AICC II

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Contents

List of lectures

Chapter 1

Introduction

1.1 About this course

In this course, there will be three main topics that will be studied:

- Communication
- Information and Data science
- Cryptography, Secrecy, Privacy

1.2 Cours Grading

- 90% Final exam during exam period
- 10 % Quizzes (online on Moodle)
  - There will be 6 quizzes. BO5
  - On the quizzes, you can update your answer as many times as you want before the deadline
- Quizzes are highly coorelated with homework.

1.2.1 How to be efficient and do well in this course

Before class:

- Browse through the slides to know what to expect
- review the background material as needed

After class:

- read the notes: they are the reference
- do the review questions

Before the exercice session

- are you up to date with the theory?
- Solve what you can ahead of time and finish during the exercice session
- write down **your** solution

February 18, 2025 — **Lecture 1 : Discrete Probability**

Chapter 2

Entropy

2.1 Initial case: Finite Ω: set of all possiblie outcomes

Definition 1 ***Sample space*** Ω *is the set of all possible outcomes*

Definition 2 ***Event*** E*: a subset of* Ω*. Since the outcomes are equally likely:*

$$p(E) = \frac{|E|}{|\Omega|}$$

2.2 Conditional Probability

Conditional probability

Definition 3 *The* ***conditional probability*** $p(E|F)$ *is the probability that* E *occurs, given that* F *has occured (hence assuming that* $|F| \neq 0$*) :*

$$p(E|F) = \frac{|E \cap F|}{|F|}$$

Independent Events

Event E and F are called **independent** if $p(E|F) = p(E)$

Personal remark

this means that even if we know that F has occured the probability of E is still the same.

General Case: Finite Ω, arbitary $p(\omega)$

Having equally likely outcomes is pretty rare in real life, juste take two dices and do the sum of the result and you will se that all the possible outcome doesn't have the same probability. In order to express those types of distribution we use the probability mass function:

Definition 4 ***Sample space*** Ω*: set of all possiblie outcomes*
Probability distribution (probability mass function) p *:*
A function $p : \Omega \to 1$ *such that:*

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

If we sum up all the probablity it gives us 1.

muss function to a subset

Given $E \subset \Omega$ we can define the domain of the probability mass function p is extended to the power set of Ω :

$$p(E) = \sum_{\omega \in E} p(\omega)$$

2.3 Conditional probability and Independent Events

General form

The general form for the conditional probability is:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

for F such that $p(F) \neq 0$

Independet events

As before E and F are called independent if $p(E|F) = p(E)$, Equivalently, E and F are independent iff $p(E \cap F) = p(E)p(F)$.

Disjoin event

if E_1 and E_2 are disjoint event then:

$$p(E_1 \cup E_2) = p(E_1) + p(E_2)$$

Law of total probability

For any $F \subseteq \Omega$ and its complement F^c,

$$p(E) = p(E|F)p(F) + p(E|F^c)p(F^c)$$

which sounds very intuitive because by definition F and F^c are disjoint.

Generally

Theorem 1 *If* Ω *is the union of disjoint event* $F_1, F_2, \ldots, F_n$ *then:*

$$p(E) = p(E|F_1)p(F_2) + p(E|F_2)p(F_2) + \cdots + p(E|F_n)p(F_n)$$

Proof

We prove the law of total probability for $\Omega = F \cup F^c$ (the general case follows straighforwardly)

$$\begin{aligned} p(E) &= p((\underbrace{E \cap F) \cup (E \cap F^c)}_{\text{union of disjoint sets}}) \\ &= p(E \cap F) + p(E \cap F^c) \\ &= \frac{p(E \cap F)}{p(F)} p(F) + \frac{p(E \cap F^c)}{p(F^c)} p(F^c) \\ &= p(E|F)p(F) + p(E|F^c)p(F^c) \end{aligned}$$

Bays' Rule

Theorem 2

$$p(F|E) = \frac{p(E|F)p(F)}{p(E)}$$

Proof We use the definition of conditional probability to write $p(E \cap F)$ two ways and solve for $p(F|E)$:

$$p(F|E)p(E) = p(E \cap F) = p(E|F)p(F)$$

2.4 Random variable

Random variable

Definition 5 *A Random variable is a function X such as $X : \Omega \to \mathbb{R}$*

Probability distribution

p_x, $p_x(X = x)$ or $p_x(x)$ is the probability that $X = x$, i.e, the probability of the event

$$E = \{\omega \in \Omega : X(\omega) = x\}$$

Hence,

$$p_x(x) = \sum_{w \in E} p(\omega)$$

Example You rolle a dice.
if the outcome is 6, you receive 10CHF. Otherwise, you pay 1 CHF.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$\text{For each } \omega, p(\omega) = \frac{1}{6}$$

Then define:

$$X(\omega) = \begin{cases} 10, & \omega = 6 \\ -1, & \omega \in \{1, 2, 3, 4, 5\} \end{cases}$$

Hence, we have

$$p_x(X) = \begin{cases} \frac{1}{6}, & x = 10 \\ \frac{5}{6}, & x = -1 \end{cases}$$

2.4.1 Two random variables

Two random variables

Definition 6 *Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be two random variables. The probability of the event $E_{x,y} = \{w \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}$ is:*

$$p_{x,y}(x, y) = \sum_{w \in E_{x,y}} p(\omega)$$

- p_x is called **marginal distribution** (of $p_{x,y}(x, y)$ with respect to x)
- p_y can be computed similarly

2.5 Expected Value

Expected value

Definition 7 *The expected value $\mathbb{E}[X]$ of a random variable $X : \Omega \to \mathbb{R}$ is :*

$$\mathbb{E}[X] = \sum_{\omega} X(\omega)p(\omega)$$
$$= \sum_{x} xp_x(x)$$

linearity

Expectation is a linear operation in the folowwing sence:
Let $X_1, X_2, \ldots, X_n$ be random variables and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be scalars. Then:

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\alpha_i\right] = \sum_{i=1}^{n} \alpha\mathbb{E}[X_i]$$

