Structural Dynamics

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Title: Free Vibration Systems (SDOF)

Question: Viscously damped system has, m = 10 kg, k = 1000 N/m, If the system is subjected to initial displacement and velocity of 0.1 m and 10 m/s respectively,

(a) Plot the undamped free vibration response of the system.

(b) Plot the damped free vibration response of the system considering under-damped, critical and over- damped conditions.

Answer:

Given system parameters are:

1. Mass (m) = 10 kg
2. Stiffness (k) = 1000 N/m
3. Initial Displacement (u_o) = 0.1 m
4. Initial Velocity (v_o) = 10 m/s

- Natural Frequency $(\omega_n) = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10\ rad/s$

- Critical Damping $(c_{cr}) = 2 \cdot \sqrt{m \cdot k} = 2 \cdot 100 = 200\ kg/s$

$$\boxed{\begin{array}{l}\text{General Equation} : m\ddot{x} + c\dot{x} + kx = 0 \\ \text{General Soution} : x(t) = e^{st}\end{array}}$$

Case 1: Undamped Free Vibrations (c = 0)

Substituting c = 0 and values of m & k, we get

$$\rightarrow \quad s^2 + \omega_n^2 = 0$$

$$\rightarrow \quad s = \pm i\omega_n$$

General Solution of this case can be given as

$\rightarrow\ u(t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t)$

where, C_1 and C_2 are constants.

Applying boundary conditions,

a) Initial displacement (u_o) = 0.1 m

$\rightarrow\ u(0) = C_1 = 0.1\text{ m}$

$\boxed{\therefore C_1 = 0.1}$

b) Initial Velocity (v_o) = 10 m/s

$\rightarrow \dot{u}(0) = C_2\omega_n = 10$

$\rightarrow C_2 = 10/\omega_n = 1$

$\boxed{\therefore C_2 = 1}$

Final Equation for Undamped Free Vibration system is:

$$\boxed{\rightarrow\ u(t) = 0.1\cos(10t) + \sin(10t)}$$

Case 2: Damped Free Vibrations – Critical Condition (c = 200)

$$\rightarrow\ \zeta = \frac{c}{c_{cr}} = \frac{200}{200} = 1$$

$$\rightarrow\ \omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-0.25} = 8.66\,\text{rad/s}$$

Substituting c = 200 and values of m & k, we get

$\rightarrow\ s^2 + 2\omega_n s + \omega_n^2 = 0$

$\rightarrow\ s = -\,\omega_n\ (Double\ root)$

General Solution of this case can be given as

$\rightarrow\ u(t) = (C_1 + C_2 t)e^{-\omega_n t}$

$\rightarrow\ \dot{u}(t) = \big(C_2 - \omega_n(C_1 + C_2 t)\big)e^{-\omega_n t}$

where, C_1 and C_2 are constants.

Applying boundary conditions,

a) Initial displacement (u_o) = 0.1 m

$$\rightarrow u(0) = C_1 = 0.1 \text{ m}$$

$$\boxed{\therefore C_1 = 0.1}$$

b) Initial Velocity (v_o) = 10 m/s

$$\rightarrow \dot{u}(0) = C_2 - C_1\omega_n = 10$$

$$\rightarrow C_2 = 10 + 1 = 11$$

$$\boxed{\therefore C_2 = 11}$$

Final Equation for Damped Free Vibration system **(Critical Condition)** is:

$$\boxed{\rightarrow u(t) = (0.1 + 11t)e^{-10t}}$$

Case 3: Damped Free Vibrations – Underdamped Condition (c = 100)

$$\rightarrow \zeta = \frac{c}{c_{cr}} = \frac{100}{200} = 0.5$$

$$\rightarrow \omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-0.25} = 8.66 \, \text{rad/s}$$

Substituting c = 100 and values of m & k, we get

$$\rightarrow s^2 + \omega_n s + \omega_n^2 = 0$$

General Solution of this case can be given as

$$\rightarrow u(t) = e^{-\zeta\omega_n t}(C_1\cos(\omega_d t) + C_2\sin(\omega_d t))$$

$$\rightarrow \dot{u}(t) = e^{-\zeta\omega_n t}\,[-\zeta\,\omega_n\,(C_1\cos(\omega_d\,t) \; + \; C_2\sin(\omega_d\,t))$$
$$+ \; (-C_1\omega_d\sin(\omega_d\,t) \; + \; C_2\omega_d\cos(\omega_d\,t))\,]$$

where, C_1 and C_2 are constants.

