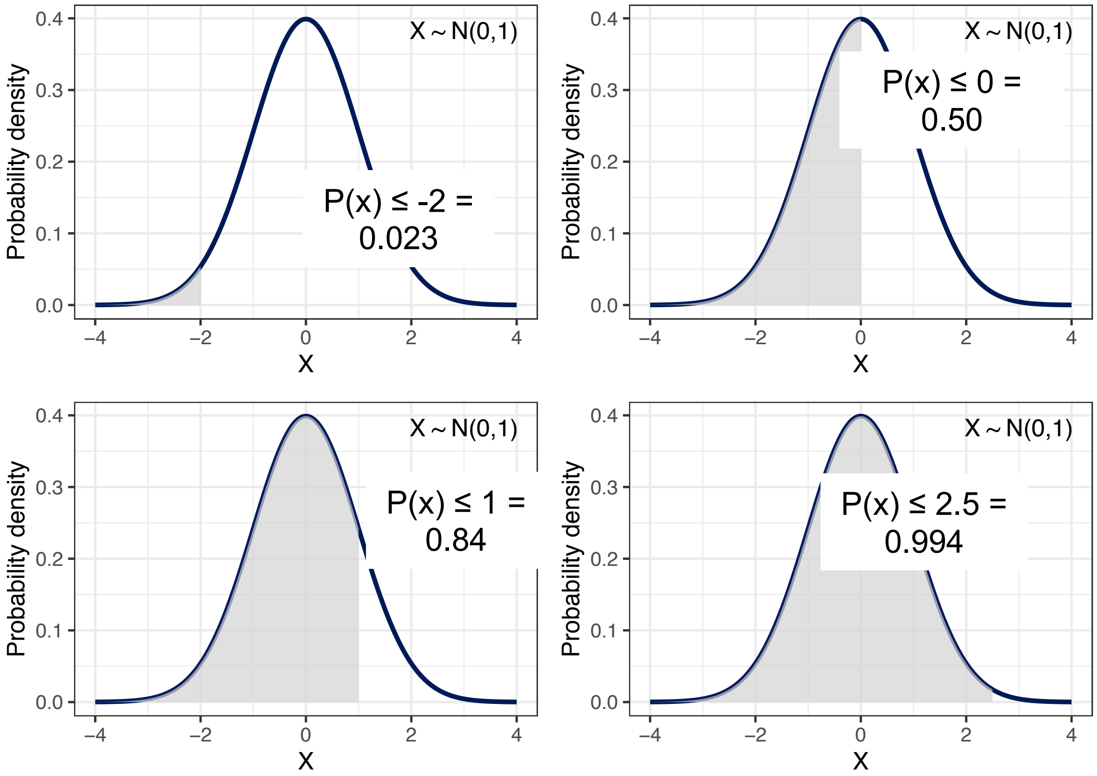
```
pnorm(q, mean, sd)

pnorm(q=-1, mean=0, sd=1)
```

$P(x \leq -1 = 0.159)$



calculate the probability that we draw a number less than or equal to X – cumulative probability



