21MAB201T - Transforms and Boundary Value Problems
Unit IV - Fourier Transforms

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FOURIER COSINE TRANSFORM

The infinite Fourier cosine transform of $f(x)$ is defined by

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ sx\ dx$$

The inverse Fourier cosine transform $F_c[f(x)]$ is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)]\ cos\ sx\ ds$$

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Properties of Fourier sine transform and Fourier cosine transform

(1) $F_s[f(ax)] = \frac{1}{a} F_s\left[\frac{s}{a}\right]$ [Change of scale property]

Proof:

$$F_s[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(ax)\ sin\ sx\ dx$$

$$\left.\begin{array}{r} \text{put } ax = t \\ a\ dx = dt \end{array}\right| \begin{array}{ll} x \to 0 & \Rightarrow t \to 0 \\ x \to \infty & \Rightarrow t \to \infty \end{array}$$

$$\begin{aligned} &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t)\ sin\ \left(\frac{st}{a}\right) \frac{dt}{a} \\ &= \frac{1}{a}\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t)\ sin\ \left(\frac{s}{a}t\right)\ dt \\ &= \frac{1}{a} F_s\left[\frac{s}{a}\right] \end{aligned}$$

$$\text{Similarly,}\quad F_c[f(ax)] = \frac{1}{a} F_c\left[\frac{s}{a}\right]$$

(2) Linear property

$$
\begin{aligned}
(i)\ F_s\ [a\ f(x) + b\ g(x)] &= a\ F_s[f(x)] + b\ F_s[g(x)] \\
(ii)\ F_c\ [a\ f(x) + b\ g(x)] &= a\ F_c[f(x)] + b\ F_c[g(x)]
\end{aligned}
$$

Proof : We know that,

$$
\begin{aligned}
F_s[f(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ sx\ dx \\
F_s\ [a\ f(x) + b\ g(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty [a\ f(x) + b\ g(x)]\ sin\ sx\ dx \\
&= a\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ sx\ dx + b\sqrt{\frac{2}{\pi}} \int_0^\infty g(x)\ sin\ sx\ dx \\
&= aF_s[f(x)] + bF_s[g(x)]
\end{aligned}
$$

(ii) We know that,

$$\begin{aligned}
F_c[f(x)] &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ cos\ sx\ dx \\
F_c\ [a\ f(x) + b\ g(x)] &= \sqrt{\frac{2}{\pi}}\int_0^\infty [a\ f(x) + b\ g(x)]\ cos\ sx\ dx \\
&= a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ cos\ sx\ dx + b\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\ cos\ sx\ dx \\
&= aF_c[f(x)] + bF_c[g(x)]
\end{aligned}$$

(3) Modulation property:

(i) $F_s[f(x)\ sin\ ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

(ii) $F_s[f(x)\ cos\ ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

(iii) $F_c[f(x)\ sin\ ax] = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$

(iv) $F_c[f(x)\ cos\ ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

Proof:

$$(i)\ \ F_s[f(x)\ sin\ ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ ax\ sin\ sx\ dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ sx\ sin\ ax\ dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \frac{1}{2}\ [cos\ (s-a)\ x - cos(s+a)\ x]\ dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ (s-a)\ x\ dx\ - \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos(s+a)\ x\ dx\right]$$

$$= \frac{1}{2}[F_c(s-a) - F_c(s+a)]$$

$$(ii)\ \ F_s[f(x)\ cos\ ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ ax\ cos\ sx\ dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ sx\ cos\ ax\ dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \frac{1}{2}\ [cos\ (s+a)\ x + cos(s-a)\ x]\ dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ (s+a)\ x\ dx\ + \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos(s-a)\ x\ dx\right]$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

$$(iii)\ \ F_c[f(x)\ sin\ ax] \quad = \quad \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ ax\ cos\ sx\ dx$$

$$= \quad \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \frac{1}{2}\ [sin\ (a+s)\ x - sin(a-s)\ x]\ dx$$

$$= \quad \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ (a+s)\ x\ dx\ + \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin(a-s)\ x\ dx\right]$$

$$= \quad \frac{1}{2}[F_s(a+s) - F_s(a-s)]$$

$$(iv)\ \ F_c[f(x)\ cos\ ax] \quad = \quad \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ ax\ cos\ sx\ dx$$

$$= \quad \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ sx\ cos\ ax\ dx$$

$$= \quad \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \frac{1}{2}\ [cos\ (s+a)\ x + cos(s-a)\ x]\ dx$$

$$= \quad \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ (s+a)\ x\ dx\ + \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos(s-a)\ x\ dx\right]$$

$$= \quad \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

(4) Derivative of transform (i) $F_s[f'(x)] = -sF_c(s)$, if $f(x) \rightarrow$ as $x \rightarrow \infty$

