11/2/22

4.5 Inhomogeneous Equations: The Method of Undetermined Coefficients

$\boxed{y'' + p(t)y' + q(t)y = f(t)}$ ← inhomogeneous linear equation

Thm 5.2: Suppose y_p is a particular solution to $y'' + py' + qy = f$ and y_1 & y_2 form a fundamental set of solutions to

$$y'' + py' + qy = f$$

the general solution is given by

$$\boxed{y = y_p + C_1 y_1 + C_2 y_2}$$

where C_1 & C_2 are constants

note: if finding general solution to $y'' + py' + qy = f$

- find general solution $y_h = C_1 y_1 + C_2 y_2$
- find particular solution y_p
- the general solution is $y = y_p + y_h$

Method of Undetermined Coefficients:

$$y'' + py' + qy = f(t) \quad p \,\&\, q \text{ are constant}$$

p & q have to be CONSTANT for this to work

key idea: If forcing term $f(t)$ has a form that is replicated under differentiation, then look for a solution w/ same general form as forcing term.

Exponential Forcing Terms: $f(t) = e^{at}$

$f'(t) = ae^{at}$

Ex: Find particular solution to $y'' - 3y' - 4y = 3e^{2t}$

look for solution

a is constant $y(t) = ae^{2t}$

$y'(t) = 2ae^{2t}$

$y''(t) = 4ae^{2t}$

SAME FORM AS FORCING TERM

$$4ae^{2t} - 3(2ae^{2t}) - 4ae^{2t} = 3e^{2t}$$

$$-6ae^{2t} = 3e^{2t}$$

$$a = -\tfrac{1}{2}$$

$$\boxed{y(t) = -\tfrac{1}{2}e^{2t}}$$

Trigonometric Forcing Terms:

$$\boxed{\begin{aligned} f(t) &= A\cos\omega t + B\sin\omega t \\ f'(t) &= (-\omega A)\sin\omega t + (\omega B)\cos\omega t \end{aligned}}$$

1st Method: look for solution $\boxed{y(t) = a\cos\omega t + b\sin\omega t}$, a & b const. tbd

Ex: Find particular solution to $y'' + 2y' - 3y = 5\sin 3t$

$y(t) = a\cos 3t + b\sin 3t$

$y'(t) = -3a\sin 3t + 3b\cos 3t$

$y''(t) = -9a\cos 3t - 9b\sin 3t$

$$(-9a\cos 3t - 9b\sin 3t) + 2(-3a\sin 3t + 3b\cos 3t) - 3(a\cos 3t + b\sin 3t) = 5\sin 3t$$

$$\Downarrow$$

$$(-9a + 6b - 3a)\cos 3t + (-9b - 6a - 3b)\sin 3t = 0\cos 3t + 5\sin 3t$$

$-12a + 6b = 0$

$-6a - 12b = 5$

$a = -\frac{1}{6}$ $b = -\frac{1}{3}$

$$\boxed{y(t) = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t}$$

Complex Method:

$y'' + 2y' - 3y = 5\sin 3t = Im(5e^{3it})$

Euler's Formula

Recall $5e^{3it} = (5\cos 3t) + i(5\sin 3t)$

$$\boxed{\begin{aligned} e^{(a+ib)t} &= e^{at}(\cos bt + i\sin bt) \\ e^{(a-ib)t} &= e^{at}(\cos bt - i\sin bt) \end{aligned}}$$

look for solution to $\boxed{z'' + 2z' - 3z = 5e^{3it}}$

If $z(t) = x(t) + iy(t)$ is a solution to ↺, then

$$(x+iy)'' + 2(x+iy)' - 3(x+iy) = 5\cos 3t + i(5\sin 3t)$$

$$(x'' + 2x' - 3x) + i(y'' + 2y - 3y) = 5\cos 3t + i(5\sin 3t)$$

Equate imaginary parts $y'' + 2y' - 3y = 5\sin 3t$

So $y(t) = Im(z(t))$ is a solution

SAME FORM AS FORCING TERM

$z(t) = ae^{3it}$

$z'(t) = 3aie^{3it}$

$z''(t) = -9ae^{3it}$

$$(-9ae^{3it}) + 2(3aie^{3it}) - 3(ae^{3it}) = 5\sin 3t$$

$$(-12a + 6ai)e^{3it} = 5e^{3it}$$

$$-6a(2-i) = 5$$

$$a = -\frac{(2+i)}{6}$$

$$z(t) = -\frac{1}{6}(2+i)e^{3it}$$

$$= -\frac{1}{6}(2+i)(\cos 3t + i\sin 3t)$$

$$= -\frac{1}{6}[(2\cos 3t - \sin 3t) + i(\cos 3t + 2\sin 3t)]$$

$y = Im(z)$ so $\boxed{y(t) = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t}$