Applying boundary conditions,

a) Initial displacement (u_o) = 0.1 m

$$\rightarrow u(0) = C_1 = 0.1 \text{ m}$$

$$\boxed{\therefore C_1 = 0.1}$$

b) Initial Velocity (v_o) = 10 m/s

$$\rightarrow \dot{u}(0) = -\zeta\omega_n C_1 + C_2\omega_d = 10$$

$$\boxed{\therefore C_2 = 1.212}$$

$$\rightarrow C_2 = (10 + 5)/8.66 = 1.212$$

Final Equation for Damped Free Vibration system **(Under-damped condition)** is:

$$\boxed{\rightarrow u(t) = e^{-5t}(0.1\cos(8.66t) + 1.212\sin(8.66t))}$$

<u>Case 4: Damped Free Vibrations – Overdamped Condition (c = 400)</u>

$$\rightarrow \quad \zeta = \frac{c}{c_{cr}} = \frac{400}{200} = 2$$

Substituting c = 400 and values of m & k, we get

$$\rightarrow \quad s^2 + 4\omega_n s + \omega_n^2 = 0$$

$$\rightarrow \quad s_1 = -37.32 \;\; \& \;\; s_2 = -2.68$$

General Solution of this case can be given as

$$\rightarrow u(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

$$\rightarrow \quad \dot{u}(t) = C_1 s_1 e^{s_1 t} + C_2 s_2 e^{s_2 t}$$

where, C_1 and C_2 are constants.

Applying boundary conditions,

a) <u>Initial displacement (u_o) = 0.1 m</u>

$$\rightarrow \; u(0) = \; C_1 + C_2 = 0.1 \text{ m}$$

b) <u>Initial Velocity (v_o) = 10 m/s</u>

$$\rightarrow \dot{u}(0) = C_1 s_1 + C_2 s_2 = 10$$

$$\rightarrow C_2 = (10 + 5)/8.66 = 1.212$$

On solving both equations, we get

$$\rightarrow \quad C_1 = \frac{s_1 u_0 - \dot{u}_0}{s_1 - s_2} = -0.296$$

$$\boxed{\therefore C_1 = -0.296}$$

$$\rightarrow C_2 = \frac{\dot{u}_0 - s_2 u_0}{s_1 - s_2} = 0.396$$

$$\boxed{\therefore C_2 = 0.396}$$

Final Equation for Damped Free Vibration system **(Overdamped condition)** is:

$$\boxed{\rightarrow u(t) = -0.296e^{-37.32t} + 0.396e^{-2.679t}}$$

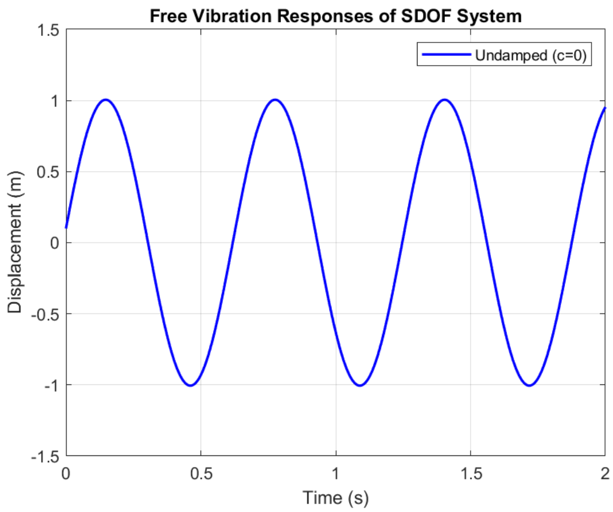

Fig.1: Undamped System (c = 0)

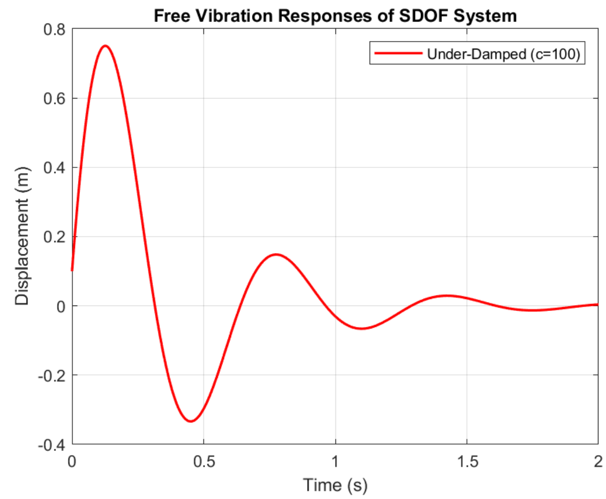

Fig.2: Under-damped System (c = 100)

