EE 102 HW 4

1)

a) $x_1(t) = \sin(2\pi 4t) + \cos(2\pi 5t) = \frac{1}{2i}\left(e^{i2\pi 4t} - e^{-i2\pi 4t}\right) + \frac{1}{2}\left(e^{i2\pi 5t} + e^{-i2\pi 5t}\right)$

$\omega_0 = 2\pi$

$T = 1$

Same

$x_1(t) = \sum_{k=-\infty}^{k=\infty} C_k e^{jw_0 kt}$

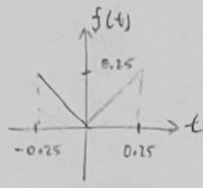
if $k = 4$, $\boxed{C_4 = \frac{1}{2i}}$ if $k=5$, $\boxed{C_5 = \frac{1}{2}}$

if $k = -4$, $\boxed{C_{-4} = -\frac{1}{2i}}$ if $k=-5$, $\boxed{C_{-5} = \frac{1}{2}}$

if $k=0$, $\frac{1}{1}\int_0^1 [\sin(2\pi 4t) + \cos(2\pi 5t)]\cancel{e^{-i2\pi kt}}^{1} dt$

$= \int_0^1 \sin(8\pi t)dt + \int_0^1 \cos(10\pi t)dt = \boxed{0 = C_k}$

b) $x_3(t) = \sum_{k=-\infty}^{\infty} f(t-k)$



$T_0 = 1$

$\omega_0 = 2\pi$

$C_0 = \frac{1}{1}\int_{-0.5}^{0.5} x_3(t)\cancel{e^{-j0\omega_0 t}}^{1} dt = \int_{-0.25}^{0} -t\,dt + \int_0^{0.25} t\,dt \boxed{= \frac{1}{16}}$

$C_k = \frac{1}{1}\int_{-0.5}^{0.5} x_3(t) e^{-j2\pi kt} = \int_{-0.25}^{0} -te^{-j2\pi kt}dt + \int_0^{0.25} te^{-j2\pi kt}dt = \left(\frac{t}{j2\pi k}e^{-j2\pi kt} - \frac{1}{4\pi^2k^2}e^{-j2\pi kt}\right)\Big|_{-0.25}^{0} - \left(\frac{t}{j2\pi k}e^{-j2\pi kt} - \frac{1}{4\pi^2k^2}e^{-j2\pi kt}\right)\Big|_{0}^{0.25}$

$= -\frac{1}{4\pi^2k^2} + e^{j\frac{\pi}{2}k}\left(\frac{1}{j8\pi k} + \frac{1}{4\pi^2k^2}\right) + e^{-j\frac{\pi}{2}k}\left(-\frac{1}{j8\pi k} + \frac{1}{4\pi^2k^2}\right) - \frac{1}{4\pi^2k^2} = -\frac{1}{2\pi^2k^2} + e^{j\frac{\pi}{2}k}\left(\frac{1}{j8\pi k} + \frac{1}{4\pi^2k^2}\right) + e^{-j\frac{\pi}{2}k}\left(-\frac{1}{j8\pi k} + \frac{1}{4\pi^2k^2}\right)$

$= -\frac{1}{2\pi^2k^2} + \frac{1}{4\pi^2k^2}\left(e^{j\frac{\pi}{2}k} + e^{-j\frac{\pi}{2}k}\right) + \frac{1}{j8\pi k}\left(e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right) = \frac{-1}{2\pi^2k^2} + \frac{\cos(\frac{\pi}{2}k)}{2\pi^2k^2} + \frac{\sin(\frac{\pi}{2}k)}{4\pi k}$

$$\boxed{C_k = \begin{cases} \frac{-1}{2\pi^2k^2} + \frac{1}{2\pi^2k^2} & \text{if } k = 0 \pmod 4 \\ \frac{-1}{2\pi^2k^2} + \frac{1}{4\pi k} & \text{if } k = 1 \pmod 4 \\ \frac{-1}{2\pi^2k^2} - \frac{1}{2\pi^2k^2} & \text{if } k = 2 \pmod 4 \\ \frac{-1}{2\pi^2k^2} - \frac{1}{4\pi k} & \text{if } k = 3 \pmod 4 \end{cases}}$$

$\cos 0 = 1$ $\sin 0 = 0$

$\cos\frac{\pi}{2} = 0$ $\sin\frac{\pi}{2} = 1$

$\cos\pi = -1$ $\sin\pi = 0$

$\cos\frac{3\pi}{2} = 0$ $\sin\frac{3\pi}{2} = -1$

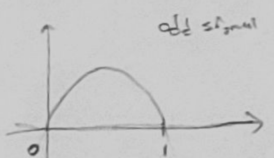
3)

a) $x_1(t) = \sin(\pi t)$ when $0 \le t \le 1$

$\omega_0 = 2\pi$

$T = 1$

odd signal



$C_k = \frac{1}{T}\int_0^T \sin(\pi t)e^{-j2\pi kt}dt$

$= \int_0^1 \sin(\pi t)e^{-j2\pi kt}dt = \int_0^1 \frac{e^{i\pi t} - e^{-i\pi t}}{2i} e^{-j2\pi kt}dt = \frac{1}{2i}\int_0^1 e^{i(\pi - 2\pi k)t}dt - \frac{1}{2i}\int_0^1 e^{-i(\pi + 2\pi k)t}dt$

$= \frac{1}{2i}\cdot\frac{e^{-i(2\pi k - \pi)t}}{-i(2\pi k - \pi)}\Big|_0^1 - \frac{1}{2i}\cdot\frac{e^{-i(2\pi k + \pi)t}}{-i(2\pi k + \pi)}\Big|_0^1 = \frac{1}{2i}\cdot\frac{e^{-i(2\pi k - \pi)} - 1}{-i(2\pi k - \pi)} - \frac{1}{2i}\cdot\frac{e^{-i(2\pi k + \pi)} - 1}{-i(2\pi k + \pi)}$

$= \frac{1}{2i}\left(\frac{e^{-i(2\pi k - \pi)} - 1}{i(\pi - 2\pi k)} + \frac{e^{-i(2\pi k + \pi)} - 1}{i(\pi + 2\pi k)}\right) = -\frac{1}{2}\left(\frac{(\pi + 2\pi k)(e^{-i(2\pi k - \pi)} - 1) + (\pi - 2\pi k)(e^{-i(2\pi k + \pi)} - 1)}{\pi^2 - 4\pi^2k^2}\right)$

$= -\frac{1}{2}\left(\frac{\pi e^{-i2\pi k}e^{i\pi} + 2\pi k e^{-i2\pi k}e^{i\pi} - \pi - 2\pi k + \pi e^{-i2\pi k}e^{-i\pi} - 2\pi k e^{-i2\pi k}e^{-i\pi} - \pi + \cancel{2\pi k}}{\pi^2 - 4\pi^2k^2}\right)$

$C_0 = \int_0^1 \sin(\pi t)\cancel{e^{0}} dt = -\frac{1}{\pi}\cos(\pi t)\Big|_0^1 = \frac{1}{\pi} - \frac{1}{\pi} \boxed{= 0}$

$= -\frac{1}{2}\left(\frac{-\pi e^{-i2\pi k} - 2\pi k e^{-i2\pi k} - \pi - \pi e^{-i2\pi k} + 2\pi k e^{-i2\pi k} - \pi}{\pi^2 - 4\pi^2k^2}\right) = -\frac{1}{2}\left(\frac{-\pi - \pi - \pi - \pi}{\pi^2 - 4\pi^2k^2}\right)$

$e^{i\pi} = \cos\pi + i\cancel{\sin\pi}^{0} = -1$

$e^{-i\pi} = \cos\pi - i\cancel{\sin\pi}^{0} = -1$

$e^{-i2\pi k} = \cos 2\pi k - i\cancel{\sin 2\pi k}^{0} = 1$

$C_k \boxed{= \frac{2}{\pi(1 - 4k^2)}}$

b)