The solution you look for has to be in SAME FORM

Polynomial Forcing Terms:

$$f(t) = a_0 t^n + a_1 t^{n-1} + \dots a_{n-1} t + a_n$$

1ST DEGREE

Ex: Find particular solution to $y'' + 3y' + 2y = 3t$, a & b tbd

SAME FORM AS FORCING TERM

$y(t) = at + b$

$y'(t) = a$

$y''(t) = 0$

$$0 + 3a + 2(at + b) = 3t$$

$$\underline{2at} + \underline{\underline{3a + 2b}} = \underline{3t} + \underline{\underline{0}}$$

$$2a = 3 \qquad\qquad a = \frac{3}{2}$$

$$3a + 2b = 0 \quad \Rightarrow \quad b = -\frac{9}{4}$$

$$\boxed{y(t) = \frac{3}{2}t - \frac{9}{4}}$$

Exceptional Cases:

Ex: $y'' - y' - 2y = 3e^{-t}$

Note that $y(t) = ae^{-t}$ is a solution to associated homogeneous equation $y'' - y' - 2y = 0$ bc $\lambda^2 - \lambda - 2 = 0$, $\lambda = -1, 2$

In exceptional cases, part or all of forcing term is a solution to associated homogeneous equation.

Multiply the usual ae^{-t} by t. If it fails multiply ae^{-t} by t^2.

Try $y(t) = ate^{-t}$

$y'(t) = ae^{-t} - ate^{-t}$

$y''(t) = -2ae^{-t} + ate^{-t}$

$$y'' - y' - 2y = 3e^{-t}$$

$$(-2ae^{-t} + \cancel{ate^{-t}}) - (ae^{-t} - \cancel{ate^{-t}}) - 2\cancel{a}te^{-t} = 3e^{-t}$$

$$-3ae^{-t} = 3e^{-t}, \quad a = -1$$

Solution: $\boxed{y(t) = -te^{-t}}$

Combining Forcing Terms:

Thm 5.22: Suppose that $y_f(t)$ is a solution to linear equation:

$$y_f'' + p y_f' + q y_f = f(t)$$

and $y_g(t)$ is a solution to

$$y_g'' + p y_g' + q y_g = g(t)$$

Then $y(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution to

$$\boxed{y'' + py' + qy = \alpha f(t) + \beta g(t)}$$

Ex: Find particular solution to $y'' - 3y' - 4y = \underbrace{3e^{2t}} + \underbrace{2\sin t} - \underbrace{8e^t \cos 2t}$

split into 3 parts

$$y_1'' - 3y_1' - 4y_1 = 3e^{2t}$$

$$y_2'' - 3y_2' - 4y_1 = 2\sin t$$

$$y_3'' - 3y_3' - 4y_3 = \underbrace{-8e^t \cos 2t}$$

more complicated, try $y_3(t) = ae^t \cos 2t + be^t \sin 2t$

Solutions:

$y_1(t) = -\frac{1}{2}e^{2t}$

$y_2(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$

$y_3(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin t$

$y(t) = y_1(t) + y_2(t) + y_3(t)$

$$\boxed{= -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin t}$$

Ex: Find particular solution to $y'' + 4y = \underbrace{\cos 2t} - \underbrace{\sin 2t}$

split

Problem: $\cos 2t$ & $\sin 2t$ are solutions to associated homogeneous equation

$y_1'' + 4y_1 = \cos 2t$

$y_2'' + 4y_2 = -\sin 2t$

$y(t) = y_1(t) + y_2(t)$

Complex Method: $\cos 2t = Re(e^{2it})$

$$z'' + 4z = e^{2it}$$

solution to associated homogeneous equation

$z(t) = ate^{2it}$

$z'(t) =$

$z''(t) =$

$\dots \quad a = -\frac{1}{4}i$

$z(t) = -\frac{1}{4}ite^{2it} = -\frac{1}{4}it(\cos 2t + i\sin 2t)$

$= \frac{1}{4}t\sin 2t + i(-\frac{1}{4}t\cos 2t)$

$y_1(t) = Re(z(t)) \boxed{= \frac{1}{4}t\sin 2t}$

note: $\sin 2t = \operatorname{Im}(e^{2it})$ so $\mu(t) = \operatorname{Im}(z(t)) = -\frac{1}{4}t\cos 2t$

solves $u'' + 4u = \sin 2t$

$-2(u'' + 4u = \sin 2t)$

$(-2u)'' + 4(-2u) = -2\sin 2t$

$y_2(t) = -2u = \frac{1}{2}t\cos 2t$

$y(t) = y_1(t) + y_2(t) \boxed{= \frac{1}{4}t\sin 2t + \frac{1}{2}t\cos 2t}$

More Complex Forcing Terms:

$f(t) = e^{rt}P(t)\cos\omega t + e^{rt}Q(t)\sin\omega t$ $\quad$ $P(t)$ & $Q(t)$ are polynomials