10/5/22

Structure of the Solution:

For $y' = ay + f$

$y_h(t) = e^{\int a(t)\,dt}$ particular solution to associated homogeneous equation

Can show using method of variation of parameters solution is:

$$y(t) = y_h(t) \int f(t) e^{-\int a(t)\,dt}\,dt + C y_h(t)$$

particular solution $y_p(t)$ associates w/ particular constant value C_p

$$y_p(t) = y_h(t) \int f(t) e^{-\int a(t)\,dt}\,dt + C_p y_h(t)$$

Difference of 2 solutions to inhomogeneous equation

$$y(t) - y_p(t) = \underbrace{(C - C_p)}\, y_h(t)$$

set $A = C - C_p$

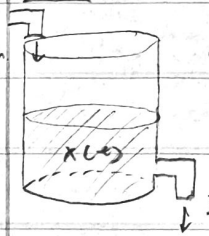
constant multiple of y_h, solution to associated homogeneous equation

Thm: If y_p is a particular solution to inhomogeneous equation: $y' = a(t)y + f(t)$ & $y_h(t)$ is a particular solution to associated homogeneous equation, then every solution to the inhomogeneous equation is of form: $y(t) = y_p(t) + A y_h(t)$

A is an arbitrary constant

2.5 Mixing Problems

Ex: Initially, water tank holds 100 gal. of pure water. Then add a 2 lbs. salt/gal mixture at 3 gal/min. At the same time, drain opened at bottom of tank so volume of solution remains constant.



How much salt is in the tank after 60 min?

Let $x(t)$ be # of lbs. of salt in tank at time t

$\frac{dx}{dt}$ = rate of change of salt in respect to time

volume rates: flow in/out of tank

concentration of solution in lbs. per gallon

rate of change = rate in − rate out

rate in = volume × concentration $= 3\frac{gal}{min} \times 2\frac{lb}{gal} = 6\frac{lb}{min}$

Assume salt dist. is uniform

rate out = volume rate × concentration $= 3\frac{gal}{min} \times \frac{x(t)}{100}\frac{lb}{gal}$

$= \frac{3x(t)}{100}\frac{lb}{min}$

So $\frac{dx}{dt} = 6 - \frac{3x}{100}$ $\quad a = -\frac{3}{100}, \; f = 6$

$x' + \frac{3x}{100} = 6$ $\quad w(t) = e^{+\int \frac{3}{100}dx} = e^{\frac{3t}{100}}$

$x'e^{\frac{3t}{100}} + e^{\frac{3t}{100}}\frac{3x}{100} = 6e^{\frac{3t}{100}}$

$\left(xe^{\frac{3t}{100}}\right)' = 6e^{\frac{3t}{100}}$

$e^{\frac{3t}{100}}x = 200e^{\frac{3}{100}t} + C$

general sol: $x(t) = 200 + Ce^{-\frac{3}{100}t}$, $x(0) = 0$

$x(0) = 200 + Ce^0 = 0$

$C = -200$

$$\boxed{x(t) = 200\left(1 - e^{-\frac{3}{100}t}\right)}$$

At $t = 60$, $x(60) = 200\left(1 - e^{-\frac{180}{100}}\right) \approx 167 \text{ lbs.}$

of pure water

Ex: 300 gallons in tank, spigot opens above tank so salt solution of 1.5 lbs./gal flows in at rate of 3 gal/min. A drain below the tank drains 1 gal/min. What's the salt content in tank when volume = 600

$V(t) = 300 + (3-1)t = 300 + 2t$

$V(150) = 300 + 300 = 600$, $t = 150$

$\frac{dx}{dt} = 3\frac{gal}{min} \cdot 1.5\frac{lbs}{gal} - 1\frac{gal}{min} \cdot \frac{x(t)}{V(t)}\frac{lb}{gal} = 4.5 - \frac{x(t)}{V(t)}$

$x' + \frac{x}{V(t)} = 4.5 \Rightarrow x' + \frac{x}{300+2t} = 4.5$

$a = -\frac{1}{300+2t}$ $\quad w(t) =$

$x(t) = 450 + 3t + \frac{C}{\sqrt{300+2t}}$, $x(0) = 0$

$x(0) = 450 + 0 + \frac{C}{\sqrt{300}} = 0$

$C = -4500\sqrt{3}$

$$\boxed{x(t) = 450 + 3t - \frac{4500\sqrt{3}}{\sqrt{300+2t}}}$$

$x(150) = 450 + 450 - \frac{4500\sqrt{3}}{\sqrt{600}} \approx 582 \text{ lbs.}$