Random variable and indepedecy

Two random variable X and Y are independent if and only if, for all realizations x and y:

$$p(\{X = x\} \cap \{Y = y\}) = p(\{X = x\})p(\{Y = y\})$$

Or, more concisely, iff

$$p_{x,y}(x, y) = p_x(x)p_y(y)$$

Generalization

Theorem 3 *Given n random variables, $X_1, \ldots, X_n$ are independent if and only if:*

$$p_{x_1,\ldots,x_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_{x_i}(x_i)$$

Condition probability

The conditional distrivution of Y given X is the function:

$$p_{x,y}(x|y) = \frac{p_{x,y}(x, y)}{p_x(x)}$$

Independent random variables

The folowig statement are equivalent to the statemant that X and Y are two indepedent random variables:

- $p_{x,y} = p_x p_y$
- $p_{y|x}(y|x) = p_y(y)$
- $p_{y|x}(y|x) = p_y(y)$ is not a function of x
- $p_{x|y}(x|y) = p_x(x)$

- $p_{x|y}(x|y)$ is not a function of y

Summary 1
- *Random Variable*
- *Probability distribution*
  - *Joint distribution of multiple variables*
  - *Marginal distribution*
  - *Conditional distribution*
- *Independence*

February 19, 2025 — **Lecture 2 : Source and entropy**

Expected value and operation

The addition works well with Expectation such that

$$\mathbb{E}[X+Y] = \mathbb{E}[x] + \mathbb{E}[Y]$$

However, the product doesn't work well,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

if and only if X and Y are independent random variables.

2.6 Entropy

Introduction

We communicate be revealing the value of sequence of variables that we call (**Symbols**), **Information**
In modern language, Hartley was saying that the value of a symbole provides information if and only if the symbol is a **random variable**.
How much information is carried by a symbol such as S?

- Suppose that $S \in \mathcal{A}$ is a symbol that can take $|\mathcal{A}|$ possible values
- The amount of information conveyed by n such symbol should be n times the informations conveyed
- there are $|\mathcal{A}|^n$ possible values for n symbols
- This suggests that $\log|\mathcal{A}|^n = n\log|\mathcal{A}|$ is the appropriate mesure for information

However, this approach doesn't works:

Example Imagine having a town where there are 360 days and 5 rainy days, this leads to have only to possibilities, $|\mathcal{A}| = 2$ which make the quantity of information $\log_2 2 = 1$ bits. Which intiutively sounds kind of false, the forecast doesn't give us that information knowing that it is sunny $\frac{360}{365}$ % of the times, it is kind of excpected.

An article in 1948 from Shannon fixes the problem by defining **Entropy**

Definition

the **uncertainty** or **entropy** $H(S)$ associated to a discrete random variable S:

Definition 8

$$H_b(S) = -\sum_{S \in supp(p_s)} p_s(s) \log_b p_s(s)$$

Where $supp(s) = \{s : p_s(s) > 0\}$.

Few comments

$$H_b(S) = -\sum_{S \in supp(p_s)} p_s(s) \log_b p_s(s)$$

- The condition $S \in supp(p_s)$ is needed because $\log_b p_s(s)$ is not define when $p_s(s) = 0$ this convention allows us to use the notation :

$$H_b(S) = -\sum_{s \in \mathcal{A}} p_s(\log_b p_s(s))$$

- The choice of b determines the unit, $b = 2$ is the **bit**

We also can see this as an "*average*" of $-\log_b p_s(S)$ which is:

$$H(S) = \mathbb{E}[-\log_b p_s(S)]$$

Example

A sequence of 4 decimal digits, s_1, s_2, s_3, s_4 representing the number to open Anne's lock can be senn as the output of a source S_1, S_2, S_3, S_4 with $S_i = \{0, \ldots, 9\}$.
If Anne picks all digits at randm and indepedently, the all outcomes are equally likely:

$$p_{S_1,S_2,S_3,S_4}(S_1, S_2, S_3, S_4) = \frac{1}{10^4}$$

If we search the entropy of this we get:

$$H_2(S) = \log_2 |\mathcal{A}| = \log_2 10^4 \approx 13.3 \; bits$$

2.6.1 Information-Theory Inequality

Lemma (IT-Inequality)

lemme 1 *For a positive real number* r,

$$\log_b r \leq (r-1) \log_b(e)$$

with equality if and only if $r = 1$

This proof juste using the deriative

Entropy Bounds

Theorem 4 *The entropy of a discrete random variable $S \in \mathcal{A}$ satisfies:*

$$0 \leq H_b(S) \leq \log_b |\mathcal{A}|$$

With equality on the left if and only if $p_s(S) = 1$ and on the right if and only if $p_s(S) = \frac{1}{|\mathcal{A}|}$ for all s.

2.6.2 Random variables and Entropy

***n* random variable**

the formula for entropy can be expanded to any number of random variables. If X and Y are two discrete random variables, with (joint) probability distribution $p_{x,y}$ then:

$$H(X,Y) = -\sum_{(x,y)\in X\times Y} p_{x,y}(x,y) \log p_{x,y}(x,y)$$

1.4 of textbooks

Theorem 5 *Let $S_1, \ldots, S_n$ be discrete random variables. Then*

$$H(S_1, S_2, \ldots, S_n) \leq H(S_1) + H(S_2) + \cdots + H(S_n)$$

With equality if and only if $S_1, \ldots, S_n$ are indepedent.

