Student Research Group Report

Pricing under Rough Volatilty Models

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Abstract

In the present paper we investigate the roughness of the Russian stock market. In order to do this, we study the behavior of the Zumbach effect in real market data and non-rough stochastic volatility model Monte-Carlo simulations. After that we study the RFSV model and we obtain the estimation of the Hurst parameter for the major Russian corporations. Futhermore, we investigate the sample normalized variation statistic and see that roughness could vary depending on estimation of volatility.

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Introduction

One of the most famous models of mathematical finance was introduced by F. Black and M. Sholes in 1973's article [BS73], and a similar model for forward prices introduced in 1976 by F. Black in [Bla76]. Later there were invented some local volatility models, and stochastic volatility models (Heston, Hull and White, SVI, SABR etc.), but they still were not a perfect fit for pricing, even when first LSVMs were introduced.

Fractional Brownian motions were employed in volatility modelling by F. Comte and E. Renault in [CR98]. Their model (called FSV) used a fractional Brownian motion with Hurst parameter $H > 0.5$ to model volatility as a long-memory process i.e. one where autocorrelation decays slowly, which used to be a widely accepted stylized fact. They thus introduced the class of fractional stochastic volatility models.

In 2014, J. Gatheral, T. Jaisson, and M. Rosenbaum showed in [GJR14] that for major American indices Hurst parameter estimations are consistently less than 0.5, They called the corresponding model (FSV, $H < 0.5$) a rough fractional stochastic volatility model (RFSV) to emphasise that the volatility is indeed rough.

However, their approach requires the use of a model, therefore, it is not perfect still. In 2022, R. Cont and P. Das [CD22] proposed a method of estimating the roughness of an asset without the need of a model, which can be used to find statistical evidence that volatility is rough even without RFSV.

In the present paper we show that the Hurst parameters of the major Russia-originated assets (stocks and depositary reciepts of Russian corporations) are less than 0.5 under RFSV, i.e. Comte and Renault's basic FSV model is not working well for the Russian stock markets, therefore, RFSV should be used instead.

Chapter 1

Basic Theoretical Aspects

1.1 Realized volatility

Consider a stochastic volatility model

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \tag{1.1.1}$$

where S_t is an asset price process, and σ_t is a stochastic volatility process representing a so-called *spot volatility*. Spot volatility, in fact, is not observable in the market, therefore, we should estimate it somehow.

Definition 1.1.1. *The* realized variance *of a price process S over time interval $[t, t+\delta]$ sampled along the time partition π^n is defined as*

$$RVar_{t,t+\delta}(\pi^n) = \sum_{\pi^n \cap [t,t+\delta]} \left(\log S_{t^n_{i+1}} - \log S_{t^n_i} \right)^2, \tag{1.1.2}$$

and realized volatility *is defined as*

$$RV_{t,t+\delta}(\pi^n) = \sqrt{\sum_{\pi^n \cap [t,t+\delta]} \left(\log S_{t^n_{i+1}} - \log S_{t^n_i} \right)^2}. \tag{1.1.3}$$

As pointed out in [Tha], realized volatility has some limitations:

1. The volume of data used influences the end results during the calculation of realized volatility. At least 20 observations are statistically required to calculate a valid value of realized volatility. Therefore, realized volatility is better used to measure longer-term price risk in the market ($\sim$ 1 month or more).

2. Realized volatility calculations are directionless. i.e., it factors in upward and downward trends in price movements.

3. It is assumed that asset prices reflect all available information while measuring volatility.

Definition 1.1.2. *Let S satisfy* (1.1.1). *Then the integrated variance is defined as*

$$IVar_t = \int_0^t \sigma_s^2 ds. \tag{1.1.4}$$

It has been shown many times (e.g. [BS02]) and mentioned in [CD22] that the realized variance converges in probability to the integrated variance as sampling frequency increases for all assets satistying the equation (3.1.1) (i.e. stochastic volatility models).

Proposition 1.1.1. *As time partition scale of π^n tends to 0, $RV_{t,t+\delta}(\pi^n) \approx \sqrt{\delta}\sigma_t$, i.e. $RV_{t,t+\delta}/\sqrt{\delta}$ could be considered as a consistent estimator of the spot volatility.*

1.2 Fractional Stochastic Processes

Definition 1.2.1. *The* fractional Brownian motion $(W_t^H)_{t\in\mathbb{R}_+}$ *with Hurst parameter $H \in (0,1)$ is a Gaussian process with the following properties:*

1. $W_0^H = 0$,
2. $\mathbb{E}\left[W_t^H\right] \equiv 0$,
3. $\mathbb{E}\left[W_s^H W_t^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$.

Definition 1.2.2. *A stationary fOU process X_t is defined as the stationary solution of the stochastic differential equation*

$$dX_t = \nu dW_t^H - \alpha(X_t - m)dt, \tag{1.2.1}$$

where $m \in \mathbb{R}$ and ν and α are positive parameters, see [CKM03].

Definition 1.2.3. *Let us define $\Delta_h f(x) := f(x-h) - f(x)$ and let us define the modulus of continuity by*

$$\omega_p^2(f,t) = \sup_{|h|\leq t} \left\|\Delta_h^2 f\right\|_p. \tag{1.2.2}$$

Let n be a non-negative integer and $s = n + \alpha$ with $\alpha \in (0,1]$. The Besov space $B_{p,q}^s(\mathbb{R})$ *contains all functions $f \in W^{n,p}(\mathbb{R})$ such that*

$$\int_0^\infty \left|\frac{\omega_p^2(f^{(n)},t)}{t^\alpha}\right|^q \frac{dt}{t} < \infty. \tag{1.2.3}$$

The Besov space $B_{p,q}^s(\mathbb{R})$ is a normed space with the standard norm defined as

$$\|f\|_{B_{p,q}^s(\mathbf{R})}^q = \|f\|_{W^{n,p}(\mathbb{R})}^q + \int_0^\infty \left|\frac{\omega_p^2(f^{(n)},t)}{t^\alpha}\right|^q \frac{dt}{t}. \tag{1.2.4}$$

1.2.1 What is a long-memory process?

Definition 1.2.4. *A process X_t is said to have a long memory, if*

$$\sum_{k=0}^{\infty} \operatorname{cov}\left[X_1, X_k - X_{k-1}\right] = \infty. \tag{1.2.5}$$

In particular, the fractional Brownian motion with $H > \frac{1}{2}$ is a long-memory process. Long-memory of the stochastic volatility process in stochastic volatilty models framework used to be a widely-accepted stylized fact [BCD98; CR98; CR96; DGE93].

1.3 Normality Statistical Tests

In the following, x_i denotes a sample of n observations, g_1 and g_2 are the sample skewness and excessed kurtosis, μ_j's are the j-th sample central moments, and $\overline{x}$ is the sample mean.

1.3.1 D'Agostino's K-squared test

The sample skewness and kurtosis are defined as

$$g_1 = \frac{m_3}{m_2^{3/2}} = \frac{\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^3}{\left(\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2\right)^{3/2}}\,, \tag{1.3.1}$$

$$g_2 = \frac{m_4}{m_2^2} - 3 = \frac{\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^4}{\left(\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2\right)^{2}} - 3\,. \tag{1.3.2}$$

Let

$$Z_1(g_1) = \delta\,\mathrm{asinh}\left(\frac{g_1}{\alpha\sqrt{\mu_2}}\right), \tag{1.3.3}$$

where constants α and δ are computed as

$$W^2 = \sqrt{2\gamma_2 + 4} - 1, \tag{1.3.4}$$

$$\delta = 1/\sqrt{\ln W}, \tag{1.3.5}$$

$$\alpha^2 = 2/(W^2 - 1), \tag{1.3.6}$$

and

$$Z_2(g_2) = \sqrt{\frac{9A}{2}}\left\{1 - \frac{2}{9A} - \left(\frac{1 - 2/A}{1 + \frac{g_2 - \mu_1}{\sqrt{\mu_2}}\sqrt{2/(A-4)}}\right)^{1/3}\right\}, \tag{1.3.7}$$

where

$$A = 6 + \frac{8}{\gamma_1}\left(\frac{2}{\gamma_1} + \sqrt{1 + 4/\gamma_1^2}\right), \tag{1.3.8}$$

$$\mu_1(g_2) = -\frac{6}{n+1}, \tag{1.3.9}$$

$$\mu_2(g_2) = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)}, \tag{1.3.10}$$

$$\gamma_1(g_2) \equiv \frac{\mu_3(g_2)}{\mu_2(g_2)^{3/2}} = \frac{6(n^2 - 5n + 2)}{(n+7)(n+9)}\sqrt{\frac{6(n+3)(n+5)}{n(n-2)(n-3)}}, \tag{1.3.11}$$

$$\gamma_2(g_2) \equiv \frac{\mu_4(g_2)}{\mu_2(g_2)^2} - 3 = \frac{36(15n^6 - 36n^5 - 628n^4 + 982n^3 + 5777n^2 - 6402n + 900)}{n(n-3)(n-2)(n+7)(n+9)(n+11)(n+13)}. \tag{1.3.12}$$

The analytical expressions for skewness and kurtosis (1.3.11) - (1.3.12) were derived by E. Pearson in [Pea31].

Definition 1.3.1. *The* D'Agostino–Pearson *statistic is defined as*

$$K^2 = Z_1(g_1)^2 + Z_2(g_2)^2 \tag{1.3.13}$$

H_0: the sample is normally distributed.

Remark. The K^2 statistic is able to detect deviations from both skewness and kurtosis. If the null hypothesis is true, then the test statistic has the χ^2 distribution with 2 degrees of freedom.

1.3.2 Shapiro–Wilk test

Definition 1.3.2. *The Shapiro–Wilk test statistic is defined as*

$$W = \frac{\left(\sum_{i=1}^{n} a_i x_{(i)}\right)^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2}, \tag{1.3.14}$$

where

$$(a_1, \ldots, a_n) = \frac{m^T V^{-1}}{C}, \quad C = \|V^{-1}m\| = (m^T V^{-1} V^{-1} m)^{1/2},$$

and $m = (m_1, \ldots, m_n)^T$ is a mean of order statistic from a normally distributed sample, V is the covariance matrix of those normal order statistics H_0: the sample is normally distributed.

Remark. The W statistic has no distinguishable name, and the cutoff values are calculated numerically by Monte-Carlo simulation.

Chapter 2

Zumbach Effect Estimation

2.1 Zumbach Effect definition and description

There are empirical features of financial time series that are not well replicated by conventional stochastic volatility models. One of them is the **Zumbach effect**. First, consider the statistic

$$\tilde{\mathcal{C}}^{(2)}(\tau) = \left\langle \left(\sigma_t^2 - \left\langle \sigma_t^2 \right\rangle\right) r_{t-\tau}^2 \right\rangle \tag{2.1.1}$$

where r_t is the open to close return of day t, σ_t is the integrated variance and $\langle \cdot \rangle$ is a sample average. We see that this is the covariance of integrated variance with past squared returns. If we substitute τ with $-\tau$, we will get

$$\tilde{\mathcal{C}}^{(2)}(-\tau) = \left\langle \left(\sigma_t^2 - \left\langle \sigma_t^2 \right\rangle\right) r_{t+\tau}^2 \right\rangle, \tag{2.1.2}$$

which is, similarly, the covariance of integrated variance with future squared returns.

Now we can consider the measure of rime-reversal asymmetry (TRA) given by

$$Z(\tau) := \tilde{\mathcal{C}}^{(2)}(\tau) - \tilde{\mathcal{C}}^{(2)}(-\tau), \quad \tau > 0 \tag{2.1.3}$$

This measure is empirically found to be positive, which can be interpreted the following way: on real data, the covariance between historical squared returns and future integrated variance is greater than the covariance between historical integrated variance and future squared returns.

On the contrary, classic continuous time stochastic volatility models, such as the Heston model, obey TRS (time-reversal symmetry) by construction and therefore cannot account for the empirical TRA of financial time series [BDB17].

We can convert $\tilde{\mathcal{C}}^{(2)}(\tau)$, which is a covariance, into correlation:

$$\tilde{\rho}(\tau) = \frac{\tilde{C}^{(2)}(\tau)}{\sqrt{\left\langle \left(\sigma_t^2 - \langle \sigma_t^2 \rangle\right)^2 \right\rangle \left\langle \left(r_{t-\tau}^2 - \left\langle r_{t-\tau}^2 \right\rangle\right)^2 \right\rangle}} \tag{2.1.4}$$

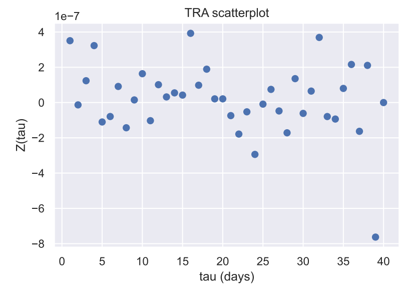
And then calculate the integrated difference (similarly to [CB14]) as another metric to estimate the Zumbach effect with:

$$\Delta(\tau) = \sum_{i=1}^{\tau} (\tilde{\rho}(i) - \tilde{\rho}(-i)) \tag{2.1.5}$$

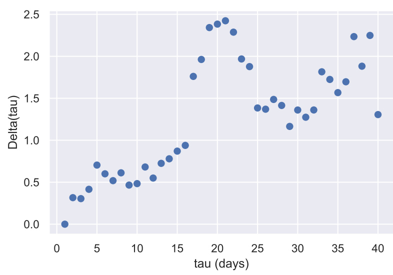
2.2 Empirical results on real data

For our empirical study, we use data on Russian stocks from Yahoo Finance (from 2021, October 7th to 2022, February 1st) and data on 31 different indices from Oxford-Man Institute of Quantitative Finance Realized Library (from 2000, January 3rd to 2020, March 3rd).

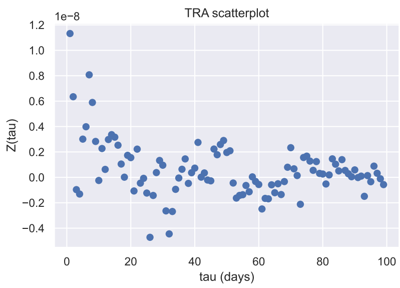
Empirical results show that most stocks and indices do exhibit Zumbach effect. In general, to conclude that the effect is observed, we need the plot of Δ to be strictly positive and have large intervals of increasing, and the points of scatter plot of Z to lie mostly above the zero line.



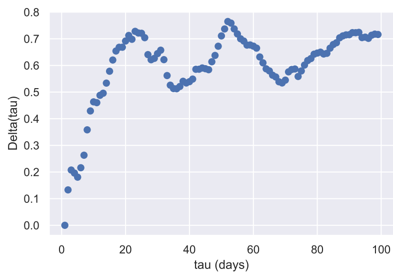
(a) $Z(\tau)$, YNDX



(b) $\Delta(\tau)$, YNDX



(c) $Z(\tau)$, .AEX



(d) $\Delta(\tau)$, .AEX

Figure 2.1: Plots of Δ and Z that exhibit Zumbach effect

It is worth saying that, of course, not all stocks and indices provide similar plots, some even showing the opposite, negative trend (all plots for other stocks and indices can be found in the appendix). Speaking about the data on Russian stocks from Yahoo Finance, we should be aware that the data is very limited, which makes it possible to estimate Δ only on the interval from 0 to 45, and this estimation can be unstable due to the low sample size.

For a stronger evidence of the existence of the Zumbach effect, we can average $\Delta(\tau)$ over all 31 indices from the Oxford-Man dataset. We will then obtain a plot very similar to the one in [El +18]:

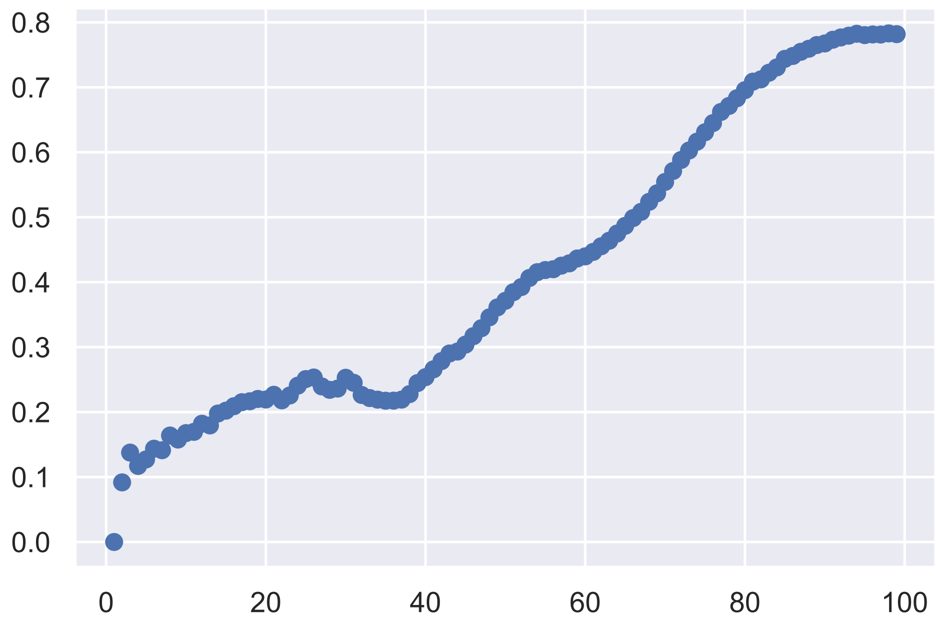


Figure 2.2: Averaged $\bar{\Delta}(\tau)$

2.3 Monte-Carlo simualtion of Heston SVM

First, let us revisit the classic Heston model. Price of the asset is given by the following process:

$$dS = \mu S dt + \sqrt{v} S dZ_1 \tag{2.3.1}$$

and the instantaneous variance is given by CIR process:

$$dv = -\lambda(v - \bar{v})dt + \eta\sqrt{v}dZ_2 \tag{2.3.2}$$

where Z_1 and Z_2 are Brownian motions and $\mathrm{cov}[dZ_1, dZ_2] = \rho dt$.

We can simulate those processes using Milstein discretization scheme:

$$v_{i+1} = v_i - \lambda\left(v_i - \bar{v}\right)\Delta t + \eta\sqrt{v_i}\sqrt{\Delta t}Z + \frac{\eta^2}{4}\Delta t\left(Z^2 - 1\right) \tag{2.3.3}$$

and then estimate Zumbach effect of this simulated data with the same methods we used before on real data.

After simulating 15 different paths, each path containing ~ 800000 points, with different parameters of correlation and mean reversion, we obtain plots that generally differ from those we observed on real data - all plots con be found in the appendix. Again, after taking the average over all 15 paths, we obtain the following figure: Thus, we can conclude that this feature is not

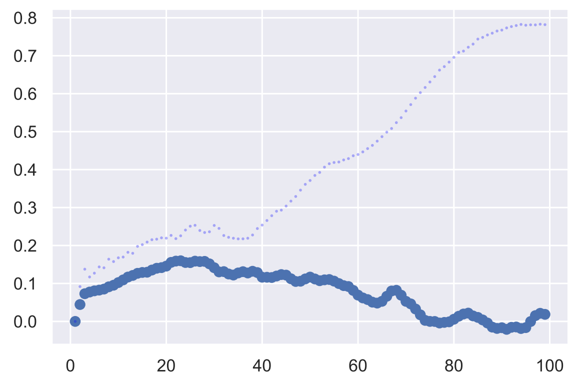


Figure 2.3: Averaged $\bar{\Delta}(\tau)$ on simulated data (with comparison to real data)

well replicated by classic Heston model.

We conclude that Zumbach effect is an argument for the roughness of volatility. Therefore, we should consider a model of volatility driven by fractional Brownian motion.

Chapter 3

Rough Fractional Stochastic Volatility Model Estimation

3.1 Model description

In [GJR14] the authors considered the following model. Let there be a riskless asset $B_t \equiv 1$, and a risky asset, whose price S_t is defined by the following equations:

$$dS_t = \alpha S_t dt + \sigma_t S_t dW_t, \tag{3.1.1}$$

$$d\log\sigma_t = \alpha(m - \log\sigma_t)dt + \nu dW_t^H. \tag{3.1.2}$$

The risky asset is being traded in the market in numeraire prices. In our case, $B_t = 1$ RUB for stocks and 1 GBP for depositary reciepts.

Definition 3.1.1. *A model* (3.1.1) – (3.1.2) *is called a* Fractional Stochastic Volatility Model *(FSV). For a special case $H < 0.5$ the model is called a* Rough Fractional Stochastic Volatility Model *(RFSV) to emphasise a so-called roughness of the trajectories of the fBm. As a stylized fact we shall demand the stationarity of log-increments.*

In [CKM03] an exact formula for the autocovariance function of the log-volatility in the RFSV model was derived:

$$\begin{aligned}&\operatorname{cov}[\log\sigma_t, \log\sigma_{t+\Delta}] = \\ &= \frac{H(2H-1)\nu^2}{2\alpha^{2H}}\left(e^{-\alpha\Delta}\Gamma(2H-1) + e^{-\alpha\Delta}\int_0^{\alpha\Delta}\frac{e^u}{u^{2-2H}}du + e^{\alpha\Delta}\int_{\alpha\Delta}^{\infty}\frac{e^u}{u^{2-2H}}du\right).\end{aligned} \tag{3.1.3}$$

Let $m(q, \Delta, \pi^n)$ be a sample q-th absolute moment of $\log RV_{t+\Delta} - \log RV_t$:

$$m(q, \Delta, \pi^n) := \frac{1}{n}\sum_t |\log RV_{t+\Delta} - \log RV_t|^q, \tag{3.1.4}$$

i.e. $m(q, \Delta, \pi^n)$ is an empirical counterpart of $\mathbb{E}[|\log RV_\Delta - \log RV_0|^q]$. In this work we shall use the uniform partition of time scale with each step being equal to 15 minutes, so we omit the π^n notation and use $m(q, \Delta)$. Via the explicit formula for the covariance function of the log-volatility in the RFSV model (3.1.3), we can write a closed-form expression for a theoretical $m(2, \Delta)$:

$$m(2, \Delta) = 2\left(\operatorname{var}\log\sigma_t - \operatorname{cov}[\log\sigma_t, \log\sigma_{t+\Delta}]\right). \tag{3.1.5}$$

3.2 Statistical Analysis

3.2.1 Data Preprocessing and Realized Volatility Estimation

In the present paper we used high-frequency data for three types of assets:

1. **Stocks**: Yandex, Sberbank, Gazprom, VTB, Moscow Exchange, Lukoil, and X5 Group;
2. **Depositary reciepts**: Sberbank, Gazprom, VTB, and Lukoil;
3. **Funds**: AEX, AORD, BFX, BVSP, DJI, FCHI, FTMIB, FTSE, GDAXI, GSPTSE, HSI, IBEX, IXIC, KS11, KSE, MXX, N225, OMXC20, OMXHPI, OMXSPI, OSEAX, RUT, SMSI, SPX, SSEC, SSMI.

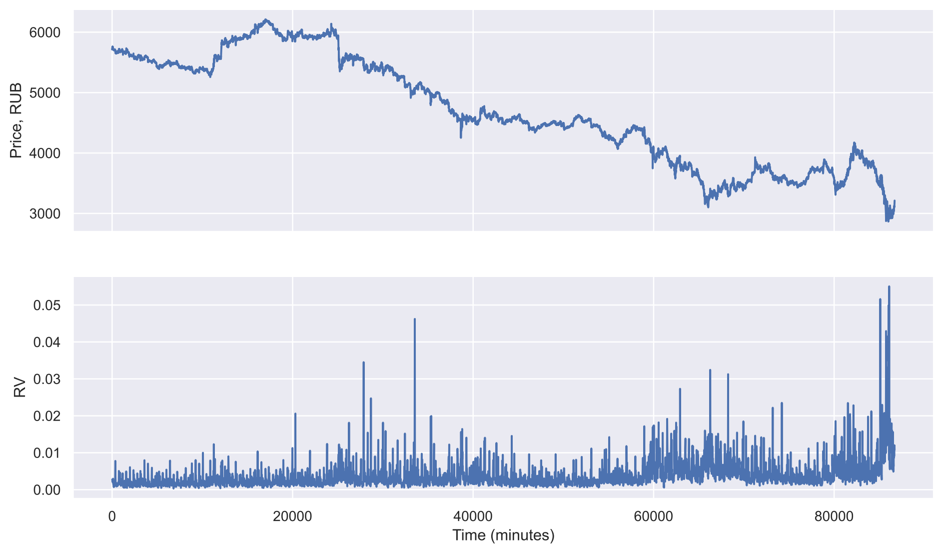


Figure 3.1: YNDX RX Equity. Price and Realized Volatility

Realized volatility is estimated by 15 minute disjoint windows (i.e. $\hat{RV}(t)$ is a piecewise constant function). Using this approach for the estimation, we can be sure that our data is correlated in the least way possible. We observe in the Figure 3.1 that as the price decreases, the rolling mean of realized volatility generally increases.

