11/4/22

# 4.6 Variation of Parameters

Applying method to $2^{nd}$ order DE:

$$y'' + p(t)y' + q(t)y = g(t)$$

$p(t)$ & $g(t)$ are functions of $t$.

For this to work, need fundamental set of solutions $y_1$ & $y_2$ to associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

$\boxed{\text{General solution is } y_h = C_1 y_1 + C_2 y_2}$ $C_1, C_2$ constant

Replace $C_1$ & $C_2$ w/ unknown functions $v_1(t)$ & $v_2(t)$

$y_p = v_1 y_1 + v_2 y_2$

$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$

$= \underbrace{(v_1' y_1 + v_2' y_2)} + (v_1 y_1' + v_2 y_2')$

set to 0 so we can impose constraint $v_1' y_1 + v_2' y_2 = 0$ and obtain simplified expression:

$\boxed{y_p' = v_1 y_1' + v_2 y_2'}$

After substituting $y_p, y_p', y_p''$, we expect 1 equation on $v_1, v_2$ + their $1^{st}$ 2 derivatives.

1 equation and 2 unknowns leads to many choices of $v_1$ & $v_2$ that works. We solve this by imposing condition so we set 2 equations and 2 unknowns.

$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$

$= v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2''$

SUB INTO $y'' + p(t)y' + q(t)y = g(t)$

$= (v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2'') + p(t)(v_1 y_1' + v_2 y_2') + q(t)(v_1 y_1 + v_2 y_2)$

$= v_1 \underline{(y_1'' + p(t) y_1' + q(t) y_1)} + v_2 \underline{(y_2'' + p(t) y_2' + q(t) y_2)} + v_1' y_1' + v_2' y_2'$

0

Because $y_1$ & $y_2$ are solutions, so

$\boxed{g(t) = v_1' y_1' + v_2' y_2'}$

$v_1' y_1 + v_2' y_2 = 0$

$v_1' y_1' + v_2' y_2' = g(t)$

$\Rightarrow \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$ can ~~solve if~~ $\cancel{\det}\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \neq 0$

~~Recall Wronskian~~ $\cancel{W(t)} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

$W(t) \neq 0$ bc $y_1$ & $y_2$ are linearly independent bc $y_1$ & $y_2$ form fundamental set of solutions.

$\therefore \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ and can be used to solve for $v_1$, $v_2$

$$v_1' = \frac{-y_2 g}{y_1 y_2' - y_1' y_2} \quad \& \quad v_2' = \frac{y_1 g}{y_1 y_2' - y_1' y_2}$$

~~Integrate:~~ $v_1(t) = \int \frac{-y_2(t) g(t)}{y_1(t) y_2'(t) - y_1'(t) y_2(t)} dt \quad v_2(t) = \int \frac{y_1(t) g(t)}{y_1(t) y_2'(t) - y_1'(t) y_2(t)} dt$

Solution: $\boxed{y_p = v_1 y_1 + v_2 y_2}$

---

To use the method,

1) Find fundamental set of solutions $y_1$, $y_2$ to $y'' + py' + qy = 0$
2) Form $y_p = v_1 y_1 + v_2 y_2$, $\left.\begin{array}{l} v_1 = v_1(t) \\ v_2 = v_2(t) \end{array}\right]$ functions TBD
3) Find $v_1$ & $v_2$

or REMEMBER FORMULAS

Follow procedure:

i) Differentiate $y_p$:

$$y_p' = \underbrace{(v_1' y_1 + v_2' y_2)}_{\text{set to } 0} + v_1 y_1' + v_2 y_2'$$

$$v_1' y_1 + v_2' y_2 = 0$$

ii) Take 2nd Derivative of $y_p$

Sub $y_p, y_p', y_p''$ into $y'' + py' + qy = g(t)$

Simplify using $y_1, y_2$ solutions to $y'' + py' + qy = 0$

and set $v_1' y_1' + v_2' y_2' = g(t)$

iii) Solve System

Solve $v_1' y_1 + v_2' y_2 = 0$

$v_1' y_1' + v_2' y_2' = g(t)$

for $v_1'$ and $v_2'$

iv) Integrate to Find $v_1$ & $v_2$

↑

exactly what it sounds like

v) Sub $v_1$ & $v_2$ into

$y_p = v_1 y_1 + v_2 y_2$

Success of method depends on being able to find fundamental set of solutions to $y'' + py' + qy = 0$ and compute integrals.

Ex: Find particular solution to $y'' + 4y = 3\csc t$

$g(t) = 3\csc t$ is <u>not</u> one of forcing terms handled by method of undetermined coefficients

Variation of Parameters:

1) $y'' + 4y = 0$

can show $y_1 = \cos 2t$, $y_2 = \sin 2t$ fundamental set of solutions

2) $y_p = v_1 \cos 2t + v_2 \sin 2t$ $\qquad v_1 = v_1(t)$

3) $y_p' = v_1' \cos 2t - 2v_1 \sin 2t + v_2' \sin 2t + 2v_2 \cos 2t$ $\qquad v_2 = v_2(t)$

$= \underbrace{(v_1' \cos 2t + v_2' \sin t)}_{\text{set } = 0} + (-2v_1 \sin 2t + 2v_2 \cos 2t)$

$y_p' = -2v_1 \sin 2t + 2v_2 \cos 2t$

$y_p'' = -2v_1' \sin 2t - 4v_1 \cos 2t + 2v_2' \cos 2t - 4v_2 \sin 2t$

Substitute: $(-2v_1' \sin 2t - 4v_1 \cancel{\cos 2t} + 2v_2' \cos 2t - 4v_2 \cancel{\sin 2t}) + 4(v_1 \cancel{\cos 2t} + v_2 \cancel{\sin 2t}) = 3\csc t$

$-2v_1' \sin 2t + 2v_2' \cos 2t = 3\csc t$

$v_1' \cos 2t + v_2' \sin 2t = 0$

$-2v_1' \sin 2t + 2v_2' \cos 2t = 3\csc t$

$\Downarrow$

$v_1' = -3\cos t \qquad v_2' = \frac{3}{2}\csc t - 3\sin t$

$v_1(t) = -3\sin t + C_1 \qquad v_2(t) = \frac{3}{2}\ln|\csc t - \cot t| + 3\cos t + C_2$

If you're looking for a particular solution, take $C_1 = C_2 = 0$

4) $\boxed{y_p = v_1 y_1 + v_2 y_2}$

$\boxed{= -3\sin t \cos 2t + \frac{3}{2}\ln|\csc t - \cot t| \sin 2t + 3\cos t \sin 2t}$

$y_h = C_1 y_1 + C_2 y_2$

$\boxed{y(t) = y_p + y_h = -3\sin t \cos 2t + \frac{3}{2}\ln|\csc t - \cot t| \sin 2t + 3\cos t \sin 2t + C_1 \cos 2t + C_2 \sin 2t}$