i) $\boxed{d_0 = c_0 = 0}$

$$\boxed{d_k = c_{-k} = \frac{2}{\pi(1-4(-k)^2)} = \frac{2}{\pi(1-4k^2)}}$$

ii) $y_2(t) = 2x_f(t-1)+3$

$\boxed{d_0 = \frac{2}{\pi}+3}$ Since it's $c_0 + 3$ due to the upshift

$$e^{-i2\pi k} = \cos 2\pi k - i\cancel{\sin 2\pi k}^{0} = 1$$

$$d_k = 2c_k e^{-i2\pi \cdot 1 \cdot k} = 2c_k e^{-i2\pi k}$$

$$\boxed{d_k = \frac{4e^{-i2\pi k}}{\pi(1-4k^2)} = \frac{4}{\pi(1-4k^2)}}$$

$$e^{-i\pi k} = \cos \pi k - i\cancel{\sin \pi k}^{0}$$

+ if k is even

- if k is odd

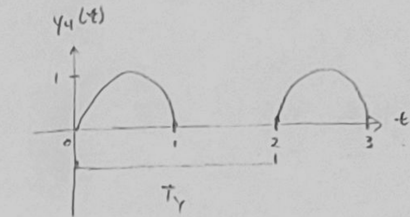
iii) $y_3(t) = 4x_f(t-0.5)$

$$d_k = 4c_k e^{-i2\pi \cdot \frac{1}{2} \cdot k} = 4c_k e^{-i\pi k} = \boxed{\begin{cases} 4c_k & \text{if } k \text{ is even} \\ -4c_k & \text{if } k \text{ is odd} \end{cases}}$$

$\boxed{d_0 = 0}$

iv)

$$y_4(t) = \begin{cases} \sin(\pi t) & \text{if } 0 \le t \le 1 \\ 0 & \text{if } 1 < t \le 2 \end{cases} \qquad T_y = 2 \qquad \omega_0 = \pi$$



$$x_f(t) + \sin(\pi t) = 2y_4(t)$$

$\frac{k}{2}$ since half ω_0

$$\Rightarrow y_4(t) = \frac{1}{2}x_f(t) + \frac{1}{2}\sin(\pi t)$$

$$d_k = \frac{1}{2}c_{\frac{k}{2}}$$

$$\frac{1}{2}\left(\frac{e^{i\pi t} - e^{-i\pi t}}{2i}\right)$$

$\boxed{d_0 = \frac{1}{2}c_0 = 0}$

$\boxed{d_k \text{ odd k is } 0}$

$\boxed{d_k = \frac{1}{\pi(1-k^2)} \text{ if k is even}}$

$\boxed{d_1 = \frac{1}{4i} \quad d_{-1} = -\frac{1}{4i}}$

4)

a)

$T_1 = \frac{2}{5}$ $\qquad T_2 = \frac{1}{2}$

i) $f(t) = \sin(5\pi t) + \frac{1}{2}\cos(4\pi t)$

$T_0 = 2$

$\omega_0 = \pi$

$$= \frac{e^{i5\pi t} - e^{-i5\pi t}}{2i} + \frac{e^{i4\pi t} + e^{-i4\pi t}}{4}$$

$$\boxed{c_k = \begin{cases} \frac{1}{2i} & \text{if } k = 5 \\ -\frac{1}{2i} & \text{if } k = -5 \\ \frac{1}{4} & \text{if } k = 4 \\ \frac{1}{4} & \text{if } k = -4 \\ 0 & \text{otherwise} \end{cases}}$$

ii) $f(t) = e^{-t}$ for $0 < t < 1$, $T = 1$ $\quad \omega = 2\pi$

$$c_0 = \frac{1}{1}\int_0^1 e^{-t}\cancel{e^{-j2\pi kt}}\,dt = \int_0^1 e^{-t}\,dt = \boxed{1 - e^{-1}}$$

$$c_k = \frac{1}{1}\int_0^1 e^{-t}e^{-j2\pi kt}\,dt = \int_0^1 e^{-j2\pi kt - t}\,dt = \left(\frac{e^{-j2\pi kt - t}}{-j2\pi k - 1}\right)\Bigg|_0^1 = \frac{e^{-j2\pi k - 1}}{-j2\pi k - 1} + \frac{1}{j2\pi k + 1} = \frac{1}{j2\pi k + 1}\left(1 - e^{-j2\pi k - 1}\right)$$

$$\boxed{c_k = \frac{1 - e^{-1}}{j2\pi k + 1}}$$

$$e^{-j2\pi k} = \cos(2\pi k) - i\cancel{\sin(2\pi k)}^{0} = 1$$

b).

i) $x(t)$ has period T_1, x_k fourier
$y(t)$ has period T_2, y_k fourier

$z(t) = x(t) - y(t)$ when $T_1 = T_2$, $z(t)$ will have same period if $T_1 = T_2$

Linearity property: $\boxed{z_k = x_k - y_k}$

ii) $T_1 = \frac{1}{2} T_2$

$w(t) = x(t) - y(t)$, $w(t)$ has period LCM of T_1 & T_2, which is T_2

$w(t)$ has same T as $y(t)$
$w(t)$ has same T as $x(2t)$

$w_k = x_{\frac{k}{2}} - y_k$, but $\frac{k}{2}$ can't be non-integer

$$\boxed{w_k = \begin{cases} -y_k & \text{if } k \text{ is odd} \\ x_{\frac{k}{2}} - y_k & \text{if } k \text{ is even} \end{cases}}$$

