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2.7 Existence & Uniqueness

Study properties of solutions even in cases when solution isn't known explicitly

1) when can we be sure a solution exists?
2) How many different solutions are there to IVP?

Ex: IVP: $tx' = x + 3t^2$ with $x(0) = 1$

But $t \neq 0$ or else $\frac{x}{0}$ → $x' = \frac{x}{t} + 3t$ ← $x' = f(t, x)$ called normal form!

⇓

$x(t) = 3t^2 + Ct$ Apart from $t=0$, every solution has this form for constant $C \in \mathbb{R}$

- these solutions defined at $t=0$, $x(0)=0$ for any C
- these solutions also solve $tx' = x + 3t^2$ for $t \in (-\infty, \infty)$
- But does not solve IVP $x(0)=1$ because $x(0) = 0 \neq 1$

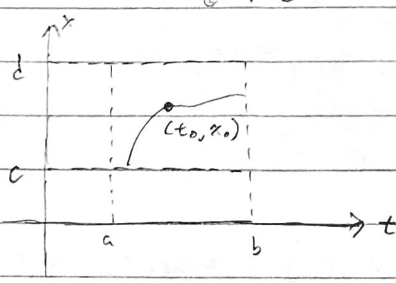
no solution to IVP

Thm: Existence of Solutions

Suppose function $f(t, x)$ is defined and continuous on rectangle R in the tx-plane. Then given any point $(t_0, x_0) \in R$, the IVP

$$x' = f(t, x) \quad \text{and} \quad x(t_0) = x_0$$

has solution $x(t)$ defined in an interval containing t_0.
Furthermore, the solution will be defined until the solution curve $t \to (t, x(t))$ leaves rectangle R.



$R = \{(t, x) \mid a < t < b, \; c < x < d\}$

In previous example, since f is discontinuous at $t=0$, thm doesn't apply in any rectangle including points (t, x) with $t=0$ in it.

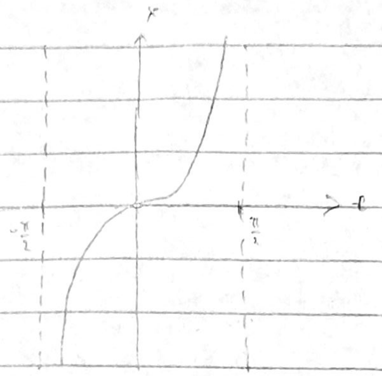
Interval of Existence of a Solution:

Ex: IVP $x' = 1 + x^2$, and $x(0) = 0$

$f(t, x) = 1 + x^2$, f continuous on $-\infty < t < \infty$, $-\infty < x < \infty$ ($a < t < b$, $c < x < d$)

So let $R = \mathbb{R}^2$

but solution to IVP is $x(t) = \tan t$.



x and x' is only defined for $-\frac{\pi}{2} < t < \frac{\pi}{2}$,

so $\boxed{\text{I of E for } x \text{ is } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

note: Even tho $f(t,x)$ is cont. for all of $\mathbb{R}^2$, the I of E is **NOT** $(-\infty, \infty)$ in this case

The solution is defined until solution curve leaves the rectangle R.

- solution curve leaves thru top of R as $t \to \frac{\pi}{2}$ from below, since $x(t) \to \infty$
- solution curve leaves thru bottom of R as $t \to -\frac{\pi}{2}$ from above, since $x(t) \to -\infty$

I of E cannot usually be found from existence thm. Only reliable way is finding explicit solution.