Proof :

$$\begin{aligned}
F_s[f'(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\ sin\ sx\ dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty sin\ sx\ d[f(x)] \\
&= \sqrt{\frac{2}{\pi}} [(sin\ sx\ f(x))_0^\infty - s \int_0^\infty f(x)\ cos\ sx\ dx] \\
&= -s\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ sx\ dx \\
& \quad [\text{Assuming } f(x) \rightarrow 0 \text{ as } x \rightarrow \infty] \\
&= -sF_c(s)
\end{aligned}$$

(ii) $F_s[xf(x)] = -\frac{d}{ds}[F_c(s)]$

Proof : We know that,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ sx\ dx$$

Differentiating both sides w.r to 's' we get

$$\begin{aligned}
\frac{d}{ds}F_c[f(x)] &= \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty f(x)\ cos\ sx\ dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \frac{\partial}{\partial s}\ (cos\ sx)\ dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ (-x\ sin\ sx)\ dx \\
&= -\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ x\ sin\ sx\ dx \\
&= -F_s[x\ f(x)] \\
\text{(i.e.,)}F_s[xf(x)] &= -\frac{d}{ds}F_c[f(x)]
\end{aligned}$$

Similarly, we have

$$F_c[x\ f(x)] = \frac{d}{ds}F_s(s)$$

Formula:

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ cos\ sx\ dx$$

Example 1: Find the Fourier cosine transform of

$$f(x) = \begin{cases} cosx, & \text{if } 0 < x < a. \\ 0, & \text{if } x \geq a. \end{cases}$$

Solution : We know that,

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ cos\ sx\ dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} cos\ x\ cos\ sx\ dx = \sqrt{\frac{2}{\pi}} \int_0^{a} cos\ sx\ cos\ x\ dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_0^{a} [cos\ (s+1)x + cos\ (s-1)x]dx$$

$$\left[\because cos\ A\ cos\ B = \frac{1}{2}[cos(A+B) + cos(A-B)]\right]$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\left[\frac{sin(s+1)x}{s+1}+\frac{sin(s-1)x}{s-1}\right]_0^a \\
&= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{sin(s+1)a}{s+1}+\frac{sin(s-1)a}{s-1}\right)-(0+0)\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{sin(s+1)a}{s+1}+\frac{sin(s-1)a}{s-1}\right]
\end{aligned}
$$

provided $s \neq 1;\ s \neq -1$

Example 2: Find the Fourier cosine transform of $\frac{e^{-ax}}{x}$ and hence, find $F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$

Solution : We know that,

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ cos\ sx\ dx$$

$$F_c\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\ cos\ sx\ dx$$

$$(\text{i.e.,})F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\ cos\ sx\ dx$$

$$
\begin{aligned}
\frac{d}{ds}F_c(s) &= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\ cos\ sx\ dx\right] \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left[\frac{e^{-ax}}{x}\ cos\ sx\right] dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\ (-x\ sin\ sx)\ dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty -e^{-ax}\ sin\ sx\ dx \\
&= -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\ sin\ sx\ dx \\
\frac{d}{ds}F_c(s) &= -\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]
\end{aligned}
$$

Formula: $\int_0^\infty e^{-ax}\ sin\ bx\ dx = \frac{b}{b^2+a^2}$ Here, $b = s$

By integrating, we get

$$F(s) = -\sqrt{\frac{2}{\pi}}\int \frac{s}{s^2+a^2}ds = -\sqrt{\frac{2}{\pi}}\frac{1}{2}\int \frac{2s}{s^2+a^2}ds$$

$$
\begin{aligned}
&= -\sqrt{\frac{2}{\pi}}\frac{1}{2}log(s^2+a^2)\\
\text{(i.e.,)}\quad F_c\left[\frac{e^{-ax}}{x}\right] &= -\frac{1}{\sqrt{2\pi}}log(s^2+a^2)\\
\text{similarly,}\ F_c\left[\frac{e^{-bx}}{x}\right] &= -\frac{1}{\sqrt{2\pi}}log(s^2+b^2)\\
\text{Now,}\ F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] &= F_c\left[\frac{e^{-ax}}{x}\right]-F_c\left[\frac{e^{-bx}}{x}\right]\\
&= -\frac{1}{\sqrt{2\pi}}log(s^2+a^2)+\frac{1}{\sqrt{2\pi}}log(s^2+b^2)\\
&= \frac{1}{\sqrt{2\pi}}\left[log(s^2+b^2)-log(s^2+a^2)\right]\\
&= \frac{1}{\sqrt{2\pi}}log\left[\frac{s^2+b^2}{s^2+a^2}\right]
\end{aligned}
$$

Example 3: Find the Fourier cosine transform of $e^{-ax} a > 0$.

Solution :

$$\text{We know that, } F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ cos\ sx\ dx$$

$$\begin{aligned} F_c[e^{-ax}] &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax}\ cos\ sx\ dx \\ &= \sqrt{\frac{2}{\pi}} \left(\frac{e^{-ax}}{a^2+s^2} (-acossx + ssinsx) \right)_0^{\infty} \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2} \right] \end{aligned}$$

$$\text{Formula: } \int_0^{\infty} e^{-ax}\ cos\ sx\ dx = \frac{a}{s^2+a^2}$$

Example 4: Find the Fourier cosine transform of the function $3e^{-5x} + 5e^{-2x}$

Solution: Let $f(x) = 3e^{-5x} + 5e^{-2x}$

$$\begin{aligned}
\text{We know that,} \quad F_c[f(x)] &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ cos\ sx\ dx \\
F_c[3e^{-5x} + 5e^{-2x}] &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} [3e^{-5x} + 5e^{-2x}]\ cos\ sx\ dx \\
&= \sqrt{\frac{2}{\pi}} 3 \int_0^{\infty} e^{-5x}\ cos\ sx\ dx \\
&\quad + \sqrt{\frac{2}{\pi}} 5 \int_0^{\infty} e^{-2x}\ cos\ sx\ dx \\
\text{We know that,} \int_0^{\infty} e^{-ax}\ cos\ bx\ dx &= \frac{a}{a^2 + b^2} \\
&= 3\sqrt{\frac{2}{\pi}} \left[\frac{5}{5^2 + s^2}\right] + 5\sqrt{\frac{2}{\pi}} \left[\frac{2}{2^2 + s^2}\right] \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{15}{25 + s^2} + \frac{10}{4 + s^2}\right]
\end{aligned}$$

Example 5: Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1, & \text{for } 0 < x < a. \\ 0, & \text{for } x > a. \end{cases}$$

Solution : We know that

$$\begin{aligned} F_c[f(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ sx\ dx \\ F_c[f(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty 1.\ cos\ sx\ dx \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{sin\ sx}{s}\right]_0^a \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{sin\ sa}{s} - 0\right] = \sqrt{\frac{2}{\pi}} \frac{sin\ sa}{s} \end{aligned}$$

Example 6: Find the Fourier cosine transform of $\frac{1}{a^2+x^2}$, Note: Ref Fourier Cosine Transform e rise to -ax, $a > 0$ and its inversion

Solution : We know that,

$$
\begin{aligned}
F_c[f(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ sx\ dx \\
F_c[\frac{1}{1+x^2}] &= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} e^{-s}\right] \\
F_c[\frac{1}{a^2+x^2}] &= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2a} e^{-as}\right] = \sqrt{\frac{\pi}{2}} \frac{1}{a} e^{-as}
\end{aligned}
$$

Example 7: Find the Fourier cosine transform of $e^{-a^2x^2}$, Note: Ref Fourier Transform of e rise to $-a^2x^2$, $a > 0$

Solution : We know that,

$$
\begin{aligned}
F_c[f(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ sx\ dx \\
F_c[e^{-a^2x^2}] &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2}\ cos\ sx\ dx \\
&= \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty e^{-a^2x^2}\ cos\ sx\ dx
\end{aligned}
$$

$$
\begin{aligned}
& \because [e^{-a^2x^2} cos\ sx \text{ is an even function}] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2}\ cos\ sx\ dx \\
&= \text{R.P. } F[e^{-a^2x^2}] \\
&= \text{R.P. } \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}} \\
F_c[e^{-a^2x^2}] &= \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}}
\end{aligned}
$$

Formula

$$F_c = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ sx\ dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(x)\ cos\ sx\ ds$$

Example 8: Solve the integral equation

$\int_0^\infty f(x) cos\lambda x dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1. \\ 0, & \lambda > 1. \end{cases}$ Hence, evaluate $\int_0^\infty \frac{sin^2 t}{t^2} dt$

Solutions : Given that,

$$\int_0^\infty f(x) cos\lambda x dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1. \\ 0, & \lambda > 1. \end{cases}$$

$$\int_0^\infty f(x) cos\ sx\ dx = \begin{cases} 1-s, & 0 \le s \le 1. \\ 0, & s > 1. \end{cases}$$

$$[\text{take}\lambda = s]$$

Multiply by $\sqrt{\frac{2}{\pi}}$ on both sides, we have

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) cos\ sx\ dx = \sqrt{\frac{2}{\pi}} \begin{cases} 1-s, & 0 \le s \le 1. \\ 0, & s > 1. \end{cases}$$

by Fourier cosine formula, $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \begin{cases} 1-s, & 0 \le s \le 1. \\ 0, & s > 1. \end{cases}$ (1)