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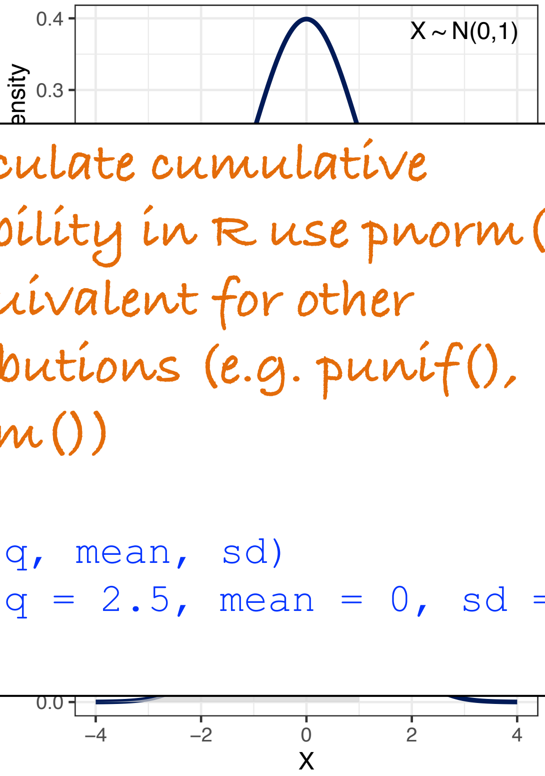
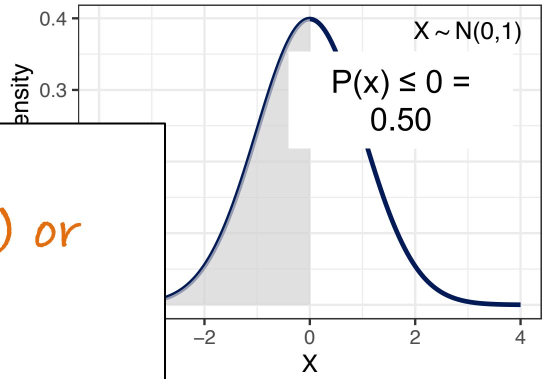
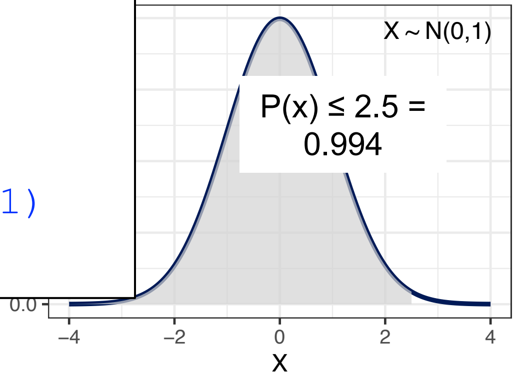






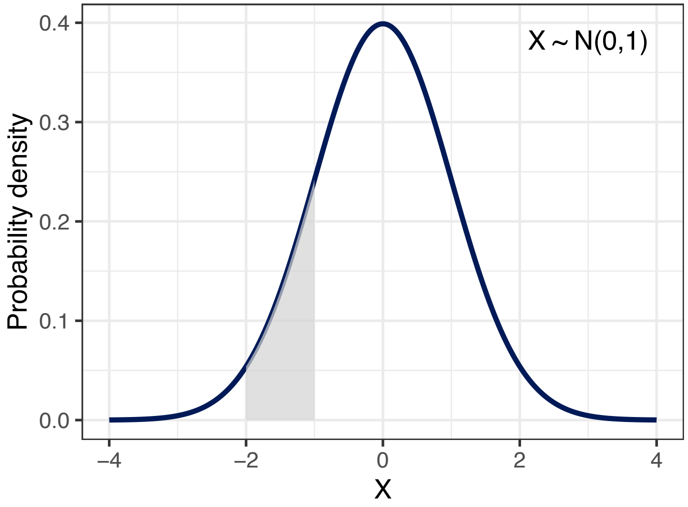
to calculate cumulative probability in R use pnorm() or its equivalent for other distributions (e.g. punif(), pbinom())

```
pnorm(q, mean, sd)
pnorm(q = 2.5, mean = 0, sd = 1)
```

normal probability distribution

- probability can be derived for area under the curve
- use `pnorm()` which calculates cumulative probability

P(-2 ≤ x ≤ -1= ???)

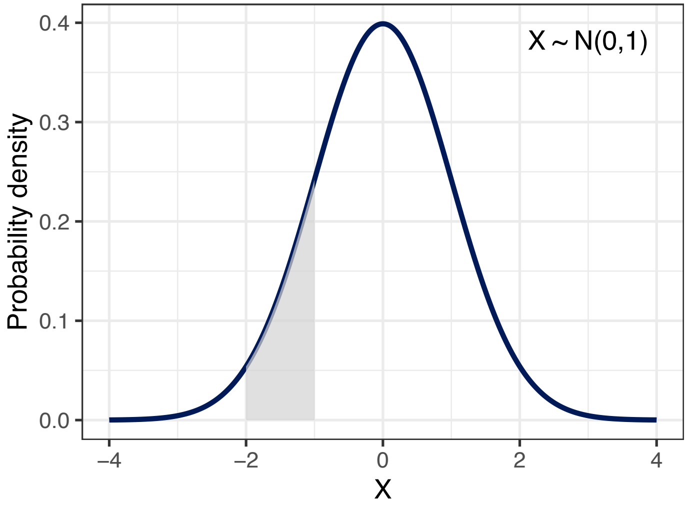


normal probability distribution

- probability can be derived for area under the curve
- use `pnorm()` which calculates cumulative probability