February 25, 2025 — **Lecture 2 : suite**

Ex hat party 1950

- n men, all have the same hat
- they throw hats in a corner
- leaving, they randomly take a hat

Solution

Let $R_i = \begin{cases} 1, & \text{if person} i \text{leaves with their own hat} \\ 0, & \text{otherwise} \end{cases}$

Entropy

$$H_2(S) = \sum_i p(s) \log \frac{1}{2p(s)} \tag{2.1}$$

$$= \frac{1}{8} \log_2 \frac{8}{2} + \frac{1}{8} \log_2 8 \tag{2.2}$$

$$\approx \frac{1}{8} + \frac{1}{8} \cdot 3 \tag{2.3}$$

personal remark

We can see it as an average of "surprise".
Where the average is the randomness. (≈ 0.55)

2.6.3 Entropy bounds

Bound

$$0 \leq H_b(S) \leq \log_b \mathcal{A}$$

2.7 Source Coding Purpose

Source coding is often seen as a way to compress the source.
More generally, the foal of source coding is to efficiently describe how much information there is to a *file*

2.7.1 Setup

Setup

The **encoder** is specified by: :

- the input alphabet$\mathcal{A}$ (the same as the source alphabet)
- the output alphabet $\mathcal{D}$(typically $\mathcal{D} = \{0, 1\}$);
- the codebook $\mathcal{C}$ Which consists of finite sequences over $\mathcal{D}$;
- By the one to one encoding map $\Gamma : \mathcal{A}^k \to \mathcal{C}$ where k is a positive integer.

For now, $k = 1$.

Example

For each code, the encoding map Γ is specified in the following table: A mettre une image.

Example Code C or B are uniquely decodable : (A mettre une image 106)

Prefix Free codes

Definition 10 *If no codeword is a prefix of another codeword, the code is said to be prefix free.*

Example The codeword **01** is a prefix of **01**1.

- A prefix free code is always uniquely decodable
- A uniquely decodable code is **not necessarily** prefix free

A prefix code A prefix free code is also called instantaneous code :

- Think of phone numbers
- Think about streaming: instantaneous codes minimize the decoding delay (for given codeword length)

Code for one random variable

We start by considering codes that encode **one single random variable** $S \in \mathcal{A}$.
To encode a sequence $S_1, S_2, \ldots$ of random variables, we encode one random variable at a time.

Complete tree of a code

Slide 113 screen.

Binary tree

- There is a root (the beginning)
- A vertex (another node)

- A **leaf** is the last vertex
- Which is like a (arbre généalogique)

Ternary Tree The same as a binary tree but with three children.

With/Without prefix slide 115.

Decoding tree

- Obtained from the complete tree by keeping only branches that form a codeword
- Useful to visualize the decoding process

Slide 116

2.7.2 Codeword length

- The codeword length is defined the obvious way:
- Example: *ct*

$\mathcal{A}$
Γ_B
codeword lengths
a
0
1
b
10
2
c
110
3
d
1110
4 height

- We would like the average codeword length to be as small as possible.

2.7.3 Kraft McMillan

Part 1. Necessary condition for the code to be uniquely decodable

Theorem 6 *If a D-ary code is uniquely decodable then its codeword length $i_1, \dots, i_M$ satisfy*

$$D^{-l_1} + \cdots + D^{-l_M} \leq i$$

Kraft's inequality

Example For code O we have :

$$2^{-2} + 2^{-2} + 2^{-2} + 2^{-2} = 1$$

Recall Kraft McMillan

Theorem 7

Example A

For code A we have $2^{-1} + 2^{-2} + 2^{-2} + 2^{-2} = 1.25 > 1$.
KRaft-McMillan's inequality is not fulfilled.
There exists no uniquely decodable code with those codeword lengths.

Proof of K-MM Part I

We prove a slightly weaker result, namely that the codeword lengths of prefix free codes satisfy K-MM inequality.
Let $L = \max_i l_i$ be the complete tree's depth.

- There are D^L terminal leaves
- There are D^{L-l_i}
- No two codewords share a terminal leaf (The code is prefix free)
- Hence $D^{L-l_i} + D^{L-l_2} + \cdots + D^{L-l_m} \leq D^L$

After dividing both sides by D^L we obtain Kraft's inequality:

$$D^{-l_1} + D^{-l_2} + \cdots + D^{-l_M} \leq 1$$

Exercice

What is the **converse** of Kraft McMillan part 1?
The **Converse** of Kraft McMillan part 1 is not true (Consider e.g. two codewords: 01 and 0101)
However, the following statement is almost as good :

Theorem 8 *If the positive integer $I_1, \ldots, I_M$ satisfy Kraft's inequality for some positive integer D,then there exists a D-ary **prefix free code** (hence uniquely decodable) that has codewords*

This says that if the inequality is true, then we **can** find D such that there exists a binary prefix which makes it decodable **and** prefix free!

2.7.4 Important Consequence of Kraft McMillan

Part I

Theorem 9 *If a **D-ary code is uniquely decodable**, then its codeword length $I_1, \ldots I_M$ satisfy Kraft's inequality :*

$$D^{-l_1} + \cdots + D^{-l_M} \leq 1$$

Part II

Theorem 10 *If the positive integer $l_1, \ldots, l_M$ satisfy Kraft's inequality for some positive integer D, then there exists a D-ary **prefix free code** that has those codeword lengths.*

The Kraft McMillan theorem implies that any uniquely decodable code can be substituted by a prefix free code of the same codeword lengths.

Prefix free codes

Our focus will be on prefix free codes. Reasons:

- No loss of optimality: codewords can be as short as for any uniquely decodable code;
- a prefix free codeword is recognized as soon as its last digit is seen:

- important, e.g. a phone number;
- advantageous to limit the decoding delay in, say streaming

Average Codeword length

- The typical use of a code is to encode a sequence of random variables
-

Example

$$\mathcal{A} = \{a, b, c, d\} \quad D = 2$$

Blackboard with table *cct*

s ∈A	a	b	c	d
$\Gamma(s)$	0	10	110	1111
$l(s)$	1	2	3	4
$p(s)$	0.05	0.05	0.1	0.8

$$\mathcal{E}[\text{length}] = 0.05 + 1 + 0.05 \cdot 2$$

Definition 11 *Let $l(\Gamma(s))$ be the length of the codeword assiociated to $s \in \mathcal{A}$ The average codeword length is:*

$$L(S, R) = \sum_i p_s(s) i(\Gamma(s))$$

Units

The unit of $L(S, \Gamma)$ are **code symbols**
When $D = 2$, the unit of $L(S, \Gamma)$ are bits.

