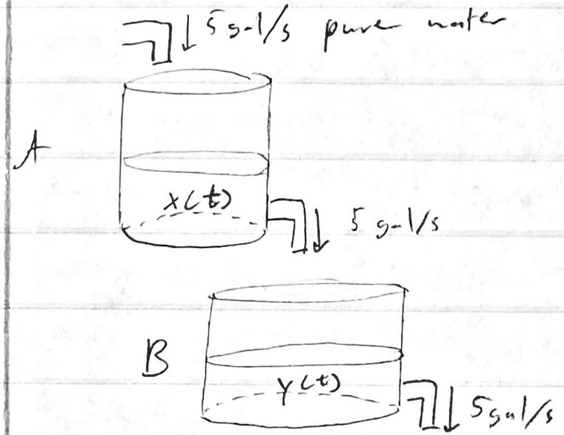
10/7/22

Ex: Tank A has 100 gal of solution w/ 20 lbs. salt. Tank B has 200 gal of solution w/ 40 lbs. of salt. Pure water goes into tank A at 5 gal/s. There's a drain at bottom of tank A that drains at 5 gal/s and flows into B. There's also a drain at tank B that drains at 5 gal/s. What's salt content in B after 60 s?



Vol. of A & B: 100 gal

$x(t)$ = amount of salt in A at time t

$y(t)$ = amount of salt in B at time t

$$\frac{dx}{dt} = \underbrace{0\,\frac{\text{lb}}{\text{s}}}_{\text{in}} - \underbrace{\left(5\,\frac{\text{gal}}{\text{s}} \times \frac{x(t)}{100}\,\frac{\text{lb}}{\text{gal}}\right)}_{\text{out}} = \frac{-x(t)}{20}\ \text{lbs/s}$$

$x(0) = 20$

$$x' = -\frac{x}{20}$$

$$x' + \frac{1}{20}x = 0 \qquad a = -\frac{1}{20},\ f = 0$$

$$x(t) = Ce^{-\frac{t}{20}}, \quad x(0) = C \cdot 1 = 20$$

$$\boxed{x(t) = 20e^{-\frac{t}{20}}}$$

$y(0) = 40$

$$\frac{dy}{dt} = \underbrace{+\frac{x}{20}}_{\text{in}} - \underbrace{\left(5 \cdot \frac{y(t)}{200}\right)}_{\text{out}} = \frac{x}{20} - \frac{y(t)}{40}$$

$$y' = e^{-\frac{t}{20}} - \frac{y(t)}{40}$$

$\vdots$

$$y(t) = -40e^{-\frac{t}{20}} + Ce^{-\frac{t}{40}}$$

$$y(0) = -40 \cdot 1 + C \cdot 1 = 40, \quad C = 80$$

$$\boxed{y(t) = -40e^{-\frac{t}{20}} + 80e^{-\frac{t}{40}}}$$

$$y(60) = -40e^{-3} + 80e^{-\frac{3}{2}} \ \boxed{\approx 15.9\ \text{lbs}}$$

2.6 Exact Differential Equations

Consider $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$

P, Q functions of independent variable x + dependent value y

Solution has to be differentiable function $y(x)$ defined for x in an interval where x satisfies each point

- necessary in some cases to define solutions implicitly by equations of form $F(x,y) = C$

Differential form: in 2 variables x and y is an expression of type: $w = P(x,y)\,dx + Q(x,y)\,dy$

P, Q functions of x and y.
dx & dy are differentials

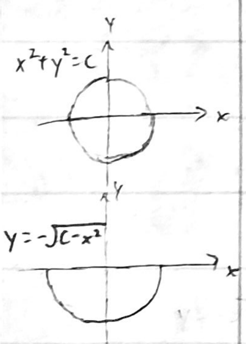
If $y = y(x)$, then $dy = y'(x)\,dx$

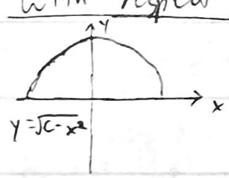
$$P(x,y)\,dx + Q(x,y)\,dy = \left(P(x,y) + Q(x,y)\frac{dy}{dx}\right)dx$$