We know that, Fourier cosine inversion formula is

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)]\ cos\ sx\ ds \\ f(x) &= \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{2}{\pi}} (1-s)\ cos\ sx\ ds \\ &= \frac{2}{\pi} \int_0^1 (1-s)\ cos\ sx\ ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi}\left[(1-s)\frac{sin\ sx}{x} - (-1)\left(\frac{-cos\ sx}{x^2}\right)\right]_{s=0}^{s=1} \\
&= \frac{2}{\pi}\left[(1-s)\frac{sin\ sx}{x} - \frac{-cos\ sx}{x^2}\right]_{s=0}^{s=1} \\
&= \frac{2}{\pi}\left[\left(0-\frac{cos\ x}{x^2}\right) - \left(0-\frac{1}{x^2}\right)\right] \\
&= \frac{2}{\pi}\left[-\frac{cos\ x}{x^2} + \frac{1}{x^2}\right] \\
f(x) &= \frac{2}{\pi}\left[\frac{1-cos\ x}{x^2}\right]
\end{aligned}
$$

We know that,

$$
\begin{aligned}
F_c[f(x)] &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ cos\ sx\ dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{2}{\pi}\left[\frac{1-cos\ x}{x^2}\right]\ cos\ sx\ dx \qquad (2)
\end{aligned}
$$

From (1) and (2), we have

$$\sqrt{\frac{2}{\pi}}\frac{2}{\pi}\int_0^\infty \frac{1-cos\ x}{x^2} cos\ sx\ dx = \begin{cases} \sqrt{\frac{2}{\pi}}\ (1-s), & 0 \le s \le 1. \\ 0, & s > 1. \end{cases}$$

Now, put $s \to 0$, we get

$$\begin{aligned}
\sqrt{\frac{2}{\pi}}\frac{2}{\pi}\int_0^\infty \frac{1-cos\ x}{x^2}\ dx &= \sqrt{\frac{2}{\pi}} \\
\frac{2}{\pi}\int_0^\infty \frac{1-cos\ x}{x^2}\ dx &= 1 \\
\int_0^\infty \frac{1-cos\ x}{x^2}\ dx &= \frac{\pi}{2} \\
\int_0^\infty \frac{2\ sin^2\ x/2}{x^2}\ dx &= \frac{\pi}{2}
\end{aligned}$$

$$\begin{array}{ll|ll} \text{put } t = \frac{x}{2} & & x \to 0 & \Rightarrow t \to 0 \\ dt = \frac{1}{2}\ dx & & x \to \infty & \Rightarrow t \to \infty \end{array}$$

$$\int_0^\infty \frac{2\ sin^2\ t}{(2t)^2}\ dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{sin^2\ t}{t^2}\ dt = \frac{\pi}{2}$$

Example 9: Find the Fourier cosine transform of $e^{-|x|}$ and deduce that $\int_0^\infty \frac{cos\ xt}{1+t^2} = \frac{\pi}{2} e^{-|x|}$

Solution : We know that,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ cos\ sx\ dx$$

$$F_c[e^{-|x|}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-|x|}\ cos\ sx\ dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x}\ cos\ sx\ dx$$

$$\because [\text{in the interval}(0, \infty), e^{-|x|} = e^{-x}]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2+1}\right]$$

Now, Fourier cosine inversion formula is

$$\begin{aligned}
f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c[f(x)]\; cos\ sx\ ds \\
f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2+1}\right]\; cos\ sx\ ds \\
&= \frac{2}{\pi} \int_0^{\infty} \frac{cos\ sx}{s^2+1}\, ds \\
\text{(i.e.,)} \int_0^{\infty} \frac{cos\ sx}{s^2+1}\, ds &= \frac{\pi}{2} f(x) \\
&= \frac{\pi}{2} e^{-|x|} \\
\text{(i.e.,)} \int_0^{\infty} \frac{cos\ sx}{s^2+1}\, ds &= \frac{\pi}{2} e^{-|x|} \quad \because (s \text{ is a dummy variable})
\end{aligned}$$

Example 10: Find the Fourier cosine transform of e^{-ax}, and deduce that $\int_0^\infty \frac{cos\ sx}{a^2+s^2}\ ds = \frac{\pi}{2a}e^{-ax}$

Solution : We know that,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ cos\ sx\ dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\ cos\ sx\ dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$$

$$\because \int_0^\infty e^{-ax}\ cos\ sx\ dx = \frac{a}{a^2+s^2}$$

Applying the inversion formula, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[e^{-ax}]\ cos\ sx\ ds$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\ cos\ sx\ ds$$

$$= \frac{2a}{\pi}\int_0^\infty \frac{cos\ sx}{a^2+s^2}\ ds$$

$$\text{(i.e.,)} \int_0^\infty \frac{cos\ sx}{a^2+s^2}\ ds = \frac{\pi}{2a}f(x) = \frac{\pi}{2a}e^{-ax}, a > 0$$

Formula :