Average codeword length: Lower Bound

Theorem 11 *Let $\Gamma : \mathcal{A} \to \mathcal{C}$ be the encoding map of a D-ary*

Proof We want to prove that:

$$\begin{aligned} H(s) - \sum_s p(s)l(s) & \\ &= -\sum_s p(s)\log p(s) - \sum_s p(s)l(s) \\ &= -\sum_s p(s)\log p(s) - \sum_s p(s)\log 2^{l(s)} \\ &= -\sum_s p(s)\log(p(s)\cdot 2^{l(s)}) \leq \dots \end{aligned}$$

Therefore:

$$\begin{aligned} &= \sum_s p(s)\log(\frac{1}{p(s)}2^{-l(s)}) \\ &\leq \sum_s p(s)\left(\frac{1}{p(s)}2^{-l(s)} - 1\right)\cdot C \\ &= (\sum_s 2^{-l(s)} - \sum_s p(s))\cdot C \\ &\leq 0 \end{aligned}$$

We know that the left side is less or equal to 1 because of the Kraft Inequality, therefore it is bounded.

February 26, 2025 — **Lecture 4 : Continue**

Key observation

The right hand side of:

$$L(S,\Gamma) = \sum_{s\in\mathcal{A}} p(s)l(\Gamma(s))$$

$$H_D(S) = \sum_{s\in\mathcal{A}} p(s)\log_D \frac{1}{p_S(s)}$$

are identical if $l(\Gamma(s))$

- Unfortunately $l(\Gamma(s)) = \log_D \frac{1}{p_S(s)}$ is often not possible (not an integer)
- How about choosing

Theorem 12 • *For every random variable $S \in \mathcal{A}$*

Theorem

Theorem 13 *The average codeword length of a D-ary Shannon-Fano code for the random variable S fulfils:*

$$H_D(S) \leq L(S,\Gamma_{SF}) < H_D(S) + 1$$

Proof it suffices to prove the upper bound (we have already proved the lower bound)
First suppose that we could use $l_i = -\log p_i$. The average

length would be:

$$L(S,\Gamma) = \sum_i p_i l_i = \sum_i p_i(-\log_D p_i) = H_D(S)$$

Instead we use $l_i = \lceil -\log p_i \rceil < -\log p_1 + 1$

March 4, 2025 — **Lecture 5 : Conditional Entropy**

Key Idea

Pack multiple symbols into " *supersymbols*"

- $(S_1, S_2, S_3, \ldots, S_n)$
- Now, apply our Main result to such supersymbols

Theorem 14 *The average codeword-length of a uniquely decodable code Γ for S must satisfy:*

$$H_D(S_1, S_2, \ldots, S_n) \leq L((S_1, S_2, \ldots, S_n), \Gamma)$$

And there exists a uniquely decodable code Γ_{SF} satisfying:

$$L((S_1, S_2, \ldots, S_n), \Gamma_{SF}) < H_D(S_1, S_2, \ldots, S_n) + 1$$

Our Next Nugget

Understand

Example

Audio recording:

- We can easily anticipate the next image in a video, there

KEy(simple) Independent

Definition 12 *The source models a seuquence $S_1, S_2, \ldots, S_n$ of n coin flips*
So $S_i \in \mathcal{A} = \{H, T\}$ where H stands for heat, T for tails.
$p_{S_i}(H) = p_{S_i}(T) = \frac{1}{2}$ for all $(s_1, S_2, \ldots, S_n) \in \mathcal{A}^n$

Not independent

Definition 13 *The source models a sequence $S_1, S_2, \ldots, S_n$ of weather conditions.*
So $S_i \in \mathcal{A} = \{S, R\}$, where S stands for sunny and R for rainy
The weather on the first day is uniformly distributed in $\mathcal{A}$.
For all other days, with probability $q = \frac{6}{7}$ the weather is as for the day before

Conditional Probability

Recall how to determine the conditional probability:

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

It gives the probability of the event $X = x$, given that the event $Y = y$ has occured.
it is defined for all y for which $p_Y(y) > 0$

Remark

There is good slide with good schema in slide 176-179

Conditional Expectation of X given $Y = y$

$$p_{X|Y}(\cdot \mid y)$$

is the probability distribution of the alphabet of X, juste like $p_x(\cdot)$

Definition 14 *The conditional expectation of X given $Y = y$ is defined as:*

$$\mathcal{E}[X \mid Y = y] = \sum_{x \in \mathcal{X}}$$

Conditional Entropy of X given $Y = y$

$p_{X|Y}(\cdot \mid y)$ is a probability distribution on the alphabet of X, juste like $p_X(\cdot)$
Every probability distribution has an entropy associated to it:

- $p_x(\cdot) \to H(X)$
- $p_{X|Y}(\cdot \mid y) \to H(X \mid Y = y)$

Definition 15 *The conditional entropy of X given $Y = y$ is defined as:*

$$H_D(X \mid Y = y) = -\sum_{x \in \mathcal{X}} p_{X|Y}($$

Example A faire

Entropy Bounds

Theorem 15 *The conditional entropy of a discrete random variable $X \in \mathcal{X}$ conditioned on $Y = y$ satisfies:*

$$0 \leq H_D(X \mid Y = y) \leq \log_D \mid \mathcal{X} \mid$$

With equality on the left iff $p_{X|Y}(x, y) = 1$ for some x, and with equality on the right iff $p_{X|Y}(x \mid y) = \frac{1}{|\mathcal{X}|}$

The proff is identical to our proof of the basic entropy bounds

Example

Question?
Do we also have the following entropy bound:

$$H_D(X \mid >= y) \overbrace{\leq}^{???} H_D(X)?$$

Answer: no.