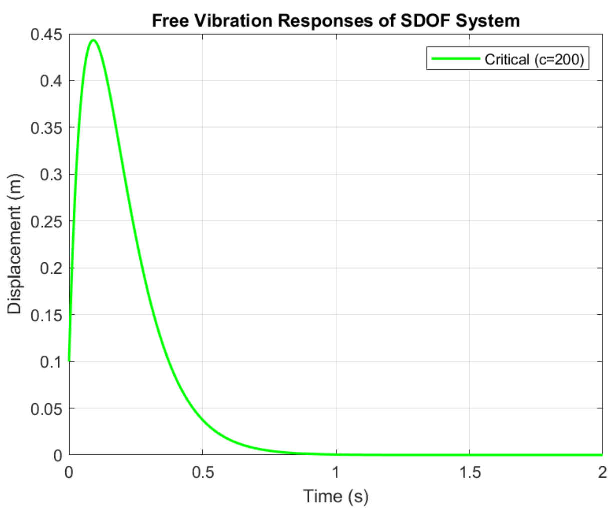

Fig.3: Critical Damped System (c = 200)

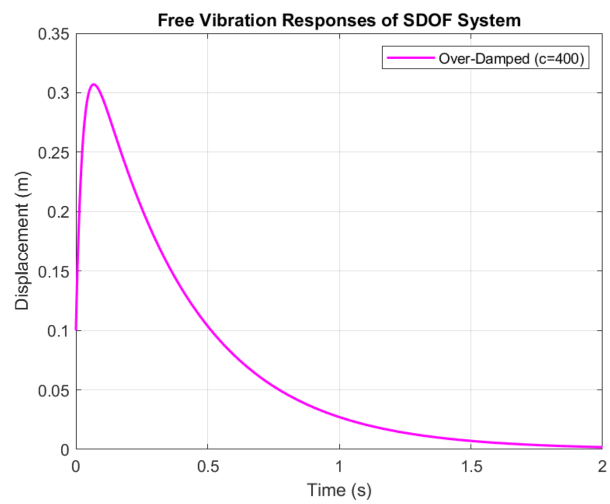

Fig.4 Over-damped System (c = 400)

