10/11/22

HW 2.5 # 4, 7, 12

$X_{(60)} = 0.01$

$x_{(0)} = 0.05$

4. rate of change = rate in − rate out

r = rate poured in / out

t is in minutes

in: $r \cdot 0 \frac{lb}{gal} = 0$ $\qquad \frac{dx}{dt} = -\frac{r\,x(t)}{500}$

out: $r \cdot \frac{x(t)}{500} = \frac{rx}{500}$

$$x' = \frac{-rx}{500}$$

$$x' + \frac{r}{500}x = 0 \qquad x_{(t)} = Ae^{-\int \frac{r}{500}dt} = Ae^{-\frac{rt}{500}}$$

$$x_{(0)} = Ae^0 = 0.05,\ A = 0.05 \qquad x_{(t)} = 0.05e^{\frac{rt}{500}}$$

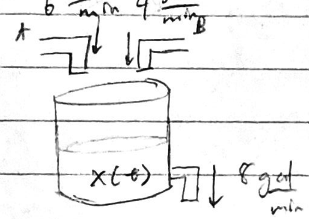
$$X_{(60)} = 0.05e^{-r\cdot\frac{60}{500}} = 0.01$$

$$e^{-\frac{3}{25}r} = 0.2 \qquad r \approx 13.412 \frac{gal}{min}$$

$$-\frac{3}{25}r = \ln 0.2$$

$$r \approx +13.412$$

7. a.



$V_{(t)} = 100 + 2t$

$x_{(0)} = 0$

rate of change = rate in − rate out

$$\text{in} = 6\frac{gal}{min}\cdot 0.5\frac{lb}{gal} + 4\frac{gal}{min}\cdot 0\frac{lb}{gal} = 3\frac{lb}{min}$$

$$\text{Out} = 8\frac{gal}{min}\cdot\frac{x(t)}{V(t)} = \frac{4x(t)}{50+t}$$

$$a = \frac{4}{50+t},\ f = 3$$

$$\frac{dx}{dt} = 3 - \frac{4x}{50+t} \quad\text{or}\quad x' + \frac{4}{50+t}x = 3 \qquad \mu = e^{+\int\frac{4}{50+t}dt} = (50+t)^4$$

$$x'(50+t)^4 + 4x(50+t)^3 = 3(50+t)^4$$

$$\left(x(50+t)^4\right)' = 3(50+t)^4$$

$$x(50+t)^4 = 3\int(50+t)^4\,dt = \frac{3}{5}(50+t)^5 + C$$

$$x_{(t)} = \frac{3}{5}(50+t) + \frac{C}{(50+t)^4},\quad x_{(0)} = \frac{3}{5}\cdot 50 + \frac{C}{50^4} = 0$$

$$\Rightarrow 30 + \frac{C}{50^4} = 0 \Rightarrow C = -30\cdot 50^4$$

$$x_{(t)} = \frac{3}{5}(50+t) - \frac{30\cdot 50^4}{(50+t)^4},\quad x_{(10)} \boxed{\approx 21.532\ lbs}$$

$V_{(10)} = 120$

$V_{(t)} = 120 - 4t$

b. $\frac{dx}{dt} = -8\cdot\frac{x(t)}{V(t)} = -8\frac{x(t)}{120-4t} = -\frac{2x(t)}{30-t}$

$$x' = -\frac{2x}{30-t}$$

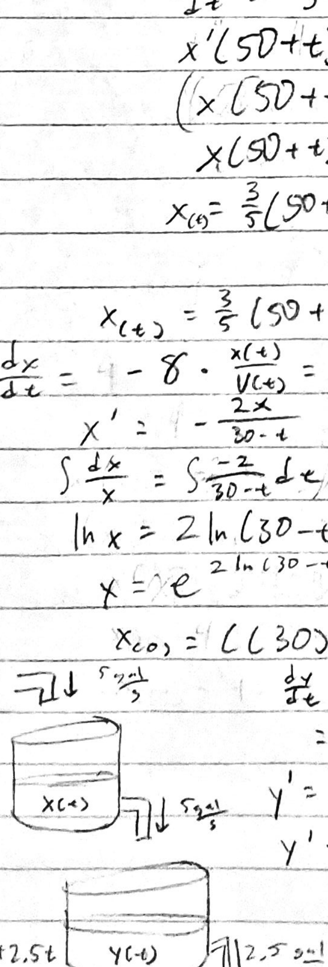
$$\int\frac{dx}{x} = \int\frac{-2}{30-t}dt \qquad x_{(t)} = 0.0239(30-t)^2$$

$$\ln x = 2\ln(30-t) + C \qquad 10.766 = 0.0239(30-t)^2$$

$$x = e^{2\ln(30-t)+C} = C(30-t)^2 \qquad 30 - t = 21.22,\ \boxed{t \approx 8.776\,min}$$

$$x_{(0)} = C(30)^2 = 21.532,\ C \Rightarrow 0.0239$$

12.