5)

a) $g(t) = 2f(t)$, $g(t)$ period is $\boxed{T_0}$

Linearity property: $\boxed{d_k = 2c_k}$

b) $g(t) = f(-2t)$, $g(t)$ period is $\boxed{\frac{T_0}{2}}$, frequency is $2\omega_0$

Time Reversal property: $\boxed{d_k = c_{-k}}$

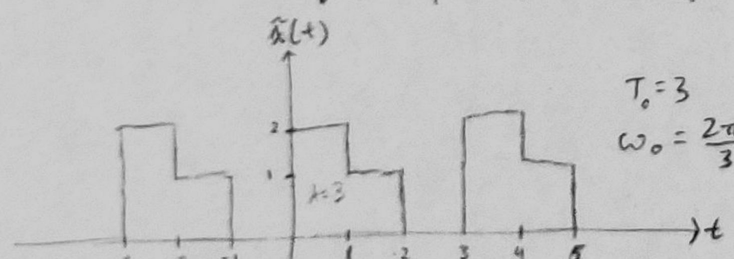
c) $g(t) = f(t - t_0)$, $g(t)$ period is $\boxed{T_0}$, frequency is $\frac{2\pi}{T_0}$

Time shift property: $\boxed{d_k = c_k e^{-j\frac{2\pi}{T_0}kt_0}}$

d) $g(t) = f(\frac{t}{a})$, $g(t)$ period is $\boxed{aT_0}$

Time scale property: $\boxed{d_k = c_k}$

6)



$T_0 = 3$
$\omega_0 = \frac{2\pi}{3}$

Area = 2 Area = 1

$$C_k = \frac{1}{3}\int_0^3 \tilde{x}(t)e^{-j\omega_0 kt}dt = \frac{1}{3}\int_0^1 2e^{-j\omega_0 kt}dt + \frac{1}{3}\int_1^2 e^{-j\omega_0 kt}dt$$

$$= \frac{2}{3}\cdot\left(\frac{e^{-j\omega_0 kt}}{-j\omega_0 k}\right)\Bigg|_0^1 - \frac{1}{3}\left(\frac{e^{-j\omega_0 kt}}{j\omega_0 k}\right)\Bigg|_1^2$$

$$= \frac{1}{-3j\omega_0 k}\left(2e^{-j\omega_0 k} - 2 + e^{-2j\omega_0 k} - e^{-j\omega_0 k}\right) = \frac{1}{-3j\omega_0 k}\left(e^{-j\omega_0 k} + e^{-2j\omega_0 k} - 2\right)$$

$$= \frac{1}{-3j\omega_0 k}\left[e^{-1.5\omega_0 k}\left(e^{\frac{j\omega_0 k}{2}} + e^{-\frac{j\omega_0 k}{2}}\right) - 2\right] = \frac{1}{-3j\omega_0 k}\left(2e^{-1.5j\omega_0 k}\cos\left(\frac{\omega_0 k}{2}\right) - 2\right)$$

$$= \frac{-1}{j\cancel{2}\pi k}\left(\cancel{2}e^{-j\pi k}\cos\left(\frac{\pi k}{3}\right) - \cancel{2}\right) = \frac{j}{\pi k}\left(e^{-j\pi k}\cos\left(\frac{\pi k}{3}\right) - 1\right)$$

$$C_0 = \frac{1}{T_0}\int_0^{T_0}\tilde{x}(t)e^{-j\omega_0 kt}dt = \frac{1}{3}\int_0^3 \tilde{x}(t)\,dt = \boxed{1}$$

$k=0$ $\cos 0 = 1$
$k=1$ $\cos\frac{\pi}{3} = \frac{1}{2}$
$k=2$ $\cos\frac{2\pi}{3} = -\frac{1}{2}$
$k=3$ $\cos\pi = -1$
$k=4$ $\cos\frac{4\pi}{3} = -\frac{1}{2}$
$k=5$ $\cos\frac{5\pi}{3} = \frac{1}{2}$

