STATS 100A HW5

Problem 1)

1) $E(Y) = E(aX + b)$
$= E(aX) + E(b)$
$= aE(X) + b$

$$\boxed{E(Y) = aE(X) + b}$$

$Var(Y) = Var(aX + b)$
$= Var(aX) + \cancel{Var(b)}$
$= a^2 Var(X)$

$$\boxed{Var(Y) = a^2 Var(X)}$$

2) $Y = g(X) = aX + b$

$g'(x) = a$

$g(y) = f(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$
$= f\left(\frac{y-a}{b}\right) \cdot \frac{d}{dy}\left(\frac{y-b}{a}\right)$

$$\boxed{g(y) = \frac{1}{a} f\left(\frac{y-b}{a}\right)}$$

Problem 2)

1) $\cancel{E(Z) = \frac{1}{2}}$
$\cancel{Var(Z) = (0-\frac{1}{2})^2 \cdot \frac{1}{2} + (1-\frac{1}{2})^2 \cdot \frac{1}{2} = \frac{1}{4}}$
$\cancel{E(|Z|) =}$

$$\boxed{\begin{aligned} E(Z) &= 0 \\ E(|Z|) &= 0 \\ Var(Z) &= 1 \end{aligned}}$$

3) $P(\mu - 2\sigma \le X \le \mu + 2\sigma) = P(-2 \le X \le 2) \boxed{= 95\%}$

2) $Z = \frac{x-\mu}{\sigma}$

$$\boxed{X \hat{\sim} N(0,1)}$$

$g(x) = f(z) \frac{\Delta z}{\Delta x} = f\left(\frac{x-\mu}{\sigma}\right)\sigma \boxed{= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$

$$\boxed{\begin{aligned} E(X) &= \mu \\ Var(X) &= \sigma^2 \end{aligned}}$$

Problem 3)

1) X = # of flips until first head

$P(X = n) = (1 - \lambda\Delta t)^{n-1} \cdot \lambda\Delta t \qquad X > 0$

$P(X > n) = (1 - \lambda\Delta t)^n$

$\boxed{E(T) = E(X\Delta t)}$

$\Downarrow$

$E(N) = \Delta t = \frac{1}{\lambda}$

$T = X\Delta t$

$X = \frac{T}{\Delta t}$

$$\boxed{P(T > t) = e^{-\lambda t}}$$

2) $x \sim \text{Binomial}\left(n = \frac{t}{\Delta t}, P = \lambda\Delta t\right)$

$P(x = k) = {}_{t/\Delta t}C_k (\lambda\Delta t)^k (1 - \lambda\Delta t)^{t/\Delta t - k}$

$E(x) = nP = \frac{t}{\Delta t} \cdot \lambda\Delta t \boxed{= \lambda t}$

Problem 4)

1) $E(X_t) = E(\Delta x \sum_{i=1}^{n} Z_i)$

$= \Delta x \, E(\sum_{i=1}^{n} Z_i)$

$= \Delta x \cdot n \cdot 0 \, \boxed{= 0}$

$Var(X_t) = Var(\Delta x \sum_{i=1}^{n} Z_i)$

$\boxed{= \frac{n}{2} \Delta x^2}$

2) $\sigma^2 t = \frac{n}{2} \Delta x^2$ $n = \frac{t}{\Delta t}$

$\sigma^2 \cancel{t} = \frac{1}{2} \frac{\cancel{t}}{\Delta t} \Delta x^2$

$\boxed{\Delta t = \frac{1}{2\sigma^2} \Delta x^2}$

3) $\boxed{X_t \sim N(\mu = 0, \sigma^2 t)}$

Problem 5) $n = 100$, $p = \frac{1}{2}$

1) $E(X) = \mu = np = 50$

$Var(X) = \sigma^2 = np(1-p) = 25$

$SD(X) = 5$

$Z = \frac{X - \mu}{\sigma} = \pm 1.96$

$X = \mu \pm 1.96\sigma \, \boxed{= [40, 60]}$

2) $E(p) = \mu = 0.2$

$Var(p) = \sigma^2 = \frac{p(1-p)}{n} = \frac{0.2 \cdot 0.8}{100} = 0.0016$

$SD = 0.04$

$Z = \frac{\hat{p} - \mu}{\sigma} = \pm 1.96$

$\hat{p} = 0.2 \pm 1.96 \cdot 0.04 \, \boxed{= [0.12, 0.28]}$

3) $p = \frac{\pi}{4}$, $n = 10k$, $p = 0.79$

$E(\pi) = np \, \boxed{= 7900}$

$Var(\pi) = np(1-p) \, \boxed{= 1659}$

$SD(\pi) \boxed{\approx 41}$

$\hat{\pi} = N(7900, 1659)$

$\hat{\pi} = \mu \pm 1.96\sigma \, \boxed{\approx [7773, 7935]}$

Problem 6)

Negative Binomial: $P(X = k) = {}_{k+r-1}C_k (1-p)^r p^k$

HyperGeometric Distribution: $P(X = k) = \frac{{}_K C_k \; {}_{N-K} C_{n-k}}{{}_N C_n}$

Chi-sq: $f(x; k) = \frac{1}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} x^{\frac{k}{2} - 1} e^{-\frac{x}{2}}$

Student t: $f(t; \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})} (1 + \frac{t^2}{\nu})^{-\frac{\nu+1}{2}}$

Cauchy: $f(x; x_0, \gamma) = \frac{1}{\pi\gamma[1 + (\frac{x - x_0}{\gamma})^2]}$

Gamma: $f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\beta x}$

Beta: $f(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$

Weibull: