EE 102 HW 5

1) $T_0 = 4$, $\omega_0 = \frac{\pi}{2}$

$$X_k = \frac{A}{|k|} + jB|k| = \frac{A}{|k|} \qquad X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\cancel{e^{-jk\omega_0 t}}^{\,1}\,dt$$

$$X_0 = C \qquad\qquad = \frac{1}{4}\int_{-2}^{2} x(t)\,dt = C$$

Since $x(t)$ is real & even, x_k is real too, so $\boxed{B = 0}$

$$\int_{-2}^{2}|x(t)|^2 dt = 2 \longleftarrow$$ sum of $[-2,0]$ and $[0,2]$ are each 1

$$\int_0^2 x(t)\,dt = 1$$

Since $x(t)$ is even, we can double $\int_0^2 x(t)dt$ to find $\int_{-2}^{2} x(t)dt$

$$\int_{-2}^{2} x(t)\,dt = 2\cdot\int_0^2 x(t)\,dt = 2$$

$$C = \frac{1}{4}\int_{-2}^{2} x(t)\,dt \boxed{= \frac{1}{2}}$$

$$\frac{1}{4}\int_{-2}^{2}|x(t)|^2 dt = \sum_{k=-\infty}^{\infty}|X_k|^2 = 2\cdot\frac{1}{4}, \quad X_k = \frac{A}{|k|}$$

$$\Rightarrow \frac{1}{2} = X_0^2 + 2\sum_{k=1}^{\infty}|X_k|^2, \quad X_0 = \frac{1}{2}$$

$$\Rightarrow \frac{1}{8} = \sum_{k=1}^{\infty}\left|\frac{A^2}{k^2}\right| = A^2\sum_{k=1}^{\infty}\frac{1}{k^2} = A^2\frac{\pi^2}{6}$$

$$\boxed{A = \frac{1}{\pi}\sqrt{\frac{3}{4}}}$$

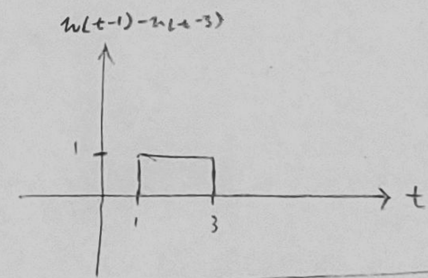
2)

a) $x_1(t) = e^{(6+4j)t}u(-t)$

$$X_1(f) = \int_{-\infty}^{\infty} x_1(t)e^{-j2\pi f t}dt = \int_{-\infty}^{0} e^{(6+4j)t}e^{-j2\pi f t}dt = \int_{-\infty}^{0} e^{(6+4j-j2\pi f)t}dt$$

$$= \left.\frac{e^{(6+4j-j2\pi f)t}}{(6+4j-j2\pi f)}\right|_{-\infty}^{0} = \frac{1}{6+4j-j2\pi f} - 0 \boxed{= \frac{1}{6+j(4-2\pi f)}}$$

b) $x_2(t) = 2\sin(3t)[u(t-1) - u(t-3)]$



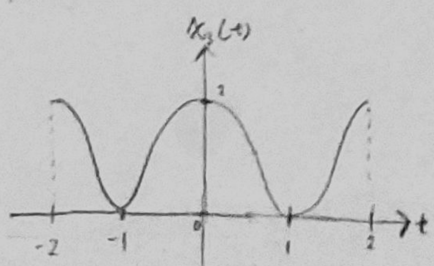
$$X_2(f) = \int_{-\infty}^{\infty} x_2(t)e^{-j2\pi f t}dt = \int_1^3 2\sin(3t)\,e^{-j2\pi f t}dt$$

$$= 2\int_1^3 \frac{e^{i3t}-e^{-i3t}}{2i}\cdot e^{-j2\pi f t}dt = \frac{1}{i}\int_1^3 e^{jt(3-2\pi f)} - e^{-jt(3+2\pi f)}dt$$

$$= \frac{1}{i}\int_1^3 e^{(3-2\pi f)jt} - \frac{1}{i}\int_1^3 e^{-(3+2\pi f)jt}dt = \frac{1}{i}\cdot\left.\frac{e^{(3-2\pi f)jt}}{(3-2\pi f)i}\right|_1^3 + \frac{1}{i}\cdot\left.\frac{e^{-(3+2\pi f)jt}}{+(3+2\pi f)j}\right|_1^3$$