Example (Or " *counterexample*" if better), Juste for ease of calculation, let us set $\delta = 0$ (but this is not necessary for the example to work). Then, we have:

$$H_D(X \mid Y = 0) h_D(\varepsilon) \text{ and } H_D(X \mid Y = 1) = 0$$

where $h_d(\cdot)$ is the binary entropy function (with $\log_D(\cdot)$). But we have:

$$H_D(X) = h_D(\frac{1 - \varepsilon}{2})$$

Conditional entropy can either go up or down (if we give the answer the entropy is 0)

Conditional Entropy of X given Y

The most useful and impactful definition is the *average* conditional entropy of X given $Y = y$, averaged over all values of y under the marginal distribution $p_Y(y)$. Formally, we thus define:

Definition 16 *The conditional entropy X given Y is defined as:*

$$H_D(X \mid Y) = \sum_{y \in \mathcal{Y}} p_Y(y) \left(- \sum_{x \in \mathcal{X}} p_{X|Y}(x \mid y) \log_D p_{X|Y}(x \mid y) \right)$$

Example

For the Bit flipper channel, we have;

$$H_D(X \mid Y) = p(Y = 0) H_D(X \mid Y = 0) + p(Y = 1) H_D(X \mid Y = 1)$$

We search now:

$$H(X \mid Y) = p(Y \text{ is Head}) H(XY \text{ is head}) + p(Y \text{ is Tail}) H(X \mid Y \text{ is tail}) = \frac{1}{2} \cdot$$

Conditional Entropy of X given Y

Theorem 16 *The conditional entropy of discrete random variable $X \in \mathcal{X}$ conditioned on Y satisfies:*

$$o \leq H_D(X \mid Y) \leq \log_D \mid \mathcal{X} \mid$$

With equality on the left iff for every y there exists and y such that $p_{X|Y}(x \mid y) = 1$ and with equality on the right iff $p_{X|Y}(x \mid y) = \frac{1}{|\mathcal{X}|}$ for all x and all y.

This follows directly from our bounds on $H_D(X \mid Y = y)$

Having $p_{X|Y}$

We know that $p(X \mid Y) = \frac{1}{|\mathcal{X}|}$ for all y.

$$\begin{aligned} p(x) &= \sum_{y \in \mathcal{Y}} p(y) p(x \mid y) \\ &= \sum_{y} p(y) \frac{1}{\mid \mathcal{X} \mid} \\ &= \frac{1}{\mid \mathcal{X} \mid} \cdot \sum_{y} \overbrace{p(y)}^{=1} \end{aligned}$$

Conditioning Reduces Entropy

The following bound is important and impactful (and also intuitively pleasing!)

Theorem 17 *For any two discrete random variables X and Y,*

$$H_D(X \mid Y) \leq H_D(X)$$

with equality iff X and Y are independent random variables

In words, **On average**, the uncertainty about X can only become smaller if we know Y.

As we have seen, this is not true point-wise: We may have $H_D(X \mid Y = y) > H_D(X)$ for some values of y.
It works only on average.

Proof

$$\begin{aligned}
H(X \mid Y) - H(X) = & \\
&= \sum_y p(y) \left(- \sum_x p(x \mid y) \log p(x \mid y) \right) + \sum_x p(x) \log p(x) \\
&= \sum_{x,y} p(y) p(x \mid y) \log \frac{1}{p(x \mid y)} + \sum_{x,y} p(y \mid x) p(x) \log p(x) \\
&= \sum_{x,y} p(x,y) \log \frac{p(x)}{p(x \mid y)} \\
&\leq \sum_{x,y} p(x,y) \left(\frac{p(x)}{p(x \mid y} - 1 \right) \cdot \log e \\
&= \sum_{x,y} (p(x)p(y) - p(x,y)) \log(e) \\
&= \left(\left(\sum_y p(x)p(y) \right) - \left(\sum_x p(x)p(y) \right) \right)
\end{aligned}$$

Conditional Entropy of $f(x)$

Let X be an arbitrary random variable. Let $f(x)$ be a (deterministic) function of x.

$$H(f(x) \mid X) = 0$$

Proof To find this conditional entropy:
Let $Y = f(x)$

$$p(y \mid y) = \begin{cases} 1, & y = f(x) \\ 0, & y \neq f(x) \end{cases}$$

the probability that y is $f(x)$ is only true if $f(x) = y$.
This implies that the entropy is equal to 0:

$$H(y \mid x) = 0$$

Conditioning reduced Entropy

A generalization of the previous bound is also interest to us:

Theorem 18 *For any three discrete random variables X, Y and Z,*

$$H_D(X \mid Y, Z) \leq H_D(X \mid Z)$$

With equality iff X and Y are conditionally independent random variables given Z (that is, if and only if $p(x, y \mid z) = p(x \mid z)p(y \mid z)$ for all x, y, z,

You can see it as make the Z fall which makes it $p(x, y) = p(x)p(y)$

Proof It is only mathematics:

$$\begin{aligned} H_D(X \mid Y, Z) - H_D(X \mid Z) &= \mathbb{E}\left[\log_D \frac{1}{p_{X|Y,Z}(X \mid Y, Z)}\right] + \mathbb{E}[\log_D p_{X|Z}(X \mid Z \\ &= \mathbb{E}\left[\log_D \frac{p_{X|Z}(X \mid Z)}{p_{X|Y}(X \mid Y, Z)}\right] \\ &= \mathbb{E}\left[\log_D \frac{p_{X|Z}(X \mid Z)p_{Y|Z}(Y \mid Z)p_Z(Z)}{niquesamere}\right. \end{aligned}$$

March 4, 2025 — **Lecture 6 : Conditional Entropy review**

Main definitions

We have here two mains definitions:
The entropy for for a "*case*" of a random variable:

$$H(X \mid Y = y) = -\sum_x p(X \mid y) \log p(X \mid y)$$

And, the conditional entropy on a random variable:

$$\begin{aligned} H(X \mid Y) &= \sum_y p(y) H(X \mid Y = y) \\ &= -\sum_y \sum_x p(x, y) \log p(x \mid y) \end{aligned}$$

The main thing to understand here is that $H(X \mid Y)$ is the *Generalization* of the first definition. It is all the possible values of Y together. This is why we sum up all possible value of y. The second way to write $H(X \mid Y)$ is like taking all the possible pairs together and calculating the entropy of each pairs.
Main Result: The main result behind this is:

$$\begin{aligned} 0 \leq H(X \mid Y = y) &\leq \log \mid \mathcal{X} \mid \\ 0 \leq H(X \mid Y) &\leq \log \mid \mathcal{X} \mid \end{aligned}$$

