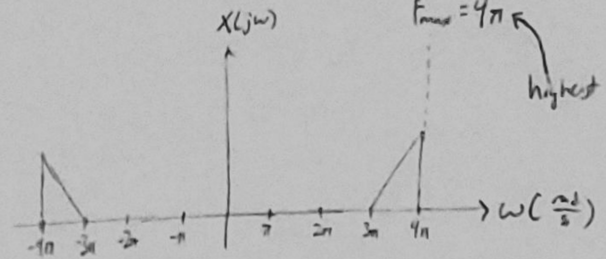
EE 102 HW6

1)

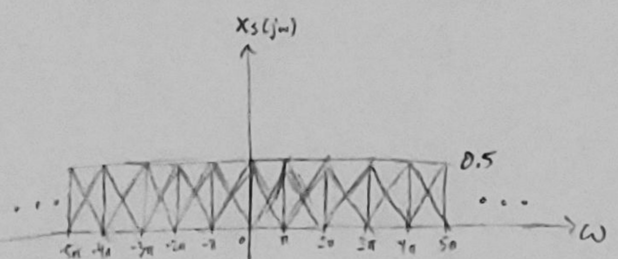
a) Nyquist Rate of Low-Pass Filter: $F_s = 2F_{max}$

$\boxed{\text{Nyquist}: 8\pi \ \frac{rad}{s} \text{ or } 4\,Hz}$

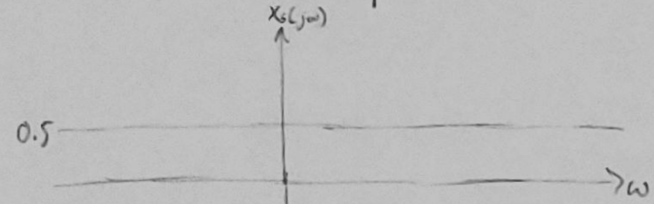


b) $F_s = 0.5\,Hz$, $\omega_s = \pi$, $T_s = 2$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - \omega_s k)) = 0.5\sum_{k=-\infty}^{\infty} X(j(\omega - \pi k))$$



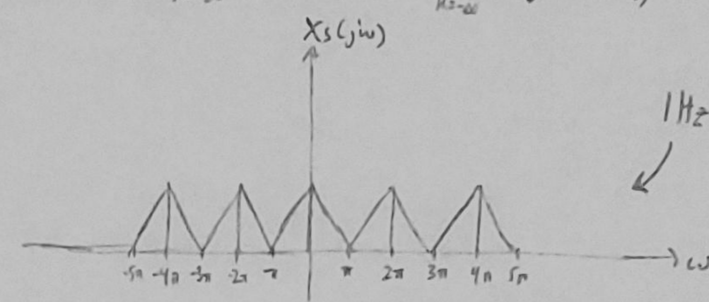
adds up to



It overlaps perfectly to $0.5 = X_s(j\omega)$, so we can't recover the original signal at all

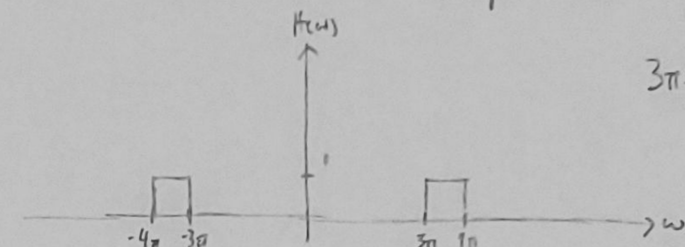
$F_s = 1\,Hz$, $\omega_s = 2\pi$, $T_s = 1$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - \omega_s k)) = \sum_{k=-\infty}^{\infty} X(j(\omega - 2\pi k))$$



1 Hz is the minimum since there's no space in between

We can use this bandpass filter



$3\pi \le |\omega| \le 4\pi$

we wanna keep the original signal too, so choose

Bandpass should allow original $x(t)$ frequencies only, $\boxed{F_{s\,min} = 1\,Hz}$

2)

a) $f(t) = te^{-at}(\cos\omega_0 t)^2 u(t)$

$$\cos\omega_0 t = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}, \quad s = \sigma + i\omega$$