$$\boxed{= -\left(\frac{e^{3(3-2\pi f)j}}{(3-2\pi f)j} - \frac{e^{(3-2\pi f)j}}{(3-2\pi f)j} + \frac{e^{-3(3+2\pi f)j}}{(3-2\pi f)j} - \frac{e^{-(3+2\pi f)j}}{(3+2\pi f)j}\right)}$$

c) $x_3(t) = \begin{cases} 1+\cos(\pi t) & \text{if } |t| < 2 \\ 0 & \text{otherwise}\end{cases}$



$$X_3(f) = \int_{-2}^{2}(1+\cos(\pi t))e^{-j2\pi f t}dt = \int_{-2}^{2}\left(1 + \frac{e^{i\pi t}+e^{-i\pi t}}{2}\right)e^{-j2\pi f t}dt$$

$$\cos(\pi t) = \frac{e^{i\pi t}+e^{-i\pi t}}{2}$$

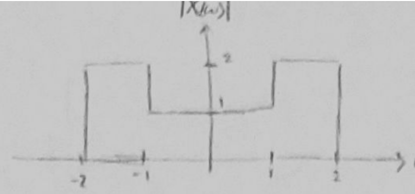
$$= \int_{-2}^{2} e^{-j2\pi f t}dt + \frac{1}{2}\int_{-2}^{2} e^{i(\pi-2\pi f)t}dt + \frac{1}{2}\int_{-2}^{2} e^{-i(\pi+2\pi f)t}dt$$

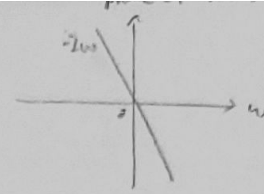
$$= \left.\frac{e^{-j2\pi f t}}{-j2\pi f}\right|_{-2}^{2} + \left.\frac{e^{i(\pi-2\pi f)t}}{2i(\pi-2\pi f)}\right|_{-2}^{2} + \left.\frac{e^{-i(\pi+2\pi f)t}}{-2i(\pi+2\pi f)}\right|_{-2}^{2}$$

$$\boxed{= \frac{1}{-j2\pi f}\left(e^{-j4\pi f} - e^{j4\pi f}\right) + \frac{1}{2i(\pi-2\pi f)}\left(e^{2i(\pi-2\pi f)} - e^{-2i(\pi-2\pi f)}\right) + \frac{1}{-2i(\pi+2\pi f)}\left(e^{-2i(\pi+2\pi f)} - e^{2i(\pi+2\pi f)}\right)}$$

3)

a) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$





$$X(\omega) = \left(2\left[u(\omega+2) - u(\omega-2)\right] - \left[u(\omega+1) - u(\omega-1)\right]\right) e^{-j2\omega}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2e^{j\omega(t-2)}\left[u(\omega+2) - u(\omega-2)\right] d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-2)}\left[u(\omega+1) - u(\omega-1)\right] d\omega$$

$$= \frac{1}{\pi}\int_{-2}^{2} e^{j\omega(t-2)}\, d\omega - \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega(t-2)}\, d\omega = \frac{1}{\pi}\cdot\frac{e^{j\omega(t-2)}}{j(t-2)}\Bigg|_{-2}^{2} - \frac{1}{2\pi}\cdot\frac{e^{j\omega(t-2)}}{j(t-2)}\Bigg|_{-1}^{1} = \frac{e^{j2(t-2)} - e^{-j2(t-2)}}{j\pi(t-2)} - \frac{e^{j(t-2)} - e^{-j(t-2)}}{j2\pi(t-2)}$$

$$= \frac{2\sin(2(t-2))}{\pi(t-2)} - \frac{\sin(t-2)}{\pi(t-2)} \boxed{= \frac{1}{\pi(t-2)}\left(4\sin(t-2)\cos(t-2) - \sin(t-2)\right)}$$

b) $X(\omega) = \omega^2\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] + \frac{2}{-j(\omega-\omega_0)+1} = \underbrace{\omega_0^2\delta(\omega-\omega_0) + \omega_0^2\delta(\omega-\omega_0)}_{\frac{\cancel{2}\omega_0^2}{\cancel{2}\pi}\cos(\omega_0 t)} + \underbrace{2\cdot\frac{1}{1+\omega_0-j\omega}}_{2e^{(1+\omega_0)t}u(-t)}$

$$\boxed{x(t) = \frac{\omega_0^2}{\pi}\cos(\omega_0 t) + 2e^{(1+\omega_0)t}u(-t)}$$

