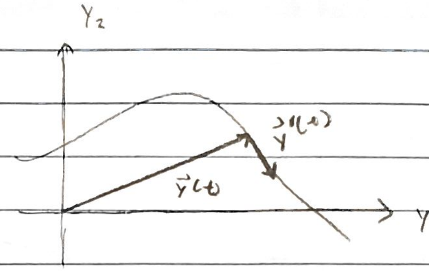
9.2 Planar Systems



$y' = Ay$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

Recall, look for solutions $\vec{y}(t) = e^{\lambda t}\vec{v}$

find λ: $\det(A - \lambda I) = 0 = \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$

$$= \lambda^2 - \underbrace{(a_{11}+a_{22})}_{T}\lambda + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{D}$$

denote $D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$
$T = tr(A) = a_{11} + a_{22}$

Definition: trace of matrix A ($tr(A)$) is sum of diagonal elements of A

characteristic equation planar system: $\lambda^2 - T\lambda + D = 0$
Eigenvalues of A: $\lambda = \frac{(T \pm \sqrt{T^2-4D})}{2}$

3 cases
1) 2 distinct real roots (when $T^2 - 4D > 0$)
2) 2 complex conjugates (when $T^2 - 4D < 0$)
3) 1 real root multiplicity 2 (when $T^2 - 4D = 0$)

Prop 2.4: Suppose λ_1 and λ_2 eigenvalues of an $n \times n$ matrix A. Suppose $\vec{v}_1 \neq \vec{0}$ is an eigenvector for λ_1 and $\vec{v}_2 = \vec{0}$ is an eigenvector for λ_2. If $\lambda_1 \neq \lambda_2$, then $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

Corollary 2.8: Suppose λ_1 & λ_2 are eigenvalues of an $n \times n$ matrix A.
$\vec{v}_1 \neq 0$ is an eigenvalue for λ_1
$\vec{v}_2 \neq 0$ is an eigenvalue for λ_2
If $\lambda_1 \neq \lambda_2$, then the solutions $\vec{y}_1(t) = e^{\lambda_1 t}\vec{v}_1$ and $\vec{y}_2(t) = e^{\lambda_2 t}\vec{v}_2$ are linearly independent on $\mathbb{R}$.

Case 1: Distinct, Real Eigenvalues ($T^2 - 4D > 0$)

$\boxed{\lambda_1 = \frac{T - \sqrt{T^2-4D}}{2} \qquad \lambda_2 = \frac{T + \sqrt{T^2-4D}}{2}}$

$\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < \lambda_2$

Let $\vec{v}_1$ and $\vec{v}_2$ be associated eigenvectors. Then $\vec{y}_1(t) = e^{\lambda_1 t}\vec{v}_1$ & $\vec{y}_2(t) = e^{\lambda_2 t}\vec{v}_2$ are 2 solutions to $\vec{y}' = A\vec{y}$ and they are linearly ind. and form fundamental set of solutions by Corollary 2.8.

Thm 2.9 A is a 2×2 matrix w/ real eigenvalues $\lambda_1 \neq \lambda_2$. Suppose $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors associated w/ the eigenvalues. The general solution to $\vec{y}' = A\vec{y}$ is $\boxed{\vec{y}(t) = C_1\vec{y}_1(t) + C_2\vec{y}_2(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2}$ where C_1 & C_2 are constants

Ex: Find the solution to IVP

$$\vec{x}' = A\vec{x}, \quad \vec{x}(0) = \begin{pmatrix}1\\1\end{pmatrix} \quad A = \begin{pmatrix}4 & 3\\3 & -4\end{pmatrix}$$

$$\vec{x}' = \begin{pmatrix}4 & 3\\3 & -4\end{pmatrix}\vec{x}$$

eigenvalues of A:

$$|A-\lambda I| = \begin{vmatrix}4-\lambda & 3\\3 & -4-\lambda\end{vmatrix} = (4-\lambda)(-4-\lambda) - 9 = \lambda^2 - 16 - 9 = \lambda^2 - 25$$

characteristic equation $\lambda^2 - 25 = 0$

$\lambda_1 = -5, \lambda = 5$

↑ real eigenvalues and $\lambda_1 \neq \lambda_2$, use thm 2.9

eigenvectors:

$\lambda_1 = 5 \quad (A-5I)\vec{v} = \begin{pmatrix}-1 & 3\\3 & -9\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow a = 3b \quad \vec{v}_1 = \begin{pmatrix}a\\b\end{pmatrix} = b\begin{pmatrix}3\\1\end{pmatrix}$

$\boxed{\vec{v}_1 = \begin{pmatrix}3\\1\end{pmatrix}}$

$\lambda_2 = -5 \quad (A+5I)\vec{v} = \begin{pmatrix}9 & 3\\3 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow b = -3a \quad \vec{v}_2 = \begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\-3\end{pmatrix}$

$\boxed{\vec{v}_2 = \begin{pmatrix}1\\-3\end{pmatrix}}$

general solution: $x(t) = C_1 x_1(t) + C_2 x_2(t)$

$$\boxed{x(t) = C_1 e^{5t}\begin{pmatrix}3\\1\end{pmatrix} + C_2 e^{-5t}\begin{pmatrix}1\\-3\end{pmatrix}}$$

Initial Condition:

$\vec{x}(0) = \begin{pmatrix}1\\1\end{pmatrix}$

$$C_1 e^{5\cdot 0}\begin{pmatrix}3\\1\end{pmatrix} + C_2 e^{-5\cdot 0}\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}3C_1 + C_2\\C_1 - 3C_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$\vdots$

$C_1 = \frac{2}{5} \quad C_2 = -\frac{1}{5}$

$$\boxed{\vec{x}(t) = \frac{2}{5}e^{5t}\begin{pmatrix}3\\1\end{pmatrix} - \frac{1}{5}e^{-5t}\begin{pmatrix}1\\-3\end{pmatrix}}$$

Case 2: Complex Eigenvalues $(T^2 - 4D < 0)$

roots of characteristic equations are complex conjugates:

$$\lambda = \frac{T + i\sqrt{4D - T^2}}{2}, \quad \bar{\lambda} = \frac{T - i\sqrt{4D - T^2}}{2}$$

Ex: Find eigenvalues & eigenvectors for $A = \begin{pmatrix}0 & 1\\-2 & 2\end{pmatrix}$

eigenvalues:

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda)+2 = \lambda^2 - 2\lambda + 2$$

$$\lambda = 1 \pm i$$

$\lambda = 1+i$ $\qquad \bar{\lambda} = 1-i$

eigenvectors:

$$(A-\lambda I)\vec{v} = (A-(1+i)I)\vec{v} = \begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

$(-1-i)a + b = 0$

$(-2a + (1-i)b = 0) \ \frac{1+i}{2}$

$b = (1+i)a$

$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ (1+i)a \end{pmatrix} = a\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

$(-1-i)a + b = 0$

$(-1-i)a + b = 0$ $\qquad$ if $a = 1$ $\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}}$

If $\vec{v}$ is an eigenvector to $\lambda = 1+i$, then

$$A\vec{v} = \lambda\vec{v}$$

take complex conjugate: $\overline{A\vec{v}} = \bar{\lambda}\bar{\vec{v}}$ $\qquad$ bc A is real matrix $\bar{A} = A$

$A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$

Complex conjugate of $\vec{v}$ is eigenvector associated to $\bar{\lambda} = 1-i$

So $\bar{\vec{v}}_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$ is an eigenvector associated to $\bar{\lambda} = 1-i$

consider $\vec{y}' = A\vec{y}$, we get solutions

$$\boxed{\vec{z}(t) = e^{\lambda t}\vec{v}_1 = e^{(1+i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}}$$

We need complex conjugate

$$\boxed{\bar{\vec{z}}(t) = e^{\bar{\lambda} t}\bar{\vec{v}}_1 = e^{(1-i)t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}}$$

Thm 2.21: A is a real 2×2 matrix w/ complex conjugate eigenvalues λ and $\bar{\lambda}$. If $\vec{w}$ is an eigenvector associated w/ λ, the general solution to $\vec{y}' = A\vec{y}$ is

$$\boxed{\vec{y}(t) = C_1 e^{\lambda t}\vec{w} + C_2 e^{\bar{\lambda} t}\bar{\vec{w}}}$$

C_1 & C_2 constants

Prop 2.22: A is an $n \times n$ matrix w/ real components. and $\vec{z}(t) = \vec{x}(t) + i\vec{y}(t)$ is a solution to system $\vec{z}' = A\vec{z}$

a) The complex conjugate $\bar{\vec{z}} = \vec{x} - i\vec{y}$ is also a solution to $\vec{z}' = A\vec{z}$

b) $\vec{x}(t)$ and $\vec{y}(t)$ are also solutions to $\vec{z}' = A\vec{z}$. If $\vec{z}$ and $\bar{\vec{z}}$ are linearly ind. so is $\vec{x}(t)$ & $\vec{y}(t)$

$\vec{x}(t) = Re(\vec{z}(t))$

$\vec{y}(t) = Im(\vec{z}(t))$

Thm 2.25: Suppose A is a 2×2 matrix w/ complex eigenvalue $\lambda = \alpha + i\beta$ and associated eigenvector $\vec{\omega} = \vec{v}_1 + i\vec{v}_2$. The general solution is:

$$\boxed{\vec{y}(t) = C_1 e^{\alpha t}(\vec{v}_1 \cos\beta t - \vec{v}_2 \sin\beta t) + C_2 e^{\alpha t}(\vec{v}_1 \sin\beta t + \vec{v}_2 \cos\beta t)}$$

C_1 & C_2 are constants

Ex: Find fundamental set of real solutions for $\vec{y}' = A\vec{y}$ where $A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$ and then find general solution.