$= te^{-at}\left(\frac{1}{2} + \frac{\cos 2\omega_0 t}{2}\right)u(t)$

$= \left[\frac{1}{2}te^{-at} + \frac{1}{2}te^{-at}\cos(2\omega_0 t)\right]u(t) = \frac{1}{2}te^{-at}u(t) + \frac{1}{2}te^{-at}\cos(2\omega_0 t)u(t) = \frac{1}{2}te^{-at} + \frac{1}{2}te^{-at}\cos(2\omega_0 t)$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

$\mathcal{L}\{e^{-at}t\} = \frac{1}{(s+a)^2}$

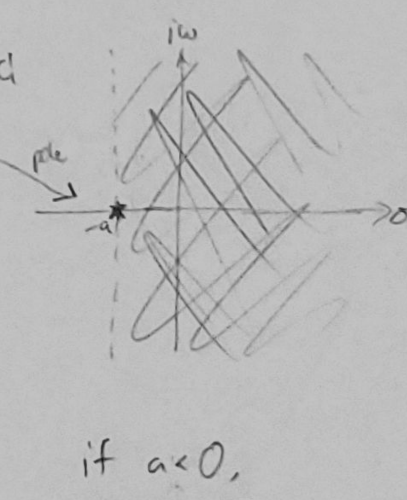
$\mathcal{L}\{e^{-at}x(t)\} = X(s+a)$

$\mathcal{L}\{t\,x(t)\} = -\frac{d}{ds}X(s)$

$\mathcal{L}\{\cos(2\omega_0 t)\} = \frac{s^2}{s^2 + 4\omega_0^2}$

$$\mathcal{L}\{\tfrac{1}{2}te^{-at}\} + \mathcal{L}\{\tfrac{1}{2}te^{-at}\cos(2\omega_0 t)\} = \boxed{\frac{1}{2(s+a)^2} + \frac{(s+a)^2 + 4\omega_0^2}{2[(s+a)^2 + 4\omega_0^2]^2}}$$

ROC



$$\int_{-\infty}^{\infty} te^{-at}\cos^2(\omega_0 t)u(t)e^{-st}\,dt = \int_0^{\infty} t\cos^2(\omega_0 t)e^{-t(a+s)}\,dt$$

will converge bc e is exponential while t is linear $\boxed{\sigma > -a}$

b) $f(t) = \int_{0^-}^{t-1} e^{-ax}(\cos(\omega_0 x) - 1)\,dx$

$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}F(s)$

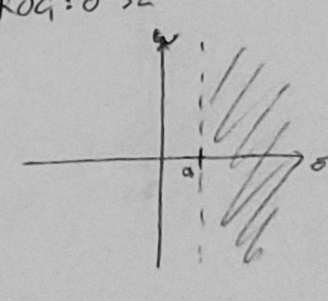
$$\mathcal{L}\{e^{-ax}\cos(\omega_0 x) - e^{-ax}\} = \frac{s+a}{(s+a)^2 + \omega_0^2} - \frac{1}{s+a}$$

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} \to \frac{F(s)}{s}$$

$$\mathcal{L}\{\int_{0^-}^{t-1} e^{-ax}(\cos(\omega_0 x) - 1)\,dx\} = \boxed{\frac{\frac{s+a}{(s+a)^2 + \omega_0^2} - \frac{1}{s+a}}{s}}$$

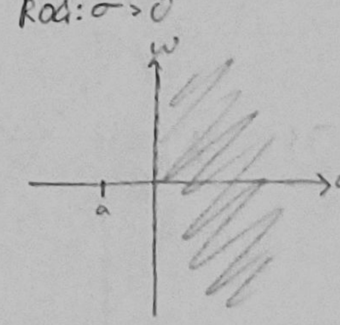
if $a > 0$,

ROC: $\sigma > a$



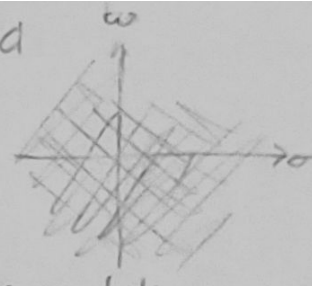
if $a < 0$,

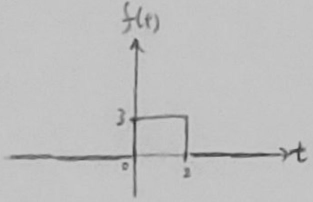
ROC: $\sigma > 0$



c) $f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases} = 3u(t) - 3u(t-2) \rightarrow \boxed{F(s) = 3\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right)}$

