STATS 100A HW4

Problem 1 Suppose we roll a die, let X be the number we get. Suppose the probability mass function $p(x) = P(X = x)$ is such that $p(1) = .1$, $p(2) = .1$, $p(3) = .1$, $p(4) = .2$, $p(5) = .2$, $p(6) = .3$.

(1) Calculate $P(X > 4)$. Calculate $P(X = 6|X > 4)$.

(2) Calculate $\mathrm{E}(X)$, $\mathrm{Var}(X)$, and $SD(X)$.

(3) Suppose the reward for x is $h(x)$, and $h(1) = -\$20$, $h(2) = -\$10$, $h(3) = \$0$, $h(4) = \$10$, $h(5) = \$20$, $h(6) = \$100$. Calculate $\mathrm{E}(h(X))$, $\mathrm{Var}(h(X))$ and $SD(h(X))$. What are the units of $\mathrm{E}(h(X))$ and $\mathrm{Var}(h(X))$?

Problem 2 Suppose $Z \in \{0, 1\}$. $P(Z = 1) = p$, $P(Z = 0) = 1 - p$. Calculate $\mathrm{E}(Z)$, $\mathrm{E}(Z^2)$ and $\mathrm{Var}(Z)$. What if we replace 0 by -1? Calculate concrete numbers for $p = 1/2$.

Problem 3 Suppose we flip a fair coin 100 times independently. Let X be the number of heads. Calculate $\mathrm{E}(X)$, $\mathrm{Var}(X)$, $SD(X)$, $\mathrm{E}(X/100)$, $\mathrm{Var}(X/100)$, $SD(X/100)$. Write down the formula for computing $P(X \in [40, 60])$.

Problem 4 Suppose within the population of voters, 20% of them support a candidate A. If we randomly sample 100 people sequentially with replacement. Let X be the number of supporters of A among these 100 people. Then what is the distribution of X? What are $\mathrm{E}(X)$, $\mathrm{Var}(X)$, and $SD(X)$? What are $\mathrm{E}(X/100)$, $\mathrm{Var}(X/100)$, and $SD(X/100)$?

Problem 5 Suppose we randomly throw 10,000 points into the unit square $[0, 1]^2$. Let A be the region $x^2 + y^2 \leq 1$. Let m be the number of points that fall into A. What is the distribution of m? Let $\hat{\pi} = 4m/10000$ be our Monte Carlo estimate of π. What are $\mathrm{E}(\hat{\pi})$, $\mathrm{Var}(\hat{\pi})$ and $SD(\hat{\pi})$?

Problem 6 Suppose X is a discrete random variable with probability mass function $p(x)$, where x takes values in a discrete set.

(1) Prove $\mathrm{E}(aX) = a\mathrm{E}(X)$.

(2) Prove $\mathrm{E}(X + b) = \mathrm{E}(X) + b$.

(3) Prove $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$.

(4) Prove $\mathrm{Var}(X + b) = \mathrm{Var}(X)$.

(5) Prove $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$.

(6) Let $\mu = \mathrm{E}(X)$, and $\sigma^2 = \mathrm{Var}(X)$. Let $Z = (X - \mu)/\sigma$. Calculate $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$.

Problem 7 (optional) (1) For $X \sim \mathrm{Binomial}(n, p)$, prove formally that $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = np(1 - p)$. (2) For $T \sim \mathrm{Geometric}(p)$, prove $\mathrm{E}(T) = 1/p$.

Problem 8 (optional) Read the slides on Jensen inequality. For a convex function $h(x)$, and for any random variable X, prove $h(\mathrm{E}(X)) \leq \mathrm{E}(h(X))$.

Problem 9 (optional) Read the slides on entropy. Explain that for a probability mass function $p(x)$, its entropy can be defined by $\mathrm{E}[-\log_2 p(X)] = -\sum_x p(x)\log_2 p(x)$. Explain that entropy can be interpreted as average number of coin flips or average code length.