9.1 Overview of Techniques

Want to find solutions to system $\boxed{\vec{y}' = A\vec{y}}$ where A is a matrix of constant entries.

If A is a 1×1 matrix, then

$y' = ay$ for some $a \in \mathbb{R}$ which is a 1st order homogeneous equation w/ constant coefficients

general solution: $\boxed{y(t) = Ce^{at}}$, C is constant

This motivates us to find equation,

$$\boxed{\vec{y}(t) = e^{\lambda t}\vec{v}}$$

$\vec{v}$ is a vector w/ constant components

Need to find $\vec{v}$ and λ

Note:
$$\vec{x}(t) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Then:
$$\vec{x}'(t) = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}$$

LHS: $\vec{y}'(t) = \lambda e^{\lambda t}\vec{v}$

RHS: $A\vec{y} = A(e^{\lambda t}\vec{v}) = e^{\lambda t}A\vec{v}$

Sub in: $\vec{y}' = A\vec{y} \implies \lambda e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v}$

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

<u>Definition</u>: If A is $n\times n$ matrix, number λ is the <u>eigenvalue</u> of A if there's a nonzero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}$$

If λ is an eigenvalue, then any nonzero vector $\vec{v}$ that satisfies $A\vec{v} = \lambda\vec{v}$ is an <u>eigenvector</u>

<u>Note:</u> If $\vec{v}$ is an eigenvector to $A\vec{v} = \lambda\vec{v}$, then any nonzero multiple of $\vec{v}$ is also an eigenvector

$r \in \mathbb{R}$ and $r \neq 0$

$A\vec{v} = \lambda\vec{v}$

$rA\vec{v} = r\lambda\vec{v}$

$A(r\vec{v}) = \lambda(r\vec{v})$ $\therefore$ $r\vec{v}$ eigenvector

<u>Thm 1.6:</u> λ is an eigenvalue of matrix A and $\vec{v}$ is eigenvector.

Then $\boxed{\vec{x}(t) = e^{\lambda t}\vec{v}}$ is a solution to the system $\vec{x}' = A\vec{x}$ and satisfies initial condition $\vec{x}(0) = \vec{y}$

Finding eigenvalues & eigenvectors

eigenvalue λ + eigenvector $\vec{v}$

$A\vec{v} = \lambda \vec{v}$

$0 = A\vec{v} - \lambda\vec{v} = A\vec{v} - \lambda I \vec{v} = (A - \lambda I)\vec{v}$

so $A\vec{v} = \lambda\vec{v}$ only if

$(A - \lambda I)\vec{v} = \vec{0}$ for some $\vec{v} \neq \vec{0}$

only if $\det(A - \lambda I) = 0$

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ \vdots & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$ $\leftarrow$ ann is gonna have $-\lambda$

Def 1.10 If A is $n \times n$ matrix, the polynomial

$$\boxed{p(\lambda) = (-1)^n \det(A - \lambda I) = \det(\lambda I - A)}$$

is the characteristic polynomial of A and $\boxed{p(\lambda) = (-1)^n \det(A - \lambda I) = 0}$ is the characteristic equation

Prop 1.11 Eigenvalues of a $n \times n$ matrix is its roots

$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$

$p(\lambda) = (-1)^2 \det(A - \lambda I) = \det\left[\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = \begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$

$\boxed{\text{eigenvalues: } \lambda = -1, 2}$

Prop 1.13 Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. The set of all eigenvectors associated w/ λ is equal to null space of $A - \lambda I$. Hence, the eigenspace of λ is a subspace of $\mathbb{R}^n$.

$$\boxed{\text{null space of } A - \lambda I = \{\vec{x} \in \mathbb{R}^n : (A - \lambda I)\vec{x} = \vec{0}\}}$$

Ex: Find an eigenvector from previous ex associated with $\lambda_2 = 2$

$A - \lambda_2 I = \begin{pmatrix} 3-2 & -1 \\ 4 & -2-2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}$

find $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \vec{0}$

$(A - 2I)\vec{v} = \vec{0}$

$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 - x_2 = 0 \\ 4x_1 - 4x_2 = 0 \end{matrix}$ $\quad x_1 = x_2$ $\quad$ so $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

If $x_1 = 1$, $\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

same works for $\lambda_1 = -1$

$$(A - (-1)I)\vec{v} = \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} 4x_1 - x_2 \\ 4x_1 - x_2 \end{matrix} \qquad 4x_1 = x_2$$

$$\vec{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 4x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \text{take } x_1 = 1 \quad \boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}}$$

Finding Solutions to DE

Ex: Find fundamental set of solutions for $\vec{y}\,' = A\vec{y}$ for $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$.

