MICROWAVE AND GUIDED PROPAGATION

Practical works : Theoretical and numerical study of channel equalization

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AERO 4 SET – 4TS1

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Context

The propagation channel is modeled by an impulse response $w(k)$ defined as :

$$w(k) = \begin{cases} \frac{1}{2}\left(1 + \cos\left(\frac{2\pi}{\beta}(k-1)\right)\right), & \text{for } k = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

- β controls the level of channel distortion
- A centered Gaussian noise $c(n)$ with variance $\sigma_u^2 = 0.001$ is added

The received signal $x(n)$ is expressed as :

$$x(n) = \sum_{k=0}^{2} w(k)s(n-k) + c(n)$$

The signal transmission and processing chain for the considered equalization problem is shown in Figure 1.

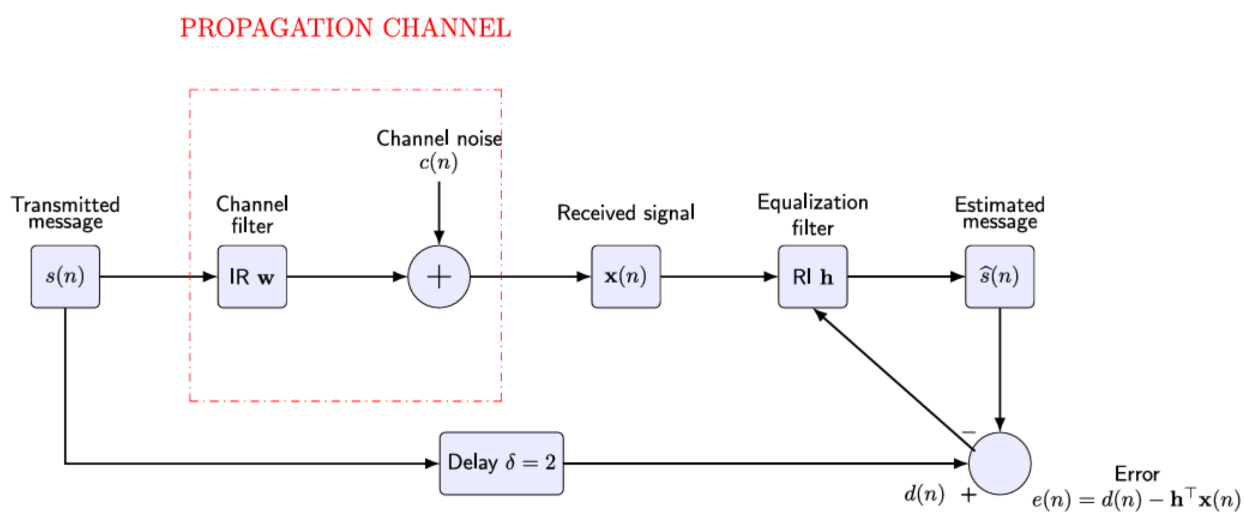

FIGURE 1 – Simulation of transmission chain

Preliminaries questions

Question 1.

We generate the message $\{s(n)\}_{n=1}^{N}$ as a random equiprobanle sequence of symbols +1 and -1. The latter are assumed independent between them and $s(n)$ is built with size N = 2000. We can plot it on MATLAB.

