9/26/22

The solution curve is TANGENT to the direction/slope field at each point. The slope at each point $(t, y(t))$ is $y'(t)$

2.2 Solutions to Separable Equations

ex: Radioactive decay

$N' = -\lambda N(t)$ where $N(t)$ is amount of radioactive substance at t

$\lambda > 0$, constant of proportionality

Often called an exponential equation b/c of the form of its solutions

Ex of Separable Equation: can be rewritten w/ variables separated then solved.

$$\frac{dN}{dt} = -\lambda N$$

Separate variables

$$\frac{1}{N} dN = -\lambda dt$$

$N \neq 0$ or else $\frac{1}{0}$

$$\int \frac{1}{N} dN = -\lambda \int dt$$

$$\ln|N| + C_1 = -\lambda t + C_2 \qquad C = C_2 - C_1$$

$$\ln|N| = -\lambda t + C$$

$$|N| = e^{C-\lambda t} = e^C e^{-\lambda t} \Rightarrow N(t) = \pm e^C e^{-\lambda t}$$

Simplify

$$A \begin{cases} e^C & \text{if } N > 0 \\ -e^C & \text{if } N < 0 \end{cases}$$

So $\boxed{N(t) = Ae^{-\lambda t}}$ where A is a constant and $A \neq$

If $A = 0$, $N(t) = 0$ which is a solution to $N' = -\lambda N$

verify by substitution

ex: Solve $y' = ty^2$

$$\frac{dy}{dt} = ty^2 \Rightarrow \frac{dy}{y^2} = t\,dt$$

$y \neq 0$ or else $\frac{1}{0}$

$$\int \frac{dy}{y^2} = \int t\,dt$$

$$-\frac{1}{y} = \frac{1}{2}t^2 + C$$

$$y = -\frac{1}{\frac{1}{2}t^2 + C} = \frac{-2}{t^2 + 2C}$$

While $y(t) = 0$ is a solution to $y'=$ C will have to be ∞ for it to work.

General Method: Equations separable if in form of:

$$\left.\begin{aligned} \frac{dy}{dt} &= \frac{g(t)}{h(y)} \\ \frac{dy}{dt} &= g(t)f(y) \end{aligned}\right] \text{separable differential equations}$$

$$\frac{dy}{dt} = g(t)f(y)$$

1) Separate variables: $\frac{dy}{f(y)} = g(t)\,dt$
2) Integrate $\int \frac{dy}{f(y)} = \int g(t)\,dt$
3) Solve for $y(t)$ if possible

Avoid divide by 0

What if $f(y) = 0$?

If y_0 makes $f(y_0) = 0$, then $y(t) = y_0$ is a solution.

LHS: $\frac{dy}{dt} = 0$

RHS: $g(t)f(y) = g(t)f(y_0) = 0$

$\therefore$ $y(t) = y_0$ is a solution

The general solution to a DE is the family of solutions depending on sufficiently many parameters to give all but a finite amount of solutions

- Doesn't always give solution to every IVP
- separable equations, problem from $\div$ by 0

$y = \frac{-2}{t^2+2C}$ is a general solution

In example $y' = ty^2$, $y(t) = 0$ is a solution but general solution $y = \frac{-2}{t^2+2C}$ is found when $y \neq 0$, because if initial condition $y(0) = 0$,

$y(0) = -\frac{2}{0^2 + 2C} = -\frac{1}{C} = 0$ and C will be an ∞ value.

Implicitly Defined Solutions:

ex: Find solutions of $y' = \frac{e^x}{1+y}$ $\qquad y \neq 1$

w/ IC's of $y(0)=1$ and $y(0)=-4$

$$\frac{dy}{dx} = \frac{e^x}{1+y}$$

$$\int (1+y)\,dy = \int e^x\,dx$$

$$y + \frac{1}{2}y^2 = e^x + C$$

$$\frac{1}{2}y^2 + y - (e^x + C) = 0$$

$$y^2 + 2y - 2(e^x + C) = 0$$

$$y = \frac{-2 \pm \sqrt{4 - 4(1)(-2(e^x+c))}}{2} = \frac{-2 \pm 2\sqrt{1+2(e^x+c)}}{2} = -1 \pm \sqrt{1+2(e^x+c)}$$

$$1 = y(0) = -1 \pm \sqrt{1+2(e^x+c)} = -1 + \sqrt{1+2(1+c)} \Rightarrow 2 = \sqrt{1+2(1+c)}$$

$$\Rightarrow 4 = 1 + 2(1+c) \Rightarrow \frac{3}{2} = 1 + c \Rightarrow c = \frac{1}{2}$$

$$\boxed{y(t) = -1 + \sqrt{2 + 2e^x}}$$

$$y(0) = -4 = -1 - \sqrt{1+2(e^x+c)} \Rightarrow -3 = -\sqrt{1+2(1+c)} \Rightarrow 9 = 1 + 2(1+c)$$

$$\Rightarrow 4 = 1 + c \Rightarrow c = 3$$

$$\boxed{y(t) = -1 - \sqrt{7 + 2e^x}}$$

An explicit solution is one where we have a form as a function of an ind. variable

- Not always possible.