Diagonalización, Hamiltoniana en espacio k.

$$\hat{H} = \sum_k \left\{ \varepsilon_k b_k^\dagger b_k - [\Delta_k b_k^\dagger b_{-k}^\dagger + \Delta_k^* b_k b_{-k}] \right\}$$

$$- \frac{U}{N} \sum_{p,q,r,s} \delta_{p+q}^{r+s}\, b_p^\dagger b_q^\dagger b_r b_s \longrightarrow \text{Despreciable,}$$

lo incluimos más adelante como perturbación.

$$\boxed{\hat{H} = \sum_k \left\{ \varepsilon_k b_k^\dagger b_k - [\Delta_k b_k^\dagger b_{-k}^\dagger + \Delta_k^* b_k b_{-k}] \right\}}$$

$$\varepsilon_k = \hbar\omega_q + 2\hbar g \cos(k)$$

$$\Delta_k = \hbar g e^{-ik} + 6\hbar\lambda$$

λ: En principio también es despreciable

Antes de remplazar, simplificamos cuentas:

$$\sum_k \Delta_k b_k^\dagger b_{-k}^\dagger + \Delta_k^* b_k b_{-k} = \frac{1}{2} \sum_k \left[(\Delta_k b_k^\dagger b_{-k}^\dagger + \Delta_k^* b_k b_{-k}) + (\Delta_{-k} b_{-k}^\dagger b_k^\dagger + \Delta_{-k}^* b_{-k} b_k) \right]$$

$$= \frac{1}{2} \sum_k (\Delta_k + \Delta_{-k}) b_k^\dagger b_{-k}^\dagger + (\Delta_k + \Delta_{-k})^* b_k b_{-k}$$

Si consideramos $\Delta_k = \hbar g e^{-ik} + 6\hbar\lambda \rightarrow \Delta_k + \Delta_{-k} \in \mathbb{R}$

lo mismo que Δ_{-k}

$\rightarrow (\Delta_k + \Delta_{-k}) = (\Delta_k + \Delta_{-k})^*$

Por tanto:

$$\boxed{\sum_k \left(\frac{\Delta_k + \Delta_{-k}}{2} \right) (b_k^\dagger b_{-k}^\dagger + b_k b_{-k})}$$

a.

Por lo cual:

$$\boxed{\hat{H} = \sum_k \left\{ \varepsilon_k b_k^\dagger b_k - \left(\frac{\Delta_k + \Delta_k^*}{2}\right)\left(b_k^\dagger b_{-k}^\dagger + b_k b_{-k}\right) \right\}}$$

El cual es resoluble exactamente por una transformación de Bogoliubov como la que se hace por:

$$H_{\mathrm{LSW}} = \frac{z\hbar^2 J}{2} \sum_{\vec{q}} \left[b_{\vec{q}}^\dagger b_{\vec{q}} + \frac{\gamma_{\vec{q}}}{2} \left(b_{\vec{q}}^\dagger b_{-\vec{q}}^\dagger + b_{\vec{q}} b_{-\vec{q}} \right) \right].$$

(Modern condensed matter physics). Girvin.

$$b_{\vec{q}} = \cosh\theta_{\vec{q}}\, \beta_{\vec{q}} + \sinh\theta_{\vec{q}}\, \beta_{-\vec{q}}^\dagger;$$

$$\beta_{\vec{q}} = \cosh\theta_{\vec{q}}\, b_{\vec{q}} - \sinh\theta_{\vec{q}}\, b_{-\vec{q}}^\dagger.$$

$$[\beta_{\vec{q}}^\dagger, \beta_{\vec{q}\,'}^\dagger] = [\beta_{\vec{q}}, \beta_{\vec{q}\,'}] = 0;$$

$$\beta_k^\dagger |\Omega\rangle$$

$$[\beta_{\vec{q}}, \beta_{\vec{q}\,'}^\dagger] = \delta_{\vec{q},\vec{q}\,'}.$$

$$H_{\mathrm{LSW}} = \sum_{\vec{q}} E_{\vec{q}} \beta_{\vec{q}}^\dagger \beta_{\vec{q}},$$

$$\beta_k |\Omega\rangle = 0, \forall k =$$

$$E_{\vec{q}} = \frac{z\hbar^2 J}{2} \sqrt{1 - \gamma_{\vec{q}}^2}, = \sqrt{\varepsilon_k^2 - (2\Delta_k)^2}$$