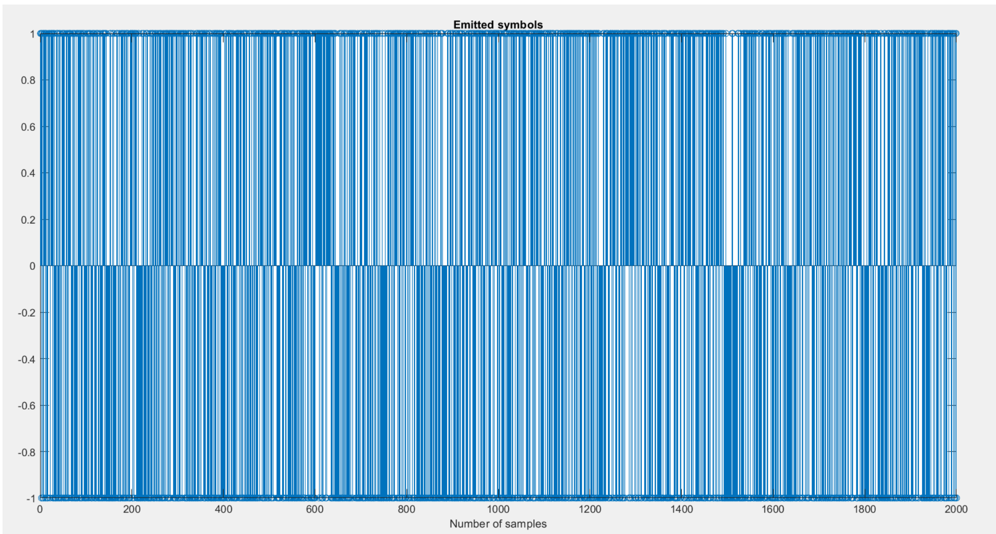

FIGURE 2 – The message $\{s(n)\}_{n=1}^{N}$

Question 2.

We draw the propagation channel for the following values of β : 0.25, 0.5, 2 and 4.

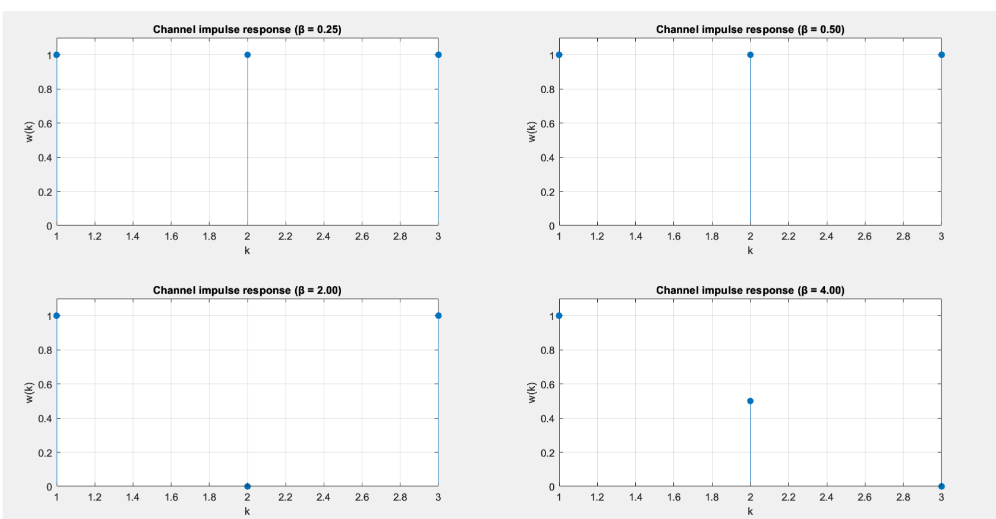

FIGURE 3 – Propagation channel for different values of β

Observations :

For $\beta = 0.25$:

- Low distorsion
- Very symmetric impulse response
- Peaks close to 1 at first and last points
- Minimal variation in channel characteristics

For $\beta = 0.5$: it's similar to $\beta = 0.25$. We can observe the same phenomena for $k = 1, 2, 3$.

For $\beta = 2$:

- Significant asymmetry in impulse response
- For $k = 2$, $w(k) = 0$
- More pronounced differences between channel weights
- Increased channel distortion

For $\beta = 4$:

- High distorsion
- Most asymmetric impulse response
- For $k = 3$, $w(k) = 0$
- Largest variation between different channel weights
- Maximum channel distortion

Finally, we can observe that the β parameter controls the level of channel distortion, with lower values preserving signal characteristics and higher values introducing more significant signal modifications.

Question 3.

Next, we can obtain the signal $x(n)$.

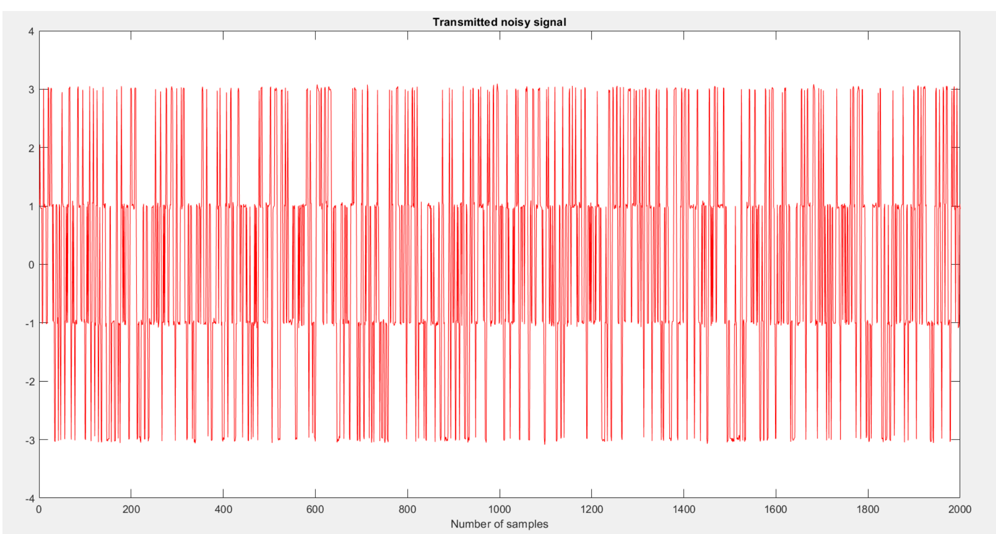

FIGURE 4 – Transmitted signal $x(n)$

Question 4.

In the following of the TP, we consider $\beta = 0.25$.

To perform the equalization, we consider a filter of order L such that :

$$y(n) = \sum_{l=1}^{L} h(l)x(n-l) = h^T x(n)$$

where :

$$x(n) = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-L+1) \end{bmatrix}, \quad h = \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(L-1) \end{bmatrix}$$

We show that the theoretical expression of the Wiener filter h_{opt}, minimizing the mean square error :

$$J(h) = \mathbb{E}\left[(d(n) - h^T x(n))^2\right]$$

is given by :

$$h_{opt} = R_{xx}^{-1} r_{dx} \tag{1}$$

where :

$$R_{xx} = \mathbb{E}\left[x(n)x(n)^T\right], \quad r_{dx} = \mathbb{E}\left[d(n)x(n)\right]$$

We want to demonstrate the expression (1).

We know that :

$$\begin{aligned} J(h) &= \mathbb{E}\left[(d(n) - h^T x(n))^2\right] \\ &= \mathbb{E}\left[d(n)^2 - 2d(n)h^T x(n) + (h^T x(n))^2\right] \\ &= \mathbb{E}[d(n)^2 - 2d(n)h^T x(n) + (h^T x(n))(h^T x(n))] \\ &= \mathbb{E}[d(n)^2 - 2d(n)h^T x(n) + h^T x(n)x(n)^T h] \\ &= \mathbb{E}[d(n)^2] - 2h^T \mathbb{E}[d(n)x(n)] + h^T \mathbb{E}[x(n)x(n)^T]h \end{aligned}$$

We minimize $J(h)$. We know that $J(h)$ is minimized iff :

$$\nabla J(h) = 0$$

We have :

$$\nabla J(h) = -2\mathbb{E}[d(n)x(n)] + 2\mathbb{E}[x(n)x(n)^T]h_{opt} = 0$$

We recognize :

$$-2r_{dx} + 2R_{xx}h_{opt} = 0$$

So, we have :

$$\begin{aligned} 2R_{xx}h_{opt} &= 2r_{dx} \\ \Rightarrow R_{xx}h_{opt} &= r_{dx} \end{aligned}$$

Finally, we obtain :

$$\boxed{h_{opt} = R_{xx}^{-1} r_{dx}}$$

We can also verify that :

$$(R_{xx})_{k,l} = \sum_{i=0}^{L-1} w(i)w(i+l-k), \quad \forall (k,l) \in \{0,\dots,L-1\}^2$$

$$r_{dx} = \sum_{i=0}^{L-1} w(i)r(i)$$

$$r(i) = [\mathbb{E}(d(n)s(n-i)),\dots,\mathbb{E}(d(n)s(n-i-L+1))]$$

Question 5.

We want to demonstrate that if every symbol $s(n)$ is independent of the others, then :

$$r_{dx} = \begin{bmatrix} w(2) & w(1) & w(0) & \underbrace{0 \quad \dots \quad 0}_{L-3} \end{bmatrix}^\top$$

To demonstrate this equation, we will start from $r(i)$:

$r(i) = [\mathbb{E}(d(n)s(n-i)), \mathbb{E}(d(n)s(n-i-1)), \mathbb{E}(d(n)s(n-i-2)), \mathbb{E}(d(n)s(n-i-3)), \dots, \mathbb{E}(d(n)s(n-i-L+1))]^T$

If we suppose that every symbol $s(n)$ is independent of the others, then we also know that $d(n) = s(n-2)$.

With this assumption, we have :

$$\begin{cases} \mathbb{E}(d(n)s(n-2)) = \mathbb{E}(s(n-2)s(n-2)) = 1, & \text{if i=2} \\ \mathbb{E}(d(n)s(n-1)) = \mathbb{E}(s(n-2)s(n-1)) = 0, & \text{if i=1} \\ \mathbb{E}(d(n)s(n-0)) = \mathbb{E}(s(n-2)s(n)) = 0, & \text{if i=0} \\ \mathbb{E}(d(n)s(n-i)) = \mathbb{E}(s(n-2)s(n-i)) = 0, & \text{if i>2} \end{cases}$$

So, we have :

$$r(i) = [\mathbb{E}(d(n)s(n-i)), \mathbb{E}(d(n)s(n-i-1)), \mathbb{E}(d(n)s(n-i-2)), 0, \dots, 0]^T$$

We put this equation into the sum and we finally obtain :

$$\begin{aligned}
r_{dx} &= \sum_{i=0}^{L-1} w(i)r(i) \\
&= \sum_{i=0}^{L-1} w(i)\left[\mathbb{E}(d(n)s(n-i)), \mathbb{E}(d(n)s(n-i-1)), \mathbb{E}(d(n)s(n-i-2)), 0, \ldots, 0\right]^T \\
&= w(0)r(0) + w(1)r(1) + w(2)r(2) + w(3)\cdot 0 + w(4)\cdot 0 + \cdots + w(L-1)\cdot 0 \\
&= w(0)\left[0,0,1,0,\ldots,0\right]^T + w(1)\left[0,1,0,0,\ldots,0\right]^T + w(2)\left[1,0,0,0,\ldots,0\right]^T \\
&= \left[w(2), w(1), w(0), 0, \ldots, 0\right]^T
\end{aligned}$$

We indeed have :

$$\boxed{r_{dx} = \left[w(2), w(1), w(0), 0, \ldots, 0\right]^T}$$

Question 6.

Now, we implement the filter on MATLAB with $L = 5$.

Question 7.

We apply the filter to the transmitted signal to obtain the equalized signal given by :

$$y_{eq}(n) = h_{opt}^T x(n)$$

We plot it and we superimpose it to the emitted signal $s(n)$.

We obtain :

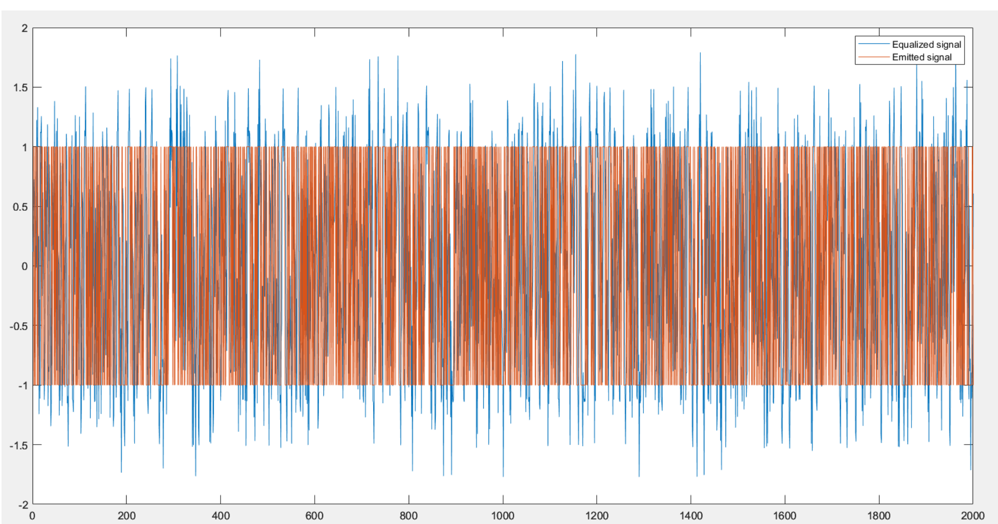

FIGURE 5 – The equalized signal $y_{eq}(n)$ and the emitted signal $s(n)$

Observations :

We can compare the received noisy signal, the equalized signal and the emitted signal.

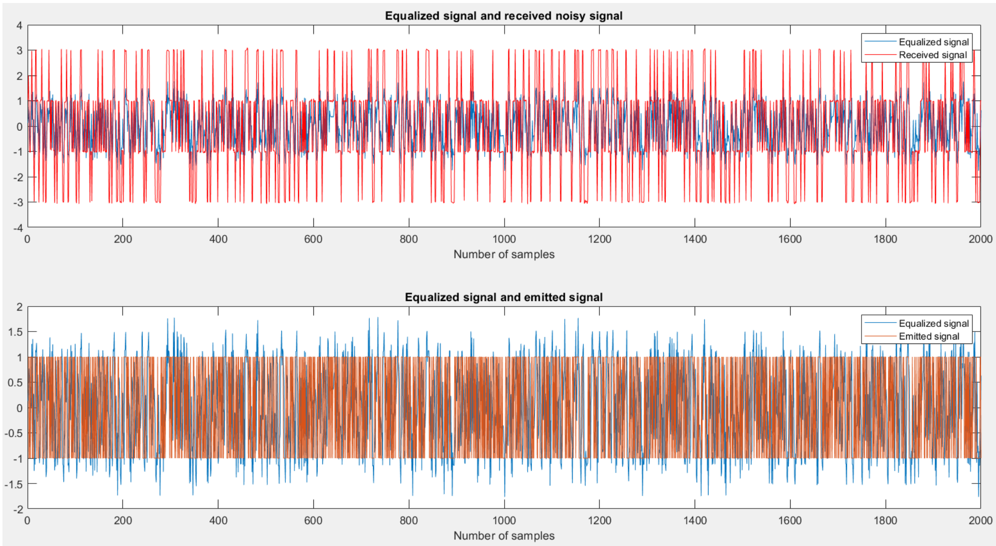

FIGURE 6 – Comparison of transmitted noisy signal, equalized signal and emitted signal

The signal processing graphs reveal the remarkable performance of the Wiener filter in reconstructing the original signal. In the upper graph, the equalized signal demonstrates significant noise reduction compared to the received noisy signal, showcasing the effectiveness of the equalization process.

The lower graph highlights the filter's ability to almost perfectly reconstruct the original emitted signal, with the equalized signal closely tracking the shape and variations of the original signal. The signals oscillate between approximate values of +1 and -1, and the equalization process preserves the binary nature of the signal while compensating for channel distortions.

This visualization confirms the technical efficacy of channel equalization, illustrating how signal processing techniques can restore degraded information with remarkable precision.

Question 8.

To obtain a more precise estimation of the message and find values in $\{-1, +1\}$, we can apply to the equalized signal the following test detection :

$$\hat{s}(n) = \begin{cases} 1 & \text{if } y_{eq}(n) > 0 \\ -1 & \text{if } y_{eq}(n) < 0 \end{cases}$$

In this case, the performance of the filtering can be determined by the Symbol Error Rate (SER) given by :

$$SER_n = \frac{\text{card}\left(\{s(n) \neq \hat{s}(n)\}\right)}{N}$$

We can implement the reconstructed message $\{\hat{s}(n)\}_{n=1}^{N}$ and assess the SER for different values of σ_u^2. The values of σ_u^2 are directly related to the chosen SNR values, as they are computed using the formula $\sigma_u = \sqrt{P_s/\text{db2pow(SNR)}}$. We will now plot our results.

By running the code several times, we obtain :

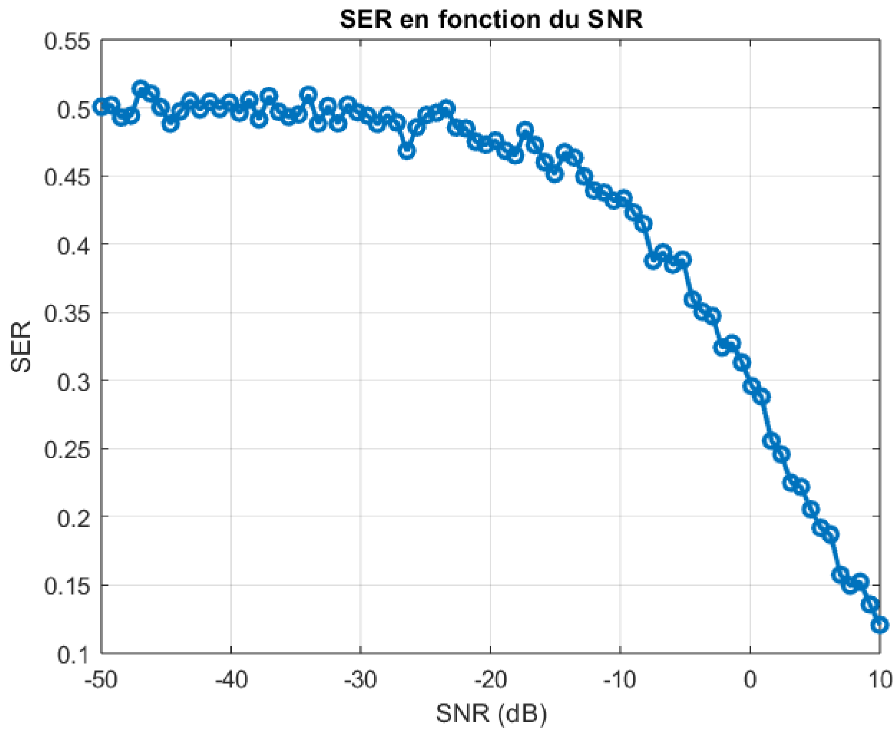

Figure 7 – SER in function of SNR(dB)

Observations :

At very low SNR (below -30 dB), the SER seems to be relatively constant around 0.5. It indicates a high error rate where roughly half the transmitted symbols are incorrectly decoded. This suggests that at extremely low signal qualities, the equalization process struggles to distinguish between transmitted symbols.

As the SNR improves, moving from left to right on the graph, there's a gradual decrease in the SER. This demonstrates the increasing effectiveness of the Wiener filter in reconstructing the original signal as the signal quality improves. The most significant performance improvement occurs between -20 dB and 0 dB, where the SER drops from around 0.5 to approximately 0.15.

At higher SNR values (above 0 dB), the SER continues to decrease. It approaches a minimum of about 0.15. This indicates that even with high-quality signals, a small percentage of symbol errors persists. It could be due to inherent limitations in the equalization process or the specific characteristics of the transmission channel.

Channel equalization with LMS algorithm

When the matrix R_{xx} is difficult to invert, the Wiener filter can be challenging to compute.
To overcome that, it is possible to iteratively solve the equalization problem by generating a sequence of filters converging to the optimal filter. That's why we apply an algorithm called LMS (Least-Mean Square). This is a recursive algorithm which minimizes the least-square criterion :

$$\mathbb{E}\left(\left(d(n) - h_{\text{opt}}^{\top} x(n)\right)^2\right)$$

and estimates the Wiener filter h_{opt} with the following recursion :

$$h_0 \in \mathbb{R}^L$$

$$h_{n+1} = h_n - \mu x(n)\left(d(n) - h_n^{\top} x(n)\right) \tag{2}$$

such as :

$$\lim_{n \to \infty} \|h_n - h_{\text{opt}}\|^2 = 0 \tag{3}$$

μ is a step size which can be numerically tuned.

Question 9.

We implement on MATLAB the LMS algorithm by using $x(n)$, $d(n)$, L and the step size μ. To begin, we fix μ with a value close to 0.

Question 10.

We compute at each iteration the filter error given by $||h_n - h_{opt}||^2$ and we plot it on MATLAB.

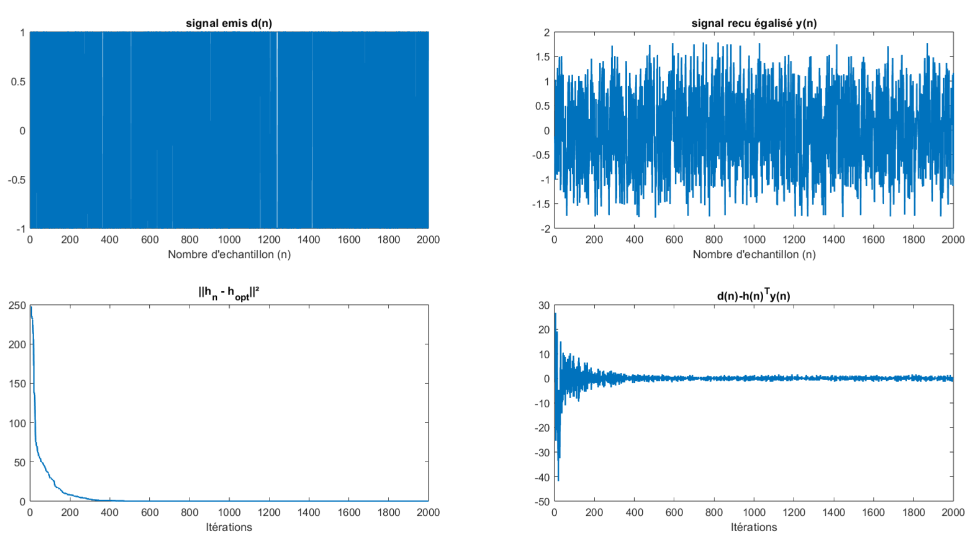

FIGURE 8 – Channel equalization with LMS algorithm , $\mu = 0.01$ and $\sigma_u^2 = 0.001$

Observations :

We can observe that the error given by $||h_n - h_{opt}||^2$ converges quite rapidly to 0. This convergence to 0 validates our implementation of the LMS (Least-Mean Square) algorithm. We can also notice that the number of errors between the transmitted and received signals decreases as our algorithm begins to converge toward zero error.

Question 11.

Finally, we can test our algorithm by computing the SER for different values of the standard deviation σ_u^2 and the step size μ.