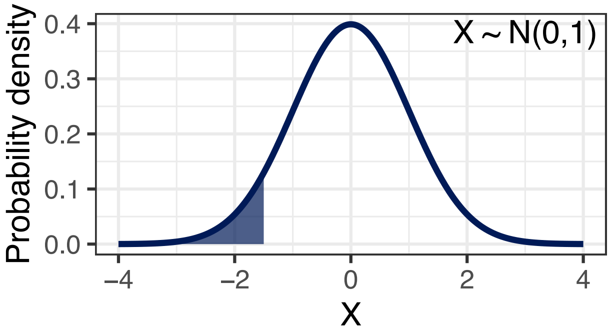
```
pnorm(-1, mean=0, sd=1) –
pnorm(-2, mean=0, sd=1)
```

$P(-2 \leq x \leq -1 = 0.136)$

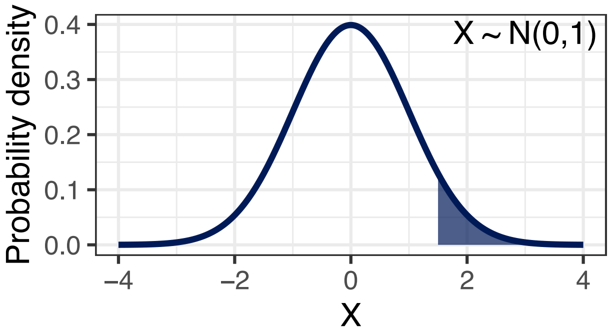


example

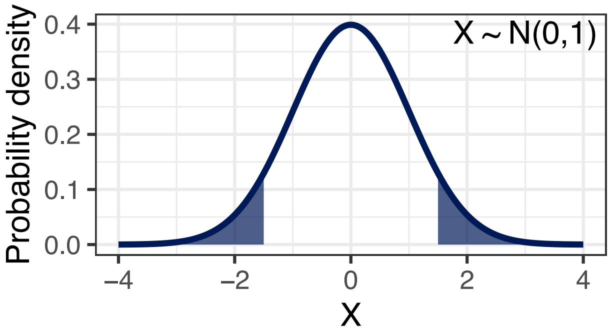
P(x ≤ -1.5) = ???



P(x ≥ 1.5) = ???



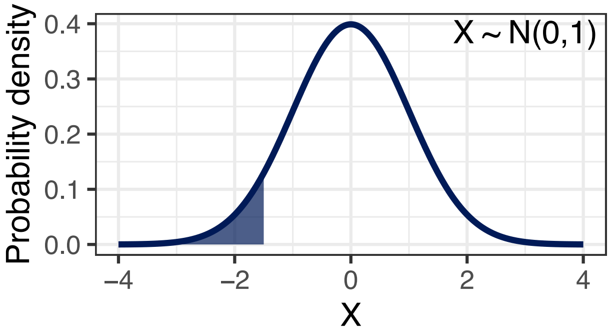
P(x ≤ -1.5) & P(x ≥ 1.5) = ???



example

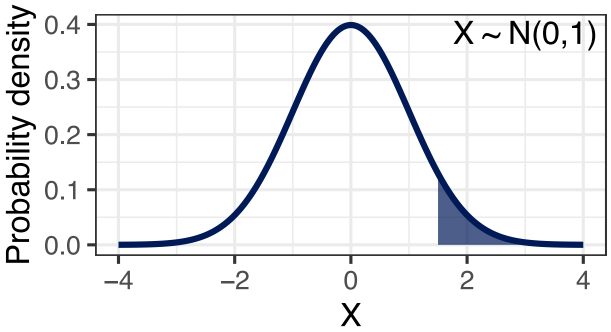
P(*x* ≤ -1.5) = 0.067

```
pnorm(q=-1.5, mean=0, sd=1)
```



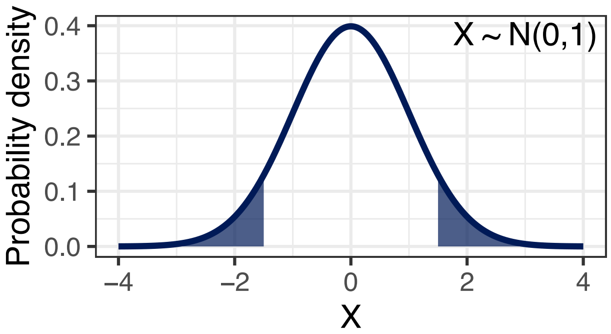
P(*x* ≥ 1.5) = 0.067

```
1 - pnorm(q=1.5, mean=0, sd=1)
```