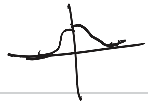
Por tanto, definiendo

$$\begin{cases} b_k = \cosh\theta_k\, \alpha_k + \sinh\theta_k\, \alpha_{-k}^\dagger \\ \alpha_k = \cosh\theta_k\, b_k - \sinh\theta_k\, b_{-k}^\dagger \end{cases}$$

Se tiene que:

$$\hat{H} = \sum_k E_k \alpha_k^\dagger \alpha_k \, ; \quad E_k = \sqrt{|\varepsilon_k|^2 - |\Delta_k + \Delta_k^*|^2}$$

Reemplazando todas las expresiones:



$$\boxed{E_k = \sqrt{\left(\hbar\omega_q + 2\hbar g\cos(k)\right)^2 - \left(2\hbar g\cos(k) + 2\hbar\lambda\right)^2}}$$

Paso a paso.

$$[a_k^\dagger, a_{k'}^\dagger] = [\cosh\theta_k b_k^\dagger - \sinh\theta_k b_{-k}^\dagger, \cdots b_{k'}^\dagger - \cdots b_{k'}^\dagger]$$
$$= 0 \checkmark$$
$$[a_k, a_{k'}] = 0 \checkmark$$
$$[a_k, a_k^\dagger] = [\cosh\theta_k b_k - \sinh\theta_k b_{-k}^\dagger, \cosh\theta_k b_k^\dagger - \sinh\theta_k b_{-k}]$$
$$= \cosh^2\theta_k - \sinh^2\theta_k = 1 \checkmark$$

Reemplazando en el Hamiltoniano:

$$b_k^\dagger b_k = (\cosh\theta_k a_k^\dagger + \sinh\theta_k a_{-k})(\cosh\theta_k a_k + \sinh\theta_k a_{-k}^\dagger)$$
$$= \cosh^2\theta_k a_k^\dagger a_k + \sinh^2\theta_k \overbrace{a_{-k}a_{-k}^\dagger}^{1+a_{-k}^\dagger a_{-k}}$$
$$+ \sinh\theta_k\cosh\theta_k (a_k^\dagger a_{-k}^\dagger + a_{-k}a_k)$$

$$b_k^\dagger b_{-k}^\dagger = (\cosh\theta_k a_k^\dagger + \sinh\theta_k a_{-k})(\cosh\theta_{-k} a_{-k}^\dagger + \sinh\theta_{-k} a_k)$$
$$= (\cosh\theta_k)(\cosh\theta_{-k}) a_k^\dagger a_{-k}^\dagger + (\sinh\theta_k)(\sinh\theta_{-k}) a_{-k}a_k$$
$$+ (\cosh\theta_k)(\sinh\theta_{-k}) a_k^\dagger a_k$$
$$+ (\sinh\theta_k)(\cosh\theta_{-k}) a_{-k}a_{-k}^\dagger$$

$$b_k b_{-k} = h.c.$$

Sumando estos dos últimos términos:

$b^{\dagger}_k b^{\dagger}_{-k} + b_k b_{-k} =$

$$= (\cosh\theta_k)(\cosh\theta_{-k})\, a^{\dagger}_k a^{\dagger}_{-k} + (\operatorname{senh}\theta_k)(\operatorname{senh}\theta_{-k})\, a_{-k} a_k$$
$$+ (\cosh\theta_k)(\operatorname{senh}\theta_{-k})\, a^{\dagger}_k a_k$$
$$+ (\operatorname{senh}\theta_k)(\cosh\theta_{-k})\, a_{-k} a^{\dagger}_{-k}$$

$$+ (\cosh\theta_k)(\cosh\theta_{-k})\, a_k a_{-k} + (\operatorname{senh}\theta_k)(\operatorname{senh}\theta_{-k})\, a^{\dagger}_{-k} a^{\dagger}_k$$
$$+ (\cosh\theta_k)(\operatorname{senh}\theta_{-k})\, a_k a^{\dagger}_k$$
$$+ (\operatorname{senh}\theta_k)(\cosh\theta_{-k})\, a^{\dagger}_{-k} a_{-k}$$

$$= \left(\cosh(\theta_k)\cosh(\theta_{-k}) + \operatorname{senh}(\theta_k)\operatorname{senh}(\theta_{-k})\right) a^{\dagger}_{-k} a^{\dagger}_k$$
$$+ \left(\cosh(\theta_k)\cosh(\theta_{-k}) + \operatorname{senh}(\theta_k)\operatorname{senh}(\theta_{-k})\right) a_k a_{-k}$$
$$+ (\cosh\theta_k \cdot \operatorname{senh}\theta_{-k})(a^{\dagger}_k a_k + a_k a^{\dagger}_k)$$
$$+ \operatorname{senh}\theta_k \cosh\theta_{-k}\,(a^{\dagger}_{-k} a_{-k} + a_{-k} a^{\dagger}_{-k})$$

