10/28/22

4.3 Linear, Homogeneous, Eqn's w/ Constant Coefficients

are equations of form $y''+py'+qy=0$ where p & q are constants

From previous class, $y''-y=0$ if $p=0, q=-1$ we found 2 linearly independent solutions: $y_1(t)=e^{t}$ & $y_2(t)=e^{-t}$

Find exponential solutions to $y''+py'+qy=0$

$y=e^{\lambda t}$ $\lambda=$ parameter tbd

$y'=\lambda e^{\lambda t}$

$y''=\lambda^2 e^{\lambda t}$ $\lambda^2 e^{\lambda t}+p\lambda e^{\lambda t}+qe^{\lambda t}=0$

$e^{\lambda t}\neq 0$ $e^{\lambda t}(\lambda^2+p\lambda+q)=0$

$\boxed{\lambda^2+p\lambda+q=0}$ ← characteristic equation for $y''+py'+qy=0$. Root is called characteristic root

If λ is characteristic root, $y=e^{\lambda t}$ is a solution

Find roots of characteristic equation using quadratic formula:

$\lambda=\frac{-p\pm\sqrt{p^2-4q}}{2}$ 3 cases

1) 2 distinct, real roots if $p^2-4q>0$
2) 2 distinct, complex roots if $p^2-4q<0$
3) 1 repeated, real root if $p^2-4q=0$

Case 1: Distinct, Real

$\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1\neq\lambda_2$

$y_1=e^{\lambda_1 t}$ & $y_2=e^{\lambda_2 t}$ are the solutions

Since $\lambda_1\neq\lambda_2$, y_1 & y_2 are linearly ind.

Prop 3.3: If characteristic equation $\lambda^2+p\lambda+q=0$ has 2 distinct, real solutions λ_1, λ_2 general solution to $y''+py'+qy=0$ is $y(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$ where C_1 & C_2 are constants

Ex: Find General Solutions to $y''+5y'+6y=0$

$y=e^{\lambda t}$

$y'=\lambda e^{\lambda t}$

$y''=\lambda^2 e^{\lambda t}$

$e^{\lambda t}(\lambda^2+5\lambda+6)=0$

$\lambda^2+5\lambda+6=0$ ← characteristic equation

$(\lambda+2)(\lambda+3)=0$

$\lambda_1=-2, \lambda_2=-3$ $\lambda_1\neq\lambda_2$

general solution: $y=C_1e^{-2t}+C_2e^{-3t}$

Ex: Find solution to IVP $y''+5y'+6y=0$, $y(0)=2$, $y'(0)=3$

$$y' = -2C_1e^{-2t} - 3C_2e^{-3t}$$

$$y(0) = C_1e^{-2(0)} + C_2e^{-3(0)} = 2 \Rightarrow C_1 + C_2 = 2 \qquad C_1 = 9$$

$$y'(0) = -2C_1e^{-2(0)} - 3C_2e^{-3(0)} = 3 \Rightarrow -2C_1 - 3C_2 = 3 \qquad C_2 = -7$$

$$\boxed{\text{solution: } y(t) = 9e^{-2t} - 7e^{-3t}}$$

Case 2: Complex Roots

If you have complex roots, they must be complex conjugates

$$\lambda = a+ib \quad \text{and} \quad \bar{\lambda} = a-ib$$

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$, solutions are

$$\left\{\boxed{\begin{aligned} z(t) &= e^{(a+ib)t} = e^{at}e^{ibt} = e^{at}(\cos(bt) + i\sin(bt)) \\ \bar{z}(t) &= e^{(a-ib)t} = e^{at}e^{-ibt} = e^{at}(\cos(bt) - i\sin(bt)) \end{aligned}}\right.$$

$$\bar{z}(t) = \underbrace{e^{-2ibt}}_{\text{not constant}} z(t) \Rightarrow$$ $\bar{z}$ & z are linearly independent and form complex-valued fundamental set of solutions

$$\boxed{\text{general solution: } y(t) = C_1 z(t) + C_2\bar{z}(t)}$$

Write $\bar{z}, z$ in real & imaginary parts

$$\boxed{z(t) = y_1(t) + iy_2(t) \qquad \bar{z}(t) = y_1(t) - iy_2(t)}$$

where $\boxed{\begin{aligned} y_1(t) &= e^{at}\cos(bt) \\ y_2(t) &= e^{at}\sin(bt) \end{aligned}}$