P(*x* ≤ -1.5) & P(*x* ≥ 1.5) = 0.132

```
pnorm(q=-1.51, mean=0, sd=1) +
(1- pnorm(q=1.5, mean=0, sd=1))
```

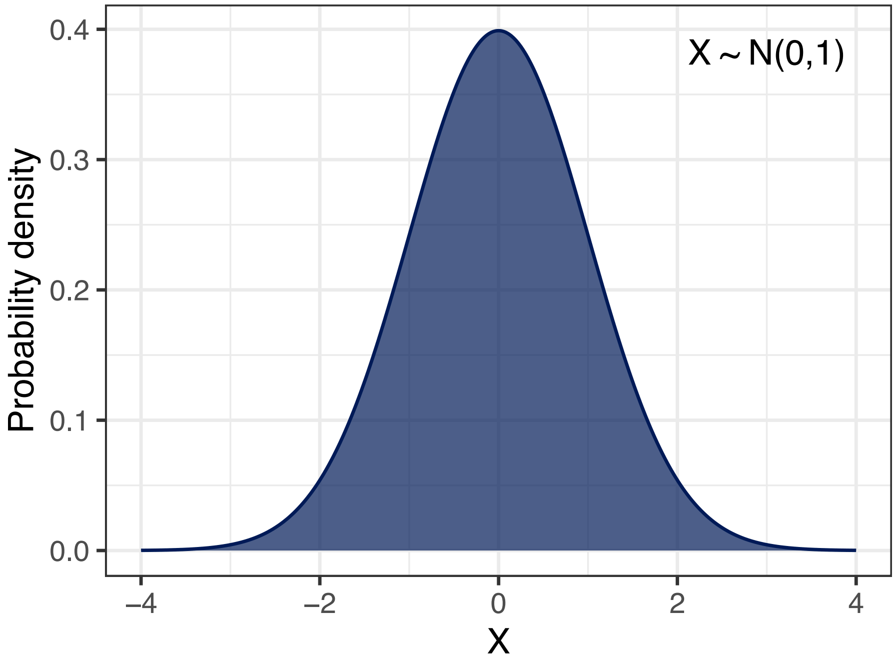


standard normal distribution

Z-distribution

$$Z \sim N(0,1)$$

$$E(Z) = 0 \text{ and } \sigma(Z) = 1$$

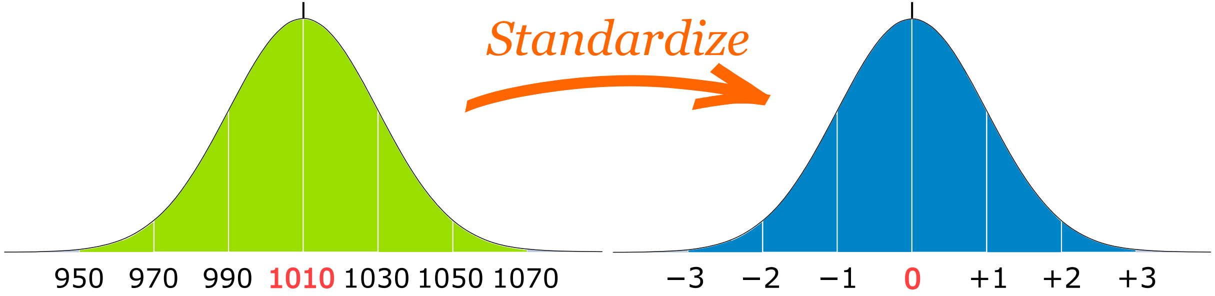


Z-score

Z-score reflects the number of σ's above or below the mean

$$Z = \frac{x - \mu}{\sigma}$$

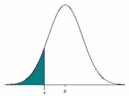
- convert any normal curve to the standard normal curve



why transform to z-distribution?