$\Rightarrow$ y is a solution to $P + Q\frac{dy}{dx} = 0$ only if it's a solution of $P\,dx + Q\,dy = 0$ $\Leftarrow$ another way to write $P + Q\frac{dy}{dx} = 0$

Ex: Consider DE $w = x\,dx + y\,dy = 0$ or $\frac{dy}{dx} = -\frac{x}{y}$

Show equation has solutions defined implicitly by $x^2 + y^2 = C$. Differentiate with respect to x





$$x^2 + y^2 = C$$
$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(C)$$
$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad 2x\,dx + 2y\,dy = 0$$
$$\Rightarrow x\,dx + y\,dy = 0$$

can solve explicitly

$y(x) = \pm\sqrt{C - x^2}$, defined for $|x| \le \sqrt{C}$

Def: Suppose solutions to $P + Q\frac{dy}{dx} = 0$ or $P\,dx + Q\,dy = 0$ are given implicitly by $F(x,y) = C$. Then the level sets defined by $F(x,y) = C$ are called integral curves of the DE.

In previous example, level $x^2 + y^2 = C$ is a circle radius $\sqrt{C}$ center $(0,0)$
- Not a graph of a function
- Contains graphs of both $y(x) = \pm\sqrt{C - x^2}$

General Idea: Suppose $y = y(x)$ is a solution to $w = P(x,y)\,dx + Q(x,y)\,dy = 0$ and it's defined implicitly by $F(x,y) = C$

differentiate w/ respect to x

$$\frac{d}{dx}(F(x,y)) = \frac{d}{dx}(C)$$
$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

$y(x)$ is a solution to $P\,dx + Q\,dx = 0 \Rightarrow \left(P + Q\frac{dy}{dx}\right)dx = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}$$

set $+\frac{P(x,y)}{Q(x,y)} = +\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$

$$\frac{1}{P}\cdot\frac{\partial F}{\partial x} = \frac{1}{Q}\cdot\frac{\partial F}{\partial y} = \mu(x,y)$$

$$\frac{\partial F}{\partial x} = \mu P \quad \& \quad \frac{\partial F}{\partial y} = \mu Q$$

To find solution, we need to find F and μ satisfying

<u>Def</u>: The <u>differential</u> of a continuously differentiable function F is the differential form: $\boxed{dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy}$

A differential form is <u>exact</u> if it's the differential of a continuously differentiable function.

ie. differential form $P\,dx + Q\,dy$ is exact only if continuously differentiable function $F(x,y)$ such that $\boxed{dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = P\,dx + Q\,dy}$

$\Rightarrow$ coefficients of dx and dy need to be equal

$$\frac{\partial F}{\partial x} = P \quad \text{and} \quad \frac{\partial F}{\partial y} = Q$$

10/10/22 If $w = P\,dx + Q\,dy$ is exact and equal to dF, then general solution to $dF = P\,dx + Q\,dy = 0$ given by $F(x,y) = 0$

<u>Ex</u>: $f(x,y) = x^2y^5$, find partial derivative w/ respect to both x and y

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^5) = y^5\frac{\partial}{\partial x}(x^2) = y^5\cdot 2x = 2xy^5$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y^5) = x^2\frac{\partial}{\partial y}(y^5) = 5x^2y^4$$