ROC





$\mathcal{L}\{u(t)\} = \frac{1}{s}$

$\mathcal{L}\{u(t-t_0)\} = \frac{e^{-t_0 s}}{s}$

$\boxed{\sigma \text{ can be any real \#}}$

$\int_{-\infty}^{\infty} 3[u(t) - u(t-2)]e^{-st}\,dt = 3\int_0^2 e^{-st}\,dt$

will never go to ∞ so no hole

d) $f(t) = \cos(t)\,u(t-3)$

$\mathcal{L}\{f(t-t_0)u(t-t_0)\} = F(s)e^{-t_0 s}$

$= \cos(t-3+3)\,u(t-3)$

$= \cos(t-3)\cos(3)\,u(t-3) - \sin(t-3)\sin(3)\,u(t-3)$

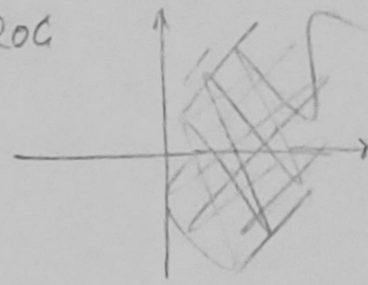
$= 0.99\cos(t-3)\,u(t-3) - 0.05\sin(t-3)\,u(t-3)$

$\boxed{F(s) = 0.99\frac{se^{-3s}}{s^2+1} - 0.05\frac{e^{-3s}}{s^2+1}}$

$\int_{-\infty}^{\infty} \cos(t)\,u(t)e^{-st}\,dt = \int_0^{\infty} \cos(t)e^{-st}\,dt$

$\boxed{\sigma > 0}$

ROC



3)

I)

a)

i) $x_1(t) = e^{-3t}u(t) \rightarrow X_1(s) = \int_0^{\infty} e^{-3t}e^{-st}\,dt = \int_0^{\infty} e^{-t(3+s)}\,dt$

$= \frac{e^{-t(3+s)}}{-3-s}\Big|_0^{\infty} \boxed{= \frac{1}{3+s}}$

$x_2(t) = \delta(t-3) + e^{-4t}u(t) \rightarrow X_2(s) = \int_{-\infty}^{\infty} \delta(t-3)e^{-st}\,dt + \int_0^{\infty} e^{-4t}e^{-st}\,dt$

$= \int_{-\infty}^{\infty} \delta(t-3)e^{-3s}\,dt + \int_0^{\infty} e^{-t(4+s)}\,dt$

$= e^{-3s} + \frac{e^{-t(4+s)}}{-4-s}\Big|_0^{\infty} \boxed{= e^{-3s} + \frac{1}{4+s}}$

ii) ROC for $X_1(s) \Rightarrow \boxed{\sigma > -3}$

ROC for $X_2(s) \Rightarrow \boxed{\sigma > -4}$

iii) $\frac{1}{3+s}\left(e^{-3s} + \frac{1}{4+s}\right) = X(s)$

$\boxed{= \frac{e^{-3s}}{3+s} + \frac{1}{s^2+7s+12}}$

iv) Common area of ROC is taken: $\boxed{\sigma > -3}$

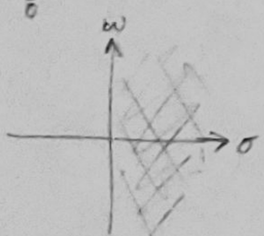
b)

i) $x_1(t) = \sin(3t)u(t) \rightarrow \boxed{X_1(s) = \frac{3}{s^2+9}}$

$x_2(t) = 2\cos(4t)u(t) \rightarrow \boxed{X_2(s) = \frac{2s}{s^2+16}}$

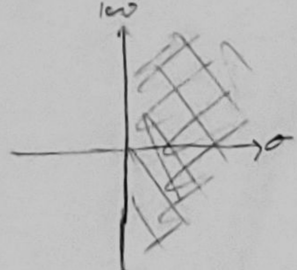
ii) $X_1(s)$ ROC: $\int_0^{\infty} \sin(3t)e^{-st}\,dt$

