Winter 2024
ECE 102
Christina Fragouli
TA: Semira Galijasevic

Systems and Signals

Homework 6
Due 1 PM Friday, Mar 15, 2024
Submit your solutions on Gradescope.

Note: Answers without justification will not be awarded any marks.

Problem 1 (14 points)[**Bandpass Sampling**]

The figure below shows the Fourier transform of a real bandpass signal, i.e., a signal whose frequencies are not centered around the origin. We want to sample this signal. Let F_s in Hz represent the sampling frequency.

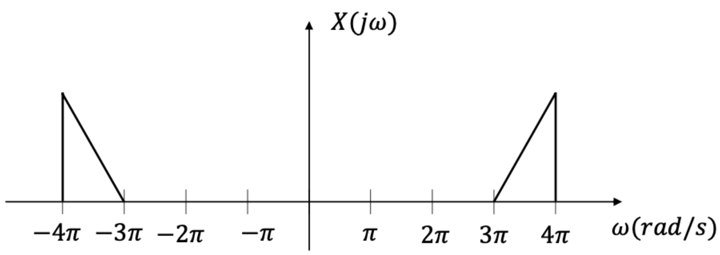

Figure 1: Problem 1

(a) *(4 points)* One option is to sample this signal at the Nyquist rate. Then to recover the signal, we pass its sampled version through a low pass filter. What is the Nyquist rate of this signal?

(b) *(10 points)* Since the signal might have high frequency components, Nyquist rate for this signal can be high. In other words, we need to have a lot of samples of the signal, which means that the sampling scheme is costy. It turns out that for this type of signal, we can sample it at a sampling frequency lower that the Nyquist rate and we can still recover the signal, however in this case, we will use a **bandpass filter**. To see this, we have the following two options for the sampling frequency:

- $F_s = 0.5\ Hz$;
- $F_s = 1\ Hz$;

For each case, draw the spectrum of the signal after sampling it. To recover the signal, which F_s can we use? How we should choose the frequencies of the bandpass filter? What is the minimum F_s we can use and still recover the signal?

Problem 2 (16 points)

Find the Laplace transforms of the following signals and determine their region of convergence.

(a) $f(t) = te^{-at}(\cos\omega_0 t)^2 u(t)$

(b) $f(t) = \int_{0^-}^{t-1} e^{-ax}\left(\cos(\omega_0 x) - 1\right)\, dx,\;\; t-1 \geq 0$

(c) $f(t) = \begin{cases} 3, & 0 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$

(d) $f(t) = \cos(t)\, u(t-3)$

Problem 3 (14 points)

(I) *(12 points)* In each of the following cases, you are give two causal signals, $x_1(t)$ and $x_2(t)$. For each case, find: [i] The Laplace transforms $X_1(s)$ and $X_2(s)$; [ii] The region of convergence (ROC) for $X_1(s)$ and $X_2(s)$;[iii] The Laplace transform $X(s)$; [iv] The region of convergence of $X(s)$.

(a) $x_1(t) = e^{-3t}u(t)$
$x_2(t) = \delta(t-3) + e^{-4t}u(t)$
$X(s) = X_1(s)X_2(s)$

(b) $x_1(t) = \sin(3t)u(t)$
$x_2(t) = 2\cos(4t)u(t)$
$X(s) = X_1(s) + X_2(s)$

(c) $x_1(t) = e^{-3t}u(t)$
$x_2(t) = \frac{d}{dt}[\delta(t) - e^{-2t}u(t)]$
$X(s) = X_1(s)X_2(s)$

(II) *(2 points)* By knowing the ROC of $X_1(s)$ and the ROC of $X_2(s)$, can we always determine the ROC of $X_3(s) = X_1(s)X_2(s)$? Can we determine the ROC of $X_4(s) = X_1(s) + X_2(s)$?

Problem 4 (28 points)

Find the inverse Laplace transform $f(t)$ for each of the following $F(s)$ (assuming $f(t)$ is a causal signal):

(a) *(5 points)* $F(s) = \dfrac{e^{-(s+3)}}{(s+3)}$

(b) *(6 points)* $F(s) = \dfrac{s^2+s+1}{(s+1)(s+2)(s+3)}$

(c) *(6 points)* $F(s) = \dfrac{s^3 e^{-s} - 1}{s(s+2)(s^2+8s+15)}$

(d) *(5 points)* $F(s) = \dfrac{s+9}{s^2+4}$

(e) *(6 points)* $F(s) = \dfrac{s^2-3}{(s+1)(s^2+2s+2)}$

Problem 5 (16 points)[**LTI system**]

Assume a causal LTI system S_1 is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = ax(t), \quad y(0) = 0,\ y'(0) = 0$$

wher a is a constant. Moreover, we know that when the input is e^t, the output of the system is $\frac{1}{2}e^t$.

(a) *(5 points)* Find the transfer function $H_1(s)$ of the system. (The answer should not be in terms of a, i.e., you should find the value of a).

(b) *(5 points)* Find the output $y(t)$ when the input is $x(t) = u(t)$.

(c) *(6 points)* The system S_1 is linearly cascaded with another causal LTI system S_2. The system S_2 is given by the following input-output pair:

$$S_2 \text{ input: } u(t) - u(t-1) \rightarrow \text{ output: } r(t) - 2r(t-1) + r(t-2)$$

Find the overall impulse response.

Problem 6 (12 points)

Suppose the transfer function of an LTI system is

$$H(s) = \frac{s}{s^2+s+1}$$

Find the unit-step response $s(t) = h(t) \star u(t)$ of the system (i.e. $s(t)$ is the output when the input is $u(t)$). Use $s(t)$ to determine the output response to the following inputs:

(a) (2 points) $x_1(t) = u(t) - u(t-1)$

(b) (4 points) $x_2(t) = \delta(t) - \delta(t-1)$

(c) (4 points) $x_3(t) = r(t)$

(d) (2 points) $x_4(t) = r(t) - 2r(t-1) + r(t-2)$