STATS 100A HW5

Problem 1 Suppose $X \sim f(x)$, and let $Y = aX + b$, where a and b are constants.

(1) Prove $\mathrm{E}(Y) = a\mathrm{E}(X) + b$, and $\mathrm{Var}(Y) = a^2\mathrm{Var}(X)$.

(2) Assuming $a > 0$, calculate the density of Y, $g(y)$.

Problem 2 Suppose $Z \sim \mathrm{N}(0, 1)$, i.e.,

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

(1) Calculate $\mathrm{E}(Z)$, $\mathrm{Var}(Z)$ and $\mathrm{E}(|Z|)$.

(2) Let $X = \mu + \sigma Z$. Calculate the density of X, $g(x)$, $\mathrm{E}(X)$, and $\mathrm{Var}(X)$ based on Problem 1.

(3) Suppose $P(Z \in [-2, 2]) = 95\%$, then what is $P(X \in [\mu - 2\sigma, \mu + 2\sigma])$?

Problem 3 Poisson process: Suppose we divide the time axis into small periods $(0, \Delta t)$, $(\Delta t, 2\Delta t)$, .., $(t, t + \Delta t)$, ... Suppose within each interval, we flip a coin independently. Suppose the probability of getting a head is $\lambda \Delta t$. Let T be the time until the first head. Let X be the number of heads within $[0, t]$.

(1) Find the probability density function of T, and calculate $\mathrm{E}(T)$ based on Geometric distribution.

(2) Calculate $\mathrm{E}(X)$ based on Binomial distribution. Find the probability mass function $P(X = k)$ as $\Delta t \to 0$.

Problem 4 Brownian motion or diffusion: Suppose a particle starts from 0, and within each period, it moves forward or backward by Δx, each with probability 1/2. Let X_t be the position at time t (assuming t is a multiple of Δt). Suppose there are n periods within $[0, t]$, i.e., $\Delta t = t/n$. Then we can write

$$X_t = \sum_{i=1}^{n} \epsilon_i \Delta x,$$

where $P(\epsilon_i = 1) = P(\epsilon_i = -1) = 1/2$, and Z_i are independent.

(1) Calculate $\mathrm{E}(X_t)$ and $\mathrm{Var}(X_t)$.

(2) What is the relationship between Δx and Δt so that $\mathrm{Var}(X_t)$ does not depend on discretization?

(3) According to the central limit theorem, what is the distribution of X_t?

Problem 5 (1) Suppose we flip a fair coin 100 times independently. Let X be the number of heads. Based on normal approximation, find the 95% probability interval for X.

(2) Suppose 20% of the population support a candidate A. Suppose we randomly sample 100 people for the population (with replacement). Let $\hat{p} = X/100$ be the proportion of people in the sample who support candidate A. Based on normal approximation, find the 95% probability interval for $\hat{p}$.

(3) Suppose we randomly throw 10,000 points into the unit square $[0, 1]^2$. Let A be the region $x^2 + y^2 \leq 1$. Let m be the number of points that fall into A. Let $\hat{\pi} = 4m/10000$ be our Monte Carlo estimate of π. What is the approximate normal distribution of $\hat{\pi}$? What is the 95% probability interval of $\hat{\pi}$?

Problem 6 Read Wikipedia about the following distributions: Negative Binomial, Hyper-Geometric, Zipf, Chi-square, student t, Cauchy, Gamma, Beta, Weibull, Gumbel, Pareto. Write down their probability mass functions or probability density functions.