$\boxed{\sigma > 0}$



$X_2(s)$ ROC: $\int_0^{\infty} 2\cos(4t)e^{-st}\,dt$

$\boxed{\sigma > 0}$



iii) $X(s) = X_1(s) + X_2(s)$

$\boxed{= \frac{3}{s^2+9} + \frac{2s}{s^2+16}}$

iv) Common area of both, so ROC is $\boxed{\sigma > 0}$

c)

1) $x_1(t) = e^{-3t} u(t)$

$\rightarrow \boxed{X_1(s) = \frac{1}{3+s}}$

$x_2(t) = \frac{d}{dt}\left[\delta(t) - e^{-2t} u(t)\right]$

$\downarrow \quad 1 \qquad \downarrow \quad \frac{1}{s+2}$

$\frac{dx(t)}{dt} \rightarrow sX(s)$, $X_2(s) = s\left(1 - \frac{1}{2+s}\right) \boxed{= s - \frac{s}{2+s}}$

$X_2(s)$ ROC: $\boxed{\sigma > -2}$

ii) $X_1(s)$ ROC: $\boxed{\sigma > -3}$

iii) $X_1(s) \cdot X_2(s) = X(s)$

$= s\left(1 - \frac{1}{s+2}\right) \cdot \frac{1}{3+s} \boxed{= \frac{s}{s+3} - \frac{s}{(s+2)(s+3)}}$

iv) ROC is common area: $\boxed{\sigma > -2}$

II) No, since although we might know $X_1(s)$ and $X_2(s)$'s ROC, the poles can cancel if you add or multiply
For example $\frac{s}{s+1} + \frac{1}{s+1} = 1$ so no pole or $\frac{s+2}{\cancel{s+1}} \cdot \frac{\cancel{s+1}}{s+50} = \frac{s+2}{s+50}$ so no $s+1$ pole

4)

a)

$F(s) = \frac{e^{-(s+3)}}{s+3}$ $\qquad e^{at} f(t) \longleftrightarrow F(s-a)$

$f(t) = e^{-3t} \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} \boxed{= e^{-3t} u(t-1)}$

b) $F(s) = \frac{s^2+s+1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} = \frac{\frac{1}{2}}{s+1} + \frac{-3}{s+2} + \frac{\frac{7}{2}}{s+3} = \boxed{\left[\frac{1}{2}e^{-t} - 3e^{-2t} + \frac{7}{2}e^{-3t}\right]u(t)}$

multiply everything by $(s+3)$ $\qquad$ multiply by $(s+2)$ $\qquad$ multiply by $(s+1)$

$\left.\frac{s^2+s+1}{(s+1)(s+2)}\right|_{s=-3} = C \Rightarrow \frac{7}{2}$ $\qquad \left.\frac{s^2+s+1}{(s+1)(s+3)}\right|_{s=-2} = B \Rightarrow -3$ $\qquad \left.\frac{s^2+s+1}{(s+2)(s+3)}\right|_{s=-1} = A \Rightarrow \frac{1}{2}$

c) $F(s) = \frac{s^3 e^{-s} - 1}{s(s+2)(s+3)(s+5)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+5} = \frac{-\frac{1}{30}}{s} + \frac{\frac{4}{3}e^2}{s+2} + \frac{\frac{1}{6}}{s+2} + \frac{-\frac{9}{2}e^3}{s+3} - \frac{\frac{1}{6}}{s+3} + \frac{\frac{25}{6}e^5}{s+5} + \frac{\frac{1}{30}}{s+5}$

Multiply all by s:

$\left.\frac{s^3 e^{-s} - 1}{(s+2)(s+3)(s+5)}\right|_{s=0} = A = -\frac{1}{30}$

Multiply by $(s+2)$:

