10/3/22

Homogeneous & Inhomogeneous Equations

Solutions to homogeneous equation: + Alternative Solution Method

$$\frac{dx}{dt} = a(t)x \Rightarrow \frac{dx}{x} = a(t)dt$$

$$\int \frac{dx}{x} = \int a(t)dt \Rightarrow \ln|x| = \int a(t)dt + C$$

$$|x| = e^{c} e^{\int a(t)dt}$$

general solution: $\boxed{x(t) = Ae^{\int a(t)dt},\ A \text{ constant},\ A \in \mathbb{R}}$

Solutions to inhomogeneous equations

ex: Newton's Law of Cooling

T = object temperature
A = ambient temp.
t = time
k = proportionality constant

$$T' = -k(T-A)$$

$$T' = -kT + kA \quad \text{(linear, inhomogeneous)}$$

$$T' + kT = kA$$

multiply by $e^{kt} > 0$

$$e^{kt}T' + ke^{kt}T = kAe^{kt}$$

LHS $(e^{kt}T)' = e^{kt}T' + ke^{kt}T$

$$(e^{kt}T)' = kAe^{kt}$$

$$\int (e^{kt}T)'dt = \int kAe^{kt}dt$$

$$e^{kt}T = Ae^{kt} + C$$

$$\boxed{T(t) = A + Ce^{-kt}}$$ General solution

General inhomogeneous linear equation:

$$x' = ax + f \qquad a(t), f(t)$$

$$x' - ax = f$$

multiply by integrating factor $w(t)$ so LHS is derivative of a product: $(wx)' = w(x' - ax)$

$$w(x' - ax) = wf$$

if w is found, will see $w \neq 0$

$$(wx)' = wf$$

$w'x + wx' = wx' - awx$

$$\int (wx)'dt = \int w(t)f(t)dt$$

$w'x = -awx$

$$w(t)x(t) = \int w(t)f(t)dt + C$$

so $w' = -aw$: same steps as linear homogeneous eqn.

$w(t) = e^{-\int a(t)dt}$

$$\boxed{x(t) = \frac{1}{w(t)}\int w(t)f(t)dt + \frac{C}{w(t)}}$$

$u(t) = e^{-\int a(t)\,dt}$

needs only one particular solution, don't need constant Choose $A=1$

General Method for solving Linear Equations:

$$x' = ax + f$$

1) Rewrite: $\boxed{x' - ax = f}$

2) Multiply by integrating factor $\boxed{u(t) = e^{-\int a(t)\,dt}}$

and get $u(x' - ax) = uf$

$(ux)' = uf$

once $u(t)$ is found, CHECK $(ux)' = u(x' - ax)$

3) Integrate $u(t)\,x(t) = \int u(t) f(t)\,dt + C$

4) Solve for $x(t)$: $\boxed{x(t) = \frac{1}{u(t)} \int u(t) f(t)\,dt + \frac{C}{u(t)}}$

ex: Find general solution to

$x' = x + e^{-t}$ we know $a(t) = 1$, $f(t) = e^{-t}$ from $x' - x = e^{-t}$:

$u(t) = e^{-\int a(t)\,dt} = e^{-\int 1\,dt} = e^{-t}$

$x' - x = e^{-t}$

multiply by e^{-t}

$e^{-t}x' - e^{-t}x = e^{-2t}$

Check LHS should be $(ux)'$

$(e^{-t}x)' = e^{-2t}$

$\int (e^{-t}x)'\,dt = \int e^{-2t}\,dt$

$e^{-t}x(t) = -\frac{1}{2}e^{-2t} + C$

$\boxed{x(t) = -\frac{1}{2}e^{-t} + Ce^{t}}$

ex: Find general solution of $x' = x\sin t + 2te^{-\cos t}$

Find solution satisfying $x(0) = 1$

$a(t) = \sin t$

$f(t) = 2te^{-\cos t}$

$u(t) = e^{-\int a(t)\,dt}$

$= e^{-\int \sin t\,dt}$

$= e^{\cos t} > 0$ for any $t \in \mathbb{R}$

$x' = (\sin t)x + 2te^{-\cos t}$

$x' - (\sin t)x = 2te^{-\cos t}$

$e^{\cos t}x' - \sin t\, e^{\cos t}x = 2t$

$e^{\cos t}(x' - x\sin t) = 2t = (e^{\cos t}x)'$

$\int (e^{\cos t} x)' dt = \int 2t dt$

$e^{\cos t} x = t^2 + C$ $\quad\quad\quad x(0) = 1$

General Solution: $\boxed{x(t) = (t^2 + C)e^{-\cos t}}$ $\quad\quad 1 = Ce^{-1} \Rightarrow C = e$

