10/12/22

Integrating Factors

Def: An integrating factor for $w = P\,dx + Q\,dy = 0$ is a function $\mu(x,y)$ such that $\boxed{\mu w = \mu(x,y)P(x,y)\,dx + \mu(x,y)\,Q(x,y)\,dy}$ is exact

Recall: need to find μ and F such that $\frac{\partial F}{\partial x} = \mu P$ and $\frac{\partial F}{\partial y} = \mu Q$

$P = x + 2y^2$
$Q = -2xy$

Ex: Show that $(x+2y^2)dx - 2xy\,dy = 0$ isn't exact and $\frac{1}{x^3}$ is an integrating factor.

$$\frac{\partial P}{\partial y} = 4y \neq \frac{\partial Q}{\partial x} = -2y \quad \text{not exact}$$

$$\mu = \frac{1}{x^3} \xrightarrow{\text{multiply}} \frac{x+2y^2}{x^3}dx - \frac{2y}{x^2}dy = 0 \qquad \tilde{P} = \mu P,\ \tilde{Q} = \mu Q$$

$$\frac{\partial \tilde{P}}{\partial y} = 4\frac{y}{x^3} = \frac{\partial \tilde{Q}}{\partial x} = \frac{4y}{x^3} \quad \text{exact}$$

REMEMBER! $\tilde{P} = \mu P = \frac{\partial F}{\partial x}$ and $\tilde{Q} = \mu Q = \frac{\partial F}{\partial y}$

FIND GENERAL EQUATION

$$\tilde{P} = \frac{\partial F}{\partial x} = \frac{1}{x^2} + \frac{2y^2}{x^3}$$
$$F = \int \frac{1}{x^2} + \frac{2y^2}{x^3}dx + \phi(y)$$
$$= -\frac{1}{x} - \frac{y^2}{x^2} + \phi(y)$$
$$\frac{\partial F}{\partial y} = -\frac{2y}{x^2} + \phi'(y)$$

$$\tilde{Q} = \frac{\partial F}{\partial y} \Rightarrow -\frac{2y}{x^2} = -\frac{2y}{x^2} + \phi'(y)$$
$$\phi'(y) = 0$$
$$\phi(y) = C_1,\ \text{take } \phi(y) = 0$$

$$F(x,y) = -\frac{1}{x} - \frac{y^2}{x^2} = C$$

~~solutions implicitly defined by~~

$$-x - y^2 = Cx^2 \Rightarrow \boxed{Cx^2 + x + y^2 = 0}$$

Types of Integrating Factors $\mu(x,y)$

→ Linear Equations: use integrating factor for linear equations: $y' = a(x)y + f(x)$

or $\underbrace{[a(x)y + f(x)]}_{P}dx \underbrace{-dy}_{Q} = 0$

$\mu(x) = e^{-\int a(x)dx}$ will make this equation exact

Strategy to find general solution to $P\,dx + Q\,dy = 0$
1) Find integrating factor μ that makes $\mu P\,dx + \mu Q\,dy$ exact (aka $\frac{\partial \tilde{P}}{\partial y} = \frac{\partial \tilde{Q}}{\partial x}$)
2) Then find F such that $dF = \mu P\,dx + \mu Q\,dy$
general solution given implicitly by $F(x,y) = C$

→ $\mu(x,y)$ that results in a differential equation w/ separated variables

$$p(x)dx + q(y)dy = 0$$

can show $F(x,y) = \int p(x)dx + \int q(x)dx$ where the solutions are $F(x,y) = C$

If such a $\mu(x,y)$ exists, original differential equation is called separable

Ex: $-y^2dx + x^3dy = 0$

We know it isn't exact bc $\frac{\partial P}{\partial y} = -2y \neq \frac{\partial Q}{\partial x} = 3x^2$, but we can separate it with $\mu(x,y) = \frac{1}{x^3y^2}$

$$\frac{1}{x^3y^2}(-y^2dx + x^3dy = 0)$$
$$-\frac{1}{x^3}dx + \frac{1}{y^2}dy = 0$$
$$F(x,y) = \int -\frac{1}{x^3}dx + \int \frac{1}{y^2}dy = C \quad \text{can solve for}$$
$$= \frac{1}{2x^2} - \frac{1}{y} = C \Longrightarrow \boxed{y(x) = \frac{2x^2}{1-2Cx^2}}$$

→ Integrating factor depends on one variable

$$\omega = P\,dx + Q\,dy = 0$$

Find μ so $\mu\omega = \mu P\,dx + \mu Q\,dy = 0$ is exact or $\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$.

If μ only depends on one variable, you can find a procedure to find μ:

If $\mu = \mu(x)$

$$\mu \frac{\partial P}{\partial y} = \frac{\partial \mu}{\partial x} Q + \mu \frac{\partial Q}{\partial x}$$

$$\Downarrow$$

$$\frac{d\mu}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mu$$

will have solution $\mu = \mu(x)$ only if $h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

Let's say $h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$\Downarrow$$

$$\frac{d\mu}{dx} = h\mu \Rightarrow \mu(x) = e^{\int h(x)\,dx}$$

Ex: Solve $(xy-2)dx + (x^2 - xy)dy = 0$ $\quad P = xy-2$, $Q = x^2 - xy$

$\frac{\partial P}{\partial y} = x$, $\frac{\partial Q}{\partial x} = 2x - y$] not exact

$$h = \frac{1}{x^2-xy}(x - 2x + y) = \frac{y-x}{-x(y-x)} = -\frac{1}{x}$$

$$\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} \Rightarrow \mu(x) = \frac{1}{x}$$

multiply by μ: $\mu(xy-2)dx + \mu(x^2-xy)dy = 0$

$$\Rightarrow \left(y - \frac{2}{x}\right)dx + (x-y)dy = 0$$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(y - \frac{2}{x}\right) = 1$, $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1$] exact!

$$\tilde{P} = \frac{\partial F}{\partial x} = y - \frac{2}{x} \qquad \tilde{Q} = \frac{\partial F}{\partial y} = x - y$$

$$F = \int y - \frac{2}{x}\,dx + \phi(y) = xy - 2\ln|x| + \phi(y)$$

$$\frac{\partial F}{\partial y} = x + \phi'(y) = x - y$$

$$\phi'(y) = -y$$

$$\phi(y) = -\frac{1}{2}y^2 + C_1$$

$$F(x,y) = xy - 2\ln|x| - \frac{1}{2}y^2 = C$$

$$\Downarrow$$

$$\boxed{y(x) = x \pm \sqrt{x^2 - 4\ln|x| - 2C}}$$

If $\mu = \mu(y)$

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$$

$$\frac{\partial \mu}{\partial y}P + \frac{\partial P}{\partial y}\mu = \mu\frac{\partial Q}{\partial x}$$

$$\frac{\partial \mu}{\partial y}P = \mu\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\frac{\partial \mu}{\partial y} = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mu$$

Let's say $g = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$, then

$$\frac{d\mu}{dy} = -g\mu \Rightarrow \boxed{\mu(x) = e^{-\int g(y)\,dy}}$$