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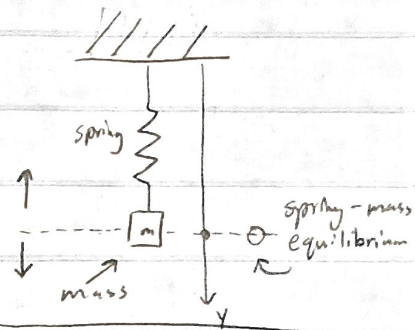
4.4 HARMONIC MOTION



Examples of 2^{nd} order, linear equations of form

$$y'' + py' + qy = g(t)$$

p, q constants and $g(t)$ forcing term

$$\boxed{my'' + \mu y' + ky = F(t)}$$

constants: m = mass, μ = dampening constant, k = spring constant

non-constants: $F(t)$ = external force, y = displacement

Equation $\boxed{x'' + 2cx' + \omega_0^2 x = f(t)}$ is for harmonic motion where

c is dampening constant

f is forcing term

start w/ homogeneous equation $f(t) = 0$. This is undefined harmonic motion:

$$\boxed{x'' + 2cx' + \omega_0^2 x = 0} \quad c \geq 0,\ \omega_0 > 0 \text{ constants}$$

Case 1: Simple Harmonic Motion w/ no dampening $(c = 0)$

$x = e^{\lambda t}$

$x' = \lambda e^{\lambda t}$

$x'' = \lambda^2 e^{\lambda t}$

$$x'' + \omega_0^2 x = 0 \Rightarrow \lambda^2 e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0$$

$$\Rightarrow \lambda^2 + \omega_0^2 = 0, \quad \lambda = \pm i\omega_0 \rightarrow \begin{matrix} \tilde{a} \pm i\tilde{b} \\ \tilde{a} = 0,\ \tilde{b} = \omega_0 \end{matrix}$$

general solution: $\boxed{x(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t)}$ a, b are constants

Let $T = \frac{2\pi}{\omega_0}$, then $\omega_0 T = 2\pi$

and $x(t+T) = x(t)$ bc period of trig functions are 2π

$\Rightarrow x(t)$ is periodic & it's period is T.

as $\omega_0 \uparrow \downarrow$
$T = \frac{2\pi}{\omega_0} \downarrow \uparrow$

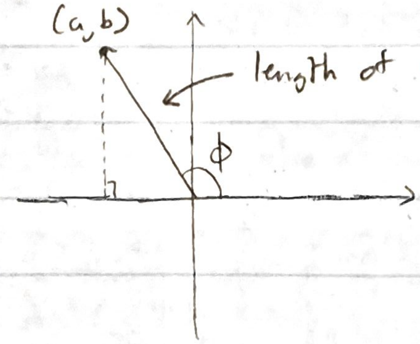
If T is in seconds, ω_0 has units in radians/seconds

$\boxed{\omega_0 \text{ is natural frequency}}$ of the spring and it's an example of angular frequency

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

assume $(a,b) \neq (0,0)$

Alternative form: Using polar coords to represent coefficients a & b



$a = A\cos\phi$, $b = A\sin\phi$

ϕ = polar angle in $[-\pi, \pi]$

$x(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t)$

$= A\cos\phi\cos(\omega_0 t) + A\sin\phi\sin(\omega_0 t)$

$$\boxed{x(t) = A\cos(\omega_0 t - \phi)}$$

general solution

Note: $x(t)$ is a pure sinusoidal oscillation in undampened harmonic motion

A = amplitude $(-A \leq x(t) \leq A)$

ϕ = phase

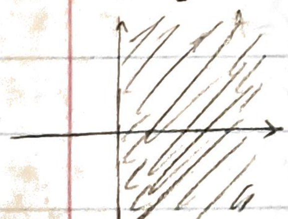
$\boxed{x(t) = A\cos(\omega_0 (t - \frac{\phi}{\omega_0}))}$ if $\phi > 0$, shift graph to right by $\frac{\phi}{\omega_0}$

$\boxed{\frac{\phi}{\omega_0} \text{ is called PHASE SHIFT}}$

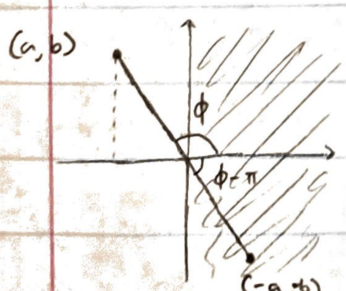
Finding A and ϕ from a and b

$A = \sqrt{a^2+b^2}$ $\tan\phi = \frac{b}{a}$

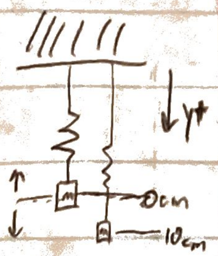
range of arctan is $(-\frac{\pi}{2}, \frac{\pi}{2})$, but ϕ can be any angle in $(-\pi, \pi]$



