Monte-Carlo Pricing under the Heston Model

Artemy Sazonov, Danil Legenkiy, Kirill Korban

Student Research Group «Stochastic Volatility Models»

Supervisors: Mikhail Zhitlukhin, Charles-Henri Roubinet

Abstract

This paper uses Monte-Carlo simulations to evaluate the pricing of exotic options under the Heston model. We apply the Euler-Maruyama scheme, along with the Andersen Quadratic-Exponential and Andersen Truncated Gaussian schemes to perform the simulations. We compare the numerical precision of the simulated European call option prices with the closed-form solutions from the Heston model. Our investigation of numerical experiments reveal that the most efficient discretization scheme is the QE one.

Objectives

- Implement the Numba/CUDA Euler and Andersen QE & TG schemes;
- Implement the pricing interface;
- Implement the variance reduction methods;
- Compare the European call option prices from the [Hes93] and MC-simulated prices.

Problem statement

The main problem with the Heston model is that there are only a few derivative and structural contract types for which the *close-form prices* exist. Therefore, we need to think of a Monte-Carlo discretization method which is both computationally efficient and accurate enough.

The central motivation of this research: *there exists a trade-off between the efficiency and accuracy of a scheme*. Furthermore, we have no way to compare these two factors of the scheme for anything but the vanilla or digital European options. Once we establish the optimal scheme, we can use it to price the exotics.

Schemes

Modified Euler Scheme

$$X_{n+1} = X_n + (\mu - 0.5 v_n^+) h_n + \sqrt{v_n^+}\sqrt{h_n}\, Z_{1,n}, \tag{1}$$

$$v_{n+1} = v_n + \kappa\,(\bar{v} - v_n^+)\, h_n + \gamma\sqrt{v_n^+}\sqrt{h_n}\, Z_{2,n}. \tag{2}$$

Truncated Gaussian Scheme

$$(v_{n+1}|\, v_n) \overset{\text{law}}{=} (\mu(\Delta, v_n) + \sigma(\Delta, v_n) Z)^+, \tag{3}$$

where Z is a standard normal random variable and μ and σ are the 'mean' and the 'standard deviation' of the desired distribution.

Schemes

Quadratic-Exponential Scheme Let $\psi = \frac{m^2}{\sigma^2}$.

- $\psi > \psi_c$: Quadratic case

$$(v_{n+1}|\, v_n) \overset{\text{law}}{=} a(\Delta, v_n)\,(b(\Delta, v_n) + Z)^2; \tag{4}$$

- $\psi \leq \psi_c$: Exponential case

$$\mathbb{P}\,(v_{n+1} \in [x, x+dx]|\, v_n) = \big(p(\Delta, v_n)\delta(0) + \beta(\Delta, v_n)(1 - p(\Delta, v_n))e^{-\beta(\Delta, v_n)x}\big)\, dx. \tag{5}$$