$e^{-j\pi k} = \cos(\pi k) - i\sin(\pi k)$
if k even, $= 1$
if k odd, $= -1$

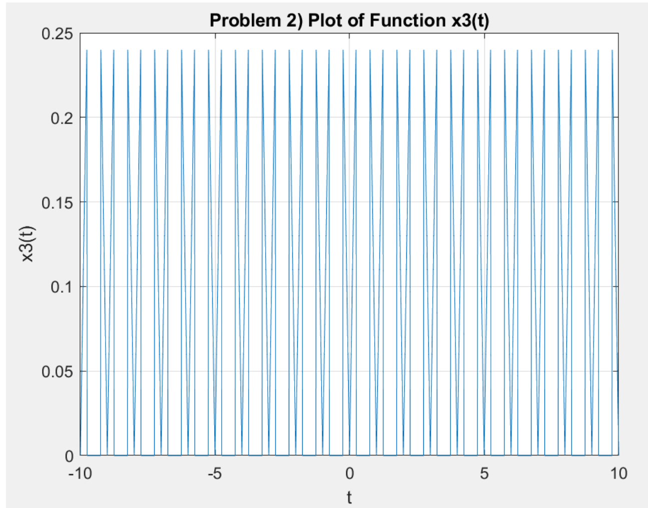
$$C_k = \begin{cases} 0 & \text{if } k = 0 \pmod 6 \\ \frac{-3j}{2\pi k} & \text{if } k = 1 \pmod 6 \\ \text{"} & \text{if } k = 2 \pmod 6 \\ 0 & \text{if } k = 3 \pmod 6 \\ -\frac{3j}{2\pi k} & \text{if } k = 4 \pmod 6 \\ \text{"} & \text{if } k = 5 \pmod 6 \end{cases} = \boxed{\begin{cases} 0 & \text{if } k = 0 \pmod 3 \\ \frac{-3j}{2\pi k} & \text{otherwise} \end{cases}}$$

Problem 2)

For $x_3(t)$ itself:

Code in main.m:

```
t = -10:0.01:10;
plot(t, x3(t));
title("Problem 2) Plot of Function x3(t)");
xlabel("t");
ylabel("x3(t)");
grid on;
```



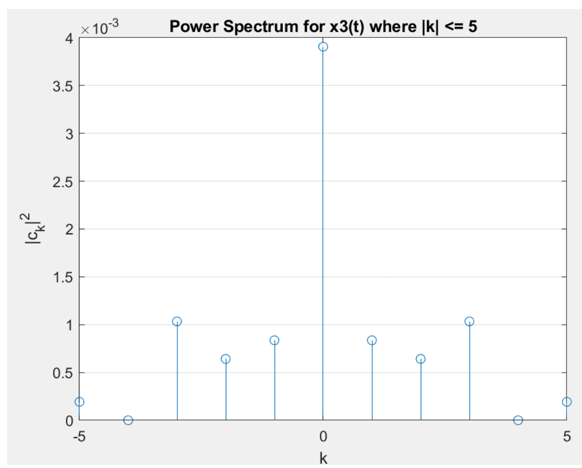
Code in x3.m:

```
function output = x3(t)
   time_array = t;
   for i = 1:length(time_array)
       temp = (mod(time_array(i) - 0.5, 1) - 0.5);
       if abs(temp) < 0.25
           time_array(i) = abs(temp);
       else
           time_array(i)= 0;
       end
   end
   output = time_array;
end
```

For |k| <= 5:

Code in main.m:

```
k = -5:1:5;
c_k = zeros(1, length(k));
for i = 1:length(k)
   c_k(i) = abs(c(k(i)))^2;
end
stem(k, c_k);
title("Power Spectrum for x3(t) where |k| <=
5");
xlabel("k");
ylabel("|c_k|^2");
grid on;
%{
Comments:
Apart from k = 0, the power is zero at all multiples of 4 which corresponds
to our answer for Problem 1). As for |k| <= 5, you can't really tell much
about this graph as we don't see enough of k yet. For |k| <= 10 and 20, it
gets far more obvious that the higher frequencies at both extremes have a
lower power. It should be noted that the odd frequencies take far slower to
decay while non-multiples-of-four decay fast.
```



```
%}
```