<u>Ex</u>: $f = x^4y + xy^{-2}$

$$\frac{\partial f}{\partial x} = 4x^3y + y^{-2}$$

$$\frac{\partial f}{\partial y} = x^4 - 2xy^{-3}$$

<u>Ex</u>: $f = \sin(4x + 3y^2)$

$$\frac{\partial f}{\partial x} = 4\cos(4x+3y^2)$$

$$\frac{\partial f}{\partial y} = 6y\cos(4x+3y^2)$$

<u>Ex</u>: $f = x\ln(x+y)$

$$\frac{\partial f}{\partial x} = \ln(x+y) + \frac{x}{x+y}$$

$$\frac{\partial f}{\partial y} = \frac{x}{x+y}$$

Ex: Solve $2x\,dx + 4y^3\,dy = 0$ to find $F(x,y)$ so that $\frac{\partial F}{\partial x} = P$ and $\frac{\partial F}{\partial x} = Q$

$$P = 2x, \quad Q = 4y^3$$

$$\frac{\partial F}{\partial x} = 2x \qquad \frac{\partial F}{\partial y} = 4y^3$$

$$\int \partial F = \int 2x\,dx \qquad \frac{\partial F}{\partial y} = h'(y) = 4y^3$$

$$F = x^2 + h(y) \qquad h(y) = y^4 + C_1$$

so $\boxed{F(x,y) = x^2 + y^4}$

Solutions are implicitly defined by $F(x,y) = C$ where $x^2 + y^4 = C$ is an integral curve at C.

Q: Is there a way to tell if $w = P\,dx + Q\,dy$ is exact?

Thm: Let $w = P(x,y)\,dx + Q(x,y)\,dy$ be a differential form where both P & Q are continuously differentiable.

a) If w is exact, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

b) If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ is true in rectangle R, then w is exact in R

Solving Exact DE's: If $P\,dx + Q\,dy = 0$ is exact, solution is given by $F(x,y) = C$ where we find F by

1) Solve $\frac{\partial F}{\partial x} = P$ by integration: $F(x,y) = \int P(x,y)\,dx + \phi(y)$

2) Solve $\frac{\partial F}{\partial y} = Q$ using $\int P(x,y)\,dx + \phi(y)$ so that $\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int P(x,y)\,dx + \phi'(y) = Q(x,y)$

Ex: Show $(2xy^2+1)\,dx + (2x^2y)\,dy = 0$ is exact and find a general solution.

=, so exact

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy^2+1) = 4xy \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2x^2y) = 4xy$$

Finding $F(x,y)$

$$\frac{\partial F}{\partial x} = 2xy^2+1 \quad \& \quad \frac{\partial F}{\partial y} = 2x^2y$$

$$F = x^2y^2+x+h(y) \qquad \frac{\partial F}{\partial y} = 2x^2y+h'(y) = 2x^2y$$

$$h'(y)=0 \text{ so } h(y)=C_1$$

Solutions given implicitly by $F(x,y) = x^2y^2+x+C_1 = \tilde{C}$ where $C=\tilde{C}-C_1$

$$F(x,y) = \boxed{x^2y^2+x=C}$$

<u>Ex:</u> Show $\sin(x+y)dx + (2y+\sin(x+y))dy=0$ is exact and find a general solution

$$\left.\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}(\sin(x+y)) = \cos(x+y)\\ \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}(2y+\sin(x+y)) = \cos(x+y)\end{aligned}\right] \text{ exact}$$

Find $F(x,y)$ $\qquad \frac{\partial F}{\partial x} = \sin(x+y) \qquad \frac{\partial F}{\partial y} = 2y+\sin(x+y)$

$$F = \int \sin(x+y)dx + \phi(y) \qquad \frac{\partial F}{\partial y} = \sin(x+y)+\phi'(y) = 2y+\sin(x+y)$$

$$= -\cos(x+y)+\phi(y) \qquad \phi'(y)=2y$$

$$\phi(y) = y^2+C_1$$

$F(x,y)=C$

solution implicitly defines by $\boxed{F(x,y) = -\cos(x+y)+y^2=C}$

<u>Def:</u> An <u>integrating factor</u> for DE $\omega = Pdx+Qdy=0$ is a function $\mu(x,y)$ such that the form

$$\mu\omega = \mu(x,y)P(x,y)dx + \mu(x,y)Q(x,y)dy \text{ is exact.}$$