In general, if we're looking for constant solutions $x(t) = C$ to $x' = f(t, x)$, note LHS $x' = C' = 0$ so RHS is $f(t, x(t)) = f(t, C) = 0$ as well. Therefore, look for constants C so that $f(t, c) = 0$ for all t.

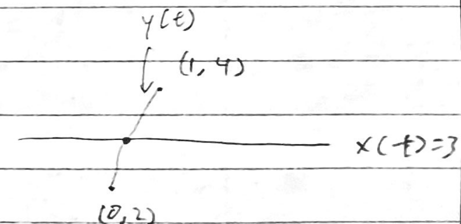
Ex: $y' = ty(3-y)$

$w(t) = 0$ and $x(t) = 3$ are constant solutions

Check conditions for uniqueness: Uniqueness Thm implies that 2 DISTINCT solution curves can't meet, since various solution curves don't meet, it's unique.

If $y(t)$ is a solution to IVP: $y' = ty(3-y)$, $y(0) = 2$, is it possible for $y(1) = 4$?

Recall $x(t) = 3$ is a solution



$y(t)$ must be continuous, but if $y(1) = 4$ that means y will cross $x(t) = 3$ at some point as said in the Intermediate Value Thm.
Since there's some $\bar{t}$ so that $y(\bar{t}) = x(\bar{t}) = 3$, and Uniqueness Thm says solution curves can't meet, $y(1) = 4$ isn't possible.

Basically $\underset{w(t)}{0} < y(t) < \underset{x(t)}{3}$. In $\mathbb{R}^2$ if solution curves can't meet, then a solution curve divides plane into 2 separate sections.

2.9 Autonomous Equations & Stability

Def: First-order autonomous equations is equation of form

$$x' = \underline{f(x)}$$

↳ independent variable doesn't explicitly show on RHS, defining feature of autonomous equations

Examples: $x' = \sin(x)$ $\quad y' = e^{y^2+1}$

Not Autonomous: $x' = \sin(tx)$ $\quad y' = ty$

In this section, we'll look at how the solution behaves as ind. variable goes to $\pm\infty$

Direction Fields & Solutions

Ex: Modeling a population graph

$x(t)$ = population at time t

Logistic Equation: $\frac{dx}{dt} = r(1-\frac{x}{k})x$

constants[r = intrinsic growth rate
k = carrying capacity

$x'(t) = f(x)$

Features: $f(x)$ doesn't depend on t explicitly, so slopes of direction lines don't change moving left or right

If $y' = f(y)$, ~~shift left or right~~
let $y_1(t) = y(t+c)$

← • A solution curve translated left or right is another solution curve

$y_1' = \frac{d}{dt}(y(t+c)) = y'(t+c) = f(y(t+c)) = f(y_1(t))$

so $\boxed{y_1' = f(y_1)}$ is also a solution

Equilibrium Points and Solutions

$x' = f(x)$

If $f(x_0) = 0$, for some $x_0 \in \mathbb{R}$ then we have constant solutions $x(t) = x_0$

Since $x' = 0 = f(x_0) = f(x(t))$, a point x_0 such that $f(x_0) = 0$ is an equilibrium point and constant function $x(t) = x_0$ is an equilibrium solution.

Previous Ex: $x' = r(1-\frac{x}{k})x = f(x)$

$f(x_0) = r(1-\frac{x_0}{k})x_0 = 0$

$x_0 = k$ or $x_0 = 0$ so equilibrium solutions are $x_1(t) = 0$ and $x_2(t) = k$

Non equilibrium Solutions:

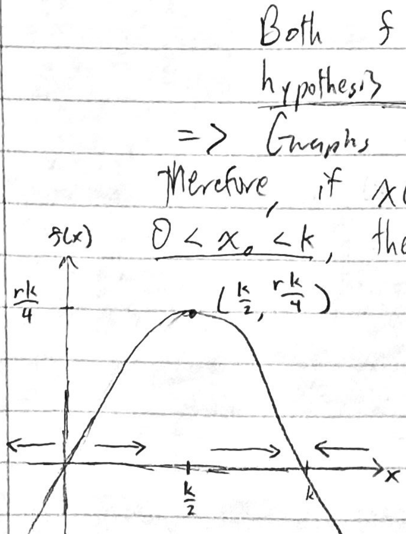
$x' = r(1-\frac{x}{k})x$ ← normal form

$f(x) = r(1-\frac{x}{k})x$ and $\frac{\partial f}{\partial x} = r - \frac{2r}{k}x$

Both f and $\frac{\partial f}{\partial x}$ are on $\mathbb{R}^2$ so $x' = r(1-\frac{x}{k})x$ satisfies hypothesis of Uniqueness Thm

=> Graphs of $x(t)$ and equilibrium solutions $x_1(t)$ & $x_2(t)$ can't cross

Therefore, if $x(t)$ is a solution to $x' = r(1-\frac{x}{k})x$ w/ $x(0) = x_0$ & $0 < x_0 < k$, then $0 < x(t) < k$ for all t.



$f(x) = r(1-\frac{x}{k})x$

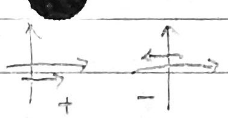
Furthermore, $x' = r(1-\frac{x}{k})x > 0$ for $0 < x < k$ so derivative $x'(t)$ is positive for all t implying that $x(t)$ is monotone increasing; since $x(t)$ is bounded above by k, $x(t)$ also approaches limit of k.

- solutions in this case w/ $0 < x_0 < k$ approach k but don't exceed it, hence it being the carrying capacity. $\lim_{t \to \infty} x(t) = k$ & $\lim_{t \to -\infty} x(t) = 0$

Phase Line: consider $y' = f(y)$ and y as "distance" from 0 along number line (y-axis)

- $y' = f(y)$ describes motion dynamics modeled by $y(t)$ along the x-axis line which is called the phase line
- Equilibrium point types:

$\leftarrow 0 \rightarrow$ $\quad$ $\rightarrow 0 \leftarrow$



- A + pointing arrow means solution increases around equilibrium points, − arrow means decreasing