$\left.\frac{s^3 e^{-s} - 1}{s(s+3)(s+5)}\right|_{s=-2} = B = \frac{4}{3}e^2 + \frac{1}{6}$

Multiply by $(s+3)$:

$\left.\frac{s^3 e^{-s} - 1}{s(s+2)(s+5)}\right|_{s=-3} = C = -\frac{9}{2}e^3 - \frac{1}{6}$

Multiply by $(s+5)$:

$\left.\frac{s^3 e^{-s} - 1}{s(s+2)(s+3)}\right|_{s=-5} = D = \frac{25}{6}e^5 + \frac{1}{30}$

$\boxed{f(t) = u(t)\left[-\frac{1}{30} + \frac{4}{3}e^{2-2t} + \frac{1}{6}e^{-2t} - \frac{9}{2}e^{3-3t} - \frac{1}{6}e^{-3t} + \frac{25}{6}e^{5-5t} + \frac{1}{30}e^{-5t}\right]}$

d) $F(s) = \frac{s+9}{(s+2i)(s-2i)} = \frac{A}{s+2i} + \frac{B}{s-2i}$

multiply by both:

$s+9 = A(s-2i) + B(s+2i) \Rightarrow A = \frac{1}{2} + \frac{9}{4}i$

$B = \frac{1}{2} - \frac{9}{4}i$

$f(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2} + \frac{9}{4}i}{s+2i} + \frac{\frac{1}{2} - \frac{9}{4}i}{s-2i}\right\} = \left(\frac{1}{2} + \frac{9}{4}i\right)e^{-2it} + \left(\frac{1}{2} - \frac{9}{4}i\right)e^{2it} = \frac{1}{2}\left(e^{2it} + e^{-2it}\right) + \frac{9}{4}\left(\frac{e^{2it} - e^{-2it}}{i}\right)$

$\boxed{= \cos(2t) + \frac{9}{2}\sin(2t)}$

e) $F(s) = \frac{s^2-3}{(s+1)(s^2+2s+2)} = \frac{-2}{s+1} + \frac{3s+1}{s^2+2s+2} = \frac{-2}{s+1} + 3 \cdot \frac{s+1}{(s+1)^2+1} - 2 \cdot \frac{1}{(s+1)^2+1}$

$\downarrow \qquad \downarrow \qquad \downarrow$

$-2e^{-t} \qquad 3e^{-t}\cos(t) - 2e^{-t}\sin(t)$

$\boxed{f(t) = -2e^{-t} + 3e^{-t}\cos(t) - 2e^{-t}\sin(t)}$

5)

a) $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = ax(t)$ IC: $y(0)=0$, $y'(0)=0$

$\frac{d^2y(t)}{dt^2} \rightarrow s^2Y(s) - sy(0^-) - y'(0^-) = s^2Y(s)$

$\frac{dy(t)}{dt} \rightarrow sY(s) - y(0^-) = sY(s)$

$x(t) = e^t \xrightarrow{*h(t)} \frac{1}{2}e^t = y(t)$

$s^2Y(s) + 4sY(s) + 3Y(s) = aX(s)$

$= Y(s)(s^2+4s+3) = aX(s)$

$\Rightarrow Y(s) = \frac{a}{s^2+4s+3}X(s)$, $H(s) = \frac{a}{s^2+4s+3}$

$\underset{X(s)}{\frac{1}{s-1}} \xrightarrow{\cdot H_1(s)} \underset{Y(s)}{\frac{1}{2}\cdot\frac{1}{s-1}}$

$H_1(s) = \frac{1}{2}$

$H_1(s)\big|_{s=1} = \frac{1}{2} \Rightarrow \frac{a}{s^2+4s+3}\Big|_{s=1} = \frac{1}{2}$

