9.3 Phase Plane Portraits

$\vec{y}' = A\vec{y}$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

since this is autonomous, use phase plane to visualize solution.

characteristic polynomial: $\lambda^2 - T\lambda + D = 0$

$D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$

$T = tr(A) = a_{11} + a_{22}$

For $\vec{y}' = A\vec{y}$, equilibrium points are points $\vec{v} \in \mathbb{R}^2$ where $A\vec{v} = \vec{0}$

In most cases we will look at, $\det A \neq 0$ and origin $\vec{0}$ is the only equilibrium pt

Note: $\vec{y}(t) = \vec{0}$ is an equilibrium solution

$\vec{y}'(t) = \vec{0} = A\vec{0} = A\vec{y}(t)$

Real, distinct eigenvalues: $T^2 - 4D > 0$

general solution: $\boxed{\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2}$

Consider when one of C_1 or C_2 is 0.

$C_1 e^{\lambda_1 t}\vec{v}_1$ and $C_2 e^{\lambda_2 t}\vec{v}_2$ are exponential solutions

If solution has form $\vec{y}(t) = Ce^{\lambda t}\vec{v}$

If $C > 0$, $\vec{y}(t)$ is a positive multiple of $\vec{v}$ bc

$\vec{y}(t) = \underbrace{(Ce^{\lambda t})}_{>0}\vec{v}$ since $e^{\lambda t} > 0$

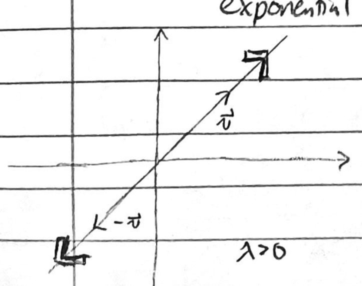
If $C < 0$, $\vec{y}(t)$ is positive multiple of $-\vec{v}$ bc

$\vec{y}(t) = \underbrace{(-Ce^{\lambda t})}_{>0}(-\vec{v})$

If $\lambda > 0$, $e^{\lambda t}$ increases from 0 to ∞ as t increases from $-\infty$ to ∞
If $\lambda < 0$, $e^{\lambda t}$ decreases from ∞ to 0 as t increases from $-\infty$ to ∞

$\vec{y}(t)$ traces half-line consisting of + multiples of $C\vec{v}$

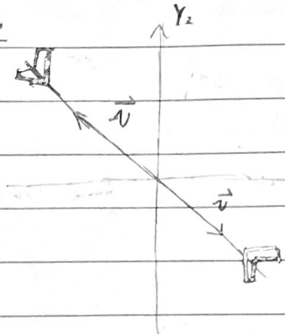
- there are 2 solution curves in phase plane depending on C's sign, these exponential solutions are sometimes called half-line solutions



when $\lambda > 0$, $e^{\lambda t}$ increases as t increases

$\vec{y}(t) = Ce^{\lambda t}\vec{v}$ move away from origin/equilibrium as t increases. Moves to origin when t gets to $-\infty$. Called <u>unstable</u> solution.

Ex:



when $\lambda < 0$, $e^{\lambda t} \to 0$ as $t \to \infty$, so solution $\vec{y}(t) = Ce^{\lambda t}\vec{v}$ approaches equilibrium pt at origin as $t \to \infty$. Called stable solutions

big arrows indicate flow direction as t increases

Saddle Point: real eigenvalues w/ different signs $\lambda_1 < 0 < \lambda_2$

$$\boxed{\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2}$$

- If $C_2 = 0$ $\quad \vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1$
  $\lambda_1 < 0 \Rightarrow$ 2 stable half-line solutions
- If $C_1 = 0$ $\quad \vec{y}(t) = C_2 e^{\lambda_2 t}\vec{v}_2$
  $\lambda_2 > 0 \Rightarrow$ 2 unstable half-line solutions

$\lambda_1 < 0$ • If $C_1 \neq 0$, $C_2 \neq 0$ $\underbrace{\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2}_{\text{superposition of 2 solutions}}$

as $t \to \infty$, $C_1 e^{\lambda_1 t}\vec{v}_1 \to \vec{0}$ so $\vec{y}(t)$ tends to $C_2 e^{\lambda_2 t}\vec{v}_2$. Solution curve goes to ∞, asymptotic to half-line generated by $C_2\vec{v}_2$

refer to figures 1 & 2

$\lambda_2 > 0$ • As $t \to -\infty$ $C_2 e^{\lambda_2 t}\vec{v}_2 \to \vec{0}$ $\vec{y}(t)$ tends to $C_1 e^{\lambda_1 t}\vec{v}_1$ as solution curve goes to ∞, asymptotic half-line generated by $C_1\vec{v}_1$

Solution $\vec{y}(t) = 0.8e^{\lambda_1 t}\vec{v}_1 + 1.2e^{\lambda_2 t}\vec{v}_2$
tends to $1.2e^{\lambda_2 t}\vec{v}_2$ as $t \to \infty$
tends to $0.8e^{\lambda_1 t}\vec{v}_1$ as $t \to -\infty$

Ex: Find general solution, sketch half-line solutions & rough approximation of a solution in each region determined by half-line solutions

$$\vec{y}' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\vec{y}$$

$\vdots$

$\lambda_1 = -3 \quad \lambda_2 = 3$

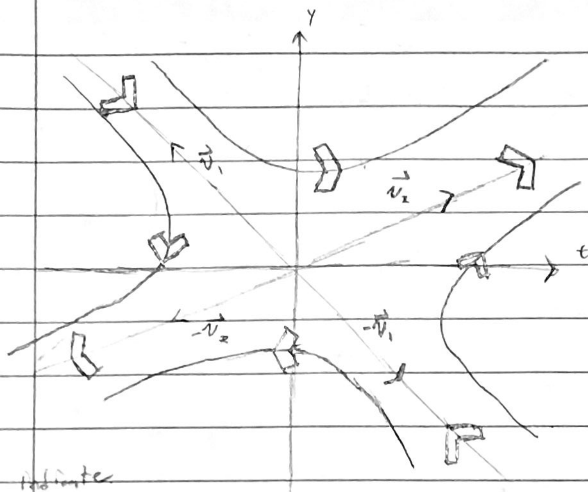
$\downarrow \quad \downarrow$

$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Since $\lambda_1 \neq \lambda_2$, $\vec{y}_1(t) = e^{-3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
$\vec{y}_2(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

fundamental set

$$\boxed{\vec{y}(t) = C_1 e^{-3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$



When $C_1 = 0$, $\vec{y}(t) = C_2 e^{3t}\binom{2}{1}$ increases as t increases, half-line solutions move away from origin. Traces half lines along line $2y_2 = y_1$ or $y_2 = \frac{y_1}{2}$ from origin

$C_2 = 0$, $\vec{y}(t) = C_1 e^{-3t}\binom{-1}{1}$, as $t \to \infty$ half-line solutions approach origin. Traces half-lines decaying to origin w/ equation $-y_2 = y_1$ or $y_2 = -y_1$

Big arrows indicate direction of flow as t increases

$\boxed{\text{The solution trajectories of } \vec{y}' = A\vec{y} \text{ CANNOT cross one another, so half-line solutions separate phase plane into 4 regions.}}$

Plot trajectories w/ initial conditions in each of 4 regions.

- For these solutions, $C_1 \neq 0$ and $C_2 \neq 0$

As t increases, solutions tend towards $C_2 e^{3t}\binom{2}{1}$

As $t \to -\infty$, solutions tend towards $C_1 e^{-3t}\binom{-1}{1}$

Exponential (half-line) solutions separate phase plane into 4 regions where solution curves have different behavior

The curve (half-lines) are called <u>separatrices</u>

2 separatrices are stable solution curves that approach equilibrium point as $t \to \infty$

2 separatrices are unstable solutions that approach equilibrium point as $t \to -\infty$

Any equilibrium point w/ this property is a <u>saddle point</u>

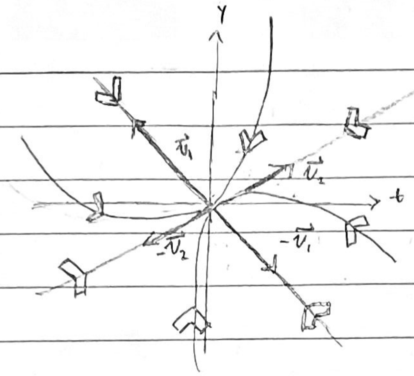
<u>Nodal Sink</u>

both eigenvalues are negative $\lambda_1 < \lambda_2 < 0$

general solution: $\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$

When one of C_1 or C_2 is 0, we get stable exponential solution since both eigenvalues are negative

↓ example next page



Ex: $\vec{y}(t) = 0.7e^{\lambda_1 t}\vec{v}_1$ & ~~$\vec{y}(t) = 0.5e^{\lambda_2 t}\vec{v}_2$~~
as $t \to \infty$, both solutions decay to origin along half-lines generates by $0.7\vec{v}_1$ and $0.5\vec{v}_2$

$\lambda_1 < \lambda_2 < 0$
As $t \to \infty$, solution $\vec{y}(t) = 0.7e^{\lambda_1 t}\vec{v}_1 + 0.5e^{\lambda_2 t}\vec{v}_2$ decays to origin in direction parallel to $0.5\vec{v}_2$.
As $t \to -\infty$, solution moves to ∞ in direction parallel to $0.7\vec{v}_1$

Therefore as $t \to \infty$, $\vec{y}(t) = \vec{0}$ but solution curve becomes to half-line generated by $C_2\vec{v}_2$

$\vec{y}(t) = e^{\lambda_1 t}(C_1\vec{v}_1 + C_2 e^{(\lambda_2-\lambda_1)t}\vec{v}_2)$, since $\lambda_1 < 0$, as $t \to -\infty$ $e^{\lambda_1 t} \to \infty$ and $\lambda_2 - \lambda_1 \to 0$ as $t \to -\infty$ so $C_2 e^{(\lambda_2-\lambda_1)t}\vec{v} \to 0$. Therefore as $t \to -\infty$ $\|\vec{y}(t)\|$ becomes ∞ large.

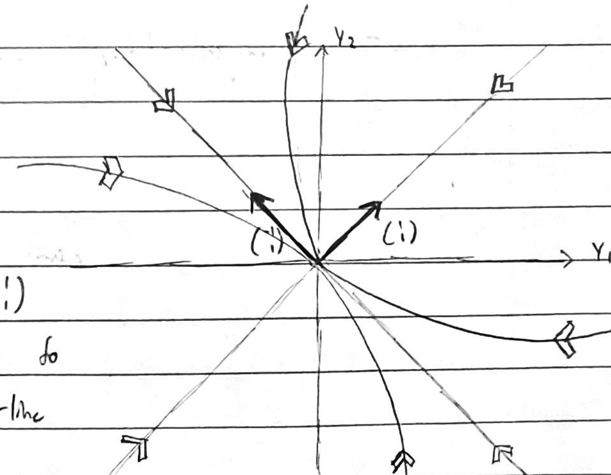
Planar linear system w/ 2 negative eigenvalues, all solution curves approach origin as $t \to \infty$ w/ tangent line. $e^{\lambda_1 t}\vec{v}_1$ approaches origin along $\vec{v}_1$, everything else goes along $\vec{v}_2$

- $\lambda_1 < \lambda_2 < 0$ Any equilibrium point for planar system that has solution curves going to it as $t \to \infty$, it's called a nodal sink
- $0 < \lambda_1 < \lambda_2$ If all solution curves approach equilibrium as $t \to -\infty$ w/ well defined tangent, equilibrium point called a nodal source
- Planar linear system w/ 2 negative eigenvalues has nodal sink at origin

Ex: $\vec{y}' = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix}\vec{y}$

eigenvalues: $-2, -4$

eigenvectors: $\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



general solution: $\vec{y}(t) = C_1 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $\vec{y}(t) = C_1 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ half-line solutions decay to origin along $y_2 = -y_1$ half-line as $t \to \infty$
- $\vec{y}(t) = C_2 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ solutions decay to origin along $y_2 = y_1$ as $t \to \infty$
- If $C_1 \neq 0$ & $C_2 \neq 0$ as $t \to \infty$ $\vec{y}(t) = C_1 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ decays to origin // to $C_1 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. As $t \to -\infty$ $\vec{y}(t)$ approaches ∞ in direction // to $C_2 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

If both eigenvalues have $0<\lambda_1<\lambda_2$, we get <u>nodal source</u>

If both eigenvalues have $\lambda_1<\lambda_2<0$ we get <u>nodal sink</u>

<u>Ex:</u> $\vec{y}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\vec{y}$ source

eigenvalues 2, 4

eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{\vec{y}(t) = C_1 e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

<u>Complex Eigenvalues</u>

$\lambda = \alpha + ib$ and $\bar{\lambda} = \alpha - ib$

If λ has associated eigenvector $\vec{w} = \vec{v}_1 + i\vec{v}_2$, then general solution is

$$\boxed{\vec{y}(t) = C_1 e^{\alpha t}(\cos bt\, \vec{v}_1 - \sin bt\, \vec{v}_2) + C_2 e^{\alpha t}(\sin bt\, \vec{v}_1 + \cos bt\, \vec{v}_2)}$$

<u>Center:</u> pure imaginary eigenvalues $\alpha = 0$

$$\vec{y}(t) = C_1(\cos bt\, \vec{v}_1 - \sin bt\, \vec{v}_2) + C_2(\sin bt\, \vec{v}_1 + \cos bt\, \vec{v}_2)$$

$\vec{y}(t)$ is periodic w/ period $T = \frac{2\pi}{|b|}$

<u>Ex:</u> $\vec{y}' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\vec{y}$

eigenvalues $\lambda = 2i$ & $\bar{\lambda} = -2i$

eigenvector $\vec{w} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

general solution: $\boxed{\vec{y}(t) = C_1\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + C_2\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}}$

solution curves are circles

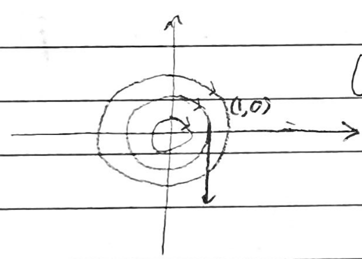
System: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$y_1' = 2y_2$

$y_2' = -2y_1$

Since can show $(y_1^2 + y_2^2)' = 0 \Rightarrow y_1^2 + y_2^2 = \text{constant}$

$\Rightarrow$ solution curves are circles around origin



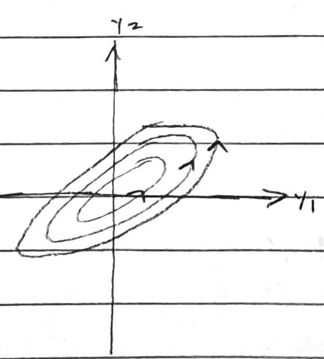
$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

compute one vector in vector field

An equilibrium point for any planar system that has the property it's surrounded by solution curves is a <u>center</u>

A PLANAR SYSTEM W/ PURELY IM EIGENVALUES HAVE CENTER AT ORIGIN

- Solution curves of center not necessarily circles
- For linear systems, centers have orbits that are similar ellipses



<u>Ex:</u> $\vec{y}' = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}\vec{y}$

eigenvalues: $\pm 2i$

eigenvector: $2i$ has $\begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

equilibrium point at origin is a center, but curves are ellipses

<u>Spiral Sink</u>: Real part of complex conjugate negative ($\alpha < 0$)