$V_{x(t)} = 100$

$x_{(0)} = 20$

$V_{y(t)} = 200 + 2.5t$

$y_{(0)} = 40$

$V_{y(20)} = 250$

$$\frac{dy}{dt} = \text{in} - \text{out} = -\frac{dx}{dt} - 2.5\frac{y(t)}{V(t)}$$

$$= \frac{x}{20} - 2.5\frac{y}{200+2.5t}$$

$$y' = e^{-\frac{t}{20}} - \frac{y}{80+t}$$

$$y' + \frac{y}{80+t} = e^{-\frac{t}{20}}$$

$$a = \frac{1}{80+t},\ f = e^{-\frac{t}{20}}$$

$$\mu = e^{+\int\frac{1}{80+t}dt} = 80+t$$

$$y'(80+t) + y = e^{-\frac{t}{20}}(80+t)$$

$$(y(80+t))' = e^{-\frac{t}{20}}(80+t)$$

$$\frac{dx}{dt} = \text{in} - \text{out} = 0 - 5\frac{x(t)}{100} = -\frac{x}{20}$$

$$x' = -\frac{x}{20}$$

$$x' + \frac{x}{20} = 0 \qquad a = -\frac{1}{20}$$

$$x_{(t)} = Ce^{-\frac{t}{20}}$$

$$x_{(0)} = C\cdot 1 = 20,\ C = 20$$

$$x_{(t)} = 20e^{-\frac{t}{20}}$$

$x_{(0)} = 20$

$$y(80+t) = \int e^{-\frac{t}{20}}(80+t)\,dt = -20(t+100)e^{-\frac{t}{20}} + C$$

$$y = \frac{-20(t+100)e^{-\frac{t}{20}}}{80+t} + \frac{C}{80+t}$$

$$Y_{(0)} = \frac{-20\cdot 100}{80} + \frac{C}{80} = 40$$

$$= -2000 + C = 3200$$

$$C = 5200$$

$$Y_{(t)} = \frac{-20(t+100)e^{-\frac{t}{20}}}{80+t} + \frac{5200}{80+t}$$

At $t = 20$, $V_y = 250$

$$Y_{(20)} = \frac{-20(20+100)e^{-1}}{100} + \frac{5200}{100} \boxed{\approx 43.171 \text{ lbs}}$$

HW 2.6 #2, 7, 9, 10, 13, 19, 24, 29

2. $P = \frac{\partial F}{\partial x} = 2x - y$ $\quad \boxed{dF = (2x-y)dx + (2y-x)dy}$
$Q = \frac{\partial F}{\partial y} = 2y - x$

7. $P = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\ln(x^2+y^2)\right) + \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{2x}{x^2+y^2} + \frac{1}{y}$ $\quad \boxed{dF = \left(\frac{2x}{x^2+y^2} + \frac{1}{y}\right)dx + \left(\frac{2y}{x^2+y^2} - \frac{x}{y^2}\right)dy}$

$Q = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\ln(x^2+y^2)\right) + \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{2y}{x^2+y^2} - \frac{x}{y^2}$

9. $P = 2x + y$ $\quad \frac{\partial P}{\partial y} = 1$
$Q = x - 6y$ $\quad \frac{\partial Q}{\partial x} = 1$] exact

$\frac{\partial F}{\partial x} = 2x + y$
$\downarrow$
$F = x^2 + xy + \phi(y)$

$\frac{\partial F}{\partial y} = Q \Rightarrow x + \phi'(y) = x - 6y$
$\phi'(y) = -6y$
$\phi(y) = -3y^2 + C_1$
take $C_1 = 0$

$$\boxed{F(x,y) = x^2 + xy - 3y^2 = C}$$

10. $P = 1 - y\sin x$ $\quad \frac{\partial P}{\partial y} = -\sin x$
$Q = \cos x$ $\quad \frac{\partial P}{\partial x} = -\sin x$] exact

$\frac{\partial F}{\partial x} = 1 - y\sin x$
$F = x + y\cos x + \phi(y)$

$\frac{\partial F}{\partial y} = Q \Rightarrow \cos x + \phi'(y) = \cos x$
$\phi'(y) = 0$
$\phi(y) = C_1$

$$\boxed{F(x,y) = x + y\cos x = C}$$

13. $\frac{dy}{dx} = \frac{3x^2+y}{3y^2-x} \Rightarrow (3y^2-x)dy = (3x^2+y)dx \Rightarrow (3x^2+y)dx + (x-3y^2)dy = 0$