3.2.2 Hurst Parameter Estimation

Main assumption: for some $s_q > 0$, $b_q > 0$ and $N = \left[\frac{T}{\Delta}\right]$ (number of RV estimations via disjoint windows)

$$N^{qs_q} m(q, \Delta) \xrightarrow[\Delta \to 0+]{} b_q. \tag{3.2.1}$$

Under additional technical conditions equation (3.2.1) is equivalent to that the volatility process belongs to the Besov smoothness space $B^{s_q}_{q,\infty}$ and for all $\tilde{s}_q > s_q$ does not belong to $B^{\tilde{s}_q}_{q,\infty}$ [Ros08].

Due to the similarities in the obtained results for all assets, we shall deeply analyze the Hurst parameter estimation only for the Yandex stocks (YNDX RX Equity). Plots for other equities could be found in the appendix, whereas the Hurst parameter estimations for them could be found in the Table 3.1. Further in the paper we assume $\Delta = 1, \dots, 40$. It has been shown that under stationarity assumptions and linearity of Figure 3.2 (left)

$$\mathbb{E}\left[|\log \sigma_{t+\Delta} - \log \sigma_t|^q\right] = K_q \Delta^{\zeta_q}, \tag{3.2.2}$$

and the s_q does not depend on q. In the Figure 3.2 (right) we can see that for $q = 0.6, 0.8,$ and 1.0 the dots are very discrepant for $\log \Delta > 2.0$. However, we get a pretty decent linear fit for $q = 0.2$ and $q = 0.4$, therefore, the estimation on these two point would be the best one we can manage to extract. On the other hand, on ζ_q plot we observe a perfect linear fit for all q-s, therefore, H is its slope indeed. We note that the graphs for ζ_q are slightly concave, which correlates with

Figure 3.2: YNDX RX Equity. Plots for $\hat{H}$

[GJR14] results. They conclude that this effect takes place due to the finite statistical population size. It has been proven in [GJR14] that $\log \mathbb{E}[\sigma_t \sigma_{t+\Delta}]$ and $\log \operatorname{cov}[\log \sigma_t, \log \sigma_{t+\Delta}]$ are linear in Δ^{2H}. And we indeed observe this behaviour in the majority of plots (especially for $\Delta < 20$, where we have enough data to work with). Numerical instability occurs when Δ is too large due to the lack of HF data.

NB. We did not manage to obtain more HF data (only 5 months of 1m-tick data), therefore my estimations are not precise and could not be used for further application.

3.2.3 Smoothing Effect Estimation

Smoothing effect is throroughly discussed in the appendix of [GJR14].

We can clearly see that due to the positive slope of the plot 3.4, the hypothesis about increasing $\hat{H}$ and decreasing $\hat{\alpha}$ as δ increases is to be accepted.

3.2.4 Tests for normality of volatility's log-increments

In order to test the normality of the log-increments of the realized volatility, we used the following tests:

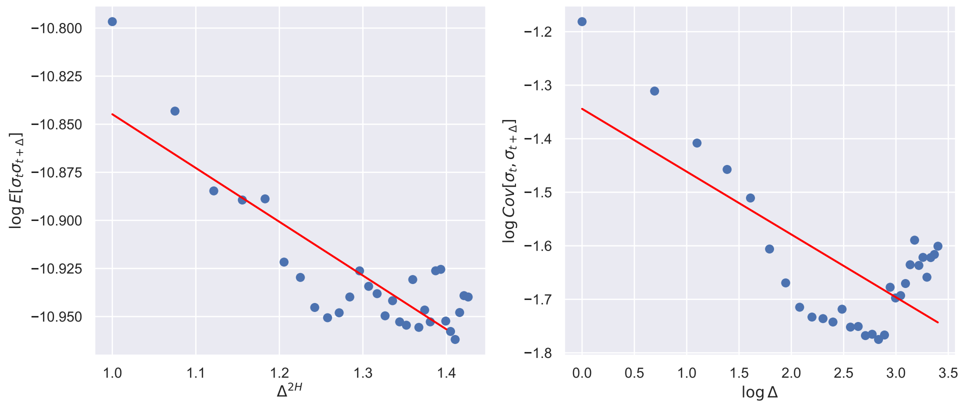


Figure 3.3: YNDX RX Equity. Empirical counterpart of $\log \mathbb{E}\left[\sigma_t \sigma_{t+\Delta}\right]$ as a function of Δ^{2H} (left) and Empirical counterpart of $\log \operatorname{cov}[\log \sigma_t, \log \sigma_{t+\Delta}]$ as a function of $\log \Delta$ (right)

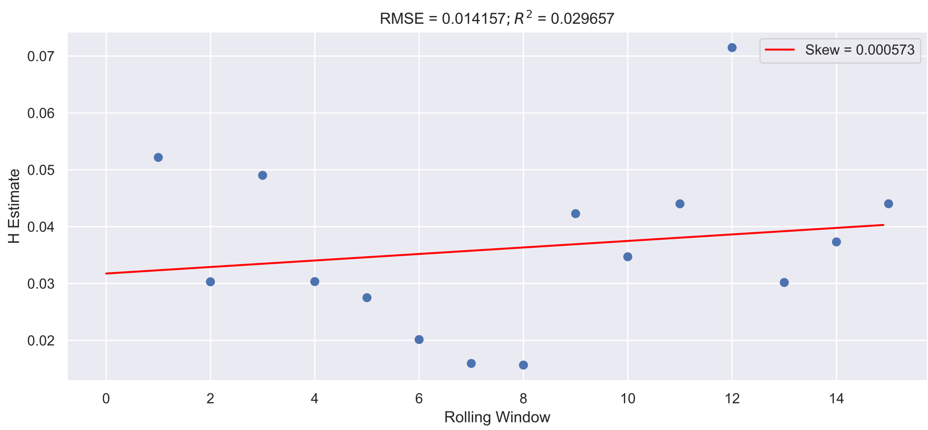


Figure 3.4: YNDX RX Equity. Smoothing Effect

Asset Type	Ticker	$\hat{H}$
Stock	YNDX	0.0521766
Stock	SBER	0.1551646
Stock	VTBR	0.0917236
Stock	MOEX	0.0853878
Stock	LKOH	0.0730521
Stock	GAZP	0.1309705
Stock	FIVE	0.0630289
Depositary reciept	OGZD	0.0523981
Depositary reciept	VTBR	0.0370185
Depositary reciept	SBER	0.0578053
Depositary reciept	LKOD	0.0352792
Index	AEX	0.1271101
Index	AORD	0.0731749
Index	BFX	0.1340391
Index	BVSP	0.1285106
Index	DJI	0.1176993
Index	FCHI	0.1300797
Index	FTMIB	0.139092
Index	FTSE	0.0958701
Index	GDAXI	0.1130176
Index	GSPTSE	0.0910194
Index	HSI	0.0893922
Index	IBEX	0.1028588
Index	IXIC	0.1278909
Index	KS11	0.1066547
Index	KSE	0.1080452
Index	MXX	0.0673153
Index	N225	0.1063503
Index	OMXC20	0.0997755
Index	OMXHPI	0.0954135
Index	OMXSPI	0.118664
Index	OSEAX	0.0987837
Index	RUT	0.1029421
Index	SMSI	0.1319457
Index	SPX	0.1328797
Index	SSEC	0.1170868
Index	SSMI	0.1469914

Table 3.1: Hurst parameter estimations

1. Visual analysis of histograms: KDE vs normal fit vs empirical fit
2. Visual analysis of excessed kurtosis plot
3. D'Agostino's K Squared normality test
4. Shapiro-Wilk normality test

In [GJR14] the authors used **only** the visual analysis of the histograms, which, as we can now say, is not surprising due to the inadequacy of results for other numerical experiments.

Visual analysis of histograms and excessed kurtosis plot

1. KDE is the *kernel density estimator* of the data.
2. *Normal fit* $NF(\Delta)$ is the normal distribution fitted to the data with the same mean and variance.
3. *Empirical fit* $EF(\Delta)$ is the scaled normal distribution:
   - $EF(1)$ is said to be same as the $NF(1)$
   - $EF(\Delta)$ for $\Delta > 1$ is said to be a scaled $NF(1)$ by the factor of $\Delta^{\hat{H}}$ (by this we test the monofractal scaling property of normal distribution)

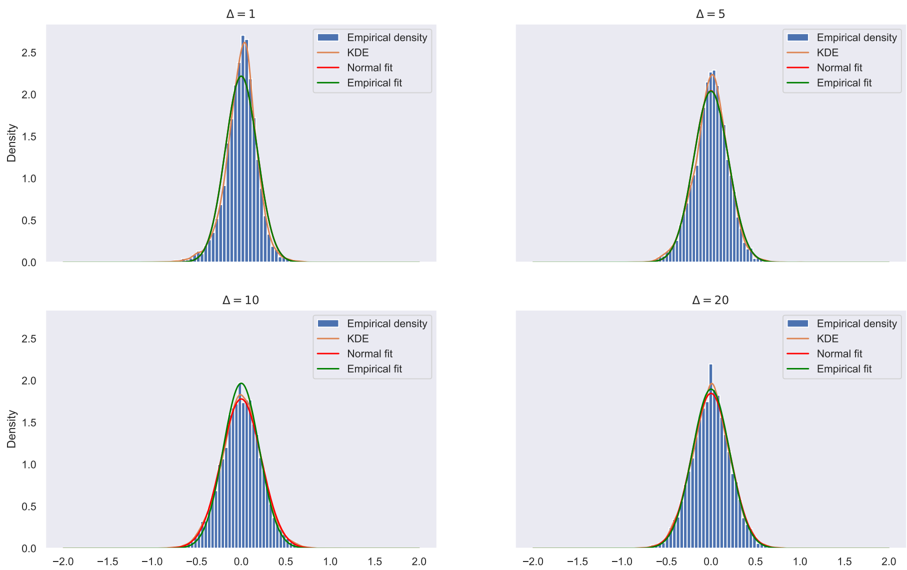


Figure 3.5: YNDX RX Equity. Empirical density of $\log \sigma_{t+\Delta} - \log \sigma_t$ for $\Delta = 1, 5, 10, 20$ days.

Looking at the figure 3.5, we may form a conclusion: KDE and EF are a decent normality approximations for $\Delta = 10, 20$. For others, we don't get a fancy picture: $KDE(1)$ and $KDE(5)$ have a large kurtosis (they are too 'peaky' for them to be normally distributed). Excessed curtosis plot 3.6 confirms our visual conclusion for KDE and EF plots.

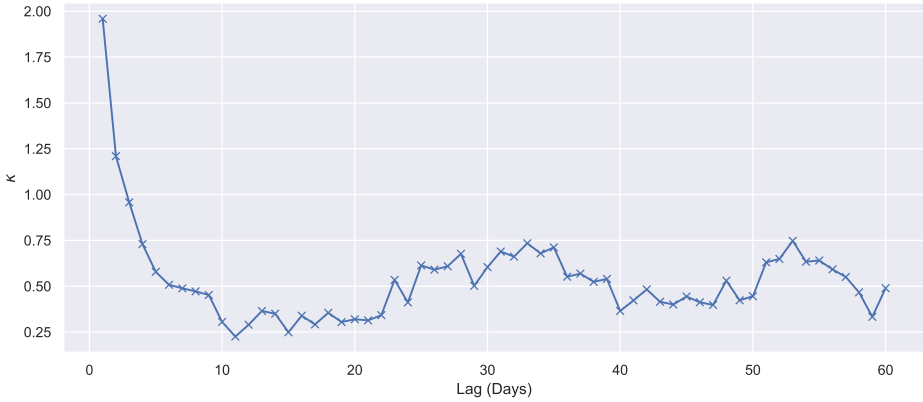


Figure 3.6: YNDX RX Equity. Excessed kurtosis κ as a function of Δ

Statistical tests for normality

We fix the confidence level to be $\alpha = 0.05$.

NB. Both of these tests require the data to be independent, but we cannot guarantee this due to the dependence of fBm's increments. We do our best to analyse the population, but these two tests give us weak proof of normality due to possible correlations.

Looking at the tables with the results of Shapiro-Wilk and D'Agostino's K-Squared tests (Tables 4.2 – 4.12), we can see that for the majority of lags and for the majority of the considered assets, both tests showed the result "Not normal", i.e. both tests rejected the null hypothesis.

The three possible explanations are:

1. The tests are correct and the data is not normally distributed or is correlated strongly.
2. The visual analysis of the histograms show that for many lags the KDE plot, the normal fit and the empirical fit are very similar, therefore, the distribution is normal, but the data is correlated strongly. The excessed kurtosis plot shows that the data is distributed very close to the normal distribution for $\Delta > 5$, and at its closest distance for $\Delta \in [10, 22]$.
3. We get a population sampling error (not enough data).

Chapter 4

Modelless Estimation of Roughness

4.1 Sample normalized variation as a measure of roughness

4.1.1 Theoretical parameters

Let us consider a sequence of partitions π^n of $[0,T]$ with $|\pi^n| := \max_{t_i^n \in \pi^n}(t_{i+1}^n - t_i^n) \to 0$.

Definition 4.1.1. *A function $x \in C[0,T]$ is said to have the finite p-th variation along the sequence of partitions π^n if there exists a continious increasing function $[x]_\pi^{(p)}$ such that for all subpartitions $\tilde{\pi}^n(t) = \pi^n \cap [0,t]$*

$$\sum_{t_i^n \in \tilde{\pi}^n(t)} \left| x(t_{i+1}^n) - x(t_i^n) \right|^p \to [x]_\pi^{(p)}(t), \quad n \to \infty, \tag{4.1.1}$$

and the set of all functions having finite p-th variation along π we denote V_π^p.

Definition 4.1.2. *The* variation index *of a path x is defined as $p^\pi(x) := \inf\{p \geq 1 : \quad x \in V_\pi^p\}$, and the* roughness index *is defined as $H^\pi(x) := \frac{1}{p^\pi(x)}$.*

It has been proven that for fBm with Hurst parameter H $p^\pi(W^H) = \frac{1}{H}$ and $H^\pi(W^H) = H$.

Definition 4.1.3. *$x \in V_\pi^p$ is said to have p-th normalized variation if there exists such continious function $w(x,p,\pi) \colon [0,T] \to \mathbb{R}$ that*

$$\sum_{\tilde{\pi}^n(t)} \frac{\left| x(t_{i+1}^n) - x(t_i^n) \right|^p}{[x]_\pi^{(p)}(t_{i+1}^n) - [x]_\pi^{(p)}(t_i^n)} (t_{i+1}^n - t_i^n) \to w(x,p,\pi). \tag{4.1.2}$$

From Theorem 2.4 [CD22], it is known that the variation index must be found as a solution of

$$w(x,p,\pi) = t. \tag{4.1.3}$$

4.1.2 The W Statistic

Let us consider π^L, π^K – partitions with sampling frequencies $L \gg K$ ($\pi^K \subset \pi^L$).

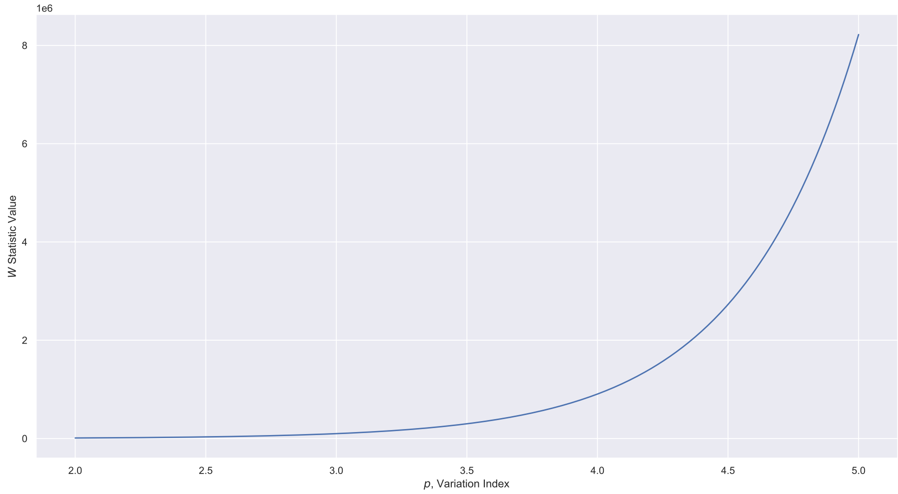


Figure 4.1: The W statistic illustration as a function of p

Definition 4.1.4. *Sample normalized p-th variation is defined as*

$$W(L,K,p,t,X) = \sum_{\tilde{\pi}^K(t)} \frac{\left|x(t_{i+1}^K) - x(t_i^K)\right|^p}{\sum_{t_i^n \in \tilde{\pi}^n(t)} \left|x(t_{i+1}^n) - x(t_i^n)\right|^p} (t_{i+1}^n - t_i^n) \tag{4.1.4}$$

4.2 Roughness estimation of Monte-Carlo simulations

4.2.1 Brownian and fractional Brownian motion

We shall test our method on those processes, whose roughness is well-known.

Brownian motion

We observe that the roughness index is equal to 0.4988, which is a pretty good approximation.

Fractional Brownian motion (Davies-Harte method)

We considered four Hurst parameters for simulation: 0.15, 0.35, 0.55, and 0.75. We used the Davies-Harte method of generating the fBm since this one is widely accepted as the most precise. We observe not the best approcsimations, but they are decent enough to be in $(\mu-\sigma, \mu+\sigma)$.

4.2.2 Heston stochastic volatility model

We observe that roughness estimations for instantaneous volatiltity and realized volatility significantly differ, which was described in the article for fOU processes.

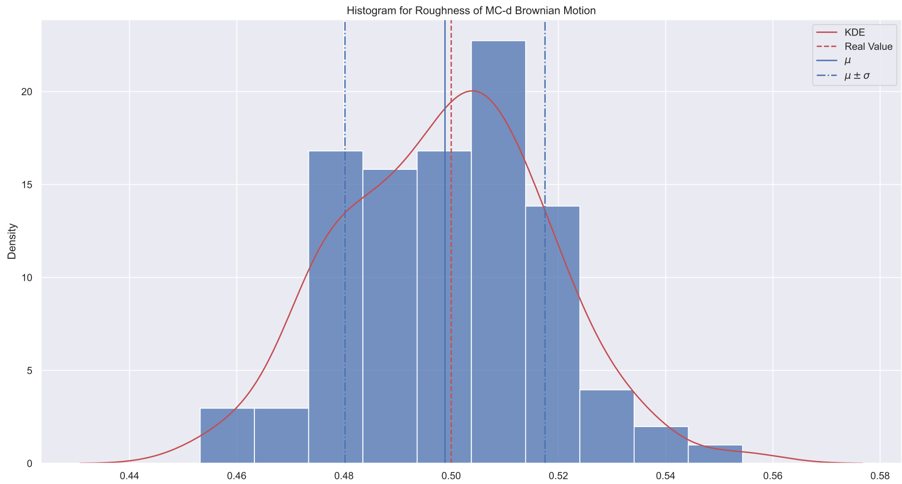


Figure 4.2: Histogram for roughness of Brownian motion

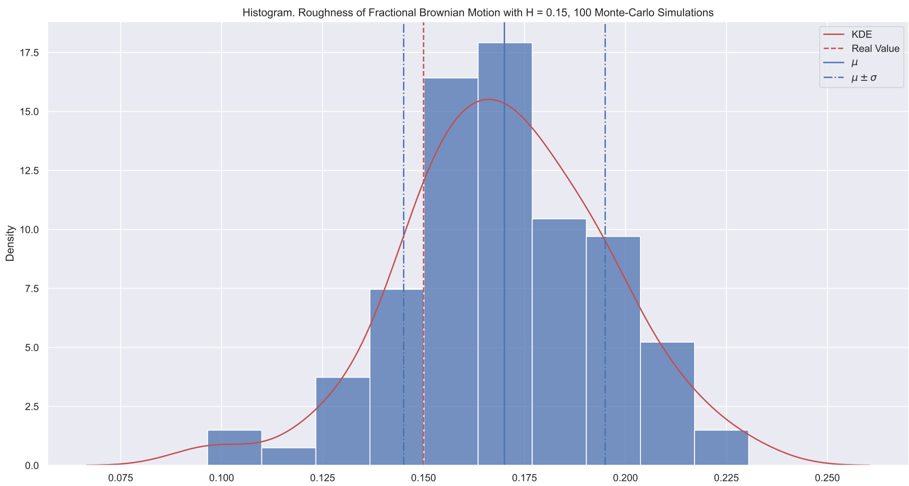


Figure 4.3: Histogram for roughness of fractional Brownian motion

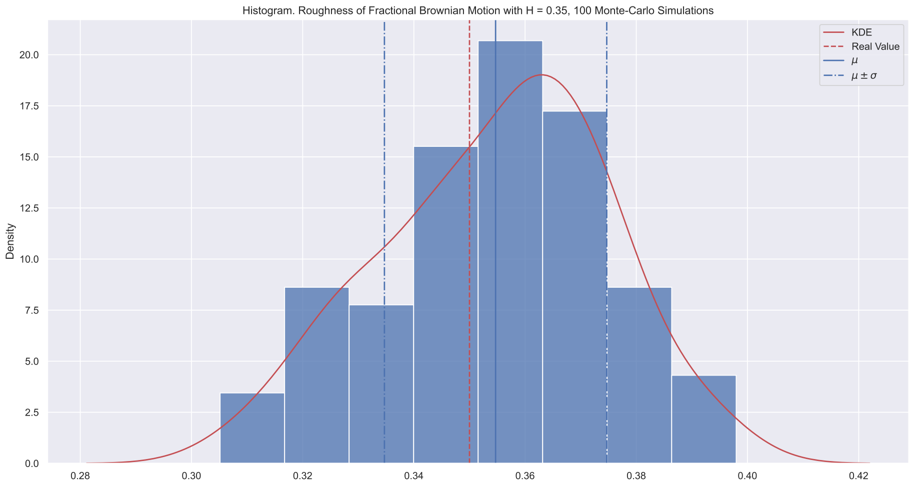


Figure 4.4: Histogram for roughness of fractional Brownian motion

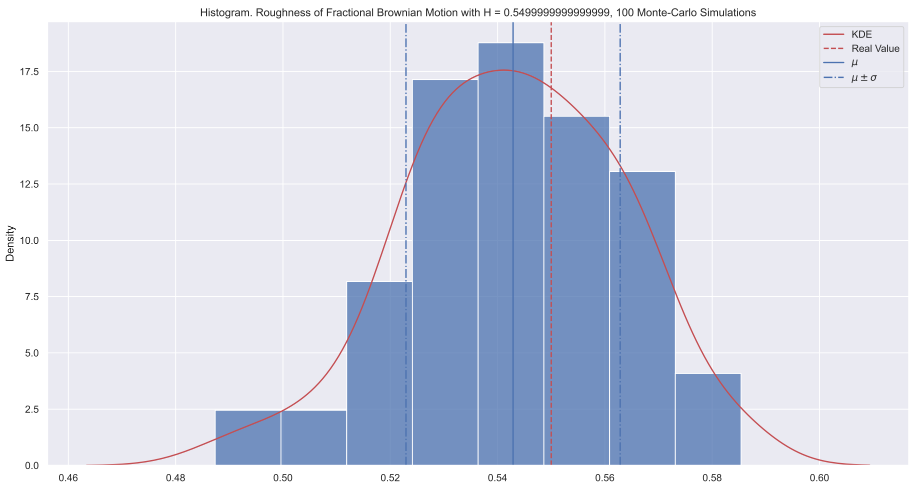


Figure 4.5: Histogram for roughness of fractional Brownian motion

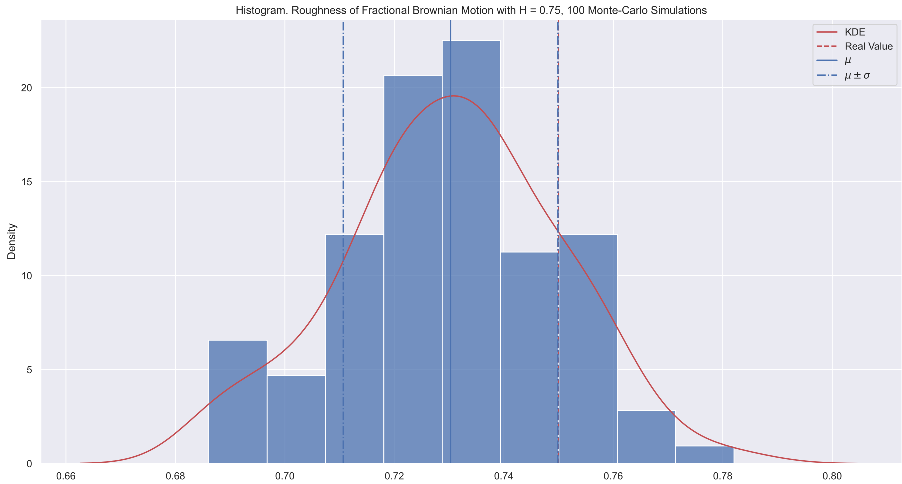


Figure 4.6: Histogram for roughness of fractional Brownian motion

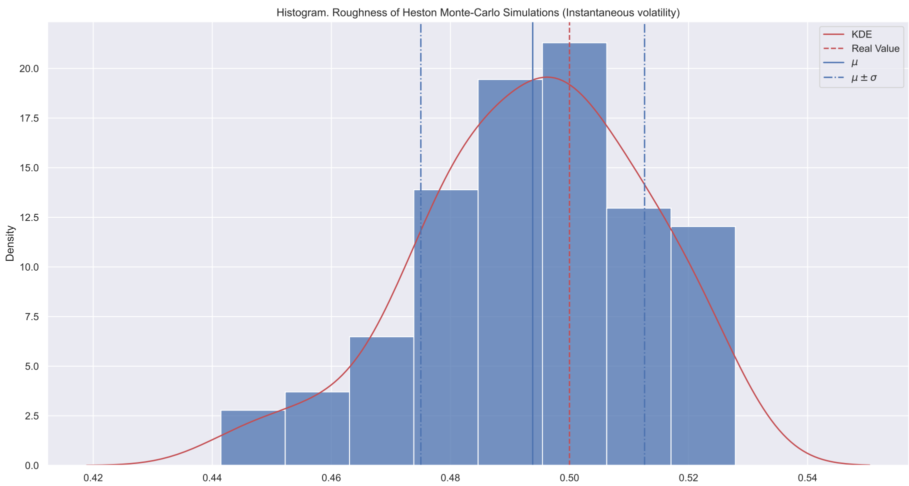


Figure 4.7: Histogram for roughness of Heston SVM

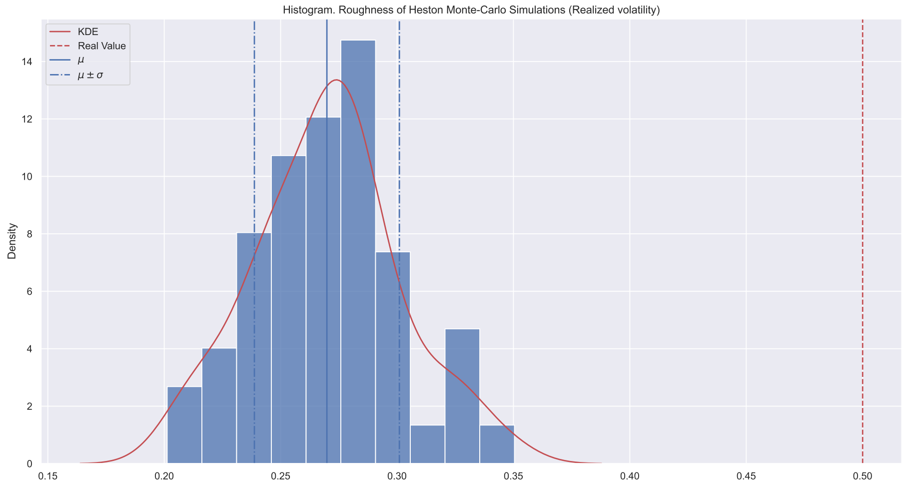


Figure 4.8: Histogram for roughness of Heston SVM

4.3 Roughness estimation of real-market data

Let us estimate the roughness of real-market data. We are using the same Bloomberg data from Table 3.1 (but without indexes).

