Student Research Group 'Stochastic Volatility Models', Project 'Heston−2'

Exotics Pricing via the Simulation of the Heston Model

Artemy Sazonov, Danil Legenky, Kirill Korban
Lomonosov Moscow State Univesity, Faculty of Mechanics and Mathematics

February 25, 2023

Heston Model Definition

Assume that the spot asset at time t follows the diffusion

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dZ_1(t), \tag{1}$$

$$dv(t) = \left(\delta^2 - 2\beta v(t)\right) dt + \sigma\sqrt{v(t)}dZ_2(t), \tag{2}$$

where Z_1, Z_2 are the correlated Wiener processes with $dZ_1 dZ_2 = \rho dt$.

Outline

Lemma 1

Let $X_1, X_2, \ldots, X_n$ be a series of independent and identically distributed random variables, and $h : \mathbb{R} \to \mathbb{R}$ be a borel function. Then $h(X_1), h(X_2), \ldots, h(X_n)$ is a series of independent and identically distributed random variables.

Thus, we could write an unbiased consistent estimator of $\mathbb{E}\left[h(X)\right]$ as follows:

$$\widehat{\mathbb{E}\left[h(X)\right]} = \frac{1}{n}\sum_{i=1}^{n} h(X_i). \tag{3}$$

Monte Carlo Simulation

Local Truncation Error

Definition 2

Monte Carlo simulation is a set of techniques that use pseudorandom number generators to solve problems that might be too complicated to be solved analytically. It is based on the central limit theorem.

Asymptotic confidence interval for $\hat{\mu} = \widehat{\mathbb{E}[X]}$ at the confidence level α:

$$\mu \in \left(\hat{\mu} - z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}, \hat{\mu} + z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}\right). \tag{4}$$

That means that the estimation error is equal to $2z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}$.

Discretization Schemes for SDEs

Strong and weak convergence as a global truncation error analogue

Definition 3

Let $\hat{X}^n(t)$ be a mesh approximation of an SDE solution $X(t)$ (we assume that there exists a unique strong solution). Then a scheme is said to have a strong convergence of order p if

$$\mathbb{E}\left[\left|\hat{X}^n(T) - X(T)\right|\right] \leq Ch^p, \quad n \to \infty. \tag{5}$$

A scheme is said to have a weak convergence of order p if for any polynomial $f : \mathbb{R} \to \mathbb{R}$ we have

$$\left|\mathbb{E}\left[f(\hat{X}^n(T))\right] - \mathbb{E}\left[f(X(T))\right]\right| \leq Ch^p, \quad n \to \infty. \tag{6}$$

Outline

Euler Scheme for the Heston Model

Modified Euler-Maruyama Discretization Scheme

Suppose we have the Heston model (1) – (2). Then it could be numerically solved by the following finite difference scheme (for the log-prices $X(t)$):

$$X_{n+1} = X_n + (\mu - 0.5v_n^+)h_n + \sqrt{v_n^+}\sqrt{h_n}Z_{1,n}, \tag{7}$$

$$v_{n+1} = v_n + \left(\delta^2 - 2\beta v_n^+\right) h_n + \sigma\sqrt{v_n^+}\sqrt{h_n}Z_{2,n}, \tag{8}$$

and then we take the exponential of the log-prices:

$$S_n = S_0 e^{X_n}. \tag{9}$$

However, the scheme is not accurate, since we ignore the $dZ_i dZ_j$ terms in the Itô-Taylor series approximation.

Outline

Quadratic-Exponential Discretization Scheme

We denote

$$m = \mathbb{E}\left[\hat{V}(t+\Delta)\middle|\hat{V}(t)\right], \tag{10}$$

$$s^2 = \mathbb{E}\left[\left(\hat{V}(t+\Delta) - m\right)^2\middle|\hat{V}(t)\right], \tag{11}$$

$$\psi = \frac{s^2}{m^2}. \tag{12}$$

Quadratic-Exponential Discretization Scheme

Idea

Andersen proposes an approximation based on moment-matching techniques. His goal is then to speed up the first step of Broadie and Kaya's method. He observes that the conditional distribution of $\hat{V}(t+\Delta)$ given $\hat{V}(t)$ visually difers when $\hat{V}(t)$ is small or large (in the variation coefficient sense). The scheme is constructed from the following two subschemes:

1. Quadratic sampling scheme ($\psi \leq 2$);
2. Exponential sampling scheme ($\psi \geq 1$).

Fortunately, these two intervals cover the whole positive real line. Furthermore, these two schemes could be applied at the same time when $\psi \in [1, 2]$. This implicates that there exist some critical value $\psi_{\text{crit}} \in [1, 2]$, which could be an indicator of which scheme is more applicable at the given value of ψ. Let us show you this.

Quadratic-Exponential Discretization Scheme

Quadratic case

For large enough $\hat{V}(t)$ we can approximate the distribution of $\hat{V}(t+\Delta)$ by the scaled non-central chi-squared distribution with 1 degree of freedom:

$$\mathrm{Law}\left(\hat{V}(t+\Delta)\middle|\,\hat{V}(t)\right) = a(\Delta, \hat{V}(t), VP)\chi_1'^2(b(\Delta, \hat{V}(t), VP)), \qquad (13)$$

where VP is the vector of parameters of the CIR variance. However, if $\hat{V}(t)$ is close to zero, then we have a problem in finding such $a = a(\Delta, \hat{V}(t), VP)$ and $b = b(\Delta, \hat{V}(t), VP)$ such that the moments of the desired conditional distribution could be properly matched.

Quadratic-Exponential Discretization Scheme

Exponential case

Therefore, we approximate the desired distribution with the following method. Let ξ and η be independent random variables and $\xi \sim Be(1-p)$, $\eta \sim Exp(\beta)$ for some $p \in (0,1)$ and $\beta > 0$. Then we have (given $\hat{V}(t)$)

$$\hat{V}(t+\Delta) = \xi \cdot \eta. \tag{14}$$

Sampling ξ and η: Smirnov's transform. Or we can use the Smirnov transform with the cdf of the desired distribution.

Truncated Gaussian Discretization Scheme

Idea

Andersen:

In this scheme the idea is to sample from a moment-matched Gaussian density where all probability mass below zero is inserted into a delta-function at the origin.

Same, but in the formular form:

$$\left(\hat{V}(t+\Delta)\middle|\, V(t)\right) = (\mu + \sigma Z)^{+}\,, \tag{15}$$

where Z is a standard normal random variable and μ and σ are the 'mean' and the 'standard deviation' of the desired distribution. We find μ and σ from the same old moment-matching techniques (see Slide 9).

Outline

Outline

Conclusion

1. We introduced the three common Heston simulation methods: Euler-Maruyama, Andersen TG and Andersen QE;
2. We compared the theoretical vanilla options prices and their Monte-Carlo counterparts;
3. We measured the performance while pricing the exotics.

To-dos

1. Implement the Exact (Broadie and Kaya) scheme;
2. Measure the presiceness of the pricers for the real market data;
3. Part of the pricing library.