$\frac{\partial F}{\partial x} = 3x^2 + y$
$F = \int 3x^2 + y\,dx + \phi(y)$
$F = x^3 + xy + \phi(y)$

$\frac{\partial F}{\partial y} = Q \Rightarrow x + \phi'(y) = x - 3y^2$
$\phi'(y) = -3y^2$
$\phi(y) = -y^3 + C_1$

$P = 3x^2 + y$ $\quad \frac{\partial P}{\partial y} = 1$
$Q = x - 3y^2$ $\quad \frac{\partial Q}{\partial x} = 1$] exact

$$\boxed{F(x,y) = x^3 + xy - y^3 = C}$$

19. $P = \sin 2t$ $\quad \frac{\partial P}{\partial t} = 2\cos 2t$

$Q = 2x\cos 2t - 2t$ $\quad \frac{\partial Q}{\partial x} = 2\cos 2t$] exact

$\frac{\partial F}{\partial x} = \sin 2t$

$F = x\sin 2t + \phi(t)$

$\frac{\partial F}{\partial t} = Q \Rightarrow 2x\cos 2t + \phi'(y) = 2x\cos 2t - 2t$

$\phi'(y) = -2t$

$\phi(y) = -t^2 + C_1$

$$\boxed{F_{(x,t)} = x\sin 2t - t^2 = C}$$

$\tilde{P} = \frac{3(y+1)^2}{x^4}$

$\tilde{Q} = -\frac{2(y+1)}{x^3}$

24. $\frac{y+1}{x^4}(3(y+1)dx - 2x\,dy = 0)$

$\frac{3(y+1)^2}{x^4}dx - \frac{2(y+1)}{x^3}dy = 0$

$\frac{\partial \tilde{P}}{\partial y} = \frac{3}{x^4}\frac{\partial}{\partial y}((y+1)^2) = \frac{6(y+1)}{x^4}$

$\frac{\partial \tilde{Q}}{\partial x} = -2(y+1)\frac{\partial}{\partial x}(\frac{1}{x^3}) = \frac{6(y+1)}{x^4}$] exact

$\tilde{P} = \frac{\partial F}{\partial x} = \frac{3(y+1)^2}{x^4}$

$F = \int \frac{3(y+1)^2}{x^4}dx = 3(y+1)^2 \int x^{-4}dx$

$= 3(y+1)^2(-\frac{1}{3x^3}) + \phi(y) = -\frac{(1+y)^2}{x^3} + \phi(y)$

$\frac{\partial F}{\partial y} = Q \Rightarrow -\frac{2(1+y)}{x^3} + \phi'(y) = -\frac{2(y+1)}{x^3}$

$\phi'(y) = 0$

$\phi(y) = C_1$

$$\boxed{F_{(x,y)} = -\frac{(1+y)^2}{x^3} = C}$$

29. $(y^2 + 2xy)dx - x^2dy = 0$

$P = y^2 + 2xy$ $\quad \frac{\partial P}{\partial y} = 2y + 2x$

$Q = -x^2$ $\quad \frac{\partial Q}{\partial x} = -2x$] not exact

$g = \frac{1}{P}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= \frac{1}{y^2+2xy}(2y + 2x + 2x) = \frac{4x+2y}{2xy+y^2} = \frac{2(2x+y)}{y(2x+y)} = \frac{2}{y}$ $\quad \frac{d\mu}{dy} = -\frac{2}{y}\mu$

$= e^{\int -\frac{2}{y}dy} = e^{-2\ln y} = \boxed{y^{-2} = \mu}$

multiply by μ: $\frac{1}{y^2}(y^2 + 2xy)dx - \frac{x^2}{y^2}dy = 0$

$\Rightarrow (1 + \frac{2x}{y})dx - \frac{x^2}{y^2}dy = 0$

$\tilde{P} = 1 + \frac{2x}{y}$ $\quad \tilde{Q} = -\frac{x^2}{y^2}$

$\frac{\partial \tilde{P}}{\partial y} = -\frac{2x}{y^2}$ $\quad \frac{\partial \tilde{Q}}{\partial x} = -\frac{2x}{y^2}$

exact!

$\tilde{P} = \frac{\partial F}{\partial x} = 1 + \frac{2x}{y}$

$F = \int 1 + \frac{2x}{y}dx + \phi(y) = x + \frac{x^2}{y} + \phi(y)$

$\frac{\partial F}{\partial y} = Q \Rightarrow -\frac{x^2}{y^2} + \phi'(y) = -\frac{x^2}{y^2}$

$\phi'(y) = 0$

$\phi(y) = C_1$

$$\boxed{F_{(x,y)} = x + \frac{x^2}{y} = C}$$