So the particular solution is $\boxed{x(t) = (t^2 + e)e^{-\cos t}}$

Alternative Solution Method:

ex: Find general solution of $y' = -2y + 3$

Let y_h be a solution to associated homogeneous equation:

$y_h' = -2y_h \rightarrow$ the solution is $y_h = Ce^{-2t}$

Replace constant C with $v = v(t) \leftarrow$ find v

$y(t) = v(t)e^{-2t}$

$y' = -2y + 3 \Rightarrow (v(t)e^{-2t})' = -2(v(t)e^{-2t}) + 3$

$v'e^{-2t} \cancel{-2ve^{-2t}} = \cancel{-2ve^{-2t}} + 3$

$v'e^{-2t} = 3$

$v' = 3e^{2t}$

$\int v' dt = \int 3e^{2t} dt$

$v(t) = \frac{3}{2}e^{2t} + C$

$y(t) = (\frac{3}{2}e^{2t} + C)e^{-2t}$

$\boxed{y(t) = \frac{3}{2} + Ce^{-2t}}$ ← general solution of equation

The general case: $\boxed{y' = a(t)y + f(t)}$

Find particular solution to associated homogeneous equation: $\boxed{y_h' = a(t)y_h}$

the solution is: $\boxed{y_h(t) = e^{\int a(t) dt}}$

where $y_h \neq 0$

If $y(t)$ is any solution to the general case, $y' = a(t)y + f(t)$

we can define $\boxed{v(t) = y(t)/y_h(t) \quad \text{so} \quad y(t) = v(t)y_h(t)}$

This method's called <u>variation of parameters</u> bc $v(t)$ is called the variable parameter

$$y' = a(t)y + f(t) \Rightarrow (v y_h)' = a(v y_h) + f \quad \text{where } y(t) = v(t) y_h(t)$$

sub $y(t)$ into general equation to solve for v

$$v' y_h + v y_h' = a v y_h + f$$

bc y_h is solution to associated homogeneous equation aka $y_h' = a(t) y_h$

$$v' y_h + \cancel{v a y_h} = \cancel{a v y_h} + f$$

$$v' y_h = f$$

$$\boxed{v' = \frac{f}{y_h}}$$

Solving $y' = ay + f$ using Method of Variation of Parameters

1) A particular solution to $y_h' = a y_h$ is $y_h(t) = e^{\int a(t)\,dt}$
2) sub in $y = v y_h$ into $y' = ay + f$ to find v or remember
$$v' = \frac{f}{y_h}$$
3) write general solution: $y(t) = v(t) y_h(t)$

ex: Use variation of parameters to find general solution of $x' = x \tan t + \sin t$ $\quad \cos t \neq 0$

$y' = a(t)y + f(t)$

find particular solution to $x_h' = x_h \tan t$

$$\hookrightarrow x_h(t) = e^{\int a(t)\,dt} = e^{\int \tan t\,dt} \Rightarrow \frac{1}{\cos t}$$

$a(t) = \tan t$

$f(t) = \sin t$

$v = \frac{x}{x_h} \Rightarrow v x_h = x \Rightarrow x = \frac{v}{\cos t}$

$$x' = x \tan t + \sin t \Rightarrow \left(\frac{v}{\cos t}\right)' = \frac{v}{\cos t} \cdot \frac{\sin t}{\cos t} + \sin t$$

$$\Rightarrow \frac{v' \cos t + v \sin t}{\cos^2 t} = \frac{v}{\cos t} \cdot \frac{\sin t}{\cos t} + \sin t$$

$$\Rightarrow \frac{v' \cos t}{\cos^2 t} = \sin t \Rightarrow v' = \sin t \cos t$$

after $\int dt$

$$v(t) = \int \sin t \cos t\,dt = -\frac{\cos^2 t}{2} + C$$

$$\text{so } x(t) = v(t) x_h(t) = \frac{-\frac{\cos^2 t}{2} + C}{\cos t} \boxed{= -\frac{\cos t}{2} + \frac{C}{\cos t}}$$