$\lambda = \alpha + \beta i \qquad \bar{\lambda} = \alpha - \beta i$

general solution: $\boxed{\vec{y}(t) = e^{\alpha t}\left[C_1(\cos\beta t\,\vec{v}_1 - \sin\beta t\,\vec{v}_2) + C_2(\sin\beta t\,\vec{v}_1 + \cos\beta t\,\vec{v}_2)\right]}$

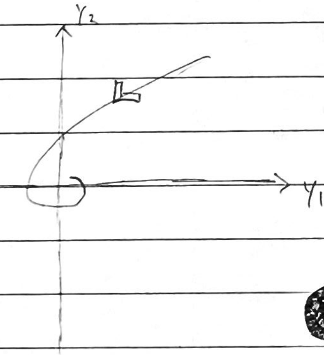
terms in bracket have period $\boxed{T = \frac{2\pi}{|\beta|}}$

factors in bracket have parametric ellipses w/ center at origin

$e^{\alpha t} \to 0$ as $t \to \infty$ bc α is negative.

All solution curves spiral to equilibrium / origin, hence the name spiral sink. $\boxed{\text{All solutions are stable}}$

Ex: $\vec{y}\,' = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix}\vec{y}$



$\lambda = -1 + 2i$, $\bar{\lambda} = -1 - 2i$

$\alpha < 0$, spiral sink

$\beta = 2$

$T = \frac{2\pi}{2} = \pi$

$\vec{w} = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$

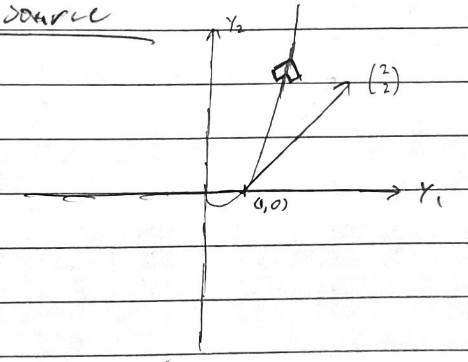
<u>Spiral Source</u>: Real part of complex eigenvalue positive ($\alpha > 0$)

$\lambda = \alpha + \beta i \qquad \bar{\lambda} = \alpha - \beta i$

$e^{\alpha t} \to \infty$ as $t \to \infty$

Amplitude of oscillation increases as solutions spiral about the origin - hence the name spiral source

Ex: $\vec{y}\,' = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}\vec{y}$



$\lambda = 1 + i \qquad \bar{\lambda} = 1 - i$

$\alpha > 0$, spiral source

$\beta = 1$

$T = \frac{2\pi}{1} = 2\pi$

To determine direction of rotation for planar system w/ complex eigenvalues, compute 1 vector in vector field using RHS of $\vec{y}\,' = A\vec{y}$

$\vec{y}\,' = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}\vec{y} \qquad \lambda = 1 + i \qquad \bar{\lambda} = 1 - i$

compute a vector at a point $\binom{1}{0}$: spiral source

$\begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Sketch vector $\binom{2}{2}$ at point $\binom{1}{0}$, rotation is counter clockwise general:

matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ w/ complex conjugate eigenvalues, compute vector field at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

- if $a_{21} > 0$, rotation is counterclockwise & vector points to upper-half plane
- if $a_{21} < 0$, vector points to lower-half plane, rotation is clockwise

9.4 The Trace-Determinant Plane

$\vec{y}\,' = A\vec{y}$

Eigenvalues & trace-determinant pairs:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$p(\lambda) = \lambda^2 - T\lambda + D$ $\qquad \lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}$

$T = tr(A) = a_{11} + a_{22}$

$D = det(A) = a_{11}a_{22} - a_{21}a_{12}$

Fundamental Thm:

$$\boxed{\begin{aligned} T &= tr(A) = \lambda_1 + \lambda_2 \\ D &= det(A) = \lambda_1 \lambda_2 \end{aligned}}$$

Classification of Equilibrium:

trace-determinant plane: coordinate plane w/ coordinates T & D