4) Note: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt = \underbrace{\int_{-\infty}^{\infty} x(t)\cos(\omega t)\, dt}_{x_e(t),\ \text{even}} - j\underbrace{\int_{-\infty}^{\infty} x(t)\sin(\omega t)\, dt}_{x_o(t),\ \text{odd}} = \int_{-\infty}^{\infty} x_e(t)\cos(\omega t)\, dt - j\int_{-\infty}^{\infty} x_o(t)\sin(\omega t)\, dt$

The Fourier cosine transform can only be applied to EVEN functions

The fourier sine transformation can only be applied to ODD functions

So $X(\omega) = X_1(\omega) + X_2(\omega)$

a) $\boxed{X_1(\omega) = \int_{-\infty}^{\infty} x_e(t)\cos(\omega t)\, dt}$, the Fourier transform of any EVEN & REAL function is always REAL & EVEN, hence $X_1(\omega)$ is real & even.

b) $\boxed{X_2(\omega) = -j\int_{-\infty}^{\infty} x_o(t)\sin(\omega t)\, dt}$, the fourier transform of any REAL & ODD function is always IMAGINARY & ODD, thus $X_2(\omega)$ is im & odd.

c) I just proved that in the note, $Re\{X(\omega)\} = X_1(\omega)$ and $j\, Im\{X(\omega)\} = X_2(\omega)$

d) If $x(t)$ is REAL & EVEN, then $X(\omega)$ is ALWAYS REAL & EVEN

If $x(t)$ is REAL & ODD, then $X(\omega)$ is ALWAYS IMAGINARY & ODD

5)

1. $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$

$\int_{-\infty}^{\infty} f(t)\, dt = F(0) = Re\{F(0)\} + Im\{F(0)\}$

$= 3 + 0\cdot j \boxed{= 3}$

2. $\int_{-\infty}^{\infty} f(t) e^{j2\pi t}\, dt = F(\omega)\big|_{\omega=2\pi} = \left(Re\{F(\omega)\} + j\, Im\{F(\omega)\}\right)\big|_{\omega=2\pi}$

$\boxed{= 2+2j}$

3. $\int_{-\infty}^{\infty} f(t)\cos(2\pi t)e^{i2\pi t}dt = \int_{-\infty}^{\infty} f(t)\left(\frac{e^{i2\pi t}+e^{-i2\pi t}}{2}\right)e^{i2\pi t}dt$

$= \frac{1}{2}\int_{-\infty}^{\infty} f(t)e^{i4\pi t}dt + \frac{1}{2}\int_{-\infty}^{\infty} f(t)dt = \frac{1}{2}F(\omega)\big|_{\omega=4\pi} + \frac{1}{2}F(\omega)\big|_{\omega=0}$

$= \frac{1}{2}(1+4j) + \frac{1}{2}\cdot 3 \boxed{= 2+2j}$

4. $f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}dt$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)dt = \frac{1}{2\pi}(2\pi + 2\pi + 6\pi + 9\pi + \pi + 2\pi)$

$\boxed{= 8.5}$

$f(t)$ $\boxed{\text{isn't real}}$, as the fourier transform of a real signal must obey $F(\omega) = F^*(-\omega)$, AKA the real component of fourier transform must be even & imaginary part must be odd. The real part in this scenario isn't even

6)

a) $x_1(t) = x(2-t) + x(-2-t)$

$x(-t) \longleftrightarrow X(-\omega)$

$x(t-t_0) \longleftrightarrow X(\omega)e^{-j\omega t_0}$

$F\{x(2-t)\} = X(-\omega)e^{-j\omega\cdot -2}$

$= X(-\omega)e^{2j\omega}$

$F\{x(-t-2)\} = X(-\omega)e^{-j\omega\cdot 2}$

$= X(-\omega)e^{-2j\omega}$

$F\{x_1(t)\} = X(-\omega)e^{2j\omega} + X(-\omega)e^{-2j\omega}$

$\boxed{X_1(\omega) = 2\cos(\omega)X(-\omega)}$

b) $x_2(t) = x(2t-4)$

$F\{x(at)\} = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$

$F\{x_2(t)\} = \frac{1}{2}X\left(\frac{\omega}{2}\right)e^{-j\omega\cdot 2}$

$\boxed{X_2(t) = \frac{e^{-2j\omega}}{2}X\left(\frac{\omega}{2}\right)}$

c) $x_3(t) = \frac{d^2}{dt^2}x(t-2)$

$F\left\{\frac{d^2x(t)}{dt^2}\right\} = (i\omega)^2X(\omega) = -\omega X(\omega)$

$F\{x_3(t)\} = (i\omega)^2X(\omega)e^{-j\omega\cdot 2}$

$\boxed{= -\omega X(\omega)e^{-2j\omega}}$