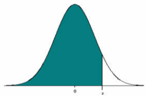
- calculate the probability of scores using the standard normal distribution (a known distribution)
- enables the comparison of two scores from different samples (which may have different means and standard deviations)

Table of Standard Normal Probabilities for Negative Z-scores



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Table of Standard Normal Probabilities for Positive Z-scores

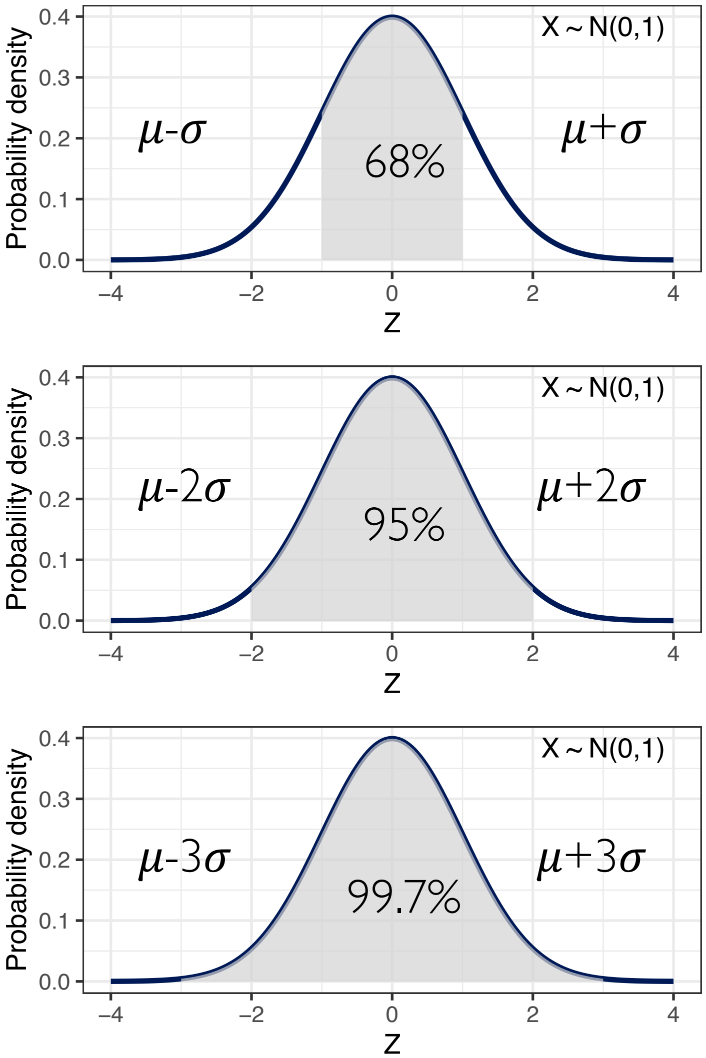


z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Note that the probabilities given in this table represent the area to the LEFT of the z-score.
The area to the RIGHT of a z-score = 1 – the area to the LEFT of the z-score

68-95-99.7 Rule

shorthand to remember the percentage of values that lie within a band around the mean in a normal distribution



example

if you scored a 70 on a test with a mean of 50 and a σ of 10, what's the probability of obtaining a score of 70 or higher?

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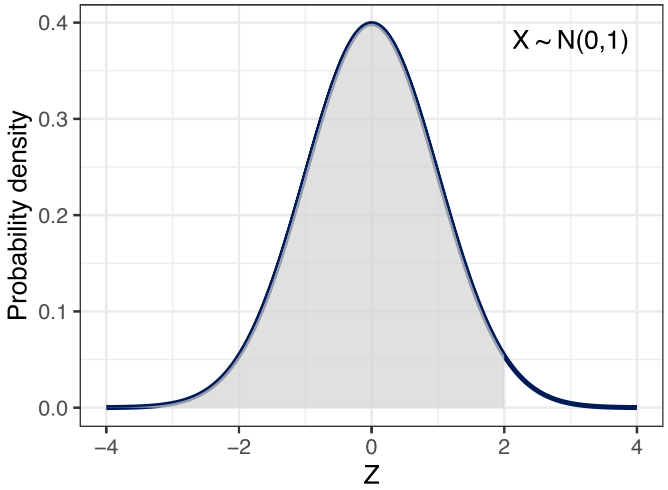
if you scored a 70 on a test with a mean of 50 and a σ of 10, what's the probability of obtaining a score of 70 or higher?