From previous example we know

$$\vec{z}(t) = e^{\lambda t}\vec{\omega} = e^{(1+i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$\bar{\vec{z}}(t) = e^{\bar{\lambda} t}\bar{\vec{\omega}} = e^{(1-i)t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

and $\lambda \neq \bar{\lambda}$ means $\vec{z}$ and $\bar{\vec{z}}$ are linearly independent. By prop 2.22, the real and imaginary parts of $\vec{z}$ are linearly independent solutions to $\vec{y}' = A\vec{y}$.

$$\vec{z}(t) = e^{(1+i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = e^t(\cos t + i\sin t)\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$= \underbrace{e^t\begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix}}_{Re(\vec{z}(t))} + i\underbrace{e^t\begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix}}_{Im(\vec{z}(t))}$$

$y_1(t) = e^t\begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix}$

$y_2(t) = e^t\begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix}$

$$\boxed{\text{General solution: } \vec{y}(t) = C_1 e^t\begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + C_2 e^t\begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix}}$$

Eigenspace of λ: subspace consisting of all eigenvectors for a given eigenvalue λ.

$\boxed{\text{dimension of subspace } V \text{ is \# of elements in any basis for } V}$

Case 3: One real eigenvalue of multiplicity 2 $(T^2-4D=0)$

characteristic polynomial:

$$p(\lambda)=\lambda^2-T\lambda+D=\lambda^2-T\lambda+\frac{T^2}{4}=\left(\lambda-\frac{T}{2}\right)^2$$

so $\lambda_1=\frac{T}{2}$ one eigenvalue, multiplicity 2

2 subcases:

3a) One real eigenvalue of multiplicity 2 - easy case

If eigenspace of λ_1 has dimension 2, every nonzero vector in $\mathbb{R}^2$ is eigenvector

λ_1 is the eigenvector

$$\boxed{\text{General solution: } \vec{x}(t)=C_1e^{\lambda_1 t}\vec{v}_1+C_2e^{\lambda_1 t}\vec{v}_2}$$

$\vec{v}_1$ & $\vec{v}_2$ linearly independent corresponding to λ_1, C_1 & C_2 constants

Ex: $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $\vec{e}_1=\begin{pmatrix}1\\0\end{pmatrix}$ $\vec{e}_2=\begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}\lambda_1\\0\end{pmatrix}=\lambda_1\vec{e}_1=A\vec{e}_1=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}a\\c\end{pmatrix}$ $\begin{pmatrix}a\\c\end{pmatrix}=\begin{pmatrix}\lambda_1\\0\end{pmatrix}$

$A=\begin{pmatrix}\lambda_1&0\\0&\lambda_1\end{pmatrix}=\lambda_1 I$

$\begin{pmatrix}0\\\lambda_1\end{pmatrix}=\lambda_1\vec{e}_2=A\vec{e}_2=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}b\\d\end{pmatrix}$ $\begin{pmatrix}b\\d\end{pmatrix}=\begin{pmatrix}0\\\lambda_1\end{pmatrix}$

system $\vec{x}'=A\vec{x}$

$\vec{x}'=\lambda_1 I\vec{x}$

$\vec{x}'=\lambda_1\vec{x} \iff \begin{pmatrix}x_1'\\x_2'\end{pmatrix}=\begin{pmatrix}\lambda_1x_1\\\lambda_2x_2\end{pmatrix}\iff \begin{matrix}x_1'=\lambda_1x_1\\x_2'=\lambda_1x_2\end{matrix}$

For any $\vec{v}$, initial condition $\vec{x}(0)=\vec{v}$ where $\vec{x}(t)=e^{\lambda_1 t}\vec{v}$

3b) one real eigenvalue of multiplicity 2 - interesting case

If eigenspace of λ_1 has dimension 1, then can only find one linearly dependent exponential solution. You need to find 2^{nd} solution that is not a multiple of $x_1(t)$

$\vec{x}'=A\vec{x}$

try to find solution $\vec{x}(t)=e^{\lambda_1 t}(\vec{v}_2+t\vec{v}_1)$

where $\vec{v}_1$ and $\vec{v}_2$ are undetermined vectors

LHS: $\vec{x}'(t)=\lambda_1e^{\lambda_1 t}(\vec{v}_2+t\vec{v}_1)+e^{\lambda_1 t}\vec{v}_1$