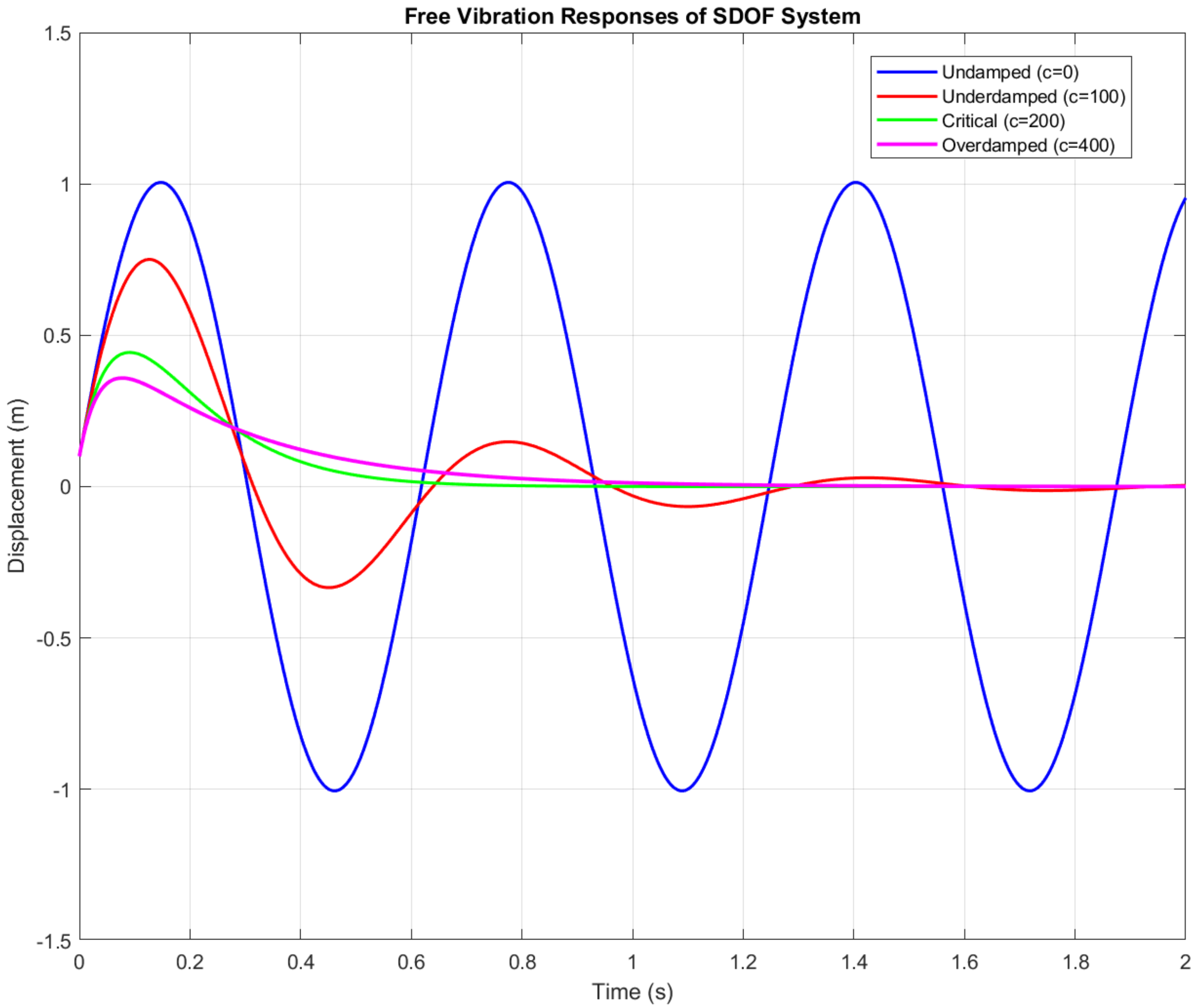

Fig.5: Comparison of all free vibrations systems

<u>MATLAB Code</u>

```
% Given Data
m = 10;                % kg
k = 1000;              % N/m
x0 = 0.1;              % initial displacement (m)
v0 = 10;               % initial velocity (m/s)

omega_n = sqrt(k/m); % natural frequency (rad/s)
c_c = 2*m*omega_n;    % critical damping (Ns/m)

% Time vector
t = linspace(0, 2, 1000);    % 0 to 2 seconds

%% 1) Undamped case (c=0, zeta=0)
x_undamped = x0*cos(omega_n*t) + (v0/omega_n)*sin(omega_n*t);
```

```
%% 2) Underdamped case (c=100)
c = 50;
zeta = c/c_c;
omega_d = omega_n*sqrt(1 - zeta^2);
x_underdamped = exp(-zeta*omega_n*t).*( x0*cos(omega_d*t) +
(v0+zeta*omega_n*x0)/omega_d * sin(omega_d*t) );

%% 3) Critically damped case (c = c_c)
c = c_c;
x_critical = ( x0 + (v0 + omega_n*x0)*t ) .* exp(-omega_n*t);

%% 4) Overdamped case (c=400)
c = 400;
zeta = c/c_c;
s1 = -omega_n*(zeta + sqrt(zeta^2 - 1));
s2 = -omega_n*(zeta - sqrt(zeta^2 - 1));
A = (v0 - s2*x0)/(s1 - s2);
B = (s1*x0 - v0)/(s1 - s2);
x_overdamped = A*exp(s1*t) + B*exp(s2*t);

%% Code for plotting various graph!!
figure;
plot(t, x_undamped, 'b-', 'LineWidth', 1.5); hold on;
plot(t, x_underdamped, 'r-', 'LineWidth', 1.5);
plot(t, x_critical, 'g-', 'LineWidth', 1.5);
plot(t, x_overdamped, 'm-', 'LineWidth', 1.8);
grid on;
xlabel('Time (s)'); ylabel('Displacement (m)');
title('Free Vibration Responses of SDOF System');
legend('Undamped (c=0)', 'Underdamped (c=100)','Critical (c=200)','Overdamped
(c=400)','Location','best');
```