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ sin\ sxdx$$

Example 11: Find the Fourier sine transform of

Solution : We know that,

$$\begin{aligned}
F_s[f(x)] &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ sin\ sx\ dx \\
F_s[e^{-x} cos\ x] &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} cos\ x\ sin\ sx\ dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} cos\ x\ sin\ sxdx \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \left[\frac{sin(s+1)x + sin(s-1)x}{2}\right] dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} e^{-x}\ sin\ (s+1)x\ dx\right] + \left[\int_0^{\infty} e^{-x}\ sin\ (s-1)x\ dx\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\left[\frac{s+1}{(s+1)^2+1}+\frac{s-1}{(s-1)^2+1}\right]\\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{s+1}{s^2+2s+1}+\frac{s-1}{s^2-2s+1}\right]\\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{s^3-2s^2+2s+s^2-2s+2+s^3+2s^2+2s-s^2-2s-2}{(x^2+2)^2-(2s)^2}\right]\\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{2s^3}{s^4+4+4s^2-4s^2}\right]\\
&= \frac{1}{\sqrt{2\pi}}\frac{2s^3}{s^2+4}
\end{aligned}
$$

Example 12: Find the Fourier sine transform of

$$
f(x)=\begin{cases} \cos x, & \text{if } 0<x<a. \\ 0, & \text{if } x\geq a. \end{cases}
$$

Solution : We know that,

$$
F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ sin\ sx\ dx
$$

$$
\begin{aligned}
F_s[f(x)] &= \sqrt{\frac{2}{\pi}} \int_0^a sinx \ sin \ sx \ dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^a sinsx \ sin \ x \ dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^a \frac{cos(s-1)x - cos(s+1)x}{2} \ dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_0^a cos(s-1)x \ dx - \int_0^a cos(s+1)x \ dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{sin(s-1)x}{s-1} \right)_0^a - \left(\frac{sin(s+1)x}{s+1} \right)_0^a \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{sin(s-1)a}{s-1} \right) - \left(\frac{sin(s+1)a}{s+1} \right) \right] \\
& \text{where } \ s \neq 1 \text{ and } s \neq -1
\end{aligned}
$$

Example 13: Find the Fourier sine transform of

$$f(x) = \begin{cases} x, & 0 < x < 1. \\ 2-x, & 1 < x < 2. \\ 0, & x > 2. \end{cases}$$

Solution : We know that,

$$F_s[f(x)]=\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ sx\ dx$$

$$\begin{aligned}
F_s[f(x)] &= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x\ sin\ sx\ dx + \int_1^2 (2-x)\ sin\ sx\ dx\right] \\
&= \sqrt{\frac{2}{\pi}} \left[x\left(\frac{-cos\ sx}{s}\right) - (1)\left(\frac{-sin\ sx}{s^2}\right)\right]_0^1 \\
&\quad + \sqrt{\frac{2}{\pi}} \left[(2-x)\left(\frac{-cos\ sx}{s}\right) - (-1)\left(\frac{-sin\ sx}{s^2}\right)\right]_1^2 \\
&= \sqrt{\frac{2}{\pi}} \left[\left(-x\frac{cos\ sx}{s} + \frac{sin\ sx}{s^2}\right)_0^1\right] + \left[\left(-(2-x)\frac{cos\ sx}{s} - \frac{sin\ sx}{s^2}\right)_1^2\right] \\
&= \sqrt{\frac{2}{\pi}} \left[\left[\left(\frac{-cos\ s}{s} + \frac{sin\ s}{s^2}\right) - (-0+0)\right] + \left[\left(-0 - \frac{sin\ 2s}{s^2}\right) - \left(\frac{-cos\ s}{s} - \frac{sins}{s^2}\right)\right]\right]
\end{aligned}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-cos\ s}{s} + \frac{sin\ s}{s^2} - \frac{sin\ 2s}{s^2} + \frac{cos\ s}{s} + \frac{sin\ s}{s^2}\right]$$
$$= \sqrt{\frac{2}{\pi}}\left[\frac{2sin\ s - sin\ 2s}{s^2}\right]$$

Example 14: Find the Fourier sine transform of $1/x$

Solution : We know that,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ sin\ sx\ dx$$
$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\ sin\ sx\ dx$$

$$\begin{array}{c|cc} \text{Let } sx = \theta & x \to 0 & \Rightarrow \theta \to 0 \\ s\ dx = d\theta & x \to \infty & \Rightarrow \theta \to \infty \end{array}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{s}{\theta}\right) sin\ \theta\ \frac{d\theta}{s}$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{sin\theta}{\theta}\ d\theta$$
$$= \sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2}\right] = \sqrt{\frac{\pi}{2}} \qquad \left[\because \int_0^\infty \frac{sin\theta}{\theta}\ d\theta = \frac{\pi}{2}\right]$$