Be careful when calculating ϕ: look at where point (a,b) is in plane.



$$\phi = \begin{cases} \arctan(\frac{b}{a}) & \text{if } a > 0 \\ \arctan(\frac{b}{a}) + \pi & \text{if } a < 0 \text{ \& } b > 0 \\ \arctan(\frac{b}{a}) - \pi & \text{if } a < 0 \text{ \& } b < 0 \end{cases}$$



Ex: Spring-mass system w/ mass 4kg attached to spring w/ spring constant $k = 169 \frac{kg}{s^2}$. Spring is stretched 10cm from equilibrium and oscillates w/ initial velocity 130 cm/s. Find frequency, amplitude, & phase of vibration.

ASSUME NO DAMPENING

$\mu = 0$

$my'' + \cancel{\mu}^{0} y' + ky = 0$

$my'' + ky = 0$ $\qquad w_0 = \sqrt{\frac{k}{m}} = \sqrt{42.25} = 6.5 \text{ rad/s}$

$y'' + \frac{k}{m}y = 0$

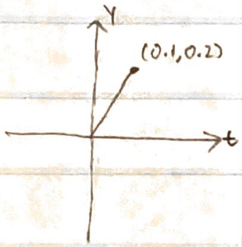
General solution: $y(t) = C_1 \cos 6.5t + C_2 \sin 6.5t$

$y'' + 42.25y = 0$ $\qquad y'(t) = -6.5C_1 \sin 6.5t + 6.5C_2 \cos 6.5t$

$y(0) = 0.1 \text{ m}$ $\qquad C_1 = 0.1$ $\qquad C_1 = 0.1$

$y'(0) = 1.3 \text{ m/s}$ $\Rightarrow$ $6.5C_2 = 1.3$ $\Rightarrow$ $C_2 = 0.2$

So $\boxed{y(t) = 0.1 \cos 6.5t + 0.2 \sin 6.5t}$



Amplitude A: $\sqrt{0.1^2 + 0.2^2} = \frac{\sqrt{5}}{10} \boxed{\approx 0.2236 \text{ m}}$

Phase: $\tan\phi = \frac{0.2}{0.1} = 2 \Rightarrow \boxed{\phi \approx 1.1071 \text{ rad}}$

$y(t) = A\cos(w_0 t - \phi) \boxed{= \frac{\sqrt{5}}{10} \cos(6.5t - \phi)}$

or

$\boxed{y(t) = \frac{\sqrt{5}}{10} \cos\left(6.5\left(t - \frac{\phi}{6.5}\right)\right)}$

Phase shift: $\frac{\phi}{\omega_0} = \frac{1.1071}{6.5} \boxed{\approx 0.1703 \text{ shift to right}}$

Period: $T = \frac{2\pi}{\omega_0} \boxed{\approx 0.9666 \text{ s}}$ $\quad$ start by graphing $y(t) = \frac{\sqrt{5}}{10}\cos(6.5t)$ then add phase shift

Case 2: Dampened Harmonic Motion $\quad c > 0, \frac{w}{c} > 0$

$$x'' + 2cx' + \omega_0^2 x = 0$$

characteristic equation: $\lambda^2 + 2c\lambda + w_0^2 = 0$

$$\lambda = \frac{-2c \pm \sqrt{4c^2 - 4w_0^2}}{2} = -c \pm \sqrt{c^2 - w_0^2}$$

roots: $\lambda_1 = -c - \sqrt{c^2 - w_0^2}$, $\lambda_2 = -c + \sqrt{c^2 - w_0^2}$

Case i) $\quad c < w_0$ underdampened case

$c^2 - w_0^2 < 0$ so 2 distinct complex roots

$\Rightarrow \underbrace{\lambda_1 = -c - i\sqrt{w_0^2 - c^2}}_{a-ib}, \underbrace{\lambda_2 = -c + i\sqrt{w_0^2 - c^2}}_{a+ib}$

$a = -c$, $b = \sqrt{w_0^2 - c^2}$

General Solution: $\boxed{x(t) = e^{-ct}(C_1 \cos wt + C_2 \sin wt) \text{ where } w = \sqrt{w_0^2 - c^2}}$

