10/19/22

2^{nd} ORDER EQUATIONS

Ch. 4. 2^{nd} Order Equations

4.1

A 2^{nd} order DE involves ind. variable t and unknown function y along with its 1^{st} and 2^{nd} derivatives

If we can write it in normal form,

$$y'' = f(t, y, y') \qquad \text{(solve for } 2^{nd} \text{ derivative)}$$

$\boxed{\text{The solution is } y(t) \text{ itself}}$

Ex: Newton's 2^{nd} Law $\quad F = ma$

If $y(t)$ is displacement, $v(t) = y'(t)$ and $a(t) = y''(t)$

So:

$$m \frac{d^2y}{dy^2} = F(t, y, \frac{dy}{dt})$$

---

linear equations have form

$$y'' + p(t)y' + q(t)y = g(t)$$

coefficients $p, q, \& g$ are functions of t

$g(t)$ is called the forcing term

if forcing term $= 0$, then $y'' + p(t)y' + q(t)y = 0$ is homogeneous

Existence & Uniqueness (2^{nd} order)

Thm 1.17 Suppose $p(t), q(t), \& g(t)$ are continuous on interval (α, β). Let $\alpha < t_0 < \beta$. For any real #'s y_0 & y_1, there is only one function $y(t)$ defined on (α, β) which is a solution to

$$\boxed{\begin{array}{c} y'' + p(t)y' + q(t)y = g(t) \quad \text{for } \alpha < t < \beta \\ \text{And satisfies initial conditions } y(t_0) = y_0 \text{ and } y'(t_0) = y_1 \end{array}}$$

Ex: $y'' - y = 0$ (linear & homogeneous)

Are $y_1(t) = e^t$ & $y_2(t) = e^{-t}$ solutions?

$y_1'(t) = e^t$ $\quad y_2'(t) = -e^{-t}$

$y_1''(t) = e^t$ $\quad y_2''(t) = e^{-t}$

$y_1'' - y_1 = e^t - e^t = 0$
So y_1 is a solution

$y_2'' - y_2 = e^{-t} - e^{-t} = 0$
So y_2 is a solution

Ex: Are $2e^t$ and $5e^{-t}$ solutions? $\quad C_1, C_2$ constant

$$y = C_1 y_1(t) + C_2 y_2(t) = C_1 e^t + C_2 e^{-t}$$

$y' = C_1 e^t - C_2 e^{-t}$

$y'' = C_1 e^t + C_2 e^{-t}$ $\Rightarrow y'' - y' = (C_1 e^t - C_2 e^{-t}) - (C_1 e^t + C_2 e^{-t}) = 0$

$\boxed{\text{Since } y'' - y' = 0, \ y \text{ is solution}}$

$\boxed{y(t) = C_1 e^t + C_2 e^{-t} \text{ is the general solution}}$

General Solutions:

Prop 1.18: Suppose y_1 & y_2 are both solutions to homogeneous linear equation:

$$y'' + p(t)y' + q(t)y = 0,$$

then $\boxed{y = C_1 y_1 + C_2 y_2}$ is also a solution for any constants C_1 and C_2.

Can plug in y, y', y'' into $y'' + p(t)y' + q(t)y = 0$ to prove

if u and v are functions of y_1 & y_2 respectively

Def: A linear combination of 2 functions u and v is any function of form $w = Au + Bv$ where A & B are constants. BUT since Prop 1.18 says ANY linear combination, $w = Au + Bv$ is also a solution of $y = C_1 y_1 + C_2 y_2$

Def 1.22 2 functions u and v are said to be linearly independent on the interval (α, B) if neither is a constant multiple of the other on that interval. Otherwise, if one is a constant multiple of the other on (α, B), they're said to be linearly dependent

Functions $w(t) = t$ and $v(t) = t^2$ are linearly independant on $\mathbb{R}$ bc

$v(t) = t\, w(t)$ and $w(t) = \frac{1}{t} v(t)$

$t, \frac{1}{t}$ are not constant factors!

but if $w(t) = 4\cos 3t$ & $v(t) = 2\cos 3t$,

$w(t) = 2v(t)$ and $v(t) = \frac{1}{2} w(t)$, 2 & $\frac{1}{2}$ are constant and therefore $v(t)$ & $w(t)$ are linearly dependent

Thm 1.23: Suppose y_1 & y_2 are linearly independent solutions to equation $y'' + p(t)y' + q(t)y = 0$.

$\boxed{\text{The general solution to it is } y_1 = C_1 y_1 + C_2 y_2 \text{ where } C_1 \,\&\, C_2 \text{ are constants}}$

2 linearly independent solutions form a <u>fundamental set of solutions</u>

<u>Note</u>: If we have 2 linearly independent solutions to $y'' + p(t)y' + q(t)y = 0$, we can form the general solution.

Thm 1.23:

Ex: $y'' - y = 0$ shows $y_1(t) = e^{t}$ and $y_2(t) = e^{-t}$ are solutions
and $y_1(t) = e^{2t} y_2(t)$ and $y_2(t) = e^{-2t} y_1(t)$

e^{2t}, e^{-2t} not constant
so y_1 & y_2 are linearly
independent.

In cases where it's difficult to tell linear ind. or dependent, use
Wronskian.

Wronskian of 2 functions u & v defined by

$$\boxed{W(t) = \det \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} = u(t)v'(t) - u'(t)v(t)}$$

Prop 1.26 Suppose functions u, v solutions to linear, homogeneous equation
$y'' + p(t)y' + q(t)y = 0$ in interval (α, β).

Then the Wronskian of u and v is either identically equal to 0 on (α, β) or it's never equal to 0 there.

Ex: $y_1(t) = e^{t}$, $y_2(t) = e^{-t}$ $\quad -\infty < t < \infty$

$$W(t) = \begin{vmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{vmatrix} = -1 - 1 = -2$$

$\Rightarrow$ wronskian never 0

IF $W(t_0) \neq 0$, THEN $y_1(t)$ & $y_2(t)$ ARE LINEARLY IND.

Prop 1.27: Suppose u, v are solutions to linear, homogeneous equation
$y'' + p(t)y' + q(t)y = 0$ on interval (α, β).

Then u and v are linearly dependent only if their Wronskian is identically 0 in (α, β)

Prop 1.29 Suppose functions u & v are solutions to linear, homogeneous equation $y'' + p(t)y' + q(t)y = 0$ in interval (α, β)

- If $W(t_0) \neq 0$ for some t_0 in interval (α, β), then u, v are linearly independent in (α, β)
- If u, v are linearly independent in (α, β), then $W(t)$ never vanishes in (α, β)

Ex: $\underbrace{y'' - y = 0 \text{ but w/ initial conditions } y(0) = 2 \text{ \& } y'(0) = -1}_{\text{IVP}}$

general solution $y(t) = C_1 e^{t} + C_2 e^{-t}$

$$y'(t) = C_1 e^{t} - C_2 e^{-t}$$

$$\text{ICs} = \begin{cases} y(0) = 2 \\ y'(0) = -1 \end{cases} \rightarrow \begin{cases} 2 = C_1 e^{0} + C_2 e^{0} \\ -1 = C_1 e^{0} - C_2 e^{0} \end{cases}$$

$$\begin{cases} 2 = C_1 + C_2 \\ -1 = C_1 - C_2 \end{cases}$$

$$\Downarrow$$

$C_1 = 0.5$
$C_2 = 1.5$

$$\boxed{\text{Solution: } y(t) = \frac{1}{2} e^{t} + \frac{3}{2} e^{-t}}$$