$\Rightarrow \frac{a}{8} = \frac{1}{2} \Rightarrow a = 4$

$\boxed{H_1(s) = \frac{4}{s^2+4s+3}}$

b) $x(t) = u(t) \xrightarrow{*h(t)} y(t)$

$X(s) = \frac{1}{s} \xrightarrow{\cdot H(s)} Y(s)$

$Y(s) = \underset{X(s)}{\frac{1}{s}} \cdot \underset{H_1(s)}{\frac{4}{s^2+4s+3}} = \frac{4}{s(s^2+4s+3)} = \frac{4}{s(s+3)(s+1)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1} = \frac{\frac{4}{3}}{s} + \frac{\frac{2}{3}}{s+3} - \frac{2}{s+1}$

Multiply by s:
$\frac{4}{(s+3)(s+1)}\Big|_{s=0} = A = \frac{4}{3}$

Multiply by (s+3):
$\frac{4}{s(s+1)}\Big|_{s=-3} = B = \frac{2}{3}$

Multiply by (s+1):
$\frac{4}{s(s+3)}\Big|_{s=-1} = C = -2$

$\boxed{y(t) = \left[\frac{4}{3} + \frac{2}{3}e^{-3t} - 2e^{-t}\right]u(t)}$

c) S_2: $\underset{x(t)}{u(t) - u(t-1)} \xrightarrow{*h(t)} \underset{y(t)}{r(t) - 2r(t-1) + r(t-2)}$

$X_2(s) = \frac{1}{s} - \frac{e^{-s}}{s}$

$Y_2(s) = \frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$

$(1-e^{-s})^2$

$H_2(s) = \frac{Y_2(s)}{X_2(s)} = \frac{\frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}}{\frac{1}{s} - \frac{e^{-s}}{s}} = \frac{1 - 2e^{-s} + e^{-2s}}{s(1-e^{-s})} = \frac{1-e^{-s}}{s}$

Cascade:

$X(s) \longrightarrow \boxed{H_1(s)} \longrightarrow \boxed{H_2(s)} \longrightarrow Y(s)$

$H_{eq}(s) = H_1(s)H_2(s)$

$= \frac{4}{s^2+4s+3}\cdot\frac{1-e^{-s}}{s} = \frac{4}{s(s^2+4s+3)} - \frac{4e^{-s}}{s(s^2+4s+3)}$

$\boxed{h_{eq}(t) = \left[\frac{4}{3} + \frac{2}{3}e^{-3t} - 2e^{-t}\right]u(t) + \left[-\frac{4}{3} - \frac{2}{3}e^{-3(t-1)} + 2e^{-(t-1)}\right]u(t-1)} \xleftarrow{\mathcal{L}^{-1}} H_{eq}(s) = \frac{\frac{4}{3}}{s} + \frac{\frac{2}{3}}{s+3} - \frac{2}{s+1} - \frac{\frac{4}{3}}{s}e^{-s} - \frac{\frac{2}{3}}{s+3}e^{-s} + \frac{2}{s+1}e^{-s}$

6) $S(s) = H(s)\cdot\mathcal{L}^{-1}\{u(t)\} = \frac{s}{s^2+s+1}\cdot\frac{1}{s} = \frac{1}{s^2+s+1} = \frac{1}{(s+0.5)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$\boxed{s(t) = \frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t)}$

a) $x_1(t) = u(t) - u(t-1)$, just a time shift so output is time shifted too.

$\boxed{s_1(t) = s(t) - s(t-1)}$

b) $x_2(t) = \delta(t) - \delta(t-1)$

$= \frac{d}{dt}(u(t) - u(t-1))$

$\frac{d}{dt}u(t) = \delta(t)$

$s_2(t) = \frac{d}{dt}(s(t) - s(t-1))$

$\boxed{= \frac{d}{dt}s(t) - \frac{d}{dt}s(t-1)}$

c) $x_3(t) = r(t) = \int_{-\infty}^{t} u(\tau)\,d\tau$

$$\boxed{s_3(t) = \int_{-\infty}^{t} s(\tau)\,d\tau}$$

d) $x_4(t) = r(t) - 2r(t-1) + r(t-2) = \int_{-\infty}^{t} u(\tau)\,d\tau - 2\int_{-\infty}^{t} u(\tau-1)\,d\tau + \int_{-\infty}^{t} u(\tau-2)\,d\tau$

$$\boxed{s_4(t) = s_3(t) - 2s_3(t-1) + s_3(t-2)}$$