iguales, creciendo son sumados

$$= \big(\underbrace{(\cosh\theta_k)\cosh\theta_{-k} + \operatorname{senh}\theta_k \operatorname{senh}\theta_{-k}}_{\cosh(\theta_k - \theta_{-k})?}\big)(a^{\dagger}_{-k} a^{\dagger}_k + a_k a_{-k})$$
$$+ 2\cosh\theta_k \operatorname{senh}\theta_{-k}\,(a^{\dagger}_k a_k + a_k a^{\dagger}_k)$$

↳ ¿Cómo proseguir?

Aproximaciones:

$$E_k = \sqrt{\left(\hbar\omega_q + 2\hbar g\cos(k)\right)^2 - \left(2\hbar g\cos(k) + 12\hbar\lambda\right)^2}$$

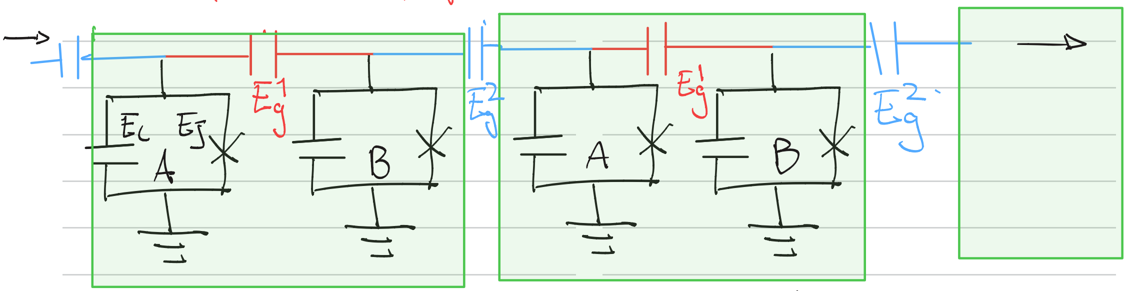
$$E_k = \sqrt{\left(\hbar\omega_q + \cancel{2\hbar g\cos(k) - 2\hbar g\cos(k)} - 12\hbar\lambda\right) \times \left(\hbar\omega_q + 2\hbar g\cos(k) + 2\hbar g\cos(k) + 12\hbar\lambda\right)}$$

$$E_k = \sqrt{\left(\hbar\omega_q - 12\hbar\lambda\right)\left(\hbar\omega_q + 12\hbar\lambda + 4\hbar g\cos(k)\right)}$$

$$E_k = \sqrt{\left(\hbar\omega_q\right)^2 - \left(12\hbar\lambda\right)^2}\sqrt{1 + \frac{4g\cos(k)}{\omega_q + 12\lambda}}$$

$$\boxed{E_k \approx \underbrace{\sqrt{(\hbar\omega_q)^2 - (12\hbar\lambda)^2}}_{\approx \hbar\omega_q}\left(1 + \frac{2g}{\omega_q + 12\lambda}\cos(k)\right)}$$

Cadena SSH de transmons.



Consideremos N celdas unidad, entonces:

$$\hat{\mathcal{H}} = \sum_{j}^{N} \hat{H}_j^A + \hat{H}_j^B + \hat{H}_j^{AB} + \hat{H}_{j,j+1}$$

Consideremos transmon con parámetros homogéneos.
Hacemos uso de la aproximación cuadrática.

$$\hat{\mathcal{H}} = \sum_j \hbar\omega_q \left(b_{j,A}^\dagger b_{j,A} + b_{j,B}^\dagger b_{j,B} \right)$$

$$- \sum_j \hbar v \left(b_{j,A}^\dagger - b_{j,A} \right)\left(b_{j,B}^\dagger - b_{j,B} \right)$$

$$- \sum_j \hbar w \left(b_{j,B}^\dagger - b_{j,B}^\dagger \right)\left(b_{j+1,A}^\dagger - b_{j+1,A} \right)$$

Despreciaremos las interacciones en una primera instancia.
Pasamos al espacio de momentos definiendo:

$$b_{k,X}^\dagger = \frac{1}{\sqrt{N}} \sum_j e^{ikj} b_{j,X}^\dagger$$

$$b_{j,X}^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ikj} b_{k,X}^\dagger \,, \quad X = A, B.$$