calculate the Z-score

$$Z = \frac{X - \mu}{\sigma} = \frac{70 - 50}{10} = 2$$



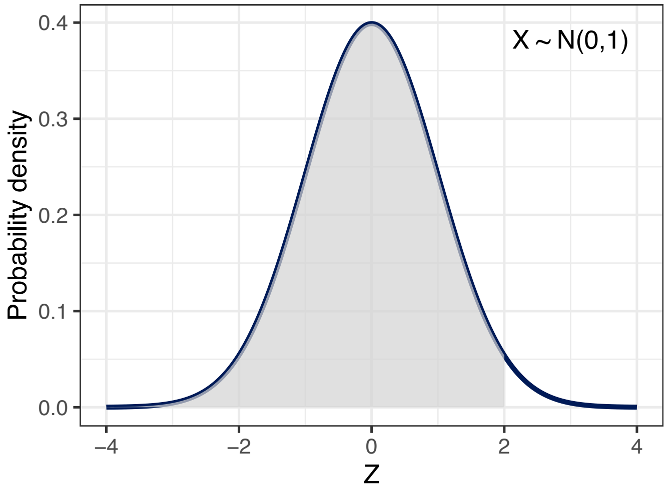
example

if you scored a 70 on a test with a mean of 50 and a σ of 10, what's the probability of obtaining a score of 70 or higher?

find the probability of obtaining a z-score of 2 or higher

```
1 - pnorm(q=2, mean=0, sd=1)
```

$P(z) \geq 2 = 0.023$



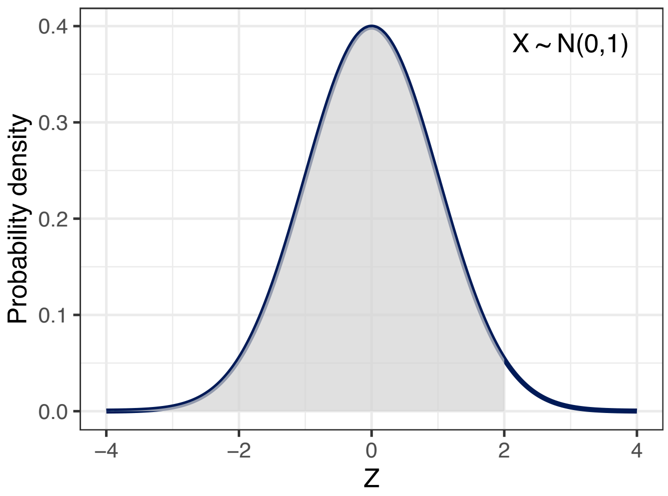
example

if you scored a 70 on a test with a mean of 50 and a σ of 10, what's the probability of obtaining a score of 70 or higher?

we could have calculated
the probability without a z-score

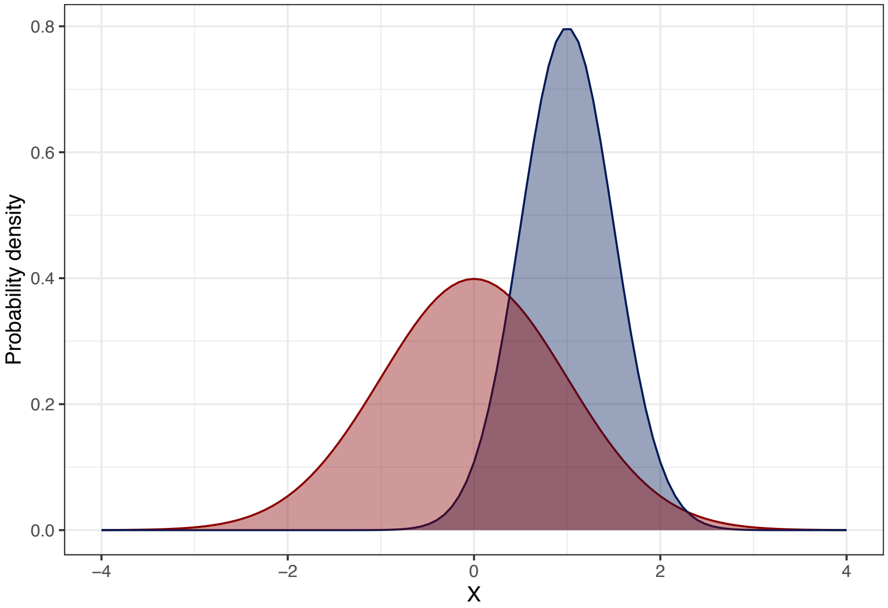
```
1 - pnorm(q=70, mean=50, sd=10)
```

$P(X) \geq 70 = 0.023$



useful properties…

- normal distributions can be transformed through shifts or "change of scale" operations
- normal distributions can be added
- transformation to standard normal distribution



Post your questions to be answered during lecture