$x(t) \to 0$ as $t \to \infty$

case ii) $c > \omega_0$ overdampened case

$c^2 - \omega_0^2 > 0$ so 2 distinct real roots

Also $\lambda_1 = -c - \sqrt{c^2-\omega_0^2} < -c + \sqrt{c^2-\omega_0^2} < \lambda_2$

so $\lambda_1 < \lambda_2$

and $\lambda_2 = -c + \sqrt{c^2-\omega_0^2} < -c + \sqrt{c^2} = -c+c = 0$

so $\lambda_2 < 0$

$\Rightarrow \lambda_1 < \lambda_2 < 0$

General solution

$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
$\lambda_1 = -c - \sqrt{c^2-\omega_0^2}$ $\lambda_2 = -c + \sqrt{c^2-\omega_0^2}$

case iii) $c = \omega_0$, critically dampened

$c^2 - \omega_0^2 = 0 \Rightarrow$ 1 double root $\lambda = -c$

$\boxed{\text{General solution: } x(t) = C_1 e^{-ct} + C_2 t e^{-ct}}$

In all cases, solution decays to 0 as $t \to \infty$ bc of the e^{-ct} and $c > 0$.

Ex: Underdampened Motion:

$m = 4\,kg$
$k = 169\,\frac{kg}{s^2}$
$\mu = 12.8\,\frac{kg}{s}$

Let's take previous question but w/ dampening constant $\mu = 12.8\,\frac{kg}{s}$

$y(0) = 0.1m \qquad y'(0) = 1.3\,m/s$

$$4y'' + 12.8y' + 169y = 0$$

$$y'' + 3.2y' + 42.25y = 0$$

$\vdots$

$$\lambda = -1.6 \pm 6.3i$$

$\boxed{\text{General sol: } y(t) = e^{-1.6t}(C_1 \cos 6.3t + C_2 \sin 6.3t)}$

$c = -1.6$
$\omega = 6.3$

$\vdots$

$y(0) = 0.1$, $y'(0) = 1.3$ $\Rightarrow$ $C_1 = 0.1$, $-1.6C_1 + 6.3C_2 = 1.3$ $\Rightarrow$ $C_1 = 0.1$, $C_2 \approx 0.2317$

Solution: $\boxed{y(t) = e^{-1.6t}(0.1\cos 6.3t + 0.2317 \sin 6.3t)}$

Amplitude: $\boxed{A = \sqrt{0.1^2 + 0.2317^2} \approx 0.2524\,m}$

Phase: $\tan\phi = \frac{0.2317}{0.1} \Rightarrow \boxed{\phi \approx 1.1634\text{ rad}}$

$$\boxed{y(t) = 0.2524 e^{-1.6t} \cos(6.3t - 1.1634)}$$

Ex: Overdampened: Same spring-mass $m = 4 kg$, $k = 169 \frac{kg}{s^2}$

but dampening constant $\mu = 77.6 \frac{kg}{s}$, $y(0) = 0.1\ m$ & $y'(0) = 1.3 \frac{m}{s}$

$4y'' + 77.6y' + 169y = 0$

$y'' + 19.4y' + 42.25y = 0$

Show work $\Longrightarrow$ $\lambda_1 = -16.9$ $\lambda_2 = -2.5$, $\lambda_1 < \lambda_2 < 0$

$\boxed{\text{General Sol: } y(t) = C_1 e^{-16.9t} + C_2 e^{-2.5t}}$ overdampened bc $c > \omega_0$

$y(0) = 0.1$
$y'(0) = 1.3$ $\longrightarrow \dots$ $C_1 = \frac{-31}{288}$, $C_2 = \frac{299}{1440}$

$$\boxed{y(t) = -\frac{31}{288} e^{-16.9t} + \frac{299}{1440} e^{-2.5t}}$$

Ex: Critically Dampened: Same spring-mass $m = 4 kg$, $k = 169 \frac{kg}{s^2}$

but dampening constant μ is critically dampened when $c = \omega_0$

$x'' + 2cx' + \omega_0^2 x = 0$

$my'' + \mu y' + ky = 0$

$y'' + 2\underbrace{\frac{\mu}{2m}}_{c} y' + \underbrace{\frac{k}{m}}_{\omega_0^2} y = 0$

If $c = \omega_0$, then $\frac{\mu}{2m} = \frac{\sqrt{k}}{\sqrt{m}}$

$\mu = 2\sqrt{mk} = 52 \frac{kg}{s}$

$4y'' + 52y' + 169y = 0$

$y'' + 13y' + 42.25y = 0$

$\lambda^2 + 13\lambda + 42.25 = (\lambda + 6.5)^2 = 0$

$\lambda = -6.5$

$y(t) = C_1 e^{-6.5t} + C_2 t e^{-6.5t}$ $y(0) = C_1 = 0.1$

$y'(t) = -6.5C_1 e^{-6.5t} - 6.5C_2 t e^{-6.5t} + C_2 e^{-6.5t}$ $y'(0) = -6.5C_1 + C_2 = 1.3$

$C_2 = 1.95$

$$\boxed{y(t) = 0.1e^{-6.5t} + 1.95te^{-6.5t}}$$