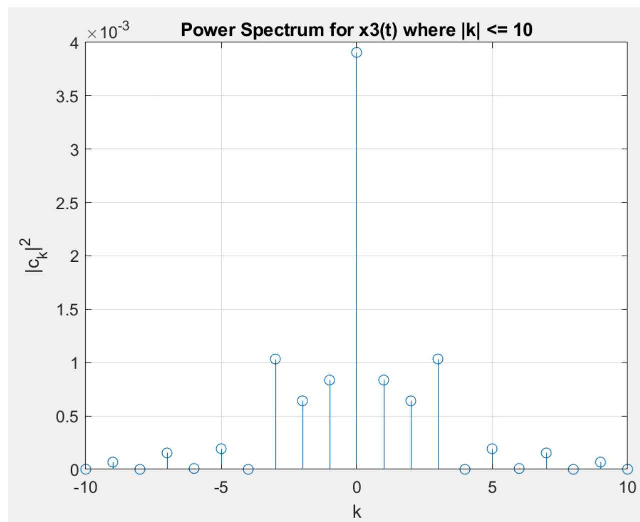
Code in c.m:

```
function out = c(k)
   if k == 0
       out = 1/16;
   elseif mod(k, 4) == 0
       out = 0;
   elseif mod(k, 4) == 1
       out = -1/(2*pi^2*k^2) + 1/(4*pi*k);
   elseif mod(k, 4) == 2
       out = -1/(pi^2*k^2);
   else
       out = -1/(2*pi^2*k^2) - 1/(4*pi*k);
   end
end
```

For |k| <= 10:

Code in main.m:

```
k = -10:1:10;
c_k = zeros(1, length(k));
for i = 1:length(k)
   c_k(i) = abs(c(k(i)))^2;
end
stem(k, c_k);
title("Power Spectrum for x3(t) where |k| <= 5");
xlabel("k");
ylabel("|c_k|^2");
grid on;
%{
Comments:
Apart from k = 0, the power is zero at all multiples of 4 which corresponds
to our answer for Problem 1). As for |k| <= 5, you can't really tell much
about this graph as we don't see enough of k yet. For |k| <= 10 and 20, it
gets far more obvious that the higher frequencies at both extremes have a
lower power. It should be noted that the odd frequencies take far slower to
decay while non-multiples-of-four decay fast.
%}
```

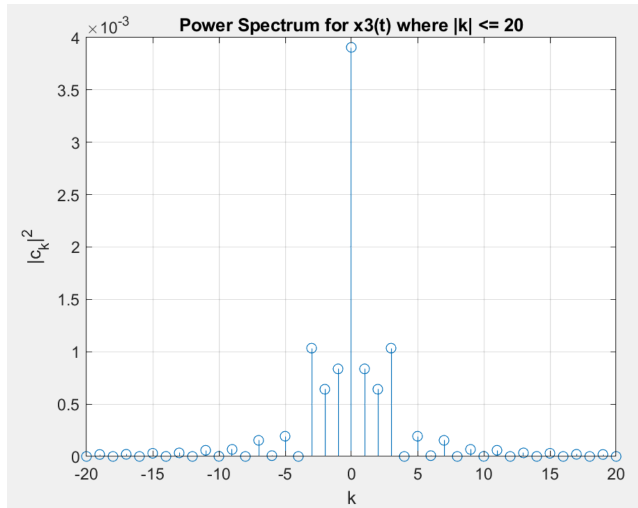


Code in c.m:

```
function out = c(k)
   if k == 0
       out = 1/16;
   elseif mod(k, 4) == 0
       out = 0;
   elseif mod(k, 4) == 1
```

```
        out = -1/(2*pi^2*k^2) + 1/(4*pi*k);
    elseif mod(k, 4) == 2
        out = -1/(pi^2*k^2);
    else
        out = -1/(2*pi^2*k^2) - 1/(4*pi*k);
    end
end
```

For |k| <= 20:



Code in `main.m`:

```
k = -20:1:20;
c_k = zeros(1, length(k));
for i = 1:length(k)
    c_k(i) = abs(c(k(i)))^2;
end
stem(k, c_k);
title("Power Spectrum for x3(t) where |k| <=
5");
xlabel("k");
ylabel("|c_k|^2");
grid on;
%{
Comments:
Apart from k = 0, the power is zero at all multiples of 4 which corresponds
to our answer for Problem 1). As for |k| <= 5, you can't really tell much
about this graph as we don't see enough of k yet. For |k| <= 10 and 20, it
gets far more obvious that the higher frequencies at both extremes have a
lower power. It should be noted that the odd frequencies take far slower to
decay while non-multiples-of-four decay fast.
%}
```

Code in `c.m`:

```
function out = c(k)
    if k == 0
        out = 1/16;
    elseif mod(k, 4) == 0
        out = 0;
    elseif mod(k, 4) == 1
        out = -1/(2*pi^2*k^2) + 1/(4*pi*k);
    elseif mod(k, 4) == 2
        out = -1/(pi^2*k^2);
    else
        out = -1/(2*pi^2*k^2) - 1/(4*pi*k);
    end
end
```