9/30/22

$\frac{dy}{dt} = \frac{g(t)}{h(y)} \Rightarrow \int h(y)dy = \int g(t)\,dt$

$\Rightarrow H(y) = G(t) + C$ $\rightsquigarrow$ implicit equation for $y(t)$,

need to find H^{-1} to get explicit solution

unless $H(y) = y$, that is $h(y) = 1$

$y(t) = H^{-1}(G(t) + C)$

Ex: Find solutions to DE

$\frac{dx}{dt} = \frac{2tx}{1+x}$, $x \neq -1$ IC: $x(0) = 1$, $x(0) = -2$, $x(0) = 0$

$\frac{1+x}{x}dx = 2t\,dt$, $x \neq 0$

$\int \frac{1}{x} + 1\,dx = \int 2t\,dt$

$\ln|x| + x = t^2 + C$

$x(0) = 1$, $\ln|1| + 1 = 0^2 + C$, $C = 1$

so $\boxed{\ln|x| + x = t^2 + 1}$ $\ln|x|$ undefined when $x = 0$, so solution can never be 0. Since $x(0) = 1 > 0$ and $x(t)$ is continuous, $x(t)$ is positive for all t. $\to |x| = x$.

Implicit solution, can't be solved for explicit

$x(0) = -2$, $\ln|-2| - 2 = 0^2 + C \Rightarrow C = \ln 2 - 2$

$\boxed{\ln|x| + x = t^2 + \ln 2 - 2}$. Since $x(0) = -2 < 0$, $x(t)$ is negative for all t so $|x| = -x$.

Implicit solution, can't solve for just x

$x(0) = 0$, can't divide by $\frac{x}{1+x}$ (in step $\frac{1+x}{x}dx = 2t\,dt$). But $x(t) = 0$ is an explicit solution. You can verify these solutions by substitution but not general method

Using Definite Integration: for IVPs for separable equations

Newton's Law of Cooling:

Rate of change of soda's temp, T, is proportional to T-A when A is ambient temp.

Ex: Can of soda at 40°F placed in a room where temp. was 70°. After 10 min, the soda is 50°F. Find temp. of soda 30 min. after the soda is placed in the room.

$$\frac{dT}{dt} = -k(T-A) \qquad \text{proportionality constant } k > 0$$

If $T < A$, then $-k(T-A) > 0$ so temp. of soda increasing

$$\frac{dT}{T-A} = -k\,dt \qquad IC: T(0) = T_0$$

If IC $T(0) = T_0$, $t=0$ corresponds to $T = T_0$, $T(t) = T$

$$\int_{T_0}^{T} \frac{dT}{T-A} = -k\int_{t_0}^{t} dt$$

$$\ln|T-A|\Big|_{T_0}^{T} = -kt\Big|_0^t$$

$$\ln\frac{|T-A|}{|T_0-A|} = -kt$$

$$\frac{|T-A|}{|T_0-A|} = e^{-kt}$$

$$|T-A| = |T_0-A|e^{-kt}$$

T-A and T_0-A have same signs

$$T-A = (T_0-A)e^{-kt}$$

$$T(t) = A + (T_0-A)e^{-kt}$$

$$= 70 + (40-70)e^{-kt}$$

$$= 70 + (-30)e^{-kt}$$

Since,

$$T(10) = 50 = 70-30e^{-10k}$$

$$\frac{2}{3} = e^{-10k}$$

$$\ln\frac{2}{3} = -10k \Rightarrow k = \frac{\ln\frac{2}{3}}{-10} \approx 0.0405$$

$$\boxed{T(t) = 70 - 30e^{-0.0405t}}$$

Plug in $t=30$, get $T(30) \approx 61.1°F$

Why separation of variables work:

$$y' = \frac{g(t)}{h(y)}$$

$$h(y)y' = g(t) \quad \text{or} \quad h(y)\,dy = g(t)\,dt$$

$$\int h(y)\, y'\, dt = \int g(t)\, dt$$

change variable of integration

$$\int \underbrace{h(y)\, dy}_{} = \int \underbrace{g(t)\, dt}_{} \qquad dy = y'(t)\, dt$$

called differential forms. Special cases dy and dt are differentials

Linear Equations

a(t) and f(t) are called the coefficients of the equation.

First-order linear equation is of form:

$$x' = a(t)x + f(t)$$

If $f(t) = 0$ $x' = a(t)x$ is called homogeneous, if $f \neq 0$, it's inhomogeneous

More general form: $b(t)x' = c(t)x + g(t)$ ← can solve for ...

Linear equations: x and x' appear alone and to 1^{st} order

Linear: $x' = \sin t\, x$, $y' = e^{2t} y + \cos t$

Nonlinear: $x' = \tan(x)$, $y' = yy'$, $z' = 1 - z^2$