Ticker	Roughness Index
YNDX RX Equity	0.372691
SBER RX Equity	0.313109
VTBR RX Equity	0.304677
MOEX RX Equity	0.295378
LKOH RX Equity	0.301795
GAZP RX Equity	0.316125
FIVE RX Equity	0.284704
OGZD LI Equity	2.968608
VTBR LI Equity	0.306763
SBER LI Equity	1.176616
LKOD LI Equity	0.306061

Table 4.1: Roughness index estimation

As we can see, modelless estimation of roughness differs from the Hurst parameter under RFSV (as Monte-Carlo simulations predicted).

Conclusion

Reproduced Hypotheses

We got aquainted with the fractional stochastic volatility models framework and studied the statistical properties of RFSV. We observed certain effects and calculated different statistical estimations. We obtained roughness estimations for major Russian companies stocks and depositary reciepts, and reproduced some effects described in [GJR14].

1. Zumbach effect is not observable for conventional SVMs (e.g. Heston);
2. The Hurst exponent of the considered assets has the order of $1e-1$ and is less than $\frac{1}{2}$.
3. The volatility of the considered assets **does not** have a property of long memory under fractional stochastic volatiltiy models.
4. Visual analysis and normality tests for the log-increments of volatility shows that for $\Delta \in [10, 25]$ the normality of log-inrements hypothesis holds.
5. The smoothing effect holds for the estimations of H and α (volatility of volatility under fOU). But **only** for VTBR LI Equity we got a negative slope of the smoothing effect. For other asset we got a nearly perfect linear fit and positive smoothing slopes.
6. Modelless estimation gives different roughness indexes for realized and instantaneous volatility under SVMs.