Existence of linear equations: $x' = a(t)x + g(t)$

If $a(t)$ and $g(t)$ are cont. on $b < t < c$, then f continuous on $R = \{(t,x) \mid b < t < c, -\infty < x < \infty\}$

- In this case, stronger thm holds that guarantee solutions exist on the entire interval $b < t < c$

Thm: Suppose $f(t,x)$ and $\frac{\partial f}{\partial x}$ are both continuous on rectangle R in tx-plane and let

$$M = \max_{(t,x) \in R} \left| \frac{\partial f}{\partial x}(t,x) \right|$$

Suppose (t_0, x_0) and (t_0, y_0) are in R and

$$x'(t) = f(t, x(t)) \quad \text{and} \quad x(t_0) = x_0$$
$$y'(t) = f(t, y(t)) \quad \text{and} \quad y(t_0) = y_0$$

Then as long as $(t, x(t))$ & $(t, y(t))$ belong to R,

$$\boxed{\underbrace{|x(t) - y(t)|}_{\text{how different solutions are depends on ...}} \leq \underbrace{|x_0 - y_0|}_{\text{how close initial values are}} e^{\underbrace{M|t - t_0|}_{\text{and how far from initial time } t_0 \text{ we go}}}}$$

$\boxed{\text{If } x_0 = y_0, \text{ then } |x(t) - y(t)| \leq 0 \implies x(t) = y(t)}$

Uniqueness of Solutions: Suppose $f(t,x)$ and $\frac{\partial f}{\partial x}$ are both continuous on rectangle R in the tx-plane. Suppose $(t_0, x_0) \in R$ and that the solutions $x(t)$, $y(t)$ to

$$x' = f(t,x) \quad \text{and} \quad y' = f(t,x)$$

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satisfy $x(t_0) = y(t_0) = x_0$

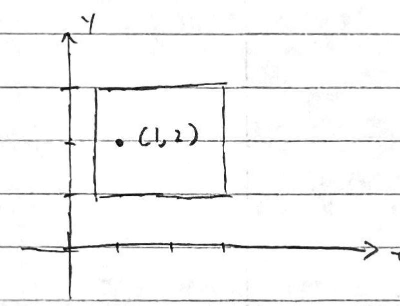
As long as $(t, x(t))$ and $(t, y(t))$ stay in R, $x(t) = y(t)$

Conclusion:

1) There's only one solution to IVP
2) Solution exists until solution curve $t \to (t, x(t))$ leaves R

- If 2 solutions to same differential equation start 2gether, they stay 2gether
  - Thru any $(t_0, x_0) \in R$, there exists only 1 solution curve
- If 2 solutions $x(t), y(t)$ to same DE in R meet at a t_1 so $x(t_1) = y(t_1)$, then t_1 can be starting point + Uniqueness Thm implies the 2 solutions agree everywhere

Ex: Consider $ty' + 2y = 4t^2$, $y(1) = 2$. Is there a solution to this IVP? If yes is solution unique?



normal form: $y' = \underbrace{-\frac{2}{t}y + 4t}_{f(t,y) = -\frac{2}{t}y + 4t}$

$f(t,y)$ is cont. everywhere except $t=0$

Choose R: $R = \{(t,y) \mid \frac{1}{2} < t < 3 \text{ and } 1 < y < 3\}$
↳ avoids $t = 0$

$f(t,y)$ continuous on R
and initial point $(1,2)$ in R, by Existence Thm
there's a solution to IVP.

Uniqueness requires one more condition: $\frac{\partial f}{\partial y}$ needs to be continuous on R. $\frac{\partial f}{\partial y} = \frac{-2}{t}$ which is cont. on R
so by Uniqueness Thm, theres 1 solution

Ex: Consider $y' = ty(3-y)$ and suppose $y(1) = 3$. Show that $y(t) = 3$ for all t.

$$\left.\begin{aligned} f(t,y) &= ty(3-y) \\ \frac{\partial f}{\partial y} &= t(3-y) - ty = 3t - 2ty \end{aligned}\right]$$ Both f and $\frac{\partial f}{\partial y}$ continuous on $\mathbb{R}^2$, take $R = \mathbb{R}^2$

Initial point $(1,3)$ in R

Do Conditions of Uniqueness Thm hold?

Can see by substitution $x(t)=3$ is solution to $x' = tx(3-x)$

Since $y(1) = x(1) = 3$

LHS: $x' = 0$

RHS: $tx(3-x) = t3(3-3) = 0$

Uniqueness Thm implies $y(t) = x(t) = 3$ for all t

so $\boxed{y(t) = 3 \text{ for all } t}$