Example 15: Find the Fourier sine transform of $3e^{-5x} + 5e^{-2x}$

Solution: Let $f(x) = 3e^{-5x} + 5e^{-2x}$

$$\text{We know that, } F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ sin\ sx\ dx$$

$$\begin{aligned}
F_s[3e^{-5x} + 5e^{-2x}] &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} [3e^{-5x} + 5e^{-2x}]\ sin\ sx\ dx \\
&= \sqrt{\frac{2}{\pi}} \left[\int_0^{\infty} 3e^{-5x}\ sin\ sx\ dx + \int_0^{\infty} 5e^{-2x}\ sin\ sx\ dx\right] \\
&= \sqrt{\frac{2}{\pi}} \left[3\int_0^{\infty} e^{-5x}\ sin\ sx\ dx + 5\int_0^{\infty} e^{-2x}\ sin\ sx\ dx\right] \\
&= \sqrt{\frac{2}{\pi}} \left[3\left[\frac{s}{s^2+25}\right] + 5\left[\frac{s}{s^2+4}\right]\right] \\
& \because [\text{Formula} : \int e^{-ax} sin\ bx\ dx = \frac{b}{a^2+b^2}] \\
&= \sqrt{\frac{2}{\pi}} s\left[\frac{3}{s^2+25} + \frac{5}{s^2+4}\right]
\end{aligned}$$

Example 16: Find the Fourier sine transform of $f(x) = e^{-ax}$.

Solution :

$$\text{We know that, } F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ sx\ dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\ sin\ sx\ dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + a^2}\right]$$

$$\text{Formula: } \int_0^\infty e^{-ax}\ sin\ bx\ dx = \frac{b}{a^2 + b^2}$$

Example 17: Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ and hence, find $F_s\left[\frac{e^{-ax} - e^{-bx}}{x}\right]$

Solution :

We know that, $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ sx\ dx$

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x}\ sin\ sx\ dx$$

Diff. w.r. to s on both sides,

$$\begin{aligned}
\frac{d}{ds} F_s\left[\frac{e^{-ax}}{x}\right] &= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x}\ sin\ sx\ dx\right] \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial s}\left[\frac{e^{-ax}}{x}\ sin\ sx\right] dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty x \frac{e^{-ax}}{x}\ cos\ sx\ dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\ cos\ sx\ dx \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2 + a^2}\right]
\end{aligned}$$

$$\left[\textbf{Formula:} \int_0^\infty e^{-ax}\ \cos\ bx\ dx = \frac{a}{a^2+b^2} \quad \text{Here, } b = s\right]$$

By integrating w. r. to 's', we get

$$\begin{aligned}
F_s\left[\frac{e^{-ax}}{x}\right] &= \sqrt{\frac{2}{\pi}}\int \frac{a}{s^2+a^2}ds \\
&= \sqrt{\frac{2}{\pi}}a.\frac{1}{a}.tan^{-1}\frac{s}{a} = \sqrt{\frac{2}{\pi}}tan^{-1}\frac{s}{a} \\
\text{Similarly,} \quad F\left[\frac{e^{-bx}}{x}\right] &= \sqrt{\frac{2}{\pi}}tan^{-1}\frac{s}{b} \\
F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right] &= F_s\left[\frac{e^{-ax}}{x}\right] - F_s\left[\frac{e^{-bx}}{x}\right] \\
&= \sqrt{\frac{2}{\pi}}tan^{-1}\frac{s}{a} - \sqrt{\frac{2}{\pi}}tan^{-1}\frac{s}{b} \\
&= \sqrt{\frac{2}{\pi}}\left[tan^{-1}\frac{s}{a} - tan^{-1}\frac{s}{b}\right]
\end{aligned}$$

Problem based on Fourier sine transform and its inversion formula.

Formula:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ sx\ dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[f(x)]\ sin\ sx\ ds$$

Example 18: Find the Fourier sine transform of e^{-ax} $a > 0$ and deduce that $\int_0^\infty \frac{s}{s^2+a^2} sin\ sxdx = \frac{\pi}{2} e^{-ax}$

Solution :

$$\text{We know that, } F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ sx\ dx$$

$$\begin{aligned} F_s\left[e^{-as}\right] &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-as}\ sin\ sx\ dx \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right] \quad \left[\because \int_0^\infty e^{-ax} sin\ bx\ dx = \frac{b}{a^2+b^2}\right] \end{aligned}$$

Applying the inversion formula, we get

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[f(x)]\ sin\ sx\ ds \\ f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\ sin\ sx\ ds \\ &= \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+a^2}\ sin\ sx\ ds \\ \int_0^\infty \frac{s}{s^2+a^2}\ sin\ sx\ ds &= \frac{\pi}{2} f(x) \\ &= \frac{\pi}{2} e^{-ax},\ a > 0 \end{aligned}$$

Example 19: Find the Fourier sine transform of e^{-x} hence show that $\int_0^\infty \frac{x\ sin\ mx}{1+x^2}dx = \frac{\pi}{2}e^{-m},\ m > 0$