A has eigenvalue $\lambda_1 = -1$ w/ eigenvector $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 2$ w/ eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We know $\vec{y}(t) = e^{\lambda t}\vec{v}$ from Thm 1.16. So the solutions to $\vec{y}\,' = A\vec{y}$ are:

$$\boxed{\vec{y}_1(t) = e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \vec{y}_2(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

To determine linear independence of $\vec{y}_1(t)$ and $\vec{y}_2(t)$ use following 2 results:

1) Let $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n \in \mathbb{R}^n$ and W be a matrix w/ these columns

$$W = \begin{pmatrix} | & | & & | \\ w_1 & w_2 & \cdots & w_n \\ | & | & & | \end{pmatrix}$$

$\boxed{w_1, w_2, \dots, w_n \text{ are linearly independent only if } \det W \neq 0}$

Prop 5.12 2) Let $\vec{y}_1(t), \vec{y}_2(t), \dots, \vec{y}_k(t)$ be solutions to n-dimensional system $\vec{y}\,' = A\vec{y}$ defined on interval $I = (\alpha, \beta)$. If for some $t_0 \in I$, vectors $\vec{y}_1(t_0), \vec{y}_2(t_0), \dots, \vec{y}_k(t_0)$ are linearly independent, then $\vec{y}_1(t), \dots, \vec{y}_k(t)$ are linearly ind. for all t.

Returning to ex:

$$\vec{y}_1(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \vec{y}_2(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left| \vec{y}_1(0) \ \vec{y}_2(0) \right| = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1 \neq 0$$

then $\vec{y}_1(0)$ & $\vec{y}_2(0)$ are linearly independent and by Prop 5.12, $\vec{y}_1(t)$ & $\vec{y}_2(t)$ are linearly independent on $(-\infty, \infty)$. Therefore, $\vec{y}_1(t)$ & $\vec{y}_2(t)$ form a fundamental set of solutions.

Note: For system of dimension n, you want to find n linearly independent solutions to form fundamental set. Bc characteristic polynomial of n^{th} degree has n roots and each has eigenvalue λ and eigenvector $\vec{v}$

Cases:
1. Distinct Real Roots
2. Complex Roots
3. Repeated Roots

Ex: Find fundamental set of solutions to

$$\vec{y}\,' = \begin{pmatrix} -5 & 0 & -6 \\ 26 & -3 & 38 \\ 4 & 0 & 5 \end{pmatrix}\vec{y} \qquad A = \begin{pmatrix} -5 & 0 & -6 \\ 26 & -3 & 38 \\ 4 & 0 & 5 \end{pmatrix}$$

eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} -5-\lambda & 0 & -6 \\ 26 & -3-\lambda & 38 \\ 4 & 0 & 5-\lambda \end{vmatrix} = (-5-\lambda)\begin{vmatrix} -3-\lambda & 38 \\ 0 & 5-\lambda \end{vmatrix} - 0\begin{vmatrix} 26 & 38 \\ 4 & 5-\lambda \end{vmatrix} + (-6)\begin{vmatrix} 26 & -3-\lambda \\ 4 & 0 \end{vmatrix} = -(\lambda+3)(\lambda+1)(\lambda-1)$$

Find eigenvalues

$$p(\lambda) = (-1)^3 |A - \lambda I| = (\lambda+3)(\lambda+1)(\lambda-1) = 0$$

$$\lambda_1 = -3 \qquad \lambda_2 = -1 \qquad \lambda_3 = 1$$

eigenvectors

$\lambda_1 = -3$ $\quad (A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\begin{pmatrix} -2 & 0 & -6 \\ 26 & 0 & 38 \\ 4 & 0 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} x_1 = 0 \\ x_2 = x_2 \\ x_3 = 0 \end{array} \quad \vec{v}_1 = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \boxed{= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

$\lambda_2 = -1$ $\quad (A - \lambda_2 I)\vec{v}_2 = \vec{0}$

$$\begin{pmatrix} -4 & 0 & -6 \\ 26 & -2 & 38 \\ 4 & 0 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} x_1 = -\frac{3}{2}x_3 \\ x_2 = -\frac{1}{2}x_3 \\ x_3 = x_3 \end{array} \quad \vec{v}_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}x_3 \\ -\frac{1}{2}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

if $x_3 = -2$, $\boxed{\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}}$

$\lambda_3 = 1$ $\quad \boxed{\vec{v}_3 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}}$

$$\boxed{\vec{Y}_1(t) = e^{-3t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad Y_2(t) = e^{-t}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \quad \vec{Y}_3(t) = e^{t}\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}}$$