We will first test different values of μ for a fixed variance. We obtain the following SER and filtering errors :

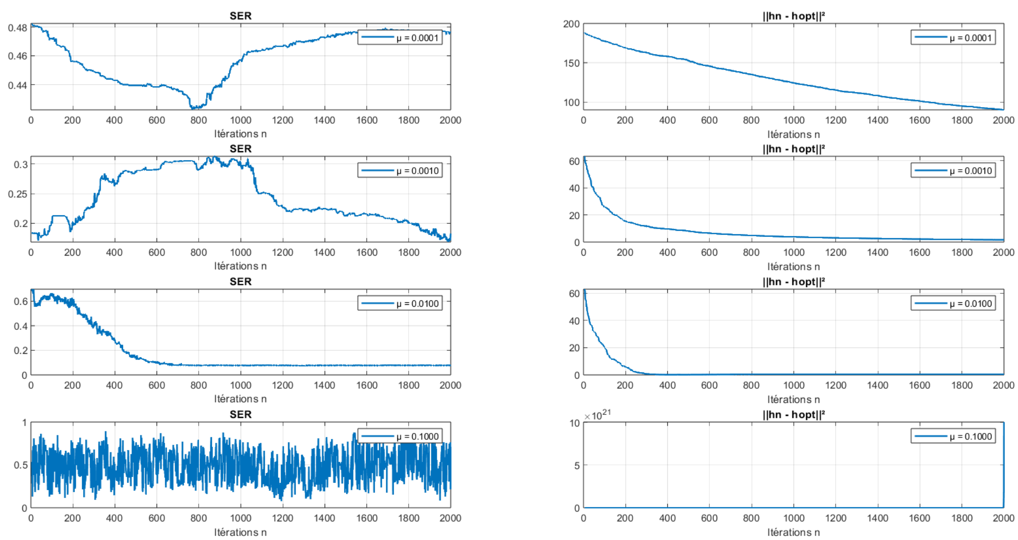

FIGURE 9 – SER and the filter error for different μ and $\sigma_u^2 = 0.001$

We observe that the larger the learning rate, the faster the algorithm converges. However, if the step size is too large, the algorithm completely diverges, and the symbols become entirely unpredictable.

We now vary the noise variance added to the signal while keeping a fixed step size of 0.01.

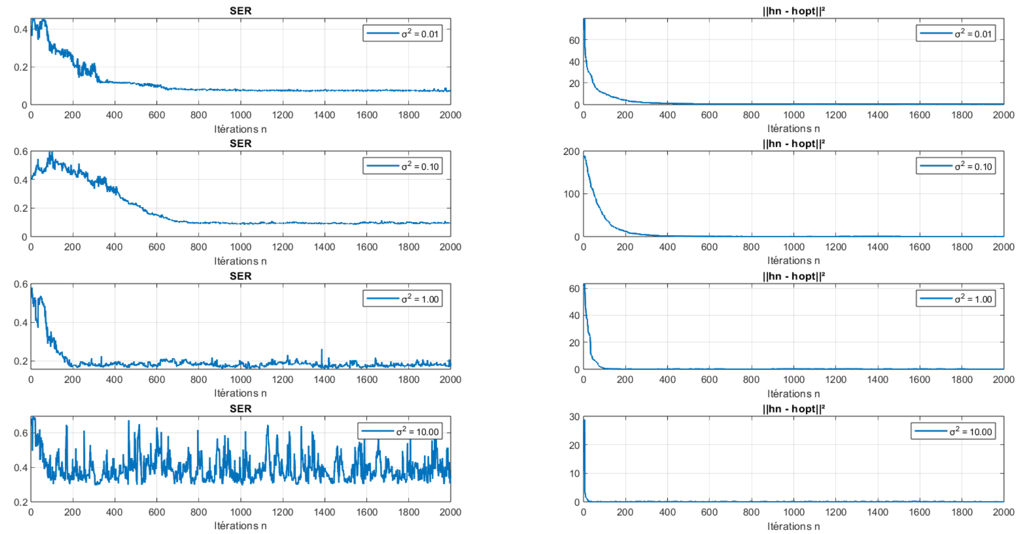

FIGURE 10 – SER and the filter error for different σ_u^2 and $\mu = 0.01$

We observe that noise does not affect the convergence of the algorithm, but it does impact the SER. For relatively low variance values, the SER does not change significantly. However, for a variance of 10, the signal becomes unpredictable.