Since $z(t) + \bar{z}(t) = y_1(t) + y_2(t) + y_1(t) - y_2(t) = 2y_1(t)$

we set $\boxed{\begin{aligned} y_1(t) &= \tfrac{1}{2}(z(t) + \bar{z}(t)) \\ &\text{and} \\ y_2(t) &= \tfrac{1}{2i}(z(t) - \bar{z}(t)) \end{aligned}}$ linear combination of solutions to $y''+py'+qy=0$

y_1 & y_2 are real-valued solutions to $y''+py'+qy=0$. They are linearly independent and form a real, fundamental set of solutions.

$$\boxed{\text{general solution: } y(t) = A_1y_1(t) - A_2y_2(t) = A_1e^{at}\cos(bt) - A_2e^{at}\sin(bt)}$$

where A_1 & A_2 are constants

Ex: Find general solution of $y''+2y'+2y=0$

Find solution corresponding to $y(0)=2$, $y'(0)=3$

$y=e^{\lambda t}$ $\qquad e^{\lambda t}(\lambda^2+2\lambda+2)=0$

$y'=\lambda e^{\lambda t}$ $\qquad \lambda^2+2\lambda+2=0$

$y''=\lambda^2 e^{\lambda t}$

$\lambda=-1\pm i$

$a\pm ib$, so $a=-1$, $b=1$

$y_1(t)=e^{-t}\cos t$ and $y_2(t)=e^{-t}\sin t$

$\boxed{\text{general solution: } y(t)=C_1e^{-t}\cos t+C_2e^{-t}\sin t=e^{-t}(C_1\cos t+C_2\sin t)}$

$y'(t)=-e^{-t}(C_1\cos t+C_2\sin t)+e^{-t}(-C_1\sin t+C_2\cos t)$

$y(0)=2$, $y'(0)=3$ $\Rightarrow$ $C_1=2$, $-C_1+C_2=3$ $\Rightarrow$ $C_1=2$, $C_2=5$ $\qquad \boxed{y(t)=e^{-t}(2\cos t+5\sin t)}$

- $e^{at}=e^{-t}$
  - since $a=-1<0$, solution is decaying
  - $\sin t$ $\cos t$ causes oscillations

Case 3: Repeat Roots

If $\lambda^2+p\lambda+q=0$ has repeat roots, discriminant $\sqrt{p^2-4q}=0$

$\lambda=-\frac{p\pm\sqrt{p^2-4q}}{2}=-\frac{p}{2}$, $\lambda_1=-\frac{p}{2}$ $\qquad q=\frac{p^2}{4}$

Issue: both roots give same solution: $\boxed{y_1=e^{\lambda_1 t}=e^{-\frac{p}{2}}}$

We need a 2^{nd} solution that's not a constant multiple of y

$0=y''+py'+qy=y''+py'+\frac{p^2}{4}y$

Method By D'Alembert:

Refer to Fig 1

Refer to Fig 1

Prop 3.18 If $\lambda^2+p\lambda+q=0$ has only one double root λ_1, the general solution to $y''+py'+qy=0$ is

$$\boxed{y(t)=C_1e^{\lambda_1 t}+C_2te^{\lambda_1 t}=(C_1+C_2t)e^{\lambda_1 t}}$$

C_1 & C_2 are arbitrary constants

Ex: Find solution of $y''-y'+0.25y=0$, $y(0)=2$, $y'(0)=\frac{1}{3}$

$y=e^{\lambda t}$

$y'=\lambda e^{\lambda t}$

$y''=\lambda^2e^{\lambda t}$

$$\lambda^2-\lambda+\frac{1}{4}=0$$

$$(\lambda-\tfrac{1}{2})^2=0$$

$\lambda=\frac{1}{2}$ repeated / double root

$\underbrace{e^{\frac{1}{2}t}}_{y_1}$ & $\underbrace{te^{\frac{1}{2}t}}_{y_2}$ form fundamental set of solutions to $y''-y'+\frac{1}{4}y=0$

$$y(t)=C_1e^{\frac{t}{2}}+C_2te^{\frac{t}{2}}$$

$$y'(t)=\frac{C_1}{2}e^{\frac{t}{2}}+C_2e^{\frac{t}{2}}+\frac{C_2}{2}te^{\frac{t}{2}}$$

$y(0)=2$, $y'(0)=\frac{1}{3}$ $\Rightarrow$ $C_1=2$, $\frac{1}{2}C_1+C_2=\frac{1}{3}$ $\Rightarrow$ $C_1=2$, $C_2=-\frac{2}{3}$ So $\boxed{y(t)=2e^{\frac{t}{2}}-\frac{2}{3}te^{\frac{t}{2}}}$