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Appendix

Appendix A. Results for Additional Assets

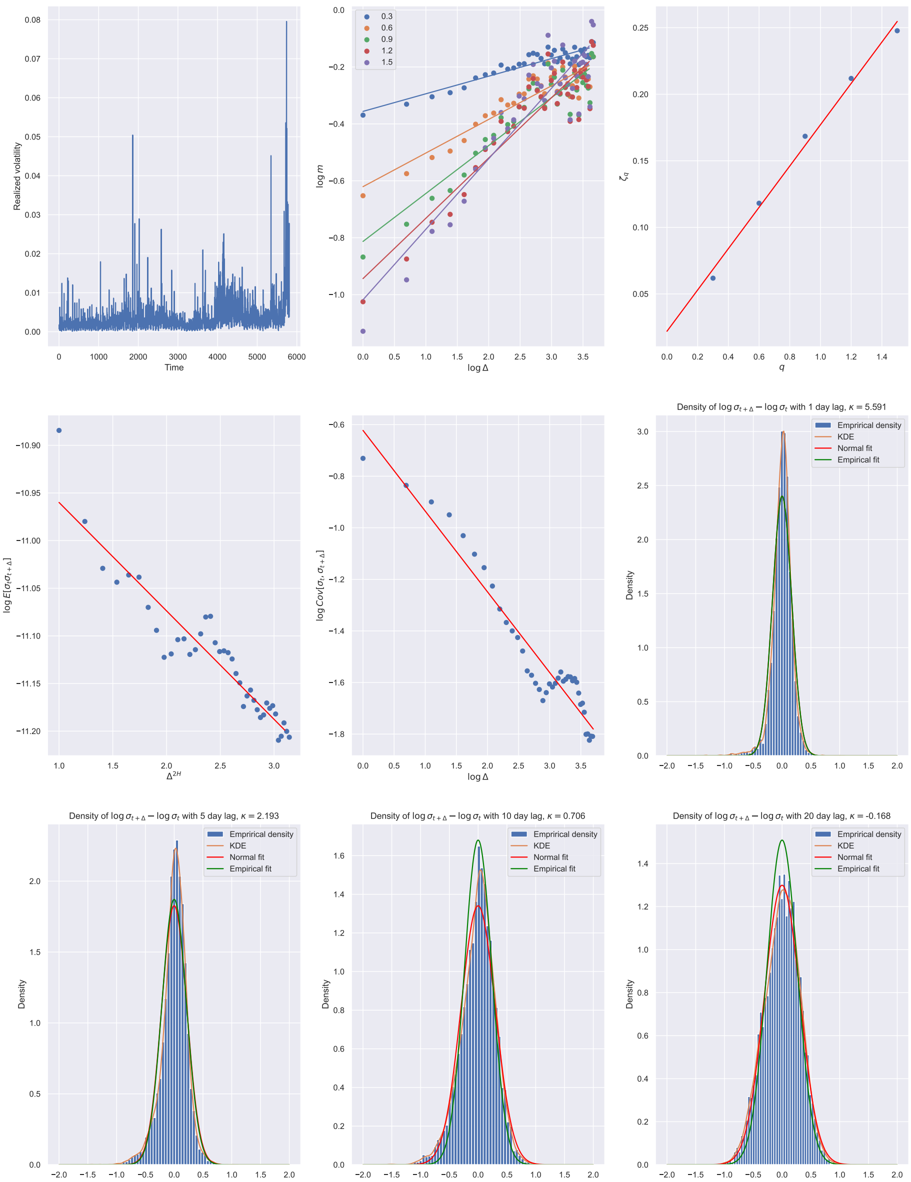

Figure 4.9: SBER RX Equity. $\hat{H}$ plots

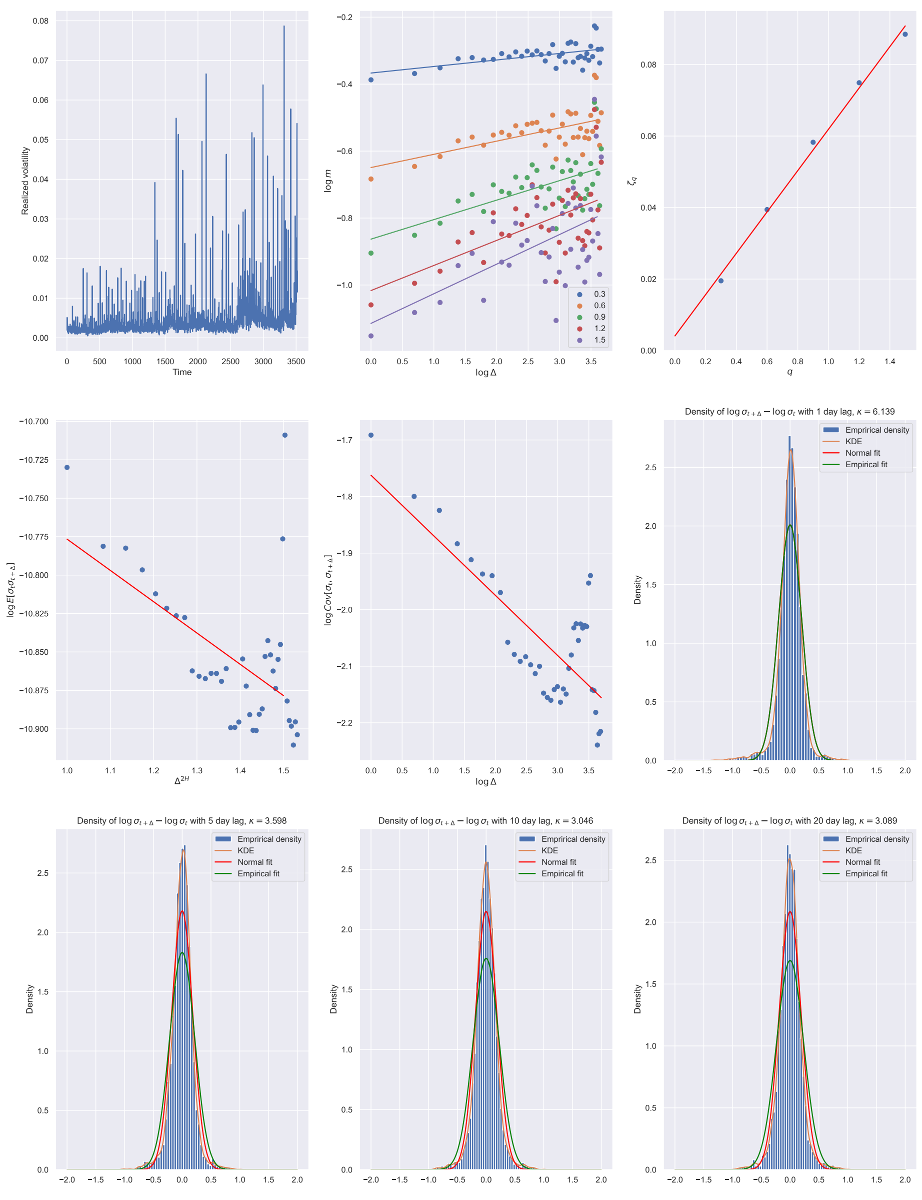

Figure 4.10: SBER LI Equity. $\hat{H}$ plots

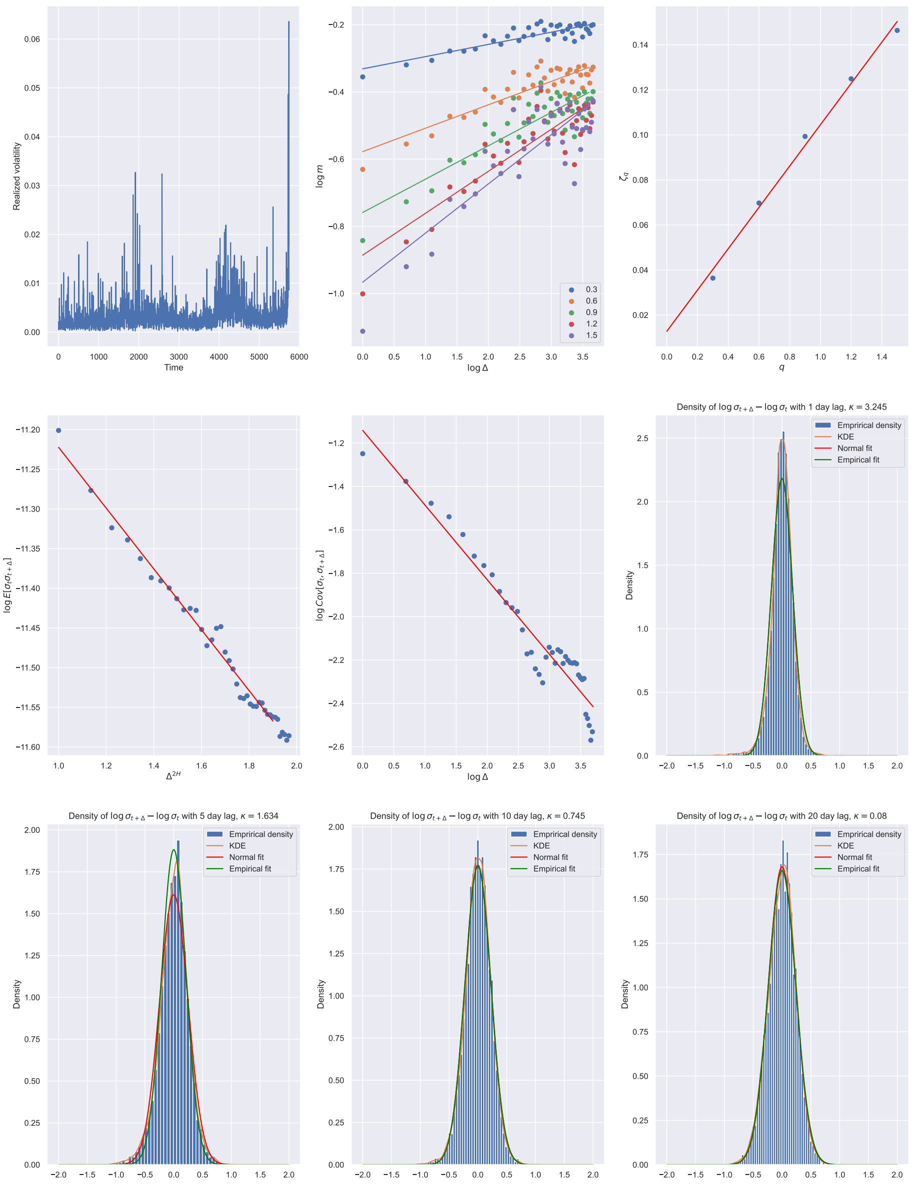

Figure 4.11: VTBR RX Equity. $\hat{H}$ plots

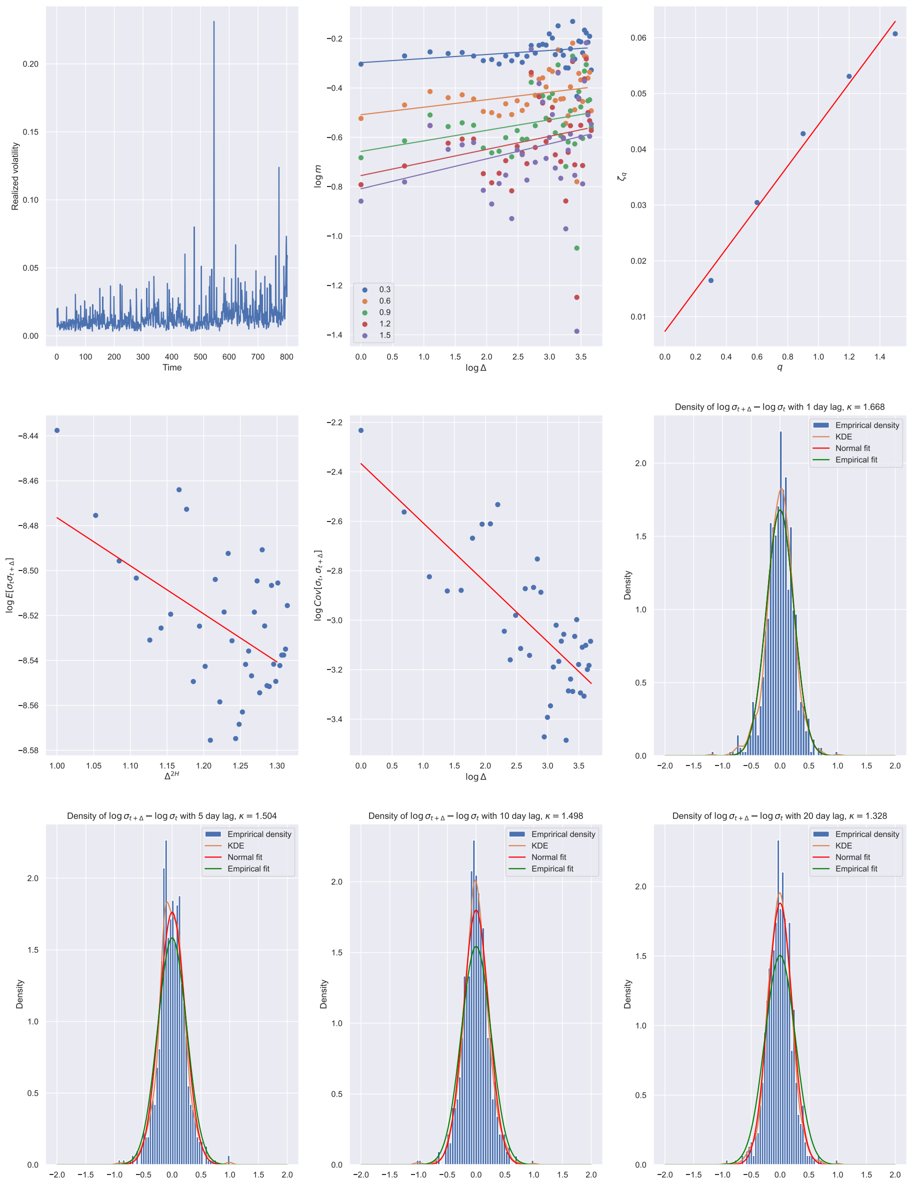

Figure 4.12: VTBR LI Equity. $\hat{H}$ plots

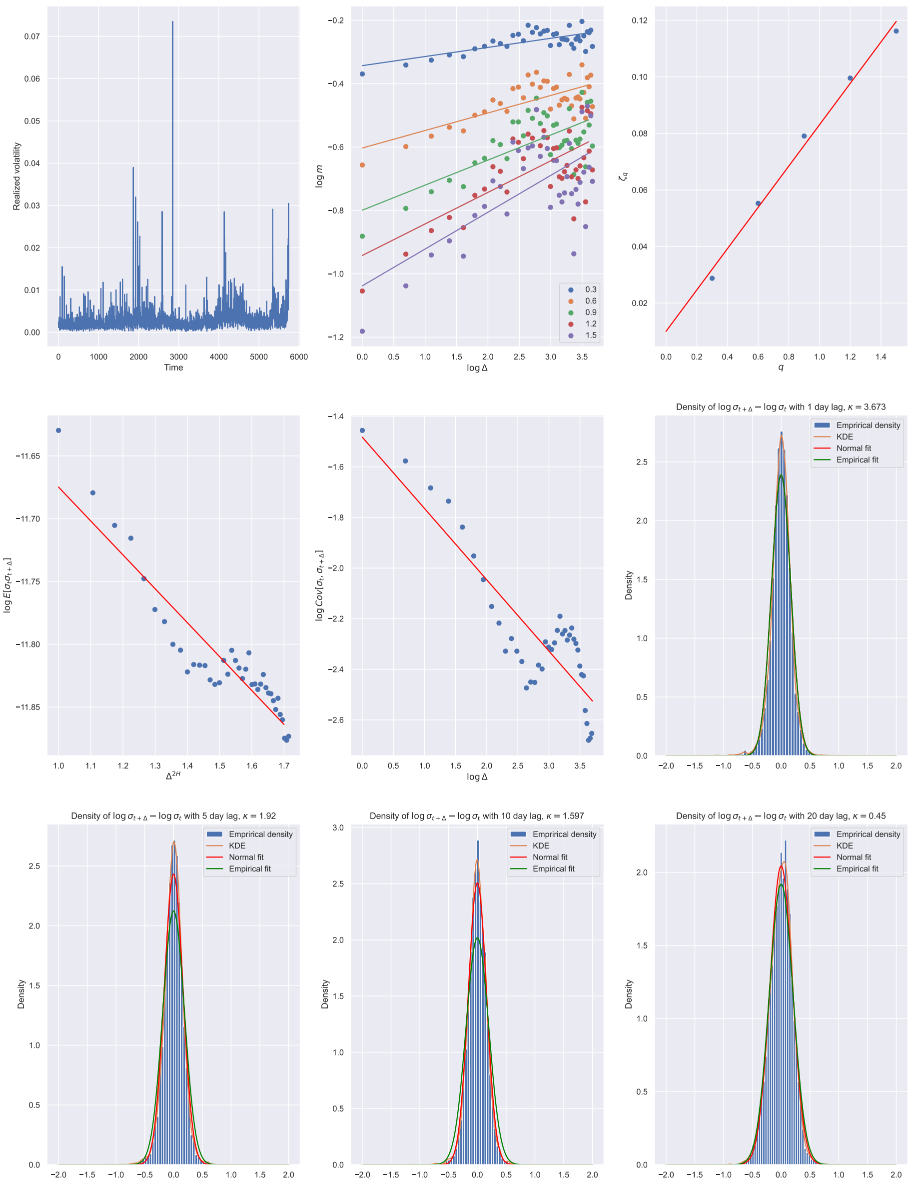

Figure 4.13: LKOH RX Equity. $\hat{H}$ plots

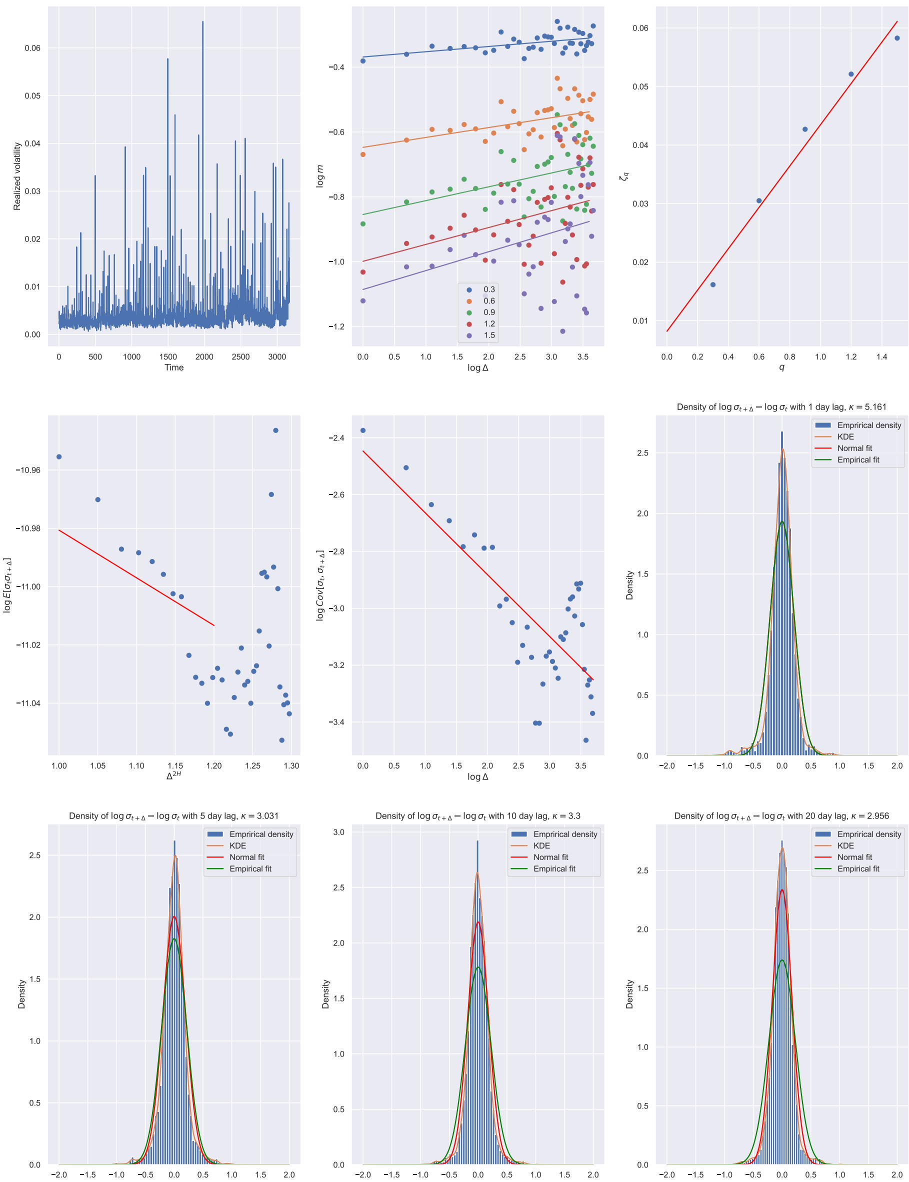

Figure 4.14: LKOD LI Equity. $\hat{H}$ plots

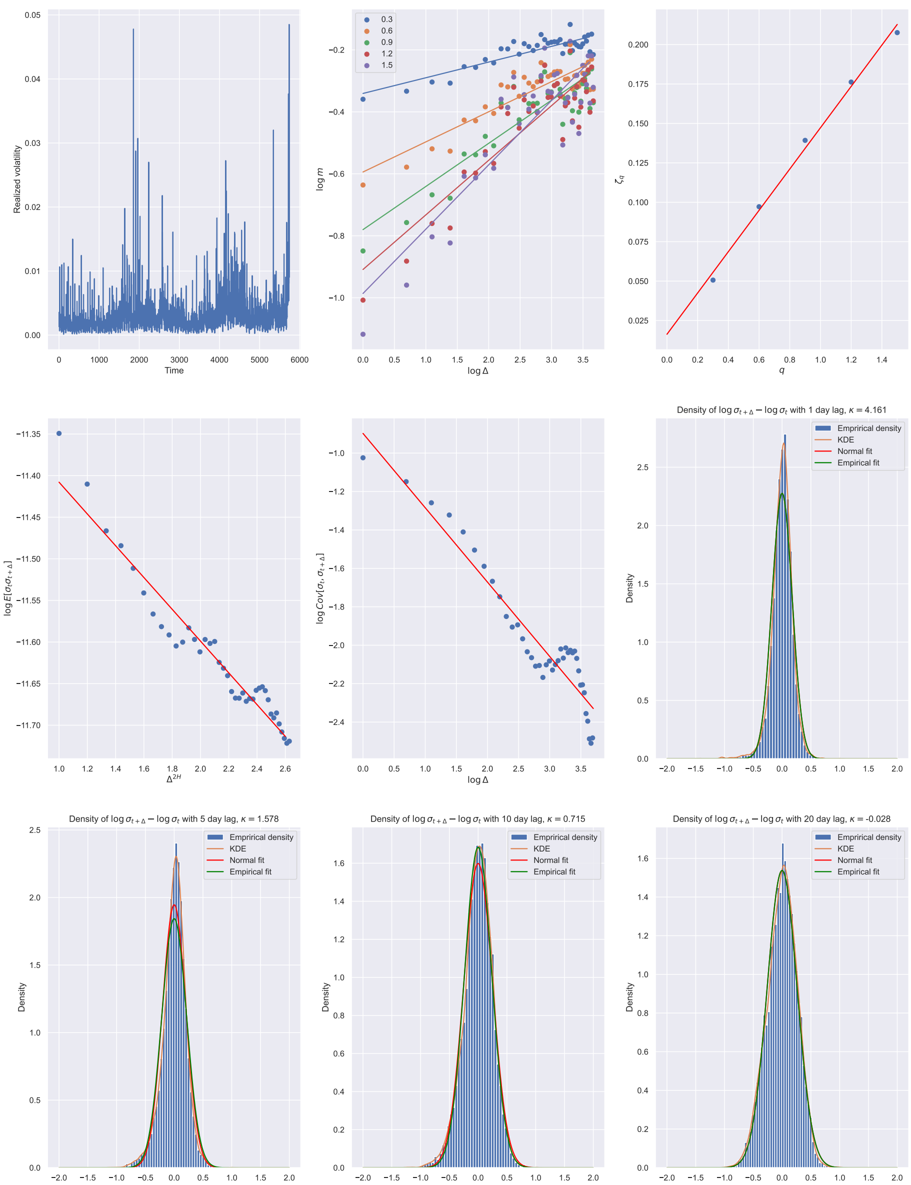

Figure 4.15: GAZP RX Equity. $\hat{H}$ plots

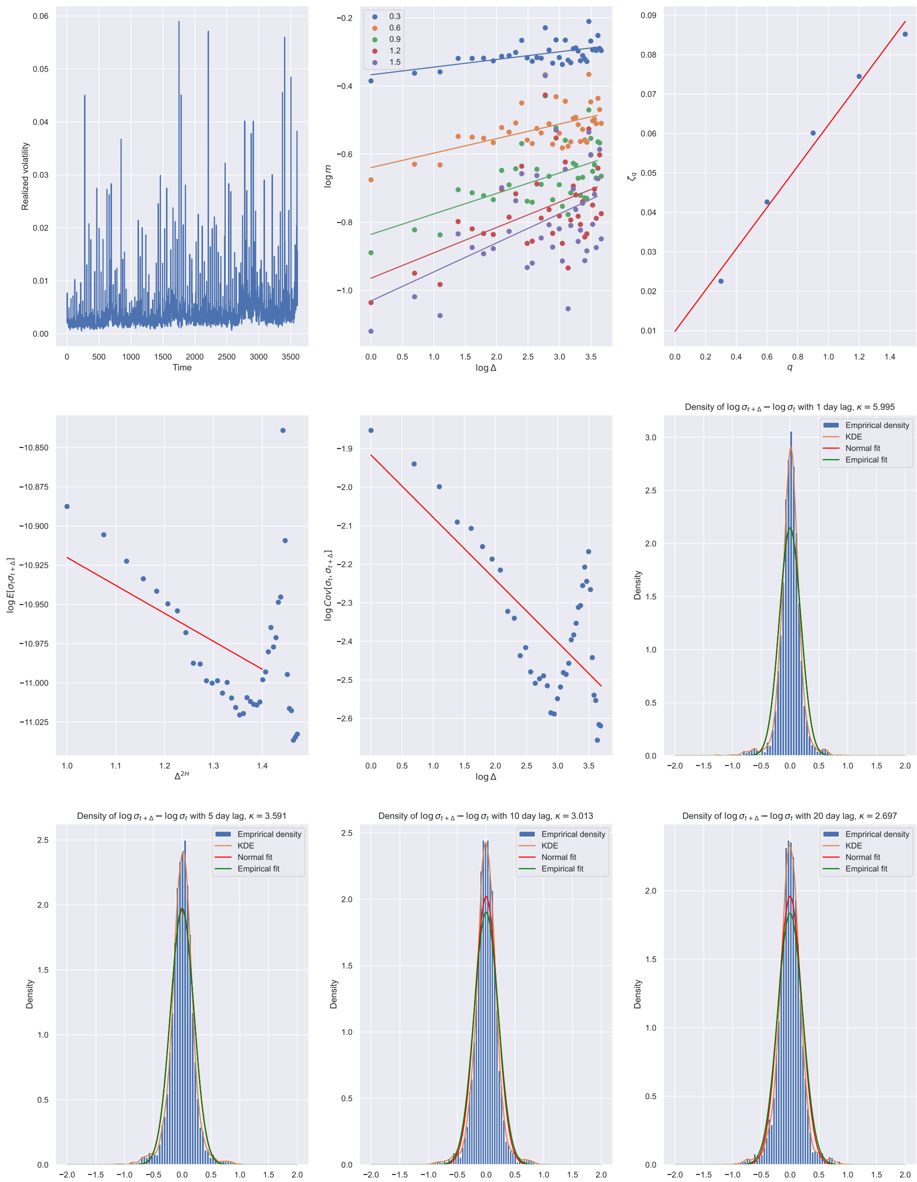

Figure 4.16: OGZD LI Equity. $\hat{H}$ plots

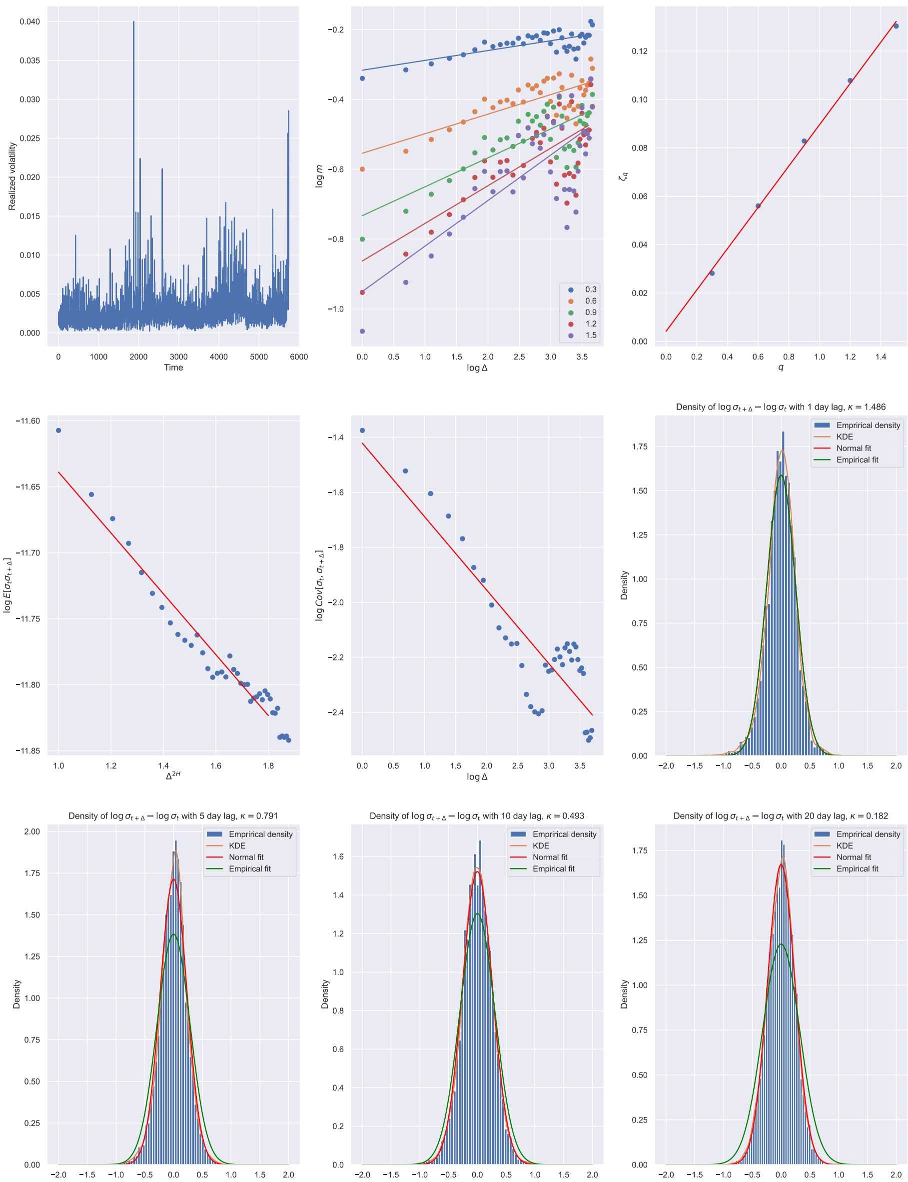

Figure 4.17: MOEX RX Equity. $\hat{H}$ plots

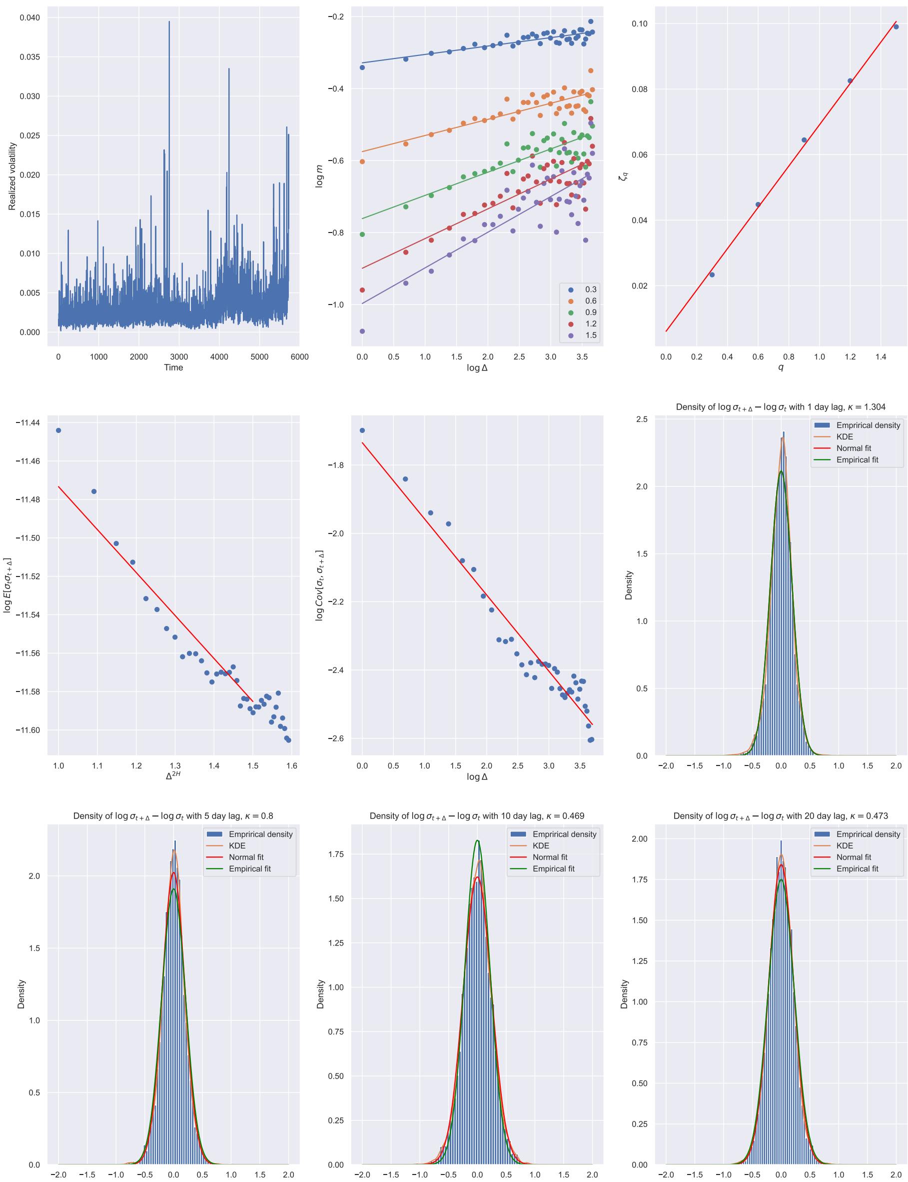

Figure 4.18: FIVE RX Equity. $\hat{H}$ plots

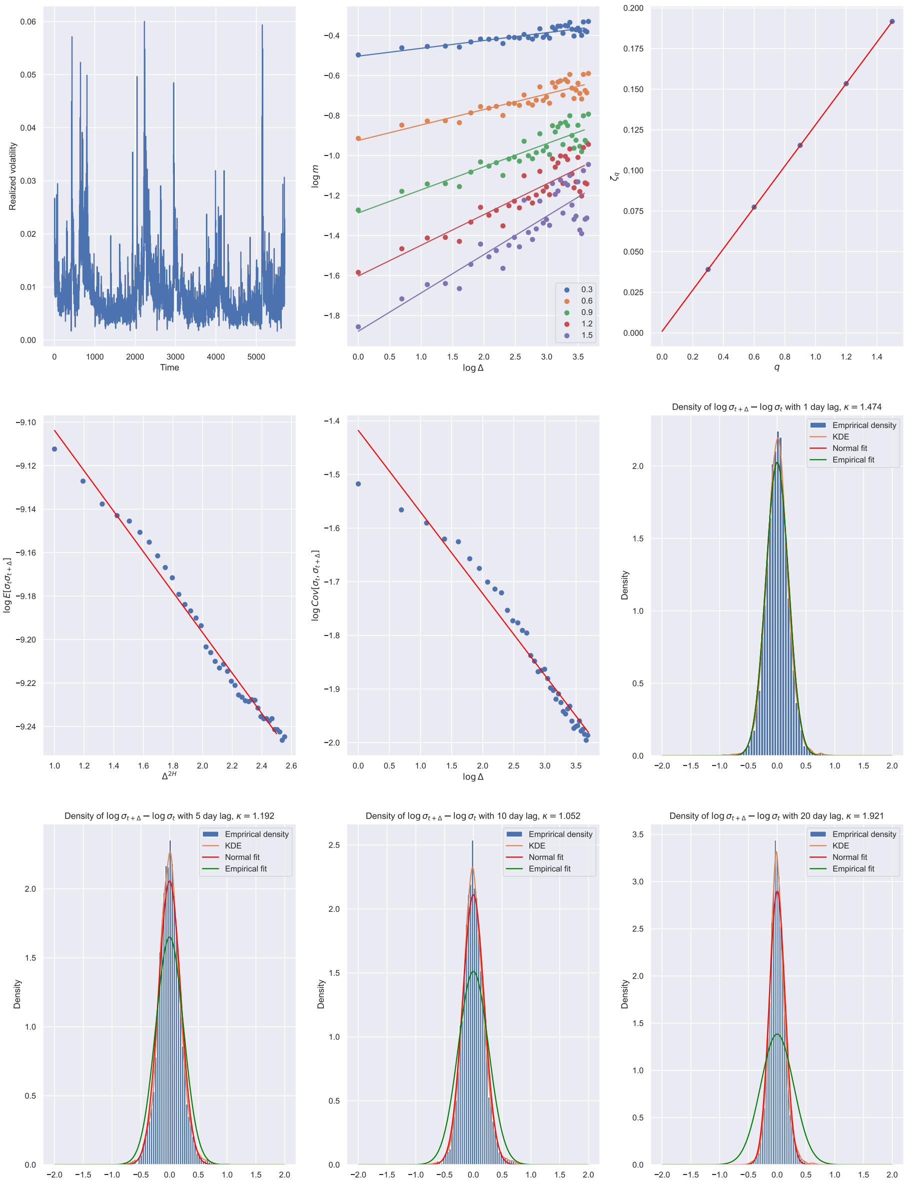

Figure 4.19: .AEX. $\hat{H}$ plots

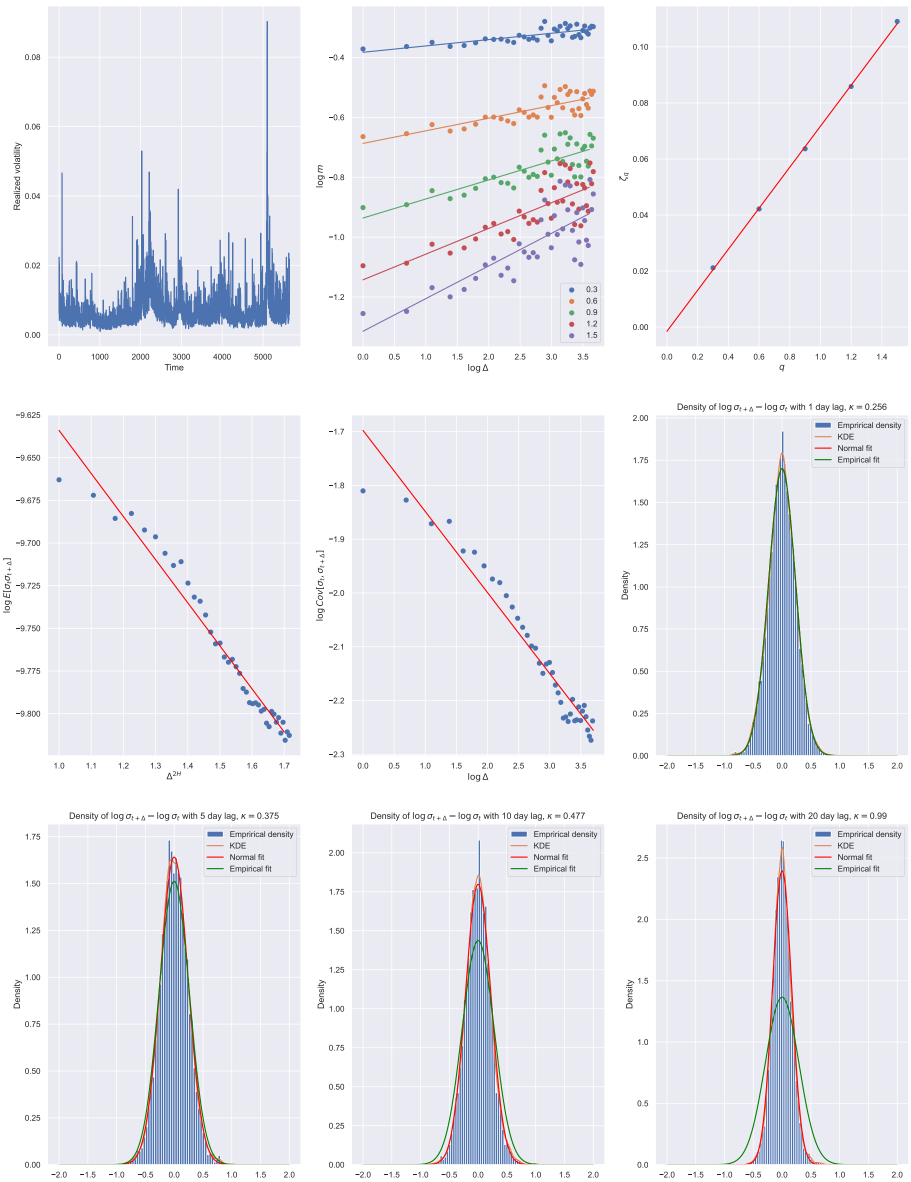

Figure 4.20: .AORD. $\hat{H}$ plots

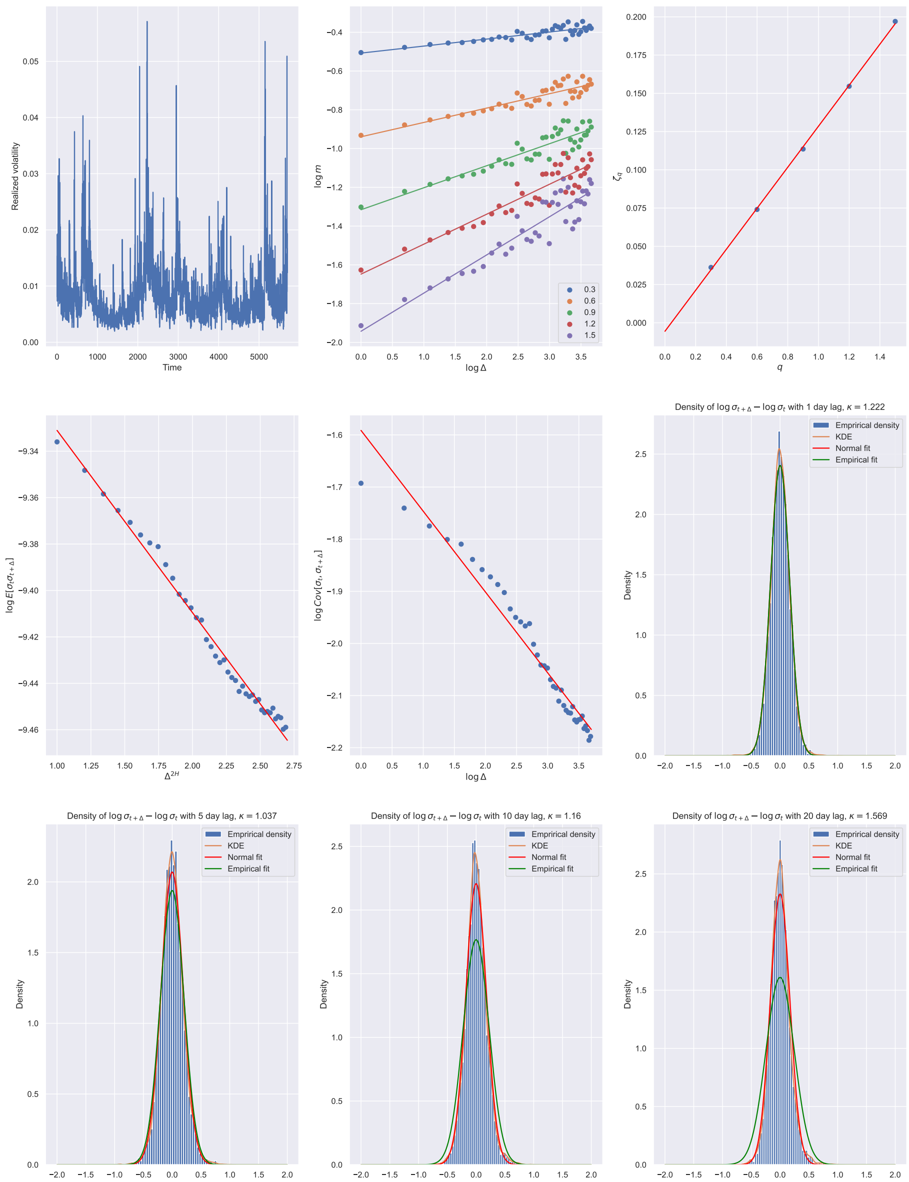

Figure 4.21: .BFX. $\hat{H}$ plots

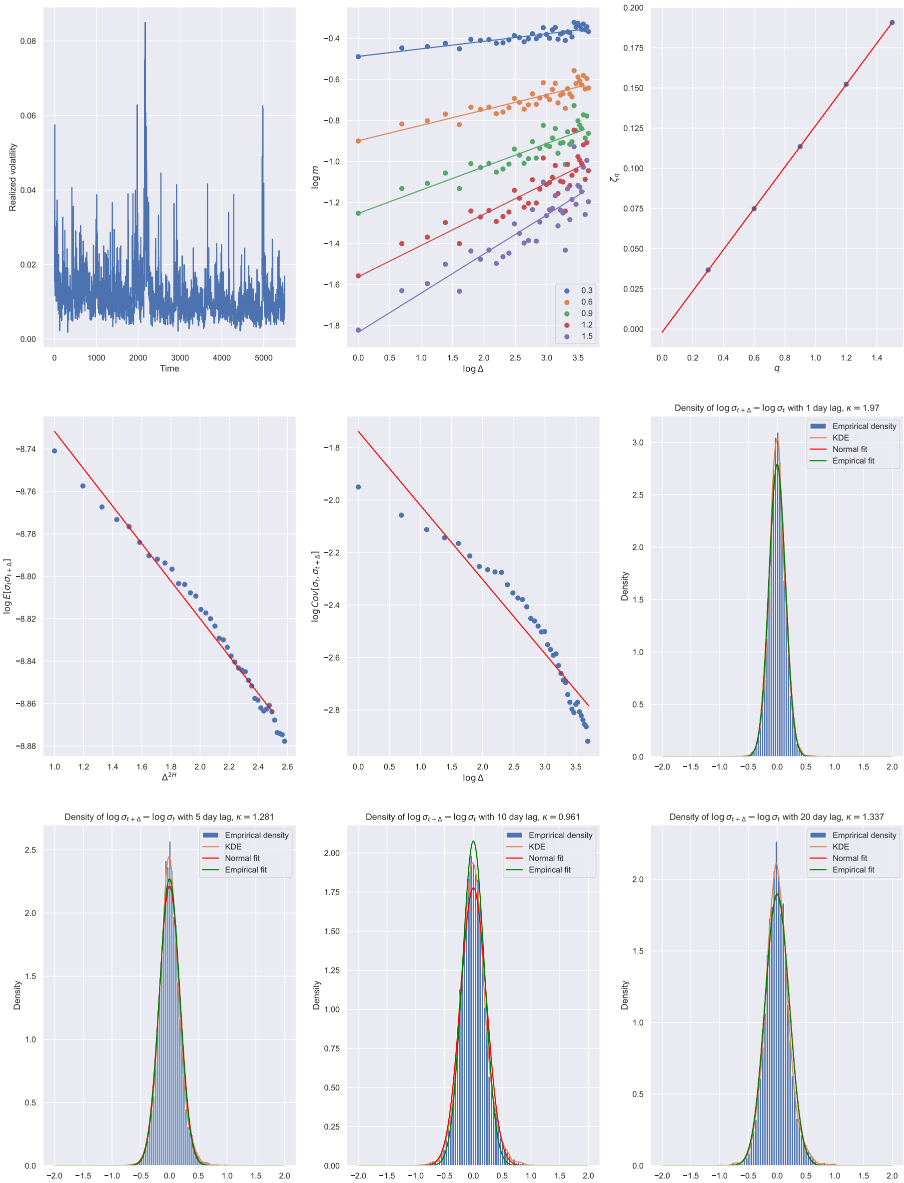

Figure 4.22: .BVSP. $\hat{H}$ plots

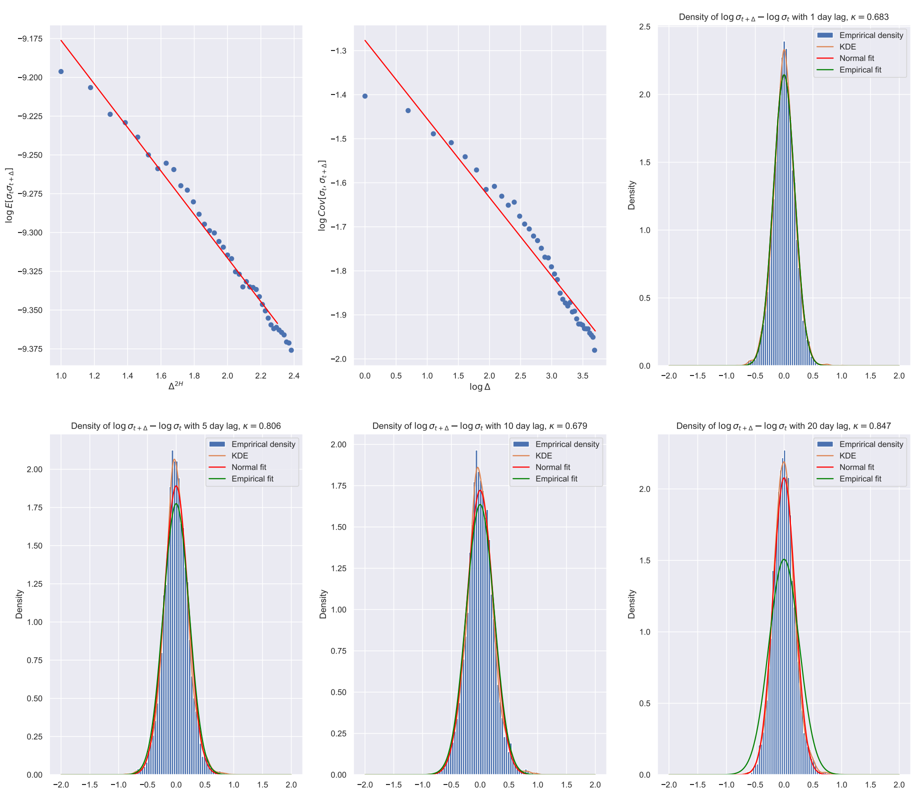

Figure 4.23: .DJI. $\hat{H}$ plots

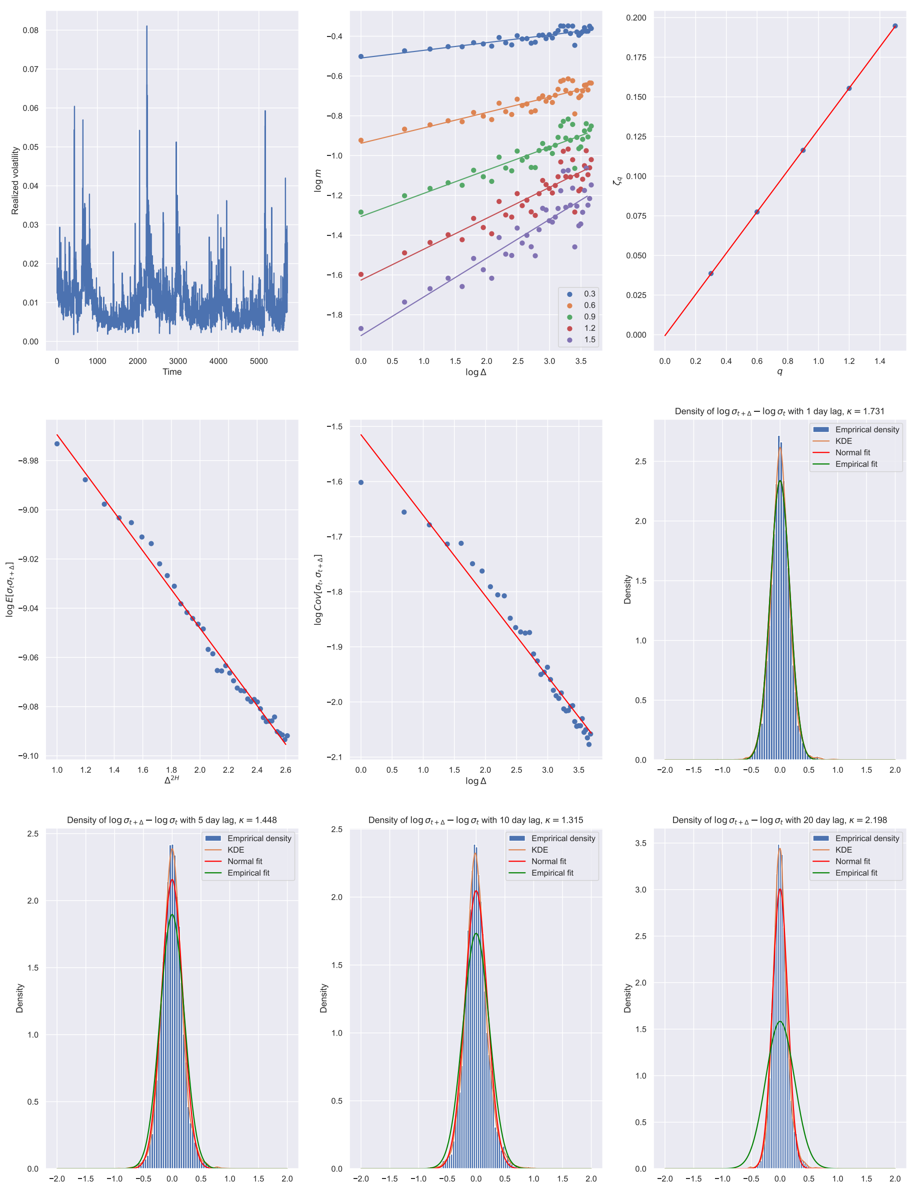

Figure 4.24: .FCHI. $\hat{H}$ plots

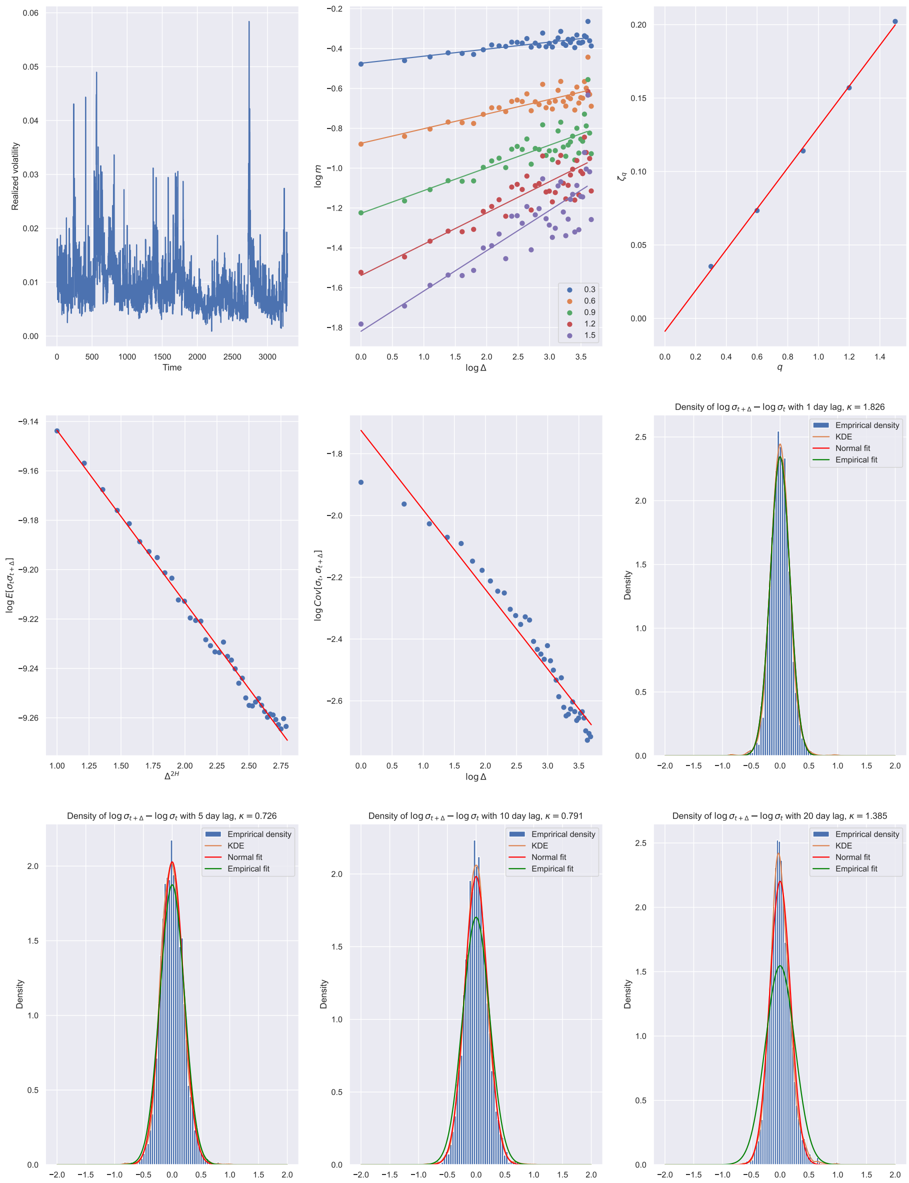

Figure 4.25: .FTMIB. $\hat{H}$ plots

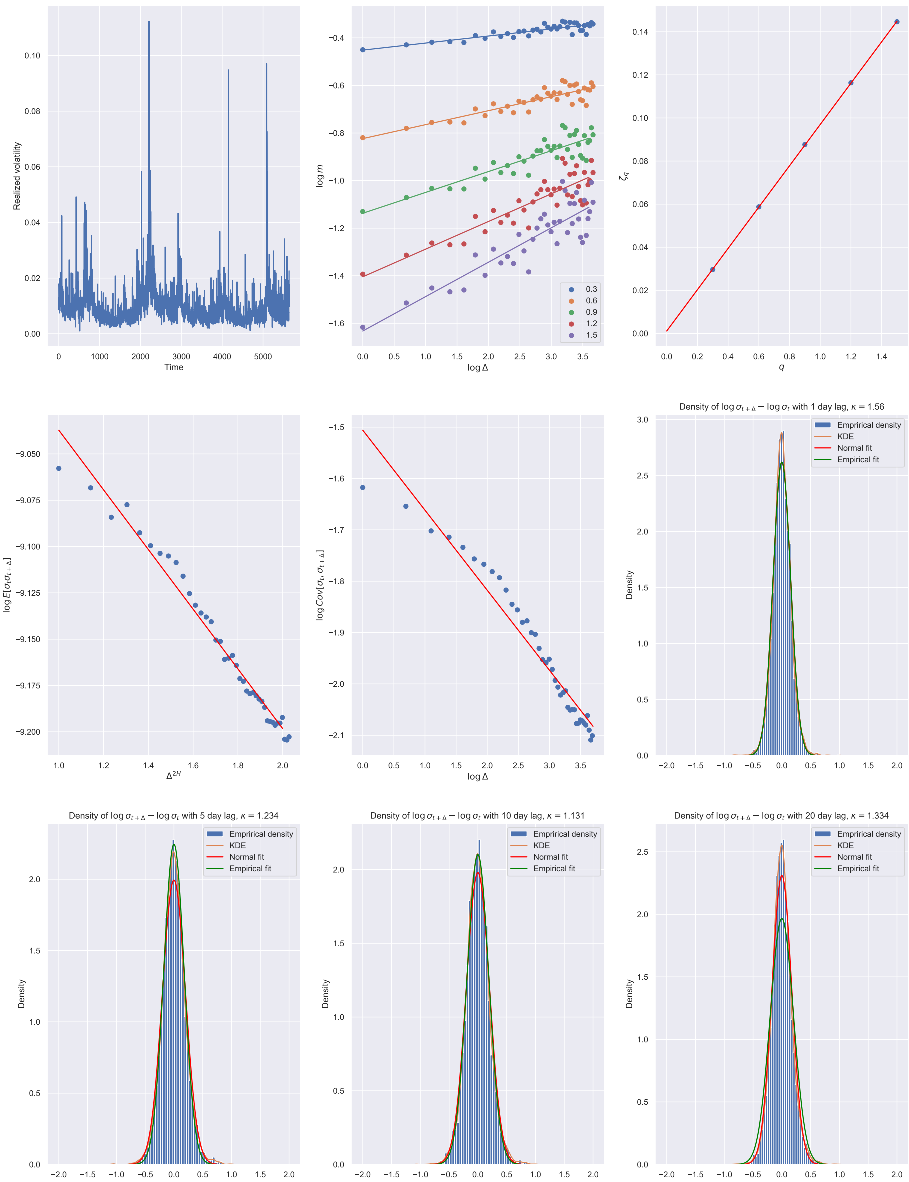

Figure 4.26: .FTSE. $\hat{H}$ plots

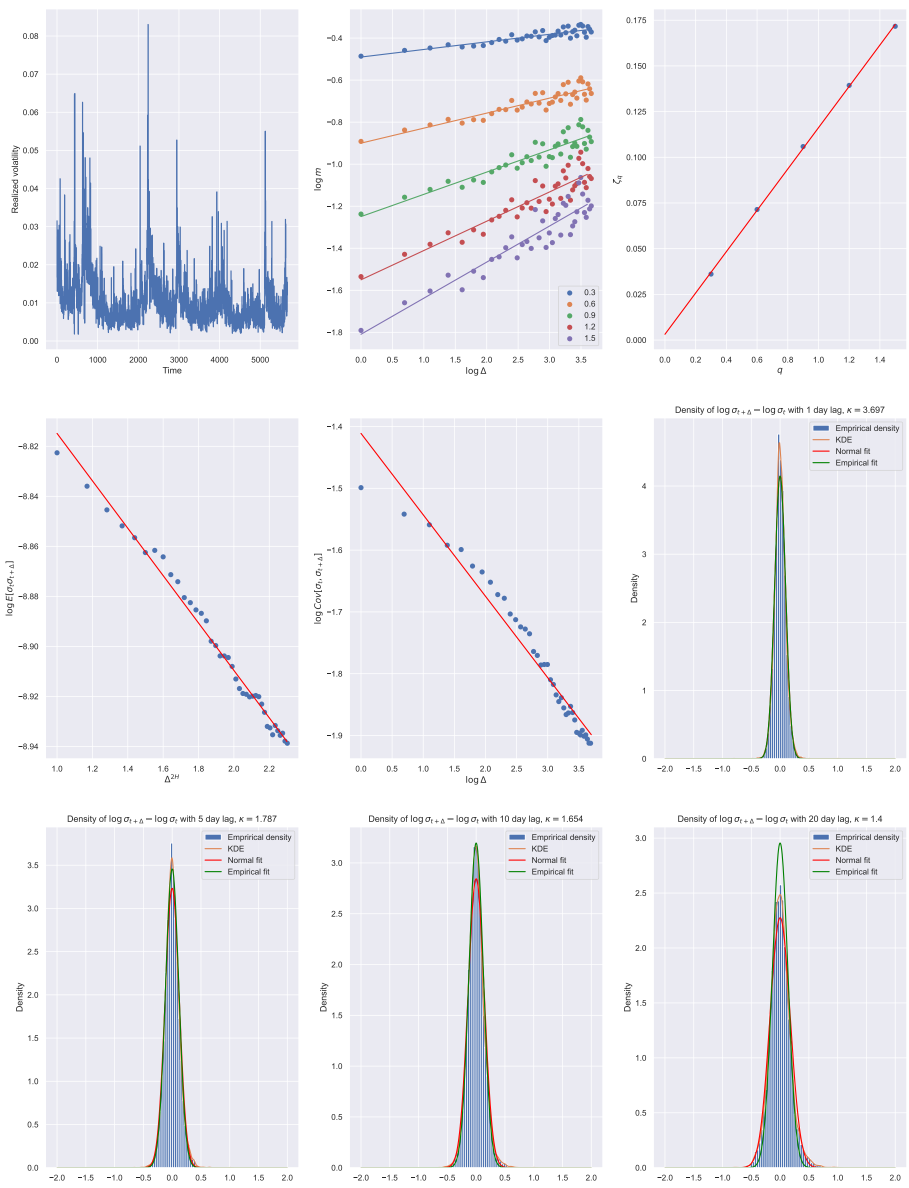

Figure 4.27: .GDAXI. $\hat{H}$ plots

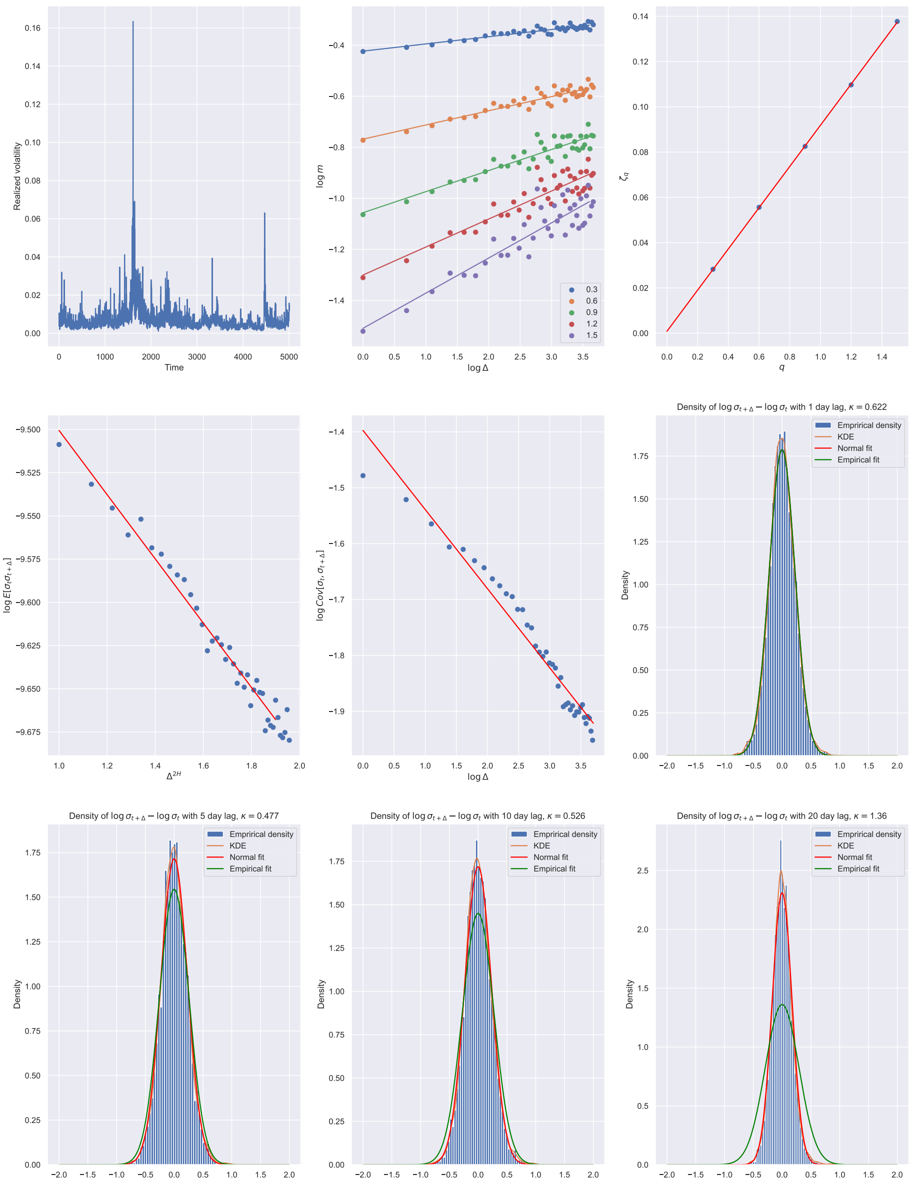

Figure 4.28: .GSPTSE. $\hat{H}$ plots

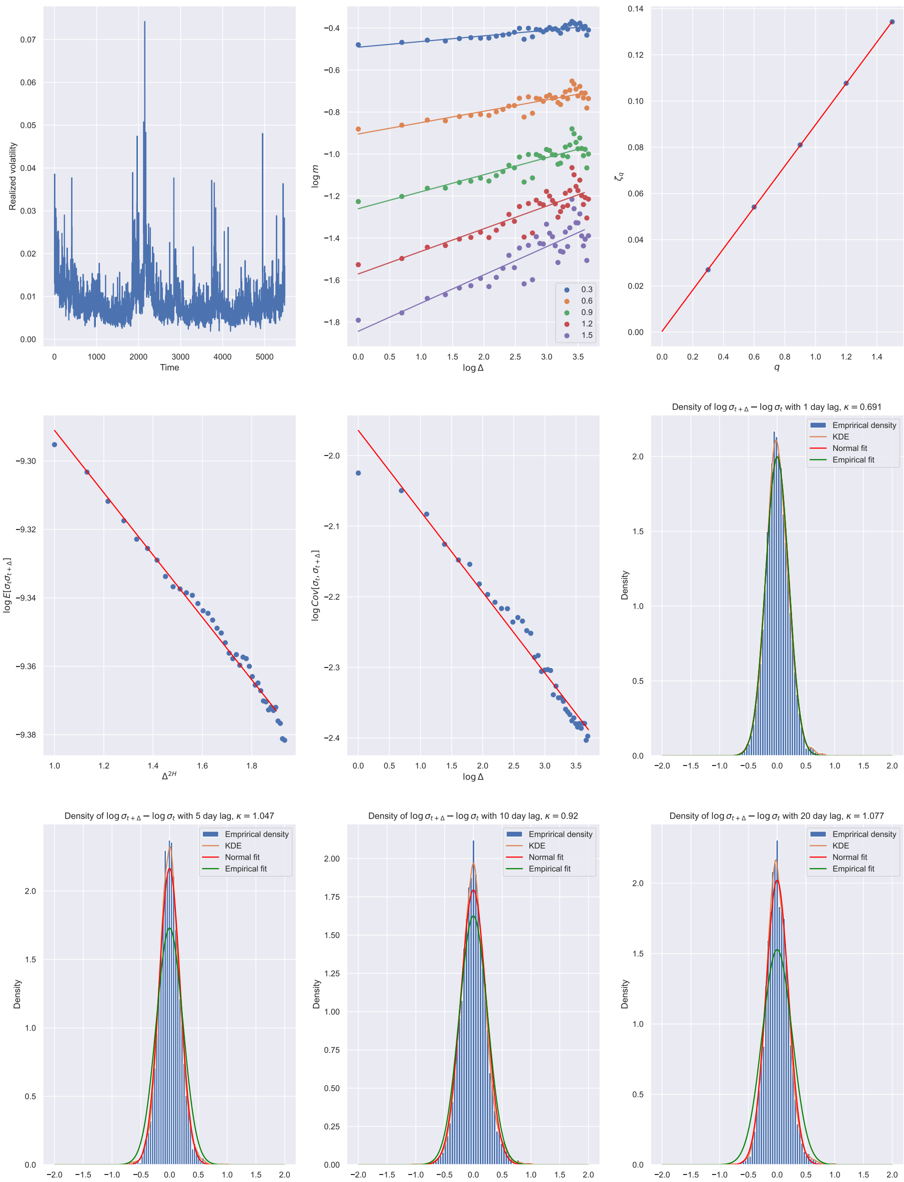

Figure 4.29: .HSI. $\hat{H}$ plots

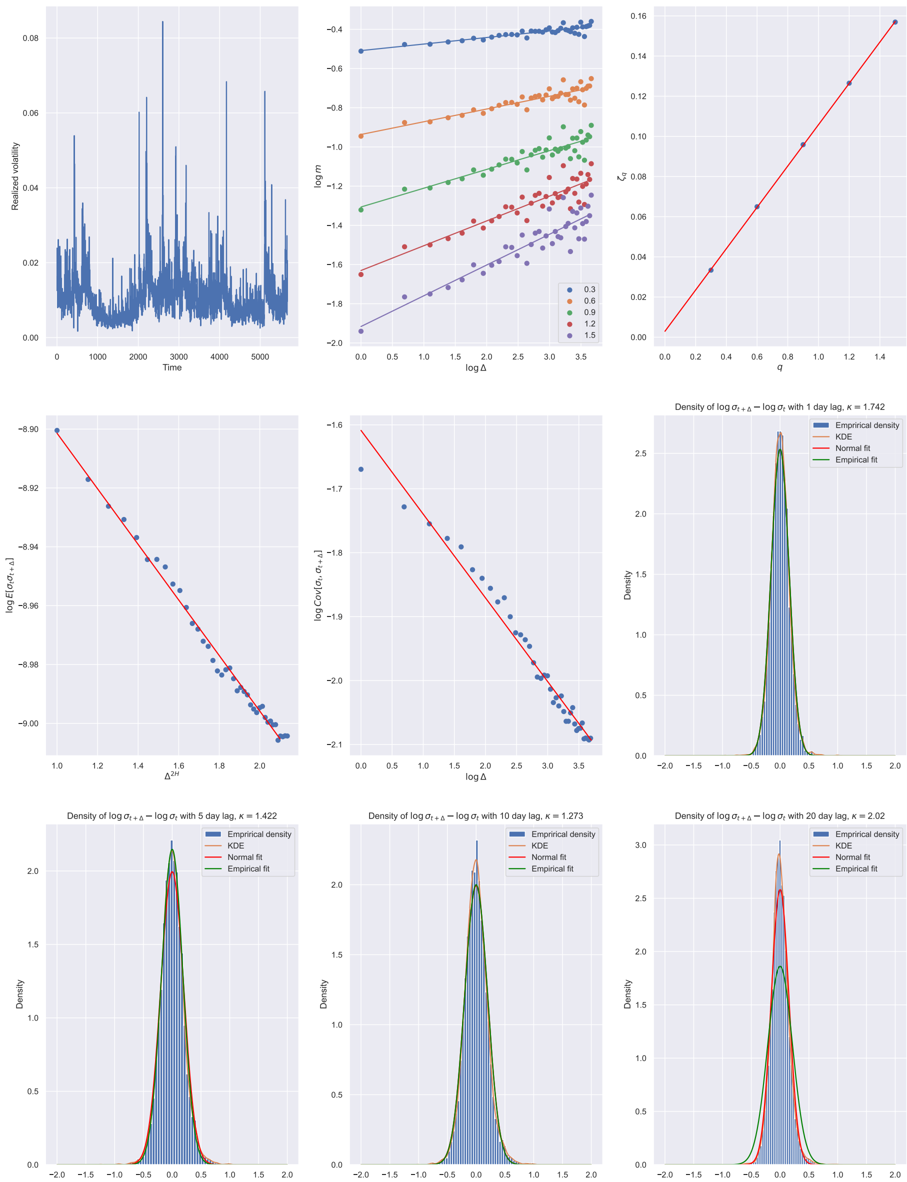

Figure 4.30: .IBEX. $\hat{H}$ plots

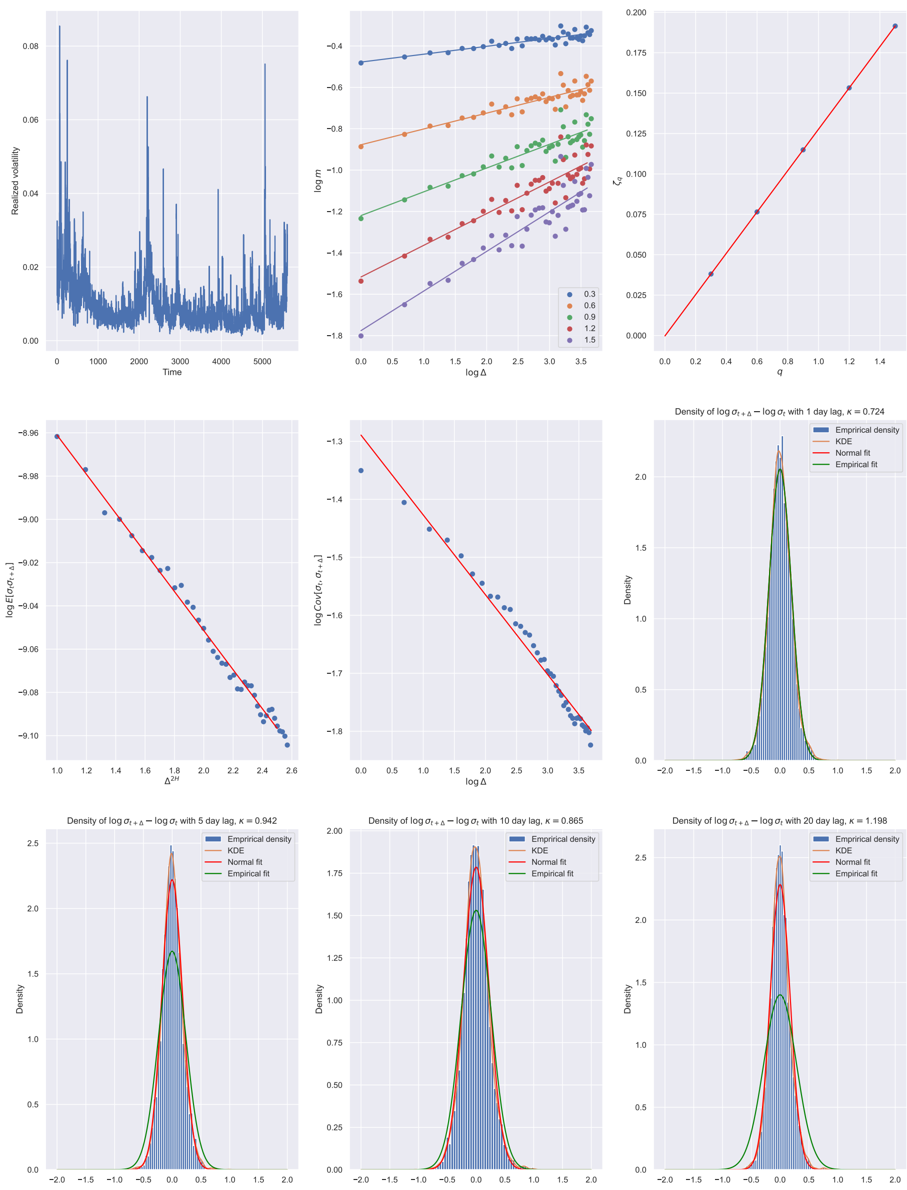

Figure 4.31: .IXIC. $\hat{H}$ plots

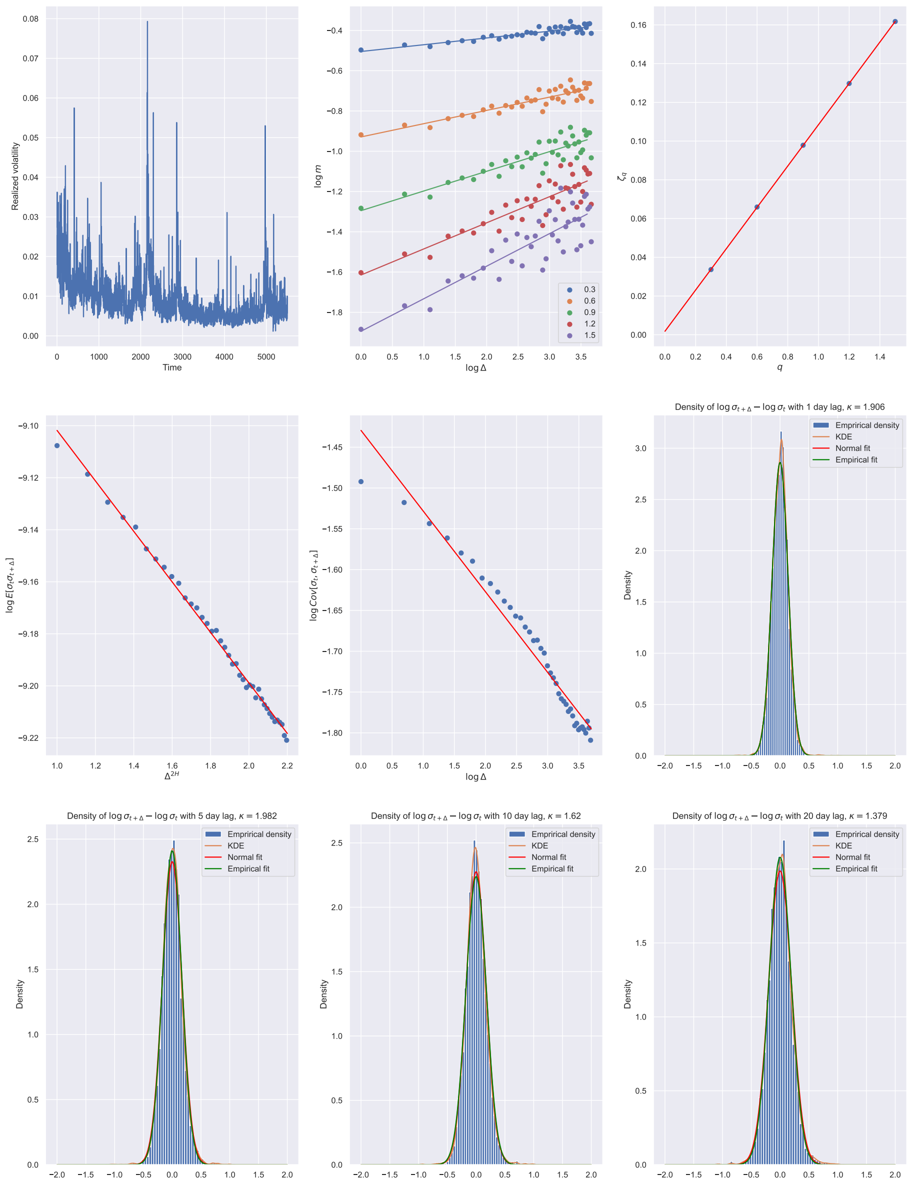

Figure 4.32: .KS11. $\hat{H}$ plots

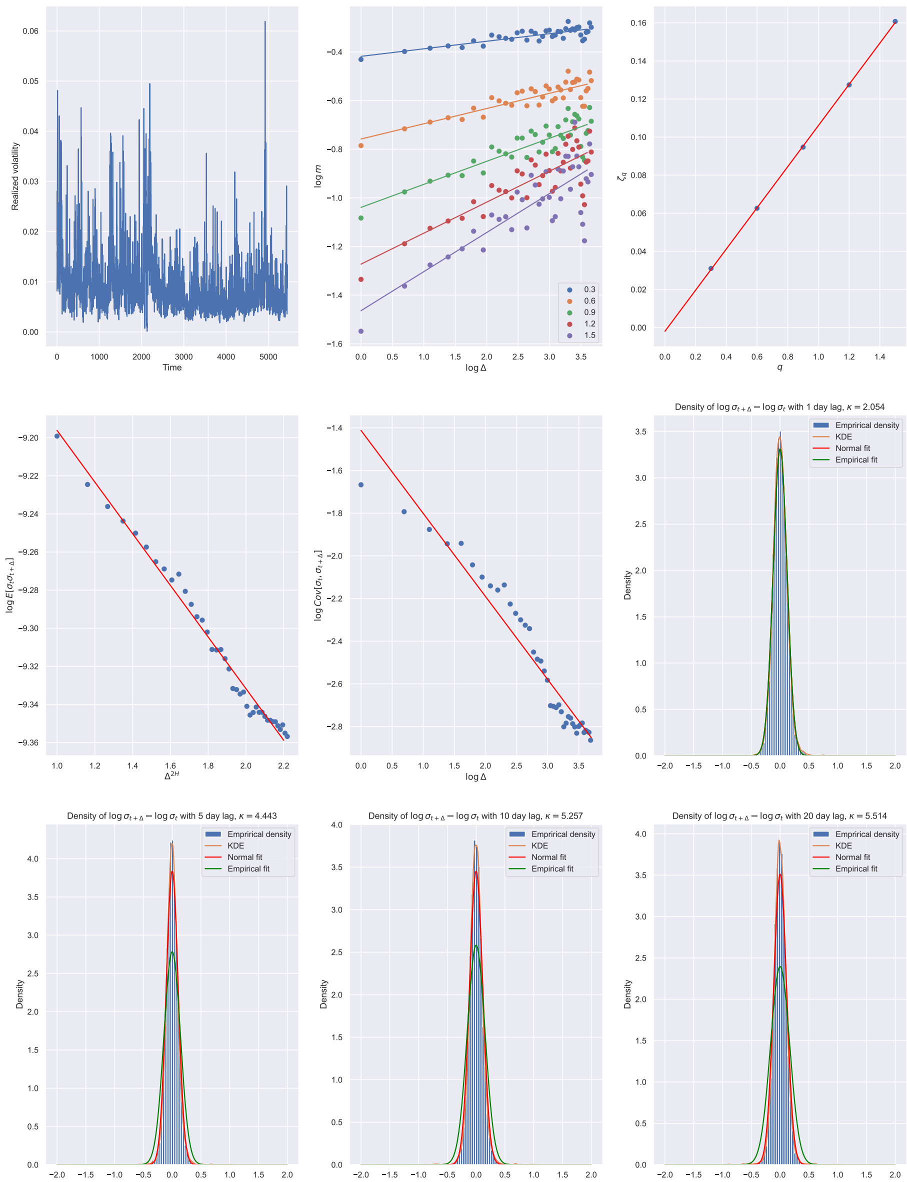

Figure 4.33: .KSE. $\hat{H}$ plots

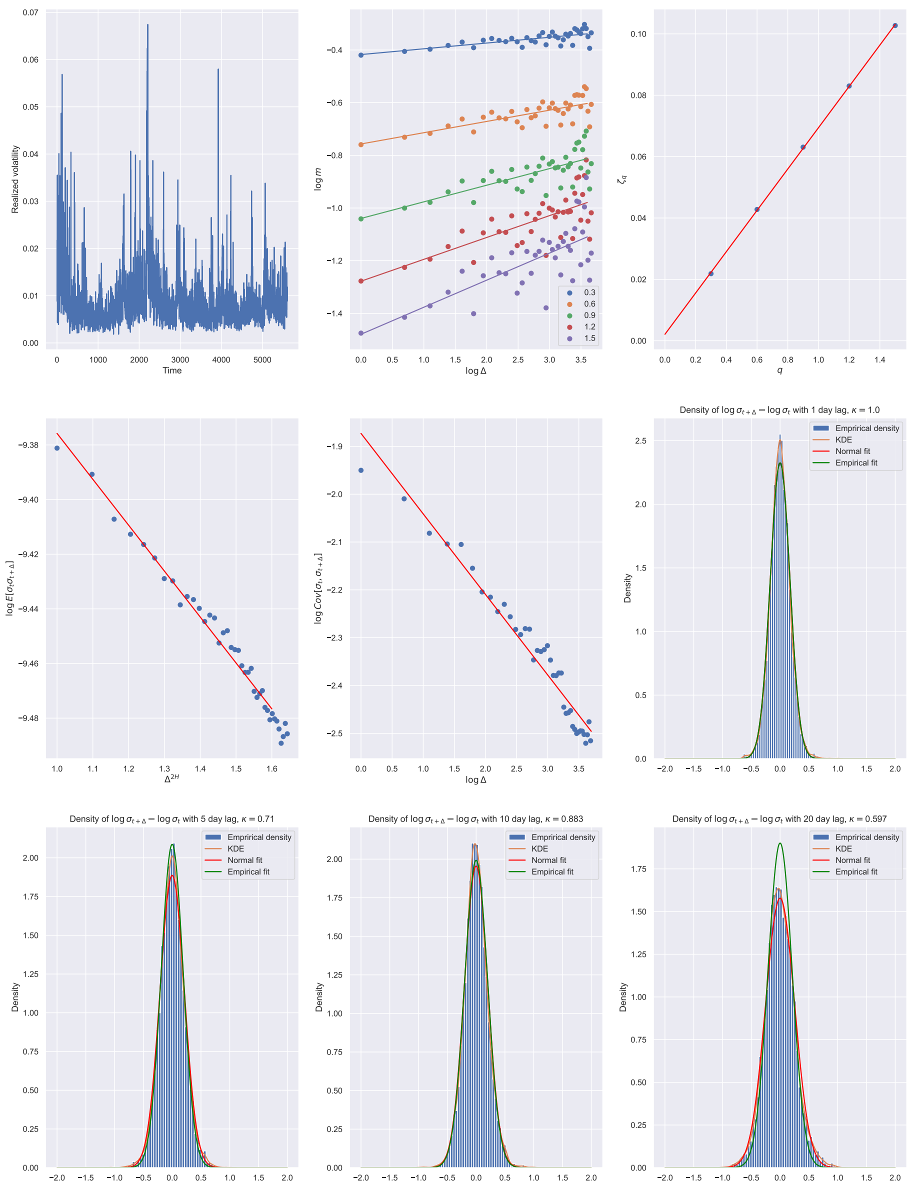

Figure 4.34: .MXX. $\hat{H}$ plots

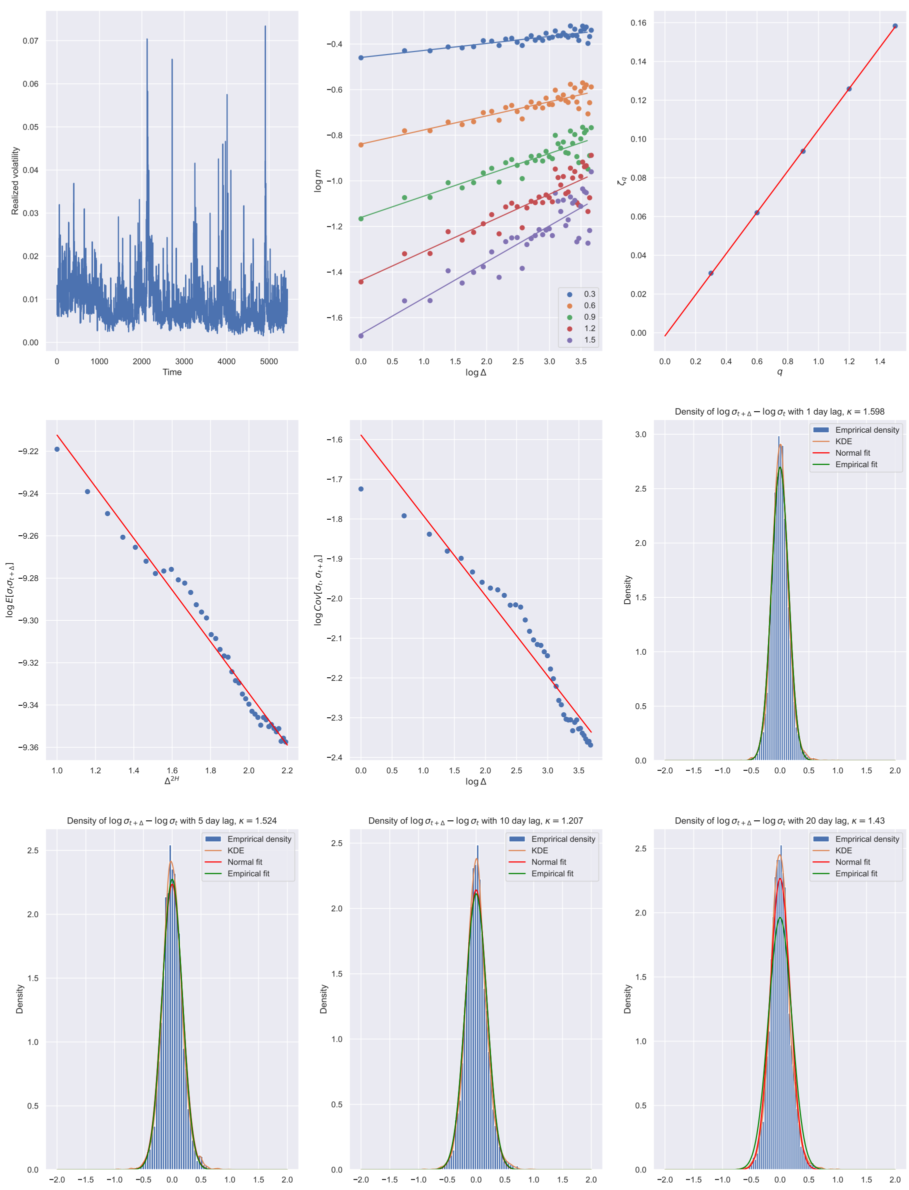

Figure 4.35: .N225. $\hat{H}$ plots

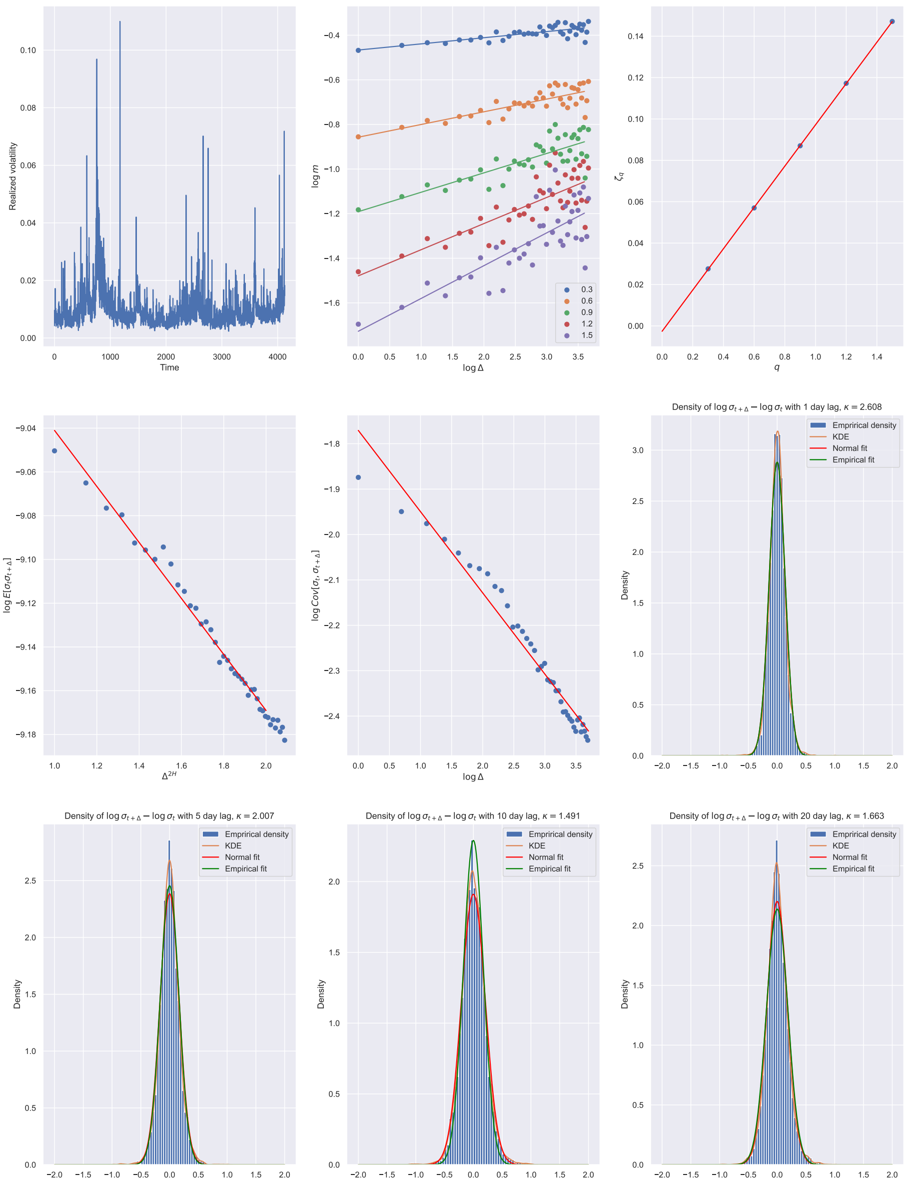

Figure 4.36: .OMXC20. $\hat{H}$ plots

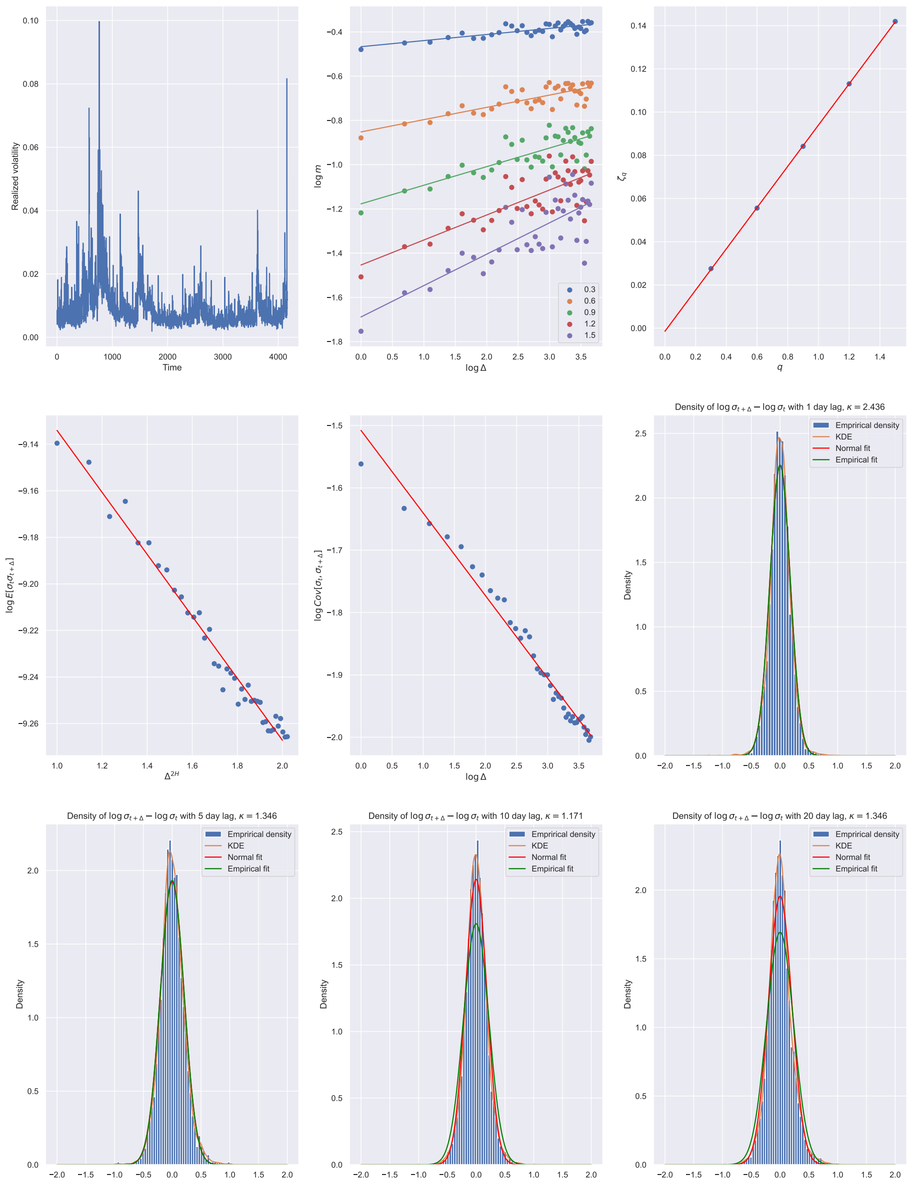

Figure 4.37: .OMXHPI. $\hat{H}$ plots

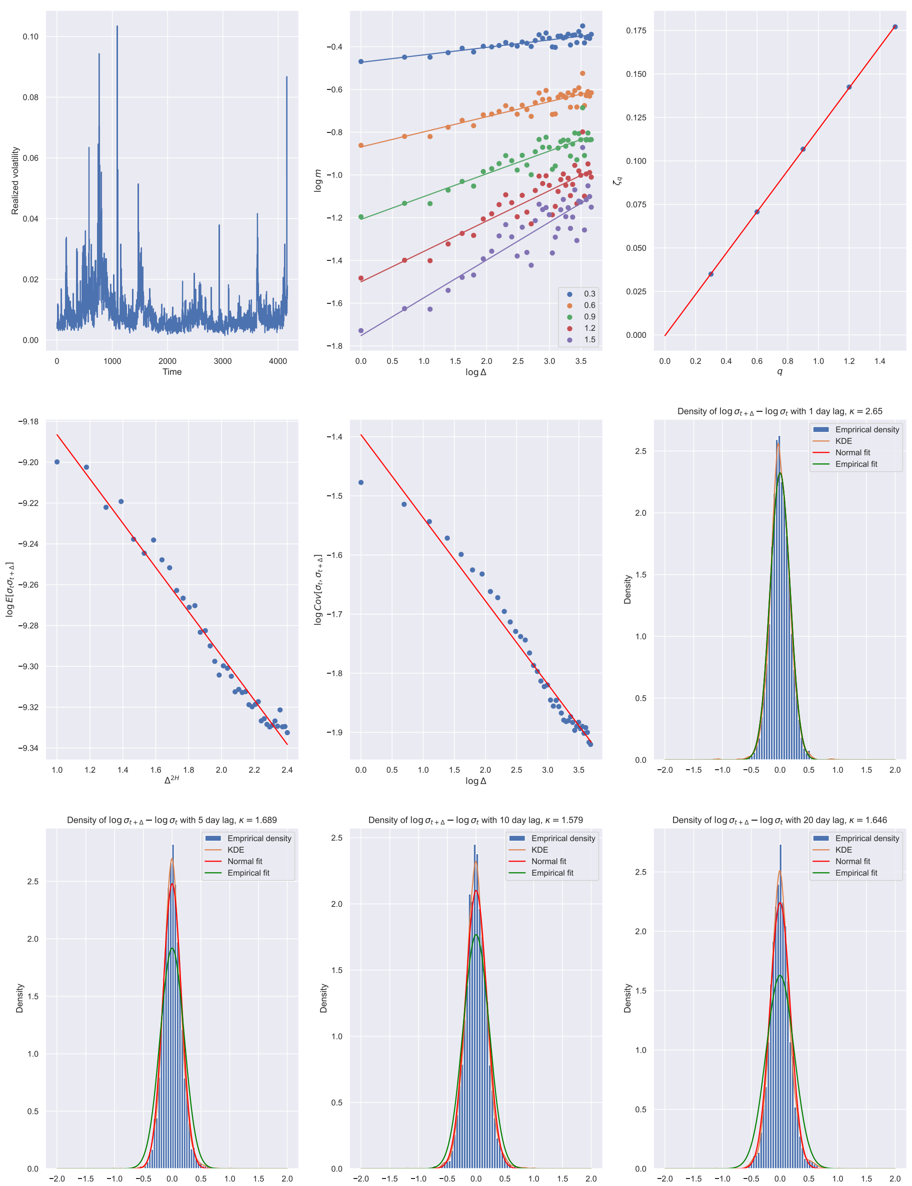

Figure 4.38: .OMXSPI. $\hat{H}$ plots

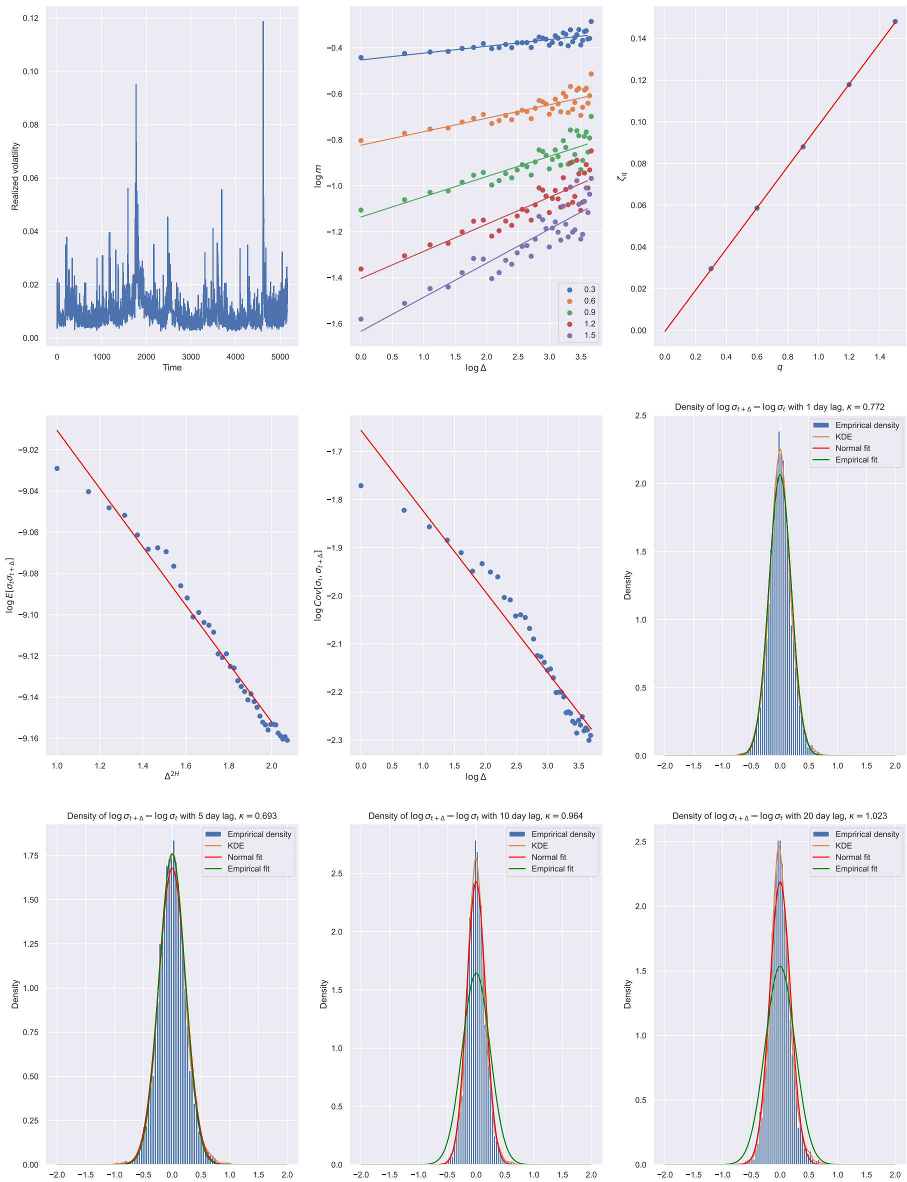

Figure 4.39: .OSEAX. $\hat{H}$ plots

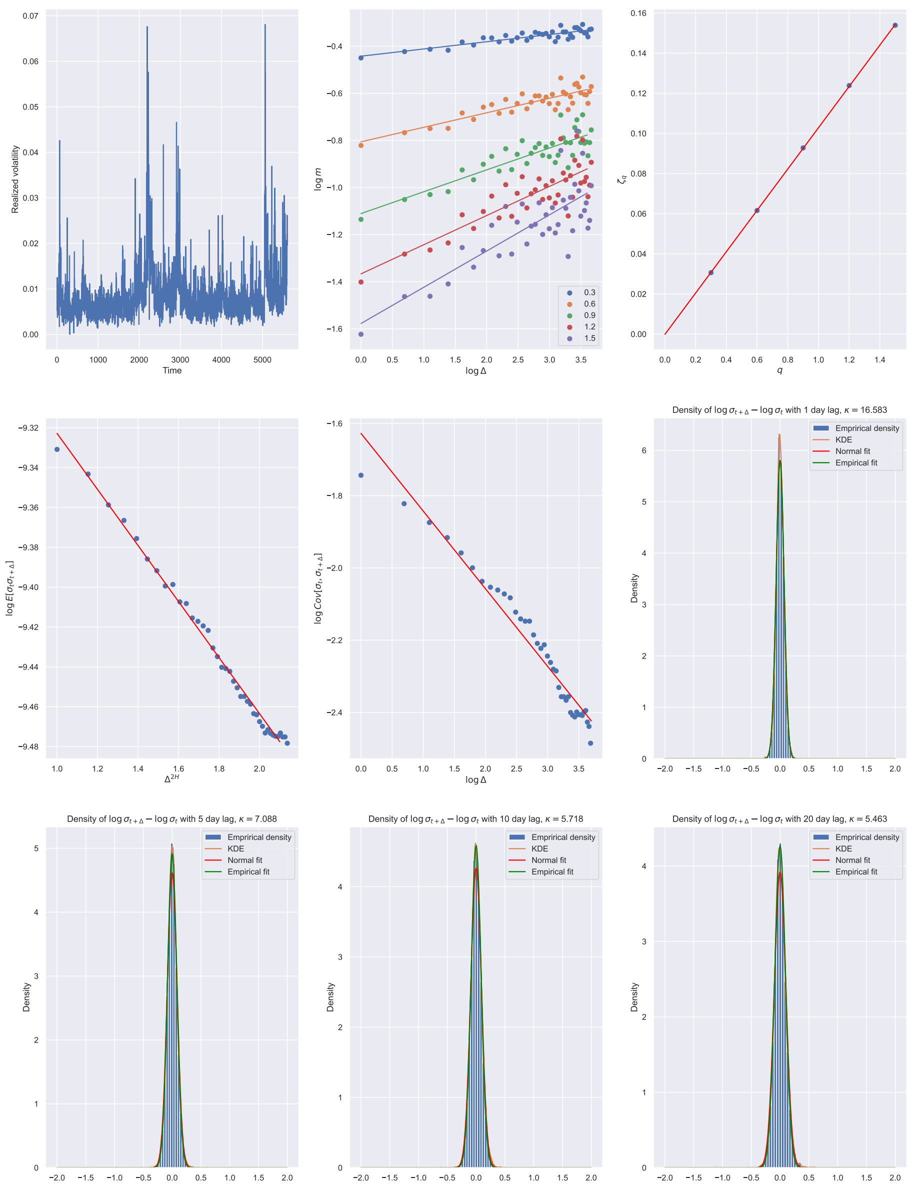

Figure 4.40: .RUT. $\hat{H}$ plots

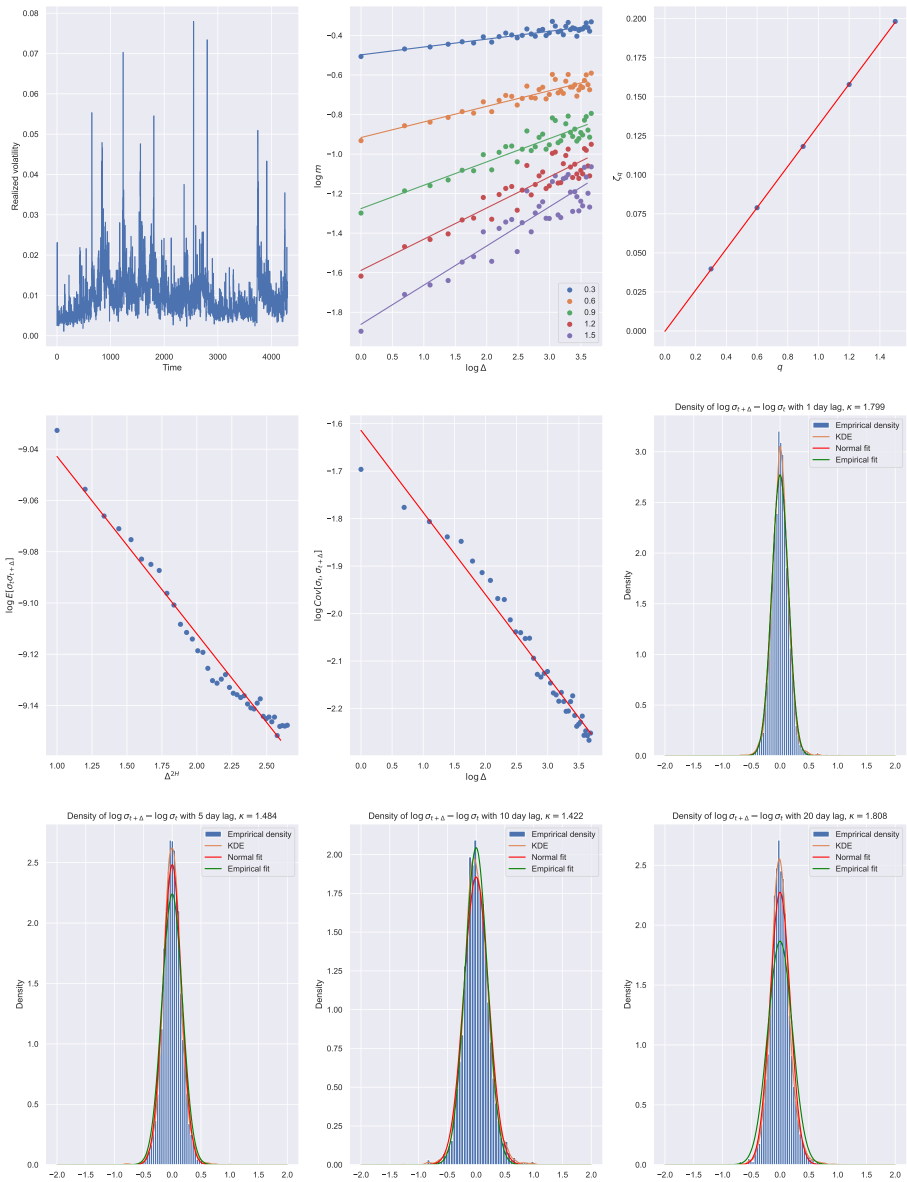

Figure 4.41: .SMSI. $\hat{H}$ plots

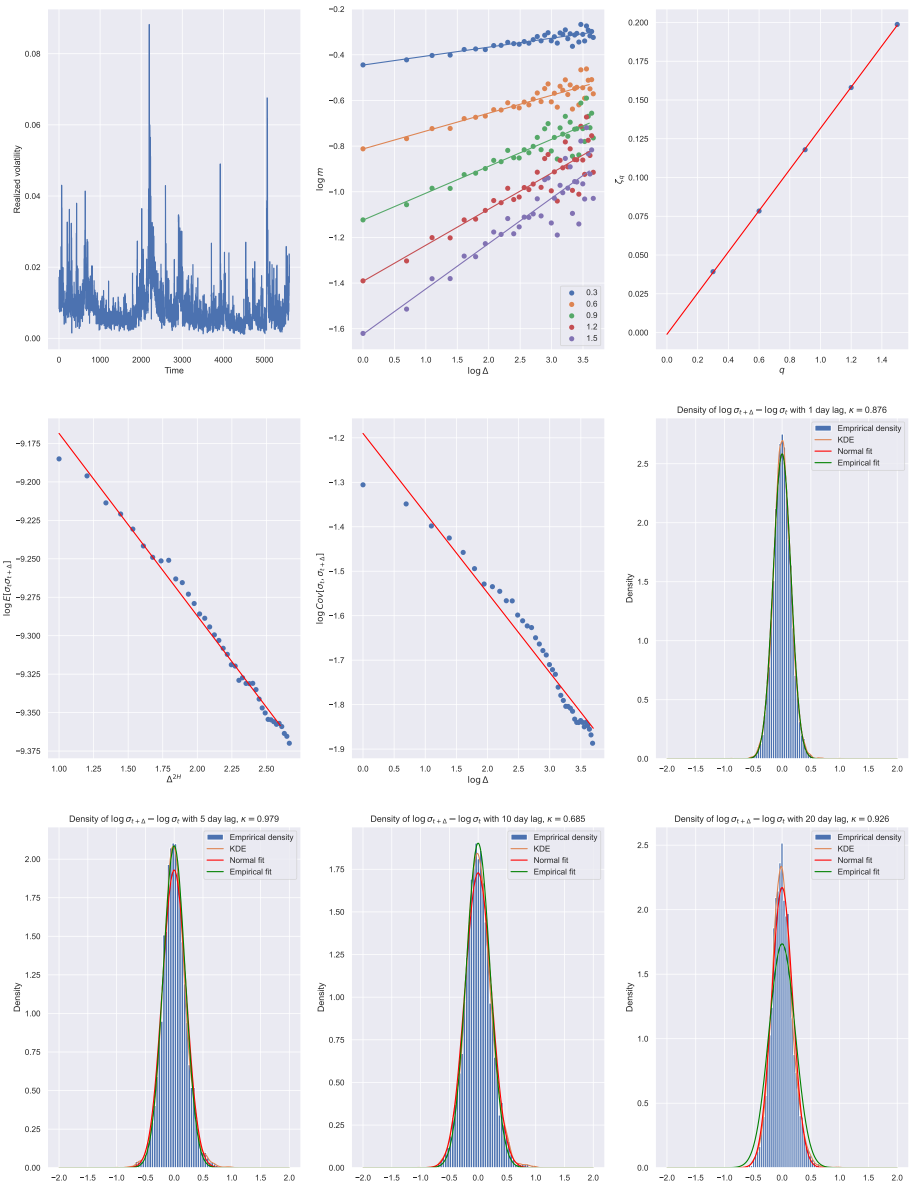

Figure 4.42: .SPX. $\hat{H}$ plots

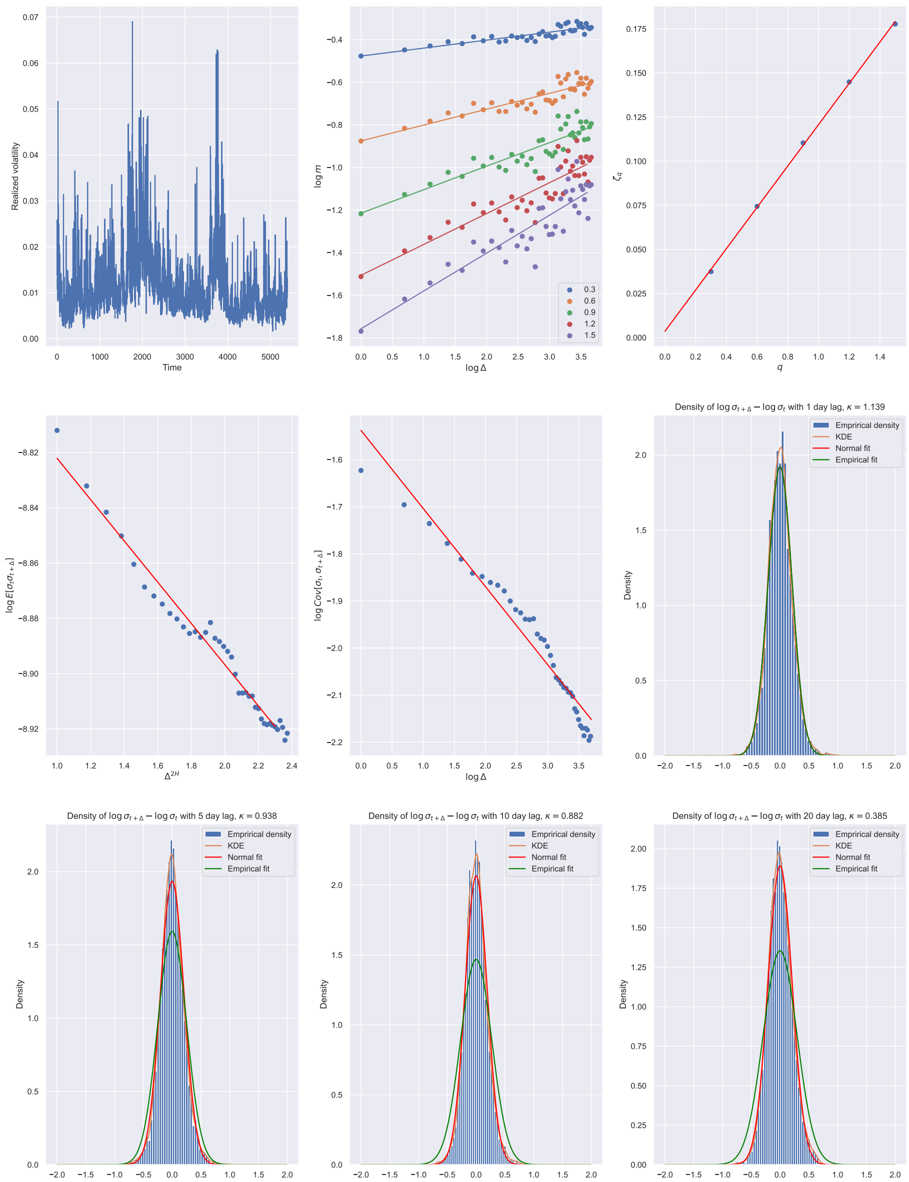

Figure 4.43: .SSEC. $\hat{H}$ plots

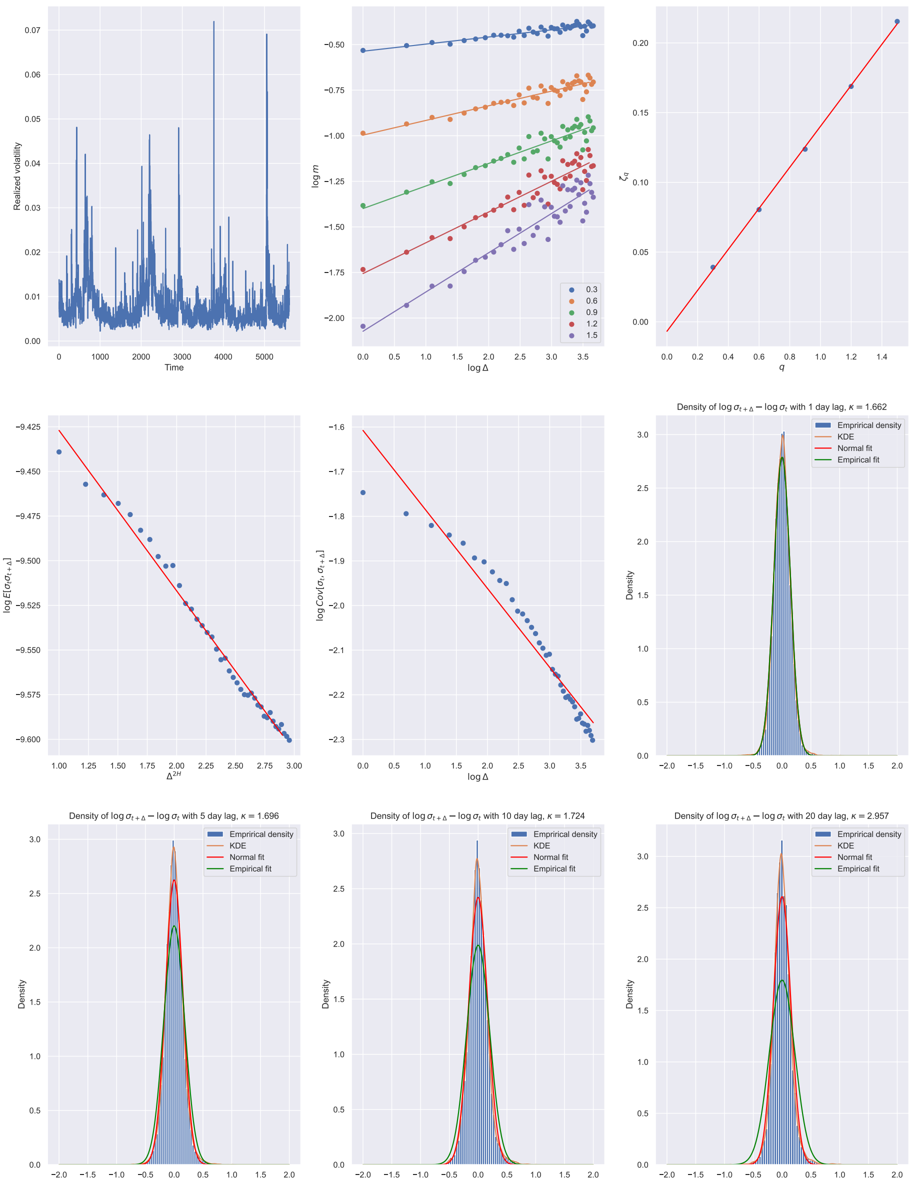

Figure 4.44: .SSMI. $\hat{H}$ plots

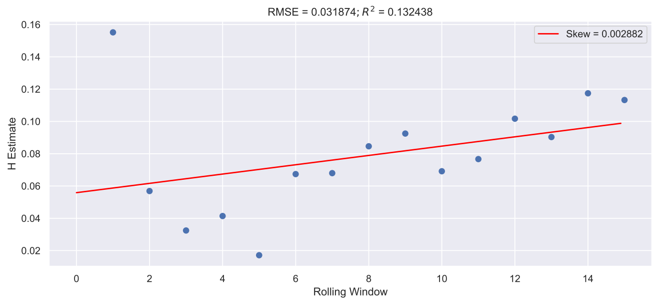

Figure 4.45: SBER RX Equity Smoothing Effect