And the inequality:

$$H(X \mid Y) \leq H(X)$$

Conditional entropy of $f(x)$

Let X be an arbitrary random variable.
Let $f(x)$ be a (deterministic) function of x:

$$H(f(x) \mid x) = 0$$

For example:

$$\begin{aligned} X &\in \{0,1,2,3\} \\ f(x) &= X \mod 2 \end{aligned}$$

Which is:

$$f(x) = \begin{cases} 0 \text{ if x is even} \\ 1 \text{ if x is odd} \end{cases}$$

Then,

$$P(f(x) \mid X) = \begin{cases} 0, \text{ if } x = 0,2 \\ 1, \text{ if } x = 1,3 \end{cases}$$

If we now compute the entropy for $X = 0$ and $X = 1$ etc..., we get:

$$\begin{aligned} H(f(x) \mid X = 0) &= 0 \\ H(f(x) \mid X = 1) &= 0 \\ &\vdots \end{aligned}$$

Lisa rolls two dice

Lisa rolls two dice and annoucnes the sum L written as a two digit number. The alphabet of $L = L_1L_2 = \{02, 03, 04, 05, 06, 07, 08, 09, 10, 11, 12\}$ Where the alphabet of $L_1 = \{0, 1\}$ and the alphabet of $L_2 = \{0, 1, 2, \ldots, 9\}$.
We are looking for the probability that $L_2 = 2$ knowing that $L_1 = 1$:

$$p_{L_2|L_2}(2 \mid 1)$$

What we are doing here is the joint distribution:

$$p_{L_2|L_1}(2 \mid 1) = \frac{P_{L_1,L_2}(1,2)}{P_{L_1}(1)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

After running over all possible values for (i, j), we obtain:

$L_2 = j$ $L_1 = i$	0	1	$p_{L_2}(j)$
0	0	$\frac{3}{36}$	$\frac{3}{36}$
1	0	$\frac{2}{36}$	$\frac{2}{36}$
2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{2}{36}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$p_{L_i}(i)$	$\frac{5}{6}$	$\frac{1}{6}$	height

now you can do the same with conditional probability and then after computing all those value:

$$H(L_2 \mid L_1) = \frac{5}{6} \cdot 2.857 + \frac{1}{6} \cdot 1.459 = 2.624 \text{ bits}$$

Now we can observe that:

$$2.624 = H(L_2 \mid L_1) \leq H(L_2) = 3.22$$

Which says that on average, knowing something takes out some randomness

The chain rule for entropy

Recall that the joint entropy of two random variables X, Y is completely naturally defined as:

$$H_D(X,Y) = -\sum_x \sum_y p_{X,Y}(x,y) \log_D p_{X,Y}(x,y)$$

Or as seen earlier:

$$H_D(X,Y) = H_D(X) + H_D(Y)$$

Using the fact the $p_{X,Y}(x,y) = p_X(x)p_{Y|X}(y \mid x)$, we can write this as:

$$\begin{aligned}
H_D(X,Y) &= -\sum_x p_X(x) \left(\sum_Y p_{Y|X}(y \mid x) \log_D (p_X(x) p_{Y|X}(y \mid x)) \right) \\
&= -\sum_x p_X(x) \left(\sum_y p_{Y|X}(y \mid x) (\log_D p_X(x) + \log_D p_{Y|X}(y \mid x)) \right) \\
&= -\sum_x p_X(x) \left[\left(\sum_y p_{Y|X}(y \mid x) \log_D p_X(x) \right) + \left(\sum_y p_{Y|X}(y \mid x) \log_D p_{Y|X}(y \mid x) \right) \right] \\
&= H(X) + H(Y \mid X)
\end{aligned}$$

Theoreme 19

$$H(X,Y) = H(X) + H(Y \mid X)$$

Professor remark

Firstly:

$$H(Y,X) = H(X,Y)$$

Which is either proved by:

$$\begin{aligned}
H(Y,X) &= H(Y) + H(X \mid Y) \\
&= H(X,Y)
\end{aligned}$$

The relation proved before in words it:
To find the joint entropy of two random variables, we can first calculate the entropy of one of the two, and then add to it the conditional entropy of the second, given the first.

The chain rule entropy

Theorem 20 *Let $S_1, \ldots, S_n$ be discrete random variables. Then:*

$$H_D(S_1, S_2, \ldots, S_n) = H_D(S_1) + H_D(S_2 \mid S_1) + \cdots + H_D(S_n \mid S_1, \ldots, S_{n-1})$$

The above result says that the uncertainty of a collection of random variables (in any order) is the uncertainty of the first, plus the uncertainty of the second when the first is known, plus the uncertainty of the third when the first two are know, etc...
Let us see how:

$$\begin{aligned}
&H(\underbrace{S_1, S_2, \ldots, S_{n-1}}_{=Z}, S_n) \\
&= H(Z) + H(S_n \mid Z) \\
&= H(\underbrace{S_1, S_2, \ldots, S_{n-2}}_{=Z'}, S_{n-1}) + H(S_n \mid S_1 \ldots, S_{n-1}) \\
&= H(Z') + H(S_{n-1} \mid Z') + H(S_n \mid S_{1,n-1}
\end{aligned}$$

Until we get $Z'^{\cdots'} = S_1$.

Example Let X, Y, Z be discrete random variables. We have:

$$\begin{aligned}
H(X, Y, Z) &= H(X) + H(Y \mid X) + H(Z \mid X, Y) \\
&= H(X) + H(Z \mid X) + H(Y \mid X, Z) \\
&= H(Y) + H(X \mid Y) + H(Z \mid X, Y) \\
&= H(Y) + H(Z \mid Y) + H(X \mid Y, Z) \\
&= H(Z) + H(X \mid Z) + H(Y \mid X, Z) \\
&= H(Z) + H(Y \mid Z) + H(X \mid Y, Z)
\end{aligned}$$

Theorem 21 *Let $S_1, \ldots, S_n$ be discrete random variables. Then:*

$$H(S_1, S_2, \ldots, S_n) \leq H(S_1) + H(S_2) + \cdots + H(S_n)$$

With equality iff, $S_1, \ldots, S_n$ are independent

Proof

$$\begin{aligned}
H(S_1, S_2, S_3) &= H(S_1) + H(S_2 \mid S_1) + H(S_3 \mid S_1, S_2) \\
&\leq H(S_1) + H(S_2) + H(S_3)
\end{aligned}$$

Another way around

Sometimes it is convenient to compute the conditional entropy using the chain rule for entropies. For instance:

$$H(X \mid Y) = H(X, Y) - H(Y)$$

It can be useful to make it easier to compute $H(X \mid Y)$ because on the right side, it is only marginal entropies with $p \log p$ which are *"easy to compute"*

corollaire 1

$$H(X,Y) \geq H(X)$$
$$H(X,Y) \geq H(Y)$$

The above inequalities follow from the chain rule for entropies and the fact that entropy (condition or not) is nonnegative.

Example

From lisa rolls two dice:

$$H(L_1, L_2) = 3.2744$$
$$H(L_1) = 0.6500$$
$$H(L_2) = 3.2188$$

We compute:

$$H(L_2 \mid L_1) = H(L_1, L_2) - H(L_1) = 3.2744 - 0.6500 = 2.6254$$
$$H(L_1 \mid L_2) = H(L_1, L_2)_H(L_2) = 3.2744 - 3.2188 = 0.056$$

And verify that indeed:

$$H(L_1 \mid L_2) \leq H(L-1) \leq H(L_1, L_2)$$
$$H(L_2 \mid L_1) \leq H(L_2) \leq H(L_1, L_2)$$

2.7.5 Random Processes

A.K.A Source models

Definition 17 *The source models a sequence $S_1, S_2, \ldots, S_n$ of n coin flips*
So $S_i \in \mathcal{A} = \{H, T\}$, where H stands for heads, T for tails, $i = 1, 2, \ldots, n$
$p_{S_i}(H) = p_{S_i}(T) = \frac{1}{2}$ for all i, and coin flips are independent.
Hence,

$$p_{S_1,S_2,\ldots,S_n}(S_1, S_2, \ldots, S_n) = \frac{1}{2^n}, \quad \forall (S_1, S_2, \ldots, S_n) \in \mathcal{A}^n$$

Definition 18 *The source models a sequence $S_1, S_2, \ldots, S_n$ of weather conditions.*
So $S_i \in \mathcal{A} = \{S, R\}$, where S stands for sunny and R for rainy, $i = 1, 2, \ldots, n$.
The weather on the first day is uniformly distributed in $\mathcal{A}$.
For all other days, with probability $q = \frac{6}{7}$ the weather is as for the day before

What we can see here that is the conditional probability, for example:

$$p(S_2 = \text{ sun} \mid S_1 = \text{ sun}) = q$$
$$p(S_2 = \text{ rain} \mid S_1 = \text{ sun}) = 1 - q$$

However:

$$\begin{aligned} &p(S_3 = \text{sun} \mid S_1 = \text{sun}, S_2 = \text{sun}) \\ &= p(S_3 = \text{ sun} \mid S_2 = \text{ sun}) = q \end{aligned}$$

More generally:

$$P(S_n \mid S_1, S_2, \ldots, S_{n-1}) = p(S_n \mid S_{n-1})$$

March 11, 2025 — **Lecture 7 : Entropy and algorithm**

Experience little play

- Think of something
- Ask yes or no question
- Find the answer

the game was called twenty questions in old U.S tv. We want to use entropy to understand this game.

Last Week

$$H_D(X) = H_D(P) - -\sum_x p(x) \log_D p(x)$$

We also saw those two bounds:

$$0 \leq H_D(X) \leq \log_D \mid \mathcal{A} \mid$$

Information is always about option, more options you have, more information (the first way to introduce *"entropy"*)
We also saw:

$$H(X \mid Y = y) = -\sum_x p(x \mid y) \log_D p(x \mid y)$$

$$H(X \mid Y = y) = -\sum_y \ldots$$

And we also saw that on average:

$$H(X \mid Y) \leq H(X)$$

$$H(X \mid Y, Z) \leq H(X \mid Y) \leq H(X)$$

We also saw the chain rule:

$$\begin{aligned} &H(S_1, S_2, S_3, S_4) \\ &= H(S_2, S_4, S_1, S_3) \end{aligned}$$

The order in entropy doesn't matter,

$$= H(S_1) + H(S_2 \mid S_1) + H(S_3 \mid S_1, S_2) + H(S_4 \mid S_1, S_2, S_3)$$

An intresting way to use this, is if we combine the inequalities and the chain rule. The equality on the right sight is true if and only if X and Y are independent. there fore:

$$H(S_1, S_2, S_3, S_4) = H(S_1) + H(S_2) + H(S_3) + H(S_4)$$

this equality is true if and only if S_1, S_2, S_3, S_4 are independent.

The 20 question problem

Let X be a random variable. What is the minimum number of "yes/no" question needed to identify X?, which question should be asked.