Calculamos cada término en espacio k:

$$\sum_j (b^\dagger_{j,A} b_{j,A} + b^\dagger_{j,B} b_{j,B}) = \sum_k (b^\dagger_{k,A} b_{k,A} + b^\dagger_{k,B} b_{k,B})$$

$$\sum_j b^\dagger_{j,A} b^\dagger_{j,B} = \sum_k b^\dagger_{k,A} b^\dagger_{-k,B} \qquad \sum_j b_{j,A} b_{j,B} = \sum_k b_{k,A} b_{-k,B}$$

$$\sum_j b^\dagger_{j,A} b_{j,B} = \sum_k b^\dagger_{k,A} b_{k,B} \;\Big|\; \sum_j b_{j,A} b^\dagger_{j,B} = \sum_k b_{k,A} b^\dagger_{k,B}$$

$$\sum_j b^\dagger_{j,B} b_{j+1,A} = \sum_k e^{ik} b^\dagger_{k,B} b_{k,A} \;\Big|\; \sum_j b_{j,B} b^\dagger_{j+1,A} = \sum_k e^{-ik} b_{k,B} b^\dagger_{k,A}$$

$$\sum_j b^\dagger_{j,B} b^\dagger_{j+1,A} = \sum_k e^{+ik} b^\dagger_{k,B} b^\dagger_{-k,A} \;\Big|\; \sum_j b_{j,B} b_{j+1,A} = \sum_k e^{-ik} b_{k,B} b_{-k,A}$$

Reemplazando:

$$\hat{H} = \sum_k \hbar\omega_q (b^\dagger_{k,A} b_{k,A} + b^\dagger_{k,B} b_{k,B})$$
$$- \hbar v \sum_k (b^\dagger_{k,A} b^\dagger_{-k,B} + b_{k,A} b_{-k,B} - b^\dagger_{k,A} b_{k,B} - b_{k,A} b^\dagger_{k,B})$$
$$- \hbar w \sum_k (e^{ik} b^\dagger_{k,B} b^\dagger_{-k,A} + e^{-ik} b_{k,B} b_{-k,A}$$
$$- e^{ik} b^\dagger_{k,B} b_{k,A} - e^{-ik} b_{k,B} b^\dagger_{k,A})$$

$$\hat{H} = \sum_k \hbar\omega_q (b^\dagger_{k,A} b_{k,A} + b^\dagger_{k,B} b_{k,B})$$
$$- \hbar \sum_k (v + w e^{-ik}) b^\dagger_{k,B} b^\dagger_{-k,A} + (v + w e^{-ik}) b_{k,B} b_{-k,A}$$
$$+ \hbar \sum_k (v + w e^{ik}) b^\dagger_{k,B} b_{k,A} + (v + w e^{-ik}) b_{k,B} b^\dagger_{k,A}.$$

Simplificamos :

$$\sum_k t_k b^\dagger_{kB} b^\dagger_{-kA} + t_k^* b_{kB} b_{-kA}$$

$$= \frac{1}{2} \sum_k t_k b^\dagger_{kB} b^\dagger_{-kA} + \underbrace{t_{-k}}_{t_k^*} b^\dagger_{-kB} b^\dagger_{+kA} + t_k^* b_{kB} b_{-kA} + \cancel{t^*_{-k}}^{\,t_k} b_{-kB} b_{k,A})$$

$$= \frac{1}{2} \sum_k t_k (b^\dagger_{kB} b^\dagger_{-k,A} + b_{-kB} b_{kA}) + t_k^* (b^\dagger_{-kB} b^\dagger_{kA} + b_{kB} b_{-kA})$$

$$(t_{k_1} b^\dagger_{k_1B} b^\dagger_{-k_1A} + \cancel{t^*_{-k_1}}^{\,t_{k_1}} b^\dagger_{-k_1B} b^\dagger_{+k_1A})$$

$$= \sum_k t_k (b^\dagger_{kB} b^\dagger_{-kA} + b_{-kB} b_{kA}) \rightsquigarrow D$$

$$\boxed{\hat{H} = \hbar\omega_q \sum_k (b^\dagger_{kA} b_{kA} + b^\dagger_{kB} b_{kB}) + \hbar \sum_k \{ (\nu + w e^{-ik}) b^\dagger_{kA} b_{kB} + h.c. \} - \hbar \sum_k \{ (\nu + w e^{-ik}) b^\dagger_{kA} b^\dagger_{-kB} + h.c. \} .}$$