$=e^{\lambda_1 t}[(\lambda_1\vec{v}_2+\vec{v}_1)+\lambda_1t\vec{v}_1]$

RHS: $A\vec{x}=A(e^{\lambda_1 t}(\vec{v}_2+t\vec{v}_1))$

$=e^{\lambda_1 t}(A\vec{v}_2+tA\vec{v}_1)$

$\vec{x}(t)=e^{\lambda_1 t}(\vec{v}_2+\vec{v}_1)$ will be a solution if LHS = RHS

$$\underline{(\lambda_1 \vec{v}_2 + \vec{v}_1) + \lambda_1 t \vec{v}_1 = A\vec{v}_2 + tA\vec{v}_1}$$

$A\vec{v}_2 = \lambda_1 \vec{v}_2 + \vec{v}_1$

$A\vec{v}_1 = \lambda_1 \vec{v}_1 \longrightarrow$ eigenvalue of A w/ eigenvector $\vec{v}_1$

$(A-\lambda_1 I)\vec{v}_2 = \vec{v}_1$

Finding $\vec{v}_2$: For any vector $\vec{w} \in \mathbb{R}^2$

$$\boxed{(A-\lambda_1 I)[(A-\lambda_1 I)\vec{w}] = (A-\lambda_1 I)^2 \vec{w} = \vec{0}}$$

so $(A-\lambda I)\vec{w}$ is in the null space of $A-\lambda_1 I$ for any $\vec{w} \in \mathbb{R}^2$

If $(A-\lambda I)\vec{w} \neq \vec{0}$, then $(A-\lambda_1 I)\vec{w}$ is an eigenvector associated to eigenvalue λ_1.

If eigenspace for λ_1 has dimension 1 and $\vec{v}_1 \neq 0$ is an eigenvector, then for any $\vec{w} \in \mathbb{R}^2$ there is a constant a such that

$$\boxed{(A-\lambda_1 I)\vec{w} = a\vec{v}_1}$$

$\longleftarrow$ multiple of eigenvector $\vec{v}_1$

If $\vec{w}$ is not a multiple of $\vec{v}_1$, then $\vec{w}$ is not an eigenvector, so $(A-\lambda_1 I)\vec{w} \neq \vec{0}$ set $\vec{v}_2 = \frac{1}{a}\vec{w}$, then

$$(A-\lambda_1 I)\vec{v}_2 = \frac{1}{a}(A-\lambda_1 I)\vec{w} = \vec{v}_1$$

This shows we can find $\vec{v}_2$ satisfying $(A-\lambda_1 I)\vec{v}_2 = \vec{v}_1$ w/ $\vec{v}_1$ & $\vec{v}_2$. We set 2nd solution: $\boxed{\vec{x}(t) = e^{\lambda_1 t}(\vec{v}_2 + t\vec{v}_1)}$

thm 2.40 A is 2×2 matrix w/ one eigenvalue λ of multiplicity 2. Eigenspace λ has dimension 1. $\vec{v}_1$ is a nonzero eigenvector and choose $\vec{v}_2$ such that $(A-\lambda I)\vec{v}_2 = \vec{v}_1$. Then $\boxed{\vec{x}_1(t) = e^{\lambda t}\vec{v}_1 \quad \vec{x}_2(t) = e^{\lambda t}(\vec{v}_2 + t\vec{v}_1)}$ form a fundamental set to system $\vec{x}' = A\vec{x}$

$$\boxed{\text{general solution: } \vec{x}(t) = C_1 \vec{x}_1(t) + C_2 \vec{x}_2(t)}$$

Ex: Find general solution to $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$, then find solution to $\vec{x}' = A\vec{x}$ w/ initial condition $\vec{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

eigenvalues: $|A-\lambda I| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 \qquad \lambda_1 = -2$

$(A-\lambda_1 I)\vec{v} = \vec{0}$

$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 = x_2, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1$

Dimension of eigenspace of λ_1 is 1, so we only have 1 linearly independent exponential solution. Any exponential solution must be a constant multiple of $\vec{x}_1(t)$

To find 2nd solution $\vec{x}_2(t)$, need to find $\vec{v}_2$ satisfying

$$(A-\lambda_1 I)\vec{v}_2 = \vec{v}_1$$

start w/ $\vec{w}$ that isn't a multiple of $\vec{v}_1$, pick one w/ zeroes to make computation easy.

$$\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (A-\lambda_1 I)\vec{w} = (A+2I)\vec{w}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \underbrace{-1}_{a=-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_2 = \frac{1}{a}\vec{w} = -1\vec{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

So:

$$\boxed{\begin{aligned} &\vec{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &\vec{x}_2(t) = e^{-2t}\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \\ &\vec{x}(t) = C_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + C_2 e^{-2t} t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}}$$

$$\vec{x}(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$C_1 = 5 \qquad , \; C_2 = 4 \qquad \boxed{\vec{x}(t) = e^{-2t}\begin{pmatrix} 5+4t \\ 1+4t \end{pmatrix}}$$

$$C_1 - C_2 = 1$$