Solution : We know that,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ sin\ sx\ dx$$

$$\begin{aligned} F_s\left[e^{-x}\right] &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\ sin\ sx\ dx \\ &= \sqrt{\frac{2}{\pi}}\left[\frac{s}{1+s^2}\right] \quad \left[\text{Formula: } \int_0^\infty e^{-ax}\ sin\ bx\ dx = \frac{b}{a^2+b^2}\right] \end{aligned}$$

by inversion formula, we get

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\ sin\ sx\ ds \\ f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{s}{1+s^2}\ sin\ sx\ ds \\ &= \frac{2}{\pi}\int_0^\infty \frac{s\ sin\ sx}{1+s^2}\ ds \\ \int_0^\infty \frac{s\ sin\ sx}{1+s^2}\ ds &= \frac{\pi}{2}f(x) = \frac{\pi}{2}e^{-x} \end{aligned}$$

Changing ‘x’ to ‘m’ and ‘s’ to ‘x’, we get

$$\int_0^\infty \frac{x\ sin\ mx}{1+x^2}\ dx = \frac{\pi}{2}e^{-m}$$

Example 20: Find the Fourier sine and cosine transform of xe^{-ax}

Solution : (i) We know that,

$$F_c[xf(x)] = \frac{-d}{ds} F_c[f(x)]$$

$$\begin{aligned}
F_s\left[xe^{-ax}\right] &= \frac{-d}{ds} F_c[e^{-ax}] \qquad \because [f(x) = e^{-ax}] \\
&= \frac{-d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}\right] \\
&= -\sqrt{\frac{2}{\pi}}.a\frac{d}{ds}\left[\frac{1}{s^2+a^2}\right] \\
&= -a\sqrt{\frac{2}{\pi}}\left[\frac{-2s}{(s^2+a^2)^2}\right] \\
&= \sqrt{\frac{2}{\pi}}\left[\frac{2as}{(s^2+a^2)^2}\right]
\end{aligned}$$

Solution : (ii) We know that,

$$F_c[xf(x)] = \frac{-d}{ds} F_s[f(x)]$$

$$\begin{aligned}
F_c\left[xe^{-ax}\right] &= \frac{-d}{ds} F_s[e^{-ax}] \\
&= \frac{-d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right] \\
&= -\sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)(1)-s(2s)}{(s^2+a^2)^2}\right] \\
&= -\sqrt{\frac{2}{\pi}}\frac{s^2+a^2-2s^2}{(s^2+a^2)^2} \\
&= -\sqrt{\frac{2}{\pi}}\frac{a^2-s^2}{(s^2+a^2)^2} \\
&= \sqrt{\frac{2}{\pi}}\frac{s^2-a^2}{(s^2+a^2)^2}
\end{aligned}$$

Example 21: Find the Fourier sine transform of e^{-ax} and hence find the Fourier cosine transform of xe^{-ax}

Solution : (ii) We know that,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ sin\ sx\ dx$$

$$\begin{aligned} F_s\left[e^{-ax}\right] &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\ sin\ sx\ dx \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right] \end{aligned}$$

To find : $F_c[xe^{-ax}]$

$$\begin{aligned} F_c[xe^{-ax}] &= \frac{d}{ds} F_s[xe^{-ax}] \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right] \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right] \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] \end{aligned}$$

Example 22: Evaluate : $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transform.

Solution : Parseval's identity is

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_c[f(x)]F_c[g(x)]ds \qquad (1)$$

Let $f(x) = e^{-ax}$	Let $g(x) = e^{-bx}$
$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) cos\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} cos\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$	$F_c[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) cos\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} cos\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \left[\frac{b}{s^2+b^2}\right]$

$$f(x)g(x) = e^{-ax}e^{-bx} = e^{(a+b)x}$$

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty e^{(a+b)x}dx = \left[\frac{e^{(a+b)x}}{-(a+b)}\right]_0^\infty = 0 - \left[\frac{1}{-(a+b)}\right] = \frac{1}{(a+b)}$$

$$F_c[f(x)]F_c[g(x)] = \left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right]\left[\sqrt{\frac{2}{\pi}}\frac{b}{s^2+b^2}\right]$$

$$= \frac{2}{\pi}\frac{ab}{(s^2+a^2)(s^2+b^2)}$$

$$
\begin{aligned}
(1) \Rightarrow \frac{1}{a+b} &= \frac{2ab}{\pi} \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)} ds \\
\Rightarrow \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)} ds &= \frac{\pi}{2ab(a+b)} \\
\Rightarrow \int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} dx &= \frac{\pi}{2ab(a+b)} \qquad \because [s \text{ is a dummy variable}]
\end{aligned}
$$

Example 23: Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$

Solution : Step 1:

$$\text{Prove that : } \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

$$\text{Step 2 : Here,} a = 1,\ b = 2$$

$$\therefore \int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{2(1)(2)(1+2)} = \frac{\pi}{12}$$

Example 24: Using transform methods, evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

