STATS 100A HW3

Problem 1)

1) $K_{ij} = P(X_{t+1} = j \mid X_t = i)$

$$\begin{array}{c|ccc} i \backslash j & 1 & 2 & 3 \\ \hline 1 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

$K_{12} = P(X_{t+1} = 2 \mid X_t = 1) = \frac{1}{6}$

$K_{13} = \frac{1}{6}$

$K_{11} = \frac{2}{3}$

$\sum_{j=1}^{3} K_{1j} = 1$

$K_{21} = \frac{1}{2}$

$K_{22} = 0$

$K_{23} = \frac{1}{2}$

$\sum_{j=1}^{3} K_{2j} = 1$

$K_{31} = \frac{1}{2}$

$K_{32} = \frac{1}{2}$

$K_{33} = 0$

$\sum_{j=1}^{3} K_{3j} = 1$

2) $p^{(0)} = \begin{array}{c|ccc} i & 1 & 2 & 3 \\ \hline & 1 & 0 & 0 \end{array}$

$p^{(1)} = p^{(0)} K_{ij}$

$= [1 \quad 0 \quad 0] \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

$\boxed{= [\frac{2}{3} \quad \frac{1}{6} \quad \frac{1}{6}]}$

$p^{(2)} = p^{(1)} K_{ij}$

$= [\frac{2}{3} \; \frac{1}{6} \; \frac{1}{6}] \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

$\boxed{= [\frac{11}{18} \quad \frac{7}{36} \quad \frac{7}{36}]}$

$p^{(3)} = p^{(2)} K_{ij}$

$= [\frac{11}{18} \; \frac{7}{36} \; \frac{7}{36}] \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

$\boxed{= [\frac{130}{216} \quad \frac{43}{216} \quad \frac{43}{216}]}$

3) $\pi_j = \sum_{i=1}^{3} \pi_i K_{ij}$

$$\begin{array}{c|ccc} i \backslash j & 1 & 2 & 3 \\ \hline 1 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

$\lim_{t \to \infty} p^{(t)}$

$p^{(3)} = [0.601 \quad 0.199 \quad 0.199] \neq \pi$

4) For $p^{(t)}$ for $t = 1, 2, 3$

we expect to see

for $t=1$, [666,666 16666 16666] $t=2$, [500k 0 500k] $t=3$, [500k, 500k, 0]

Problem 2)

1) $P(\text{fire} \mid \text{alarm}) = ?$

$P(\text{alarm} \cap \text{fire}) = \alpha \cdot \beta$

$P(\text{no fire}) = 1 - \alpha$

$P(\text{alarm} \mid \text{fire}) = \beta = \dfrac{P(\text{alarm} \cap \text{fire})}{P(\text{fire})}$

$P(\text{alarm} \mid \text{NO fire}) = \gamma$

$= \dfrac{P(\text{alarm} \cap \text{NO fire})}{P(\text{NO fire})}$ $\quad P(\text{alarm} \cap \text{NO fire}) = \gamma(1-\alpha)$

$P(\text{alarm} \cap \text{fire}) + P(\text{alarm} \cap \text{NO fire}) = P(\text{alarm}) = \alpha \cdot \beta + \gamma(1-\alpha)$

$$P(\text{fire} \mid \text{alarm}) = \frac{P(\text{fire} \cap \text{alarm})}{P(\text{alarm})} = \boxed{\frac{\alpha \cdot \beta}{\alpha \cdot \beta + \gamma(1-\alpha)}}$$

2) If we do this experiment 100,000 times,

$\alpha = \frac{1}{1000}$ $\quad \beta = \frac{99}{100}$ $\quad \gamma = \frac{2}{100}$

$P(\text{no fire}) = \frac{999}{1000} =$ 99900 times there won't be a fire

$P(\text{fire}) = \frac{1}{1000} =$ 100 times there WILL be a fire

$P(\text{alarm} \mid \text{fire}) = \frac{99}{100} =$ 99 out of 100 times that a fire occurred an alarm will go off

$P(\text{alarm} \mid \text{No fire}) = \frac{2}{100} =$ 2000 times out of 100,000 there will be an alarm but no fire

$P(\text{fire} \mid \text{alarm}) = \dfrac{\alpha \cdot \beta}{\alpha \cdot \beta + \gamma(1-\alpha)} = \dfrac{11}{233} \approx 0.047$, out of 100,000 experiments, given the alarm goes off 2100 times ~~approx. 99 times there will be a fire~~

$P(\text{alarm}) = \alpha \cdot \beta + \gamma(1-\alpha) = 0.021$, 2100 out of 100k experiments the alarm will go off

3)