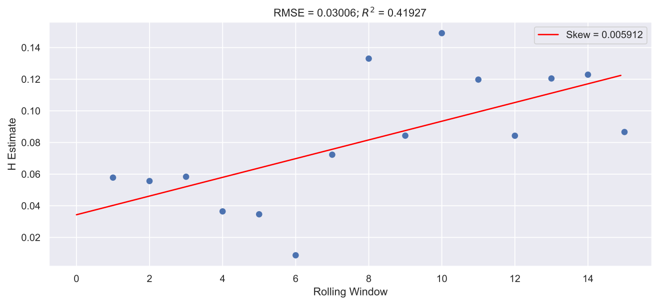

Figure 4.46: SBER LI Equity Smoothing Effect

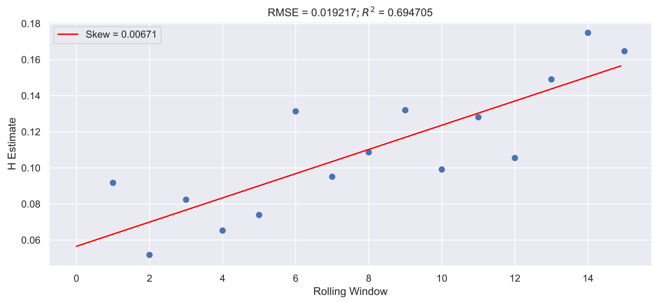

Figure 4.47: VTBR RX Equity Smoothing Effect

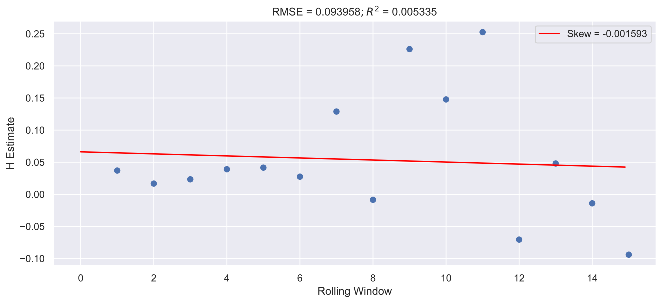

Figure 4.48: VTBR LI Equity Smoothing Effect

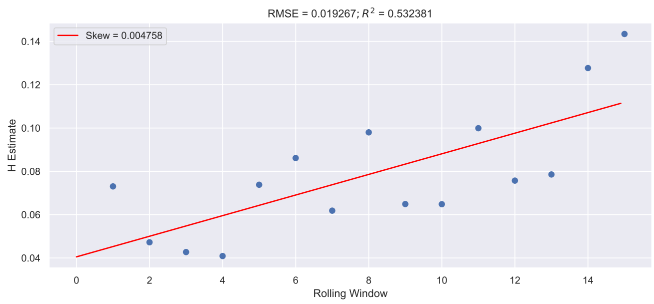

Figure 4.49: LKOH RX Equity Smoothing Effect

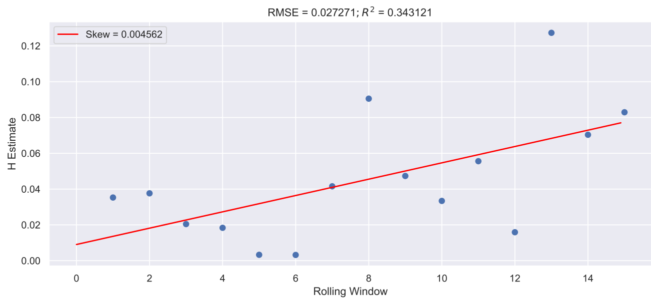

Figure 4.50: LKOD LI Equity Smoothing Effect

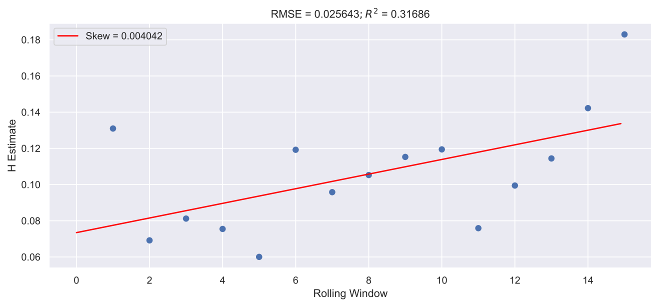

Figure 4.51: GAZP RX Equity Smoothing Effect

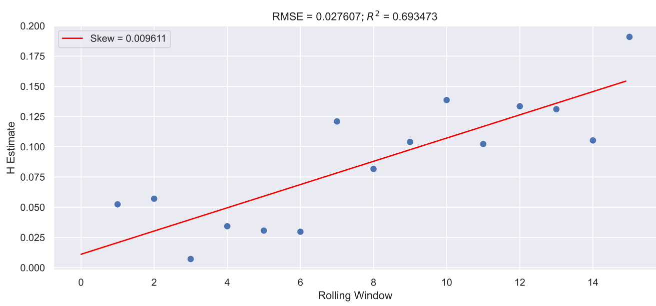

Figure 4.52: OGZD LI Equity Smoothing Effect

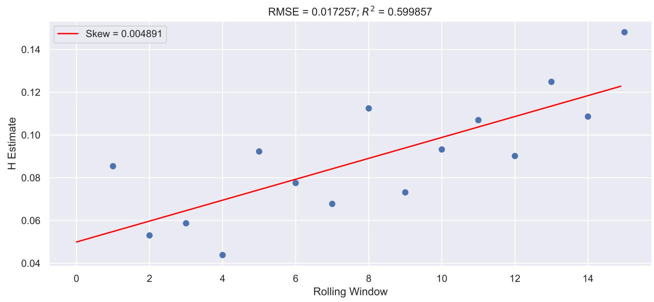

Figure 4.53: MOEX RX Equity Smoothing Effect

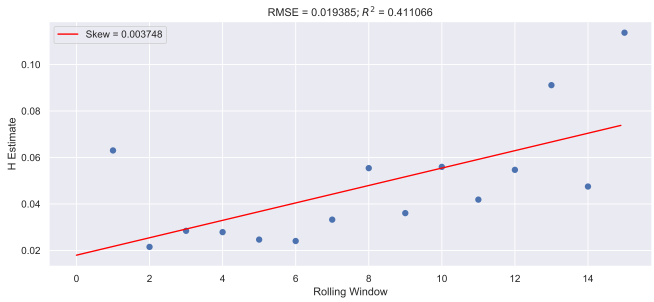

Figure 4.54: FIVE RX Equity Smoothing Effect

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0008	0.0001	Not normal
11	0.022	0.0059	Not normal
12	0.0003	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0034	0.002	Not normal
16	0.0001	0.0	Not normal
17	0.0002	0.0002	Not normal
18	0.0	0.0	Not normal
19	0.0004	0.0001	Not normal
20	0.0009	0.0001	Not normal
21	0.0003	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.2: Normality tests for YNDX RX Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0007	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.3: Normality tests for SBER RX Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0002	0.0001	Not normal
18	0.0	0.0	Not normal
19	0.0164	0.0184	Not normal
20	0.0001	0.0002	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0002	Not normal
23	0.0017	0.0183	Not normal
24	0.0025	0.0105	Not normal
25	0.0001	0.0004	Not normal
26	0.0007	0.0117	Not normal
27	0.0085	0.0096	Not normal
28	0.0166	0.0667	Normal
29	0.0152	0.0601	Normal
30	0.0636	0.0765	Normal
31	0.3774	0.272	Normal
32	0.8805	0.7112	Normal
33	0.5652	0.1653	Normal
34	0.0095	0.1165	Normal
35	0.0309	0.2913	Normal
36	0.0	0.0002	Not normal
37	0.0009	0.0282	Not normal
38	0.0	0.0007	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.4: Normality tests for VTBR RX Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0007	0.0003	Not normal
20	0.0009	0.0029	Not normal
21	0.0	0.0	Not normal
22	0.0017	0.0004	Not normal
23	0.0002	0.0	Not normal
24	0.0004	0.0001	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0014	0.0012	Not normal
28	0.0046	0.0005	Not normal
29	0.0	0.0	Not normal
30	0.0006	0.0001	Not normal
31	0.0005	0.0	Not normal
32	0.0765	0.0107	Normal
33	0.0432	0.0067	Not normal
34	0.0162	0.0018	Not normal
35	0.012	0.0273	Not normal
36	0.0001	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0002	0.0	Not normal
44	0.0007	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0008	0.0001	Not normal

Table 4.5: Normality tests for MOEX RX Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0003	0.0009	Not normal
34	0.0012	0.0031	Not normal
35	0.0009	0.0012	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.6: Normality tests for LKOH RX Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0002	0.0017	Not normal
29	0.0009	0.0016	Not normal
30	0.0007	0.0031	Not normal
31	0.0081	0.0018	Not normal
32	0.0097	0.0029	Not normal
33	0.3172	0.1282	Normal
34	0.189	0.2963	Normal
35	0.0249	0.3261	Normal
36	0.0066	0.0671	Normal
37	0.0	0.0004	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.7: Normality tests for GAZP RX Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.8: Normality tests for FIVE RX Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.9: Normality tests for OGZD LI Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0001	0.0001	Not normal
10	0.0	0.0	Not normal
11	0.0003	0.0001	Not normal
12	0.0001	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0012	0.0011	Not normal
15	0.0013	0.0007	Not normal
16	0.001	0.0005	Not normal
17	0.0038	0.0004	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0001	0.0	Not normal
21	0.0001	0.0	Not normal
22	0.0003	0.0003	Not normal
23	0.0002	0.0	Not normal
24	0.005	0.0017	Not normal
25	0.0	0.0	Not normal
26	0.0057	0.0016	Not normal
27	0.0094	0.0079	Not normal
28	0.0015	0.0046	Not normal
29	0.0004	0.0002	Not normal
30	0.0288	0.0195	Not normal
31	0.0013	0.0001	Not normal
32	0.0943	0.0363	Normal
33	0.0003	0.0001	Not normal
34	0.0126	0.0019	Not normal
35	0.0007	0.0003	Not normal
36	0.0006	0.0004	Not normal
37	0.0004	0.0001	Not normal
38	0.0032	0.0009	Not normal
39	0.0012	0.0006	Not normal
40	0.0221	0.0028	Not normal
41	0.0009	0.0001	Not normal
42	0.0054	0.0017	Not normal
43	0.0002	0.0001	Not normal
44	0.032	0.0072	Not normal
45	0.072	0.0287	Normal
46	0.0061	0.0007	Not normal
47	0.0452	0.0161	Not normal
48	0.0025	0.003	Not normal
49	0.0057	0.0008	Not normal

Table 4.10: Normality tests for VTBR LI Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.11: Normality tests for SBER LI Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.12: Normality tests for LKOD LI Equity

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.13: Normality tests for .AEX

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0053	0.0015	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.14: Normality tests for .AORD

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.15: Normality tests for .BFX

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.16: Normality tests for .BVSP

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.17: Normality tests for .DJI

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.18: Normality tests for .FCHI

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.19: Normality tests for .FTMIB

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.20: Normality tests for .FTSE

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.21: Normality tests for .GDAXI

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.22: Normality tests for .GSPTSE

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.23: Normality tests for .HSI

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.24: Normality tests for .IBEX

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.25: Normality tests for .IXIC

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.26: Normality tests for .KS11

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.27: Normality tests for .KSE

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.28: Normality tests for .MXX

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.29: Normality tests for .N225

Δ	Shapiro-Wilk (p-value)	K^2 (p-value)	Conclusion ($\alpha = 0.05$)
1	0.0	0.0	Not normal
2	0.0	0.0	Not normal
3	0.0	0.0	Not normal
4	0.0	0.0	Not normal
5	0.0	0.0	Not normal
6	0.0	0.0	Not normal
7	0.0	0.0	Not normal
8	0.0	0.0	Not normal
9	0.0	0.0	Not normal
10	0.0	0.0	Not normal
11	0.0	0.0	Not normal
12	0.0	0.0	Not normal
13	0.0	0.0	Not normal
14	0.0	0.0	Not normal
15	0.0	0.0	Not normal
16	0.0	0.0	Not normal
17	0.0	0.0	Not normal
18	0.0	0.0	Not normal
19	0.0	0.0	Not normal
20	0.0	0.0	Not normal
21	0.0	0.0	Not normal
22	0.0	0.0	Not normal
23	0.0	0.0	Not normal
24	0.0	0.0	Not normal
25	0.0	0.0	Not normal
26	0.0	0.0	Not normal
27	0.0	0.0	Not normal
28	0.0	0.0	Not normal
29	0.0	0.0	Not normal
30	0.0	0.0	Not normal
31	0.0	0.0	Not normal
32	0.0	0.0	Not normal
33	0.0	0.0	Not normal
34	0.0	0.0	Not normal
35	0.0	0.0	Not normal
36	0.0	0.0	Not normal
37	0.0	0.0	Not normal
38	0.0	0.0	Not normal
39	0.0	0.0	Not normal
40	0.0	0.0	Not normal
41	0.0	0.0	Not normal
42	0.0	0.0	Not normal
43	0.0	0.0	Not normal
44	0.0	0.0	Not normal
45	0.0	0.0	Not normal
46	0.0	0.0	Not normal
47	0.0	0.0	Not normal
48	0.0	0.0	Not normal
49	0.0	0.0	Not normal

Table 4.30: Normality tests for .OMXC20

Appendix B. Estimation Code.

```
def rlz_vol_est(df: pd.DataFrame,
                count: int,
                rolling_window: int=1) -> np.ndarray:
    log_returns = np.zeros(int(df.shape[0]/rolling_window))
    for i in range(1, log_returns.size):
        log_returns[i] =
                math.log(df["Mean"][i*rolling_window]/
                                    df["Mean"][(i-1)*rolling_window])

    rlz_vol = np.zeros(int(log_returns.size/count))

    for i in range(rlz_vol.size):
        lr_n = np.zeros(count)
        for n in range(count):
            lr_n[n] = log_returns[i*count+n]

        tmp = 0.0
        for j in range(1, lr_n.size):
            tmp += (lr_n[j] - lr_n[j-1])**2

        rlz_vol[i] = math.sqrt(tmp)

    return rlz_vol


def hurst_estimation(name: str,
                     mode: str = 'yf',
                     rolling_window: int = 1,
                     show_pics = True,
                     save_pics = False):
    if mode == 'yf':
        count = days_count
        df = yf.download(name, '2000-01-01', '2019-01-01')
        df["Mean"] = 0.5*(df["Open"]+df["Close"])

    elif mode == 'bb':
        count = minutes_count
        df = pd.read_csv('data_bloomberg/'+name+'.csv', sep="\t")
        df["Mean"] = 0.5*(df["High"]+df["Low"])

    volatility_array = rlz_vol_est(df = df,
                                   count = count,
                                   rolling_window = rolling_window)
    zetaq            = np.zeros((2, num_of_q))

    for I in range(0, num_of_q):
        graph_data = np.zeros((2, pD-sD))
        q          = step_of_q*(1+I)
```

```
    line_start = math.log(sD)
    line_stop  = math.log(pD)

    for Delta in range(sD, pD):
        graph_data[0, Delta-sD] = math.log(Delta)
        graph_data[1, Delta-sD] = math.log(m(q, Delta, volatility_array))

    linear_model    = np.polyfit(graph_data[0],graph_data[1], 1)
    linear_model_fn = np.poly1d(linear_model)
    x_s             = np.arange(line_start, line_stop, 0.1)

    skew_of_linear_model = skew(line_start,
                                line_stop,
                                linear_model_fn(line_start),
                                linear_model_fn(line_stop))

    zetaq[0, I] = q
    zetaq[1, I] = skew_of_linear_model

linear_model_H    = np.polyfit(zetaq[0], zetaq[1], 1)
linear_model_H_fn = np.poly1d(linear_model_H)
x_s               = np.arange(0, step_of_q*(num_of_q+1), step_of_q)

H_est = skew(0,
             step_of_q*(num_of_q)+1,
             linear_model_H_fn(0),
             linear_model_H_fn(step_of_q*(num_of_q)+1))

sz = 40
graph_data = np.zeros((2, sz))

for Delta in range(1, sz+1):
    graph_data[0, Delta-1] = Delta**(2*H_est)
    graph_data[1, Delta-1] = ACov(volatility_array, Delta)

linear_model    = np.polyfit(graph_data[0],graph_data[1], 1)
linear_model_fn = np.poly1d(linear_model)
x_s             = np.arange(1, (sz+1)**(2*H_est), 0.1)

for Delta in range(1, sz+1):
    graph_data[0, Delta-1] = math.log(Delta)

linear_model    = np.polyfit(graph_data[0],graph_data[1], 1)
linear_model_fn = np.poly1d(linear_model)
x_s             = np.arange(0, math.log(sz+1), 0.1)

def lag_array(Delta):
    retarr = np.zeros(volatility_array.size - Delta)
    if Delta >= 0:
```

```
            for i in range(0, volatility_array.size-Delta):
                retarr[i] = np.log(volatility_array[i+Delta]) -
                                        np.log(volatility_array[i])
        else:
            for i in range(0, volatility_array.size-math.abs(Delta)):
                retarr[i] = np.log(volatility_array[i]) -
                                        np.log(volatility_array[i-Delta])

        retarr = retarr/retarr.max()
        return retarr

    return H_est


def f(theta):
    return (1/((2*H+1)*(2*H+2)*theta**2)*((1+theta)**(2*H+2) - 2
                - 2 * theta**(2*H+2) + (1-theta)**(2*H+2)))

def smoothing_theoretical(delta: float):
    num_of_Deltas = 200
    plot = np.zeros((2, num_of_Deltas))

    Delta = np.arange(1, num_of_Deltas+1, 1)
    plot[0] = np.log(Delta)
    plot[1] = np.log(Delta**(2*H) * f(delta/Delta))

    linear_model    = np.polyfit(plot[0],plot[1], 1)
    linear_model_fn = np.poly1d(linear_model)
    x_s             = np.arange(0, 5, 0.1)

    print(skew(0, 1, linear_model_fn(0), linear_model_fn(1))*0.5)
    print(skew(0, 1, linear_model_fn(0), linear_model_fn(1))*0.5/H - 1)

def smoothing_empirical(name: str, show_pics: bool=True):
    num_of_wind = 20
    graph_data = np.zeros((2, num_of_wind))
    for i in range(1, num_of_wind+1):
        graph_data[0, i-1] = i
        graph_data[1, i-1] = analyse_volatility(name=name,
                                                mode='bb',
                                                rolling_window=i,
                                                show_pics=False)

    return [np.mean(graph_data[1]),
            np.std(graph_data[1]),
            np.min(graph_data[1])]
```