Solution : Parseval's identity is

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c[f(x)]|^2 ds \qquad (1)$$

Let $f(x) = e^{-ax}$	Let $\|f(x)\|^2 = (e^{-ax})^2 = e^{-2ax}$
$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) cos\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} cos\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$	$\|F_c[f(x)]\|^2 = \frac{2}{\pi}\left[\frac{a^2}{(s^2+a^2)^2}\right]$

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty e^{-2ax} dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = 0 - \left[\frac{1}{-2a}\right] = \frac{1}{2a}$$

$$\therefore (1) \Rightarrow \frac{1}{2a} = \frac{2a^2}{\pi} \int_0^\infty \frac{1}{(s^2+a^2)^2} ds$$

$$\Rightarrow \int_0^\infty \frac{1}{(s^2+a^2)} ds = \frac{\pi}{4a^3}$$

$$\Rightarrow \int_0^\infty \frac{1}{(x^2+a^2)} dx = \frac{\pi}{4a^3} \quad \because [s \text{ is a dummy variable}]$$

Note : Evaluate :

$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

$$\text{Step 1: Prove that : } \int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

$$\text{Step 2: Here, } a = 1$$

$$\therefore \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4(1^3)} = \frac{\pi}{4}$$

Example 25: Using transform methods, eveluate: $\int_0^\infty \frac{x^2\ dx}{(x^2+a^2)^2}$

Solution : Parseval's identity is

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s[f(x)]|^2 ds \qquad (1)$$

Let $f(x) = e^{-ax}$	Let $[f(x)]^2 = (e^{-ax})^2 = e^{-2ax}$
$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) sin\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} sin\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$	$[F_s[f(x)]]^2 = \frac{2}{\pi}\left[\frac{s^2}{(s^2+a^2)^2}\right]$

$$\int_0^\infty [f(x)]^2 dx = \int_0^\infty e^{-2ax} dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = 0 - \left[\frac{-1}{2a}\right] = \frac{1}{2a}$$

$$(1) \Rightarrow \frac{1}{2a} = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds$$

$$\Rightarrow \int_0^\infty \frac{s^2}{(s^2+a^2)} ds = \frac{\pi}{4a}$$

$$\Rightarrow \int_0^\infty \frac{x^2}{(x^2+a^2)} dx = \frac{\pi}{4a} \quad \because [s \text{ is a dummy variable}]$$

Note : Evaluate :

$$\int_0^\infty \frac{x^2}{(x^4+4)^2} dx$$

$$\text{Step 1: Prove that : } \int_0^\infty \frac{x^2}{(x^2+a^2)^2} = \frac{\pi}{4a}$$

Step 2: Here, $a = 2$

$$\therefore \int_0^\infty \frac{x^2}{(x^2+4)^2} dx = \frac{\pi}{4(2)} = \frac{\pi}{8}$$

Example 26: Using Parseval's identity of the Fourier cosine transform, eveluate: $\int_0^\infty \frac{sin\ ax}{x(a^2+x^2)^2} dx$ if

$$f(x) = \begin{cases} 1, & |x| < a. \\ 0, & |x| > a > 0. \end{cases}$$

$$g(x) = e^{-a|x|},\ a > 0$$

Solution : Parseval's identity is

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_c[f(x)]F_c[g(x)]ds \qquad (1)$$

Let $f(x) = 1,\ \|x\| < a$	Let $g(x) = e^{-a\|x\|} a > 0$
$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) cos\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \int_0^a cos\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \left[\frac{sin\ sx}{s}\right]_0^a$ $= \sqrt{\frac{2}{\pi}} \left[\frac{sin\ ax}{s}\right]$	$F_c[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) cos\ sx\ dx$ $= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} cos\ sx\ dx$ $\because [\|x\| = xin(0, \infty)]$ $= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$

$$
\begin{aligned}
f(x)g(x) = (1)e^{-ax} &= e^{-ax} \text{ in } (0, a) \\
\int_0^\infty f(x)g(x)dx &= \int_0^a e^{ax}dx = \left[\frac{e^{ax}}{-a}\right]_0^a = \left[\frac{e^{-a^2}}{-a}\right] - \frac{1}{-a} \\
F_c[f(x)]F_c[g(x)] &= \left[\frac{2}{\pi}\frac{a\sin sa}{s(s^2+a^2)}\right] \\
(1) \Rightarrow \frac{1}{a}[1 - e^{-a^2}] &= \frac{2a}{\pi}\int_0^\infty \frac{\sin sa}{s(s^2+a^2)}ds \\
\Rightarrow \int_0^\infty \frac{\sin sa}{s(s^2+a^2)}ds &= \frac{\pi}{2a^2}[1 - e^{-a^2}] \\
\Rightarrow \int_0^\infty \frac{\sin ax}{x(x^2+a^2)}dx &= \frac{\pi}{2a^2}[1 - e^{-a^2}] \\
& \because [s \text{ is a dummy variable}]
\end{aligned}
$$