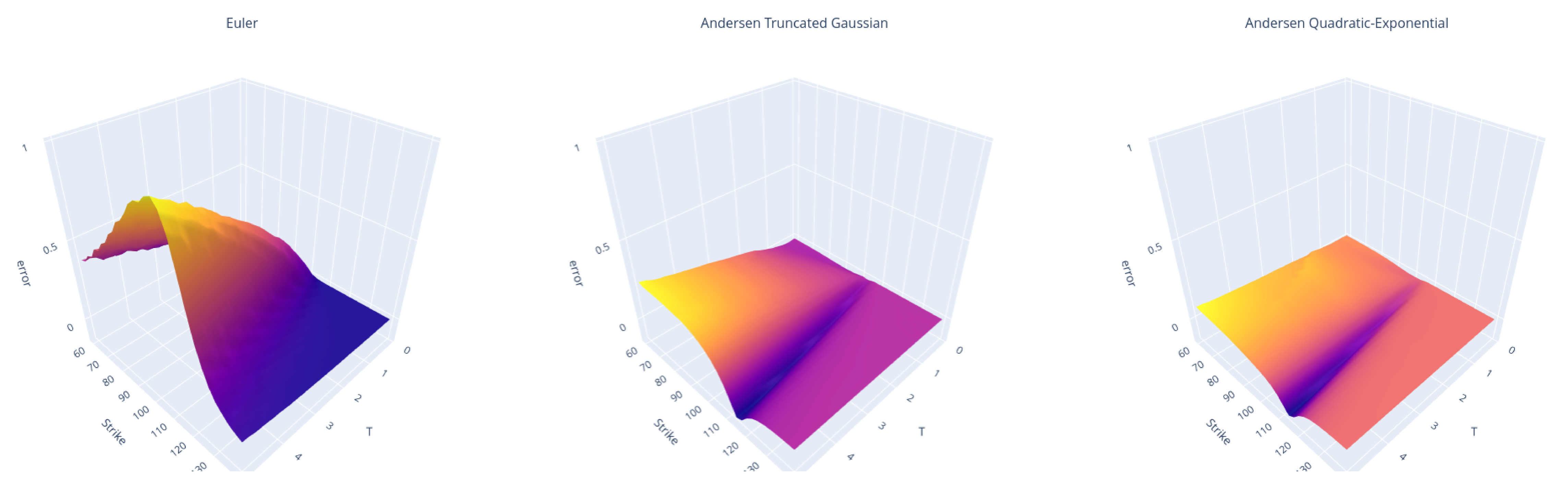

Figure 1: Absolute errors of the schemes in the following order: Euler, TG, QE.

Antithetic Variates

Suppose that we have two correlated identically distributed samples Y^1 and Y^2: $\operatorname{cov}[Y_i^1, Y_j^2] = \delta_{ij}\operatorname{cov}[Y_i^1, Y_i^2]$. Then we could introduce the following estimator:

$$\hat{\theta}_{\mathrm{AV}} = \frac{\bar{Y}^1 + \bar{Y}^2}{2}. \tag{6}$$

Again, we can see that this estimator is unbiased and consistent. The variance of this estimator is

$$\operatorname{var}\hat{\theta}_{\mathrm{AV}} = \frac{1}{4}\operatorname{var}[\bar{Y}^1] + \frac{1}{4}\operatorname{var}[\bar{Y}^2] + \frac{1}{2}\operatorname{cov}[\bar{Y}^1, \bar{Y}^2].$$

Thus, the variance reduction effect takes place when $\rho < 0$. If $Y^1 = g(U)$, then its antithetic variate is $Y^2 = g(1-U)$, where $U \sim U[0,1]$.

Control Variates

Suppose that we have another random variable Z that is correlated with Y and $\mathbb{E}[Z] = \mu$ is known. Then we could introduce the following estimator:

$$\hat{\theta}^b = \bar{Y} + b(\bar{Z} - \mu), \tag{7}$$

where b is a constant. Obviously, $\hat{\theta}^b$ is a consistent unbiased estimator of θ. How do we choose b? We need to minimize the variance of $\hat{\theta}^b$. A simple unconstrained optimization problem:

$$\operatorname{var}\hat{\theta}^b = \operatorname{var}\bar{Y} + b^2\operatorname{var}\bar{Z} - 2b\operatorname{cov}[\bar{Y}, \bar{Z}] \to \min_b.$$

The solution is

$$b^* = \frac{\operatorname{cov}[Y, Z]}{\operatorname{var} Z}. \tag{8}$$

Results

Accuracy and Performance: Euler overprices the calls, QE and TG underprice the calls. See Fig. 1.

Scheme	N_T	Time	Error
Euler	110	1.365	0.108692
Truncated Gaussian	10	0.131	-0.077350
Truncated Gaussian	110	1.525	-0.002841
Quadratic-Exponential	10	0.139	-0.107819
Quadratic-Exponential	110	1.590	-0.011618

Table 1: Accuracy-Performance comparison.

Antithetic Variates

For various combinations of both the Heston parameters and the discretization parameters, we have found that the variance reduction effect is approximately around 40% when only 1 random variable has an antithetic one, and up to around 90% for the AV estimator with 2 degrees of freedom.

Control Variates

We have found out that with the AV estimator with 2 degrees of freedom using the CV estimator speeds up the code by around 200% (while pricing the Asian options).

References

[Hes93] Steven L. Heston. "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options". In: *Review of Financial Studies* 6.2 (1993), pp. 327–343.

[And07] Leif Andersen. "Efficient Simulation of the Heston Stochastic Volatility Model". In: (2007).

[Gla13] Paul Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer New York, 2013. ISBN: 978-0-387-21617-1.

[KK22] Sergey Kobelkov and Yerkin Kitapbayev. *Numerical Methods in Finance I*. Vega Institute Foundation. 2022.