Solution

Let us consider a binary code Γ for $X \in \mathcal{X}$
Once Γ is fixed, we know $x \in \mathcal{X}$ if and only if we know the codeword $\Gamma(x)$.
The strategy consists in asking the ith question so as to obtein the ith bit of the codword $\Gamma(x)$.
The expected number of question $L(X, \Gamma)$, which is minimized if Γ is the encoding map of Huffman code

Example Suppose that we know that $\mathcal{X} = \{$ cat, dog, pony$\}$, with:

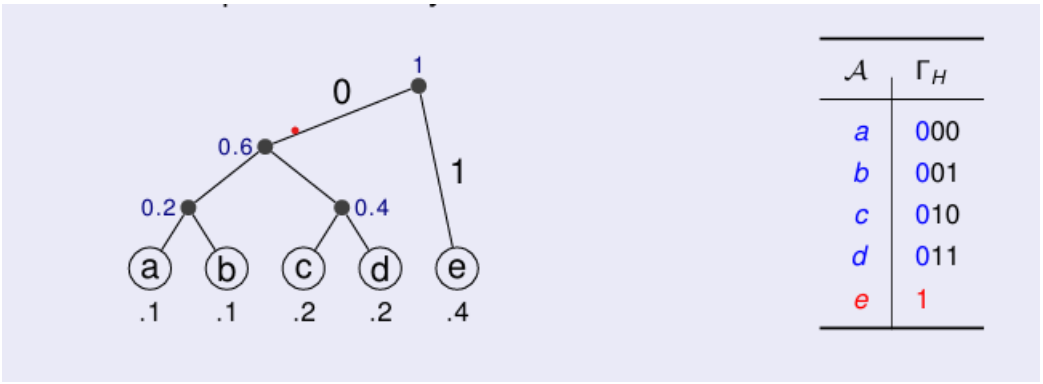
$$p(cat) = \frac{1}{2}$$
$$p(dog) = \frac{1}{4}$$
$$p(pony) = \frac{1}{4}$$

We want to make it the best way, the question we should ask is:

- is the animal a cat?

X a b c d e height

We then do a Huffman tree:



$\mathcal{A}$	Γ_H
a	000
b	001
c	010
d	011
e	1

We know here that this, will be optimal

Optimality

We have seen that a prefix free code for $X \in \mathcal{X}$ leads to a querying strategy to find the realization of X.
Similarly, a deterministic querying strategy leads to a binary prefix-free code for X. Here is why:

- Before the first question we know that $x \in \mathcal{X}$
- Without loss of generality, the first question can be formulated in terms of "is $x \in \mathcal{A}$"? for some $\mathcal{A} \subset \mathcal{X}$, (The choice of $\mathcal{A}$ is determined from the strategy, that we fix once and for all)

- Is the answer is YES, the we know that $x \in \mathcal{A} \subset \mathcal{X}$. Otherwise $x \in A^c \subset \mathcal{X}$. Either way we have reduced the size of the set that contains x.
- We continue asking similar questions until the value of x is fully determined, the we stop.

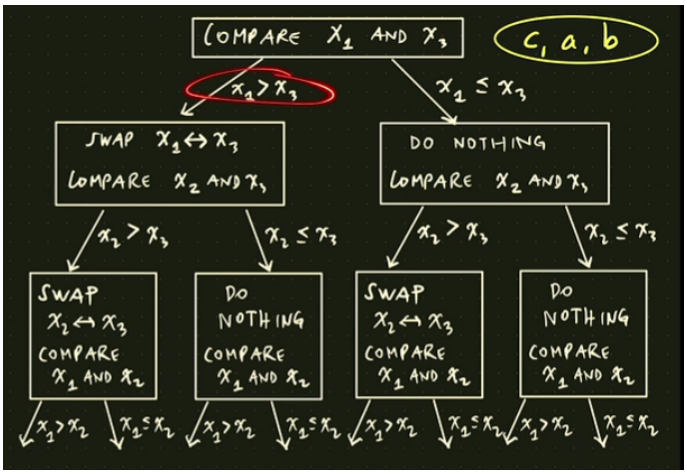
Here, the sequence of Yes or no answers is a binary codeword associated to x. The code obtained when we consider all possible values of x is a binary prefix-free code. Since the tree is prefix free, its averag codeword-length cannot be smaller than that of a Huffman code.

Sorting via pairwise comparisons

Given an **unsorted** List with n elements.
For example $l = [c, a, b]$ with $n = 3$
Repeat:

1. Select two position $1 \leq i \leq j \leq n$
2. Compare and swap:
   - If $x_i > x_j$
   - Then swap elements $x_i \iff x_j$
   - Else do nothing

One way to understands how it works:



The first observation:
The sequence of pairwise comparisons must identify the exact order of the unsorted list.
The second observation:
The sequence of pairwise comparisons in a uniquely decodable (actually, prefix-free) binary code for x.
There fore, we must have:

$$\mathbb{E}[\text{number of comparisons}] \geq H_2(X)$$

However what is the X? We see it as a random variable because we don't really know what the unsorted list is.
For example $n = 3$ we have $\mathcal{X} = \{abc, acb, bac, bca, cab, cba\}$ where $\mathcal{X}$ is the set of all permutations.
However what is $p(x)$? Here, we want to talk about our algorithm working

for all $p(x)$.

$$\begin{aligned} E \geq max_{p(x)} H_2(X) &= \log_2 \mid \mathcal{X} \mid \\ &= \log_2 n! \end{aligned}$$

We already know a bounds on factorial:

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}}$$

Therefore:

$$\begin{aligned} H_2(x) &\approx \log_2 \frac{n^n}{e^{n-1}} \\ &= n \log_2 n - (n-1) \log_2 e \end{aligned}$$

Which is *"dominated"* by $n \log_2 n$

Billard Balls

There are 14 billards balls numbered as shown:

Among balls $1 - 13$, at most one **could** be heavier/lighter than the others. What is the minimum number of weightings to simultaneously determine:

- If one ball is different
- if there is such a ball which one,
- And whether the different ball is heavier/lighter

Here we want to use entropy to solve this problem. The goal here is to associated the number of weightings to code. The goal is to see it as a tree.

The steps of picking two sets is *"mandatory"* we have to pick two sets in order to compare something, and in order to compare something, you have to compare something...
From this comparisons, there will be three possibilities. with three possibilites, We are specifying a Ternary code. The issue here is that we are losing information, yes we only get a binary tree however we wouldn't be able to have the same amount of information as with a ternary tree.
What we are saying here is, with any strategy to solve this problem **can** be written in this way. Hence we can read this tree as a ternary code.

*But a code for **What**?*

What are we finding with this code?
A code for X:

- $X = 0$: all balls are equals
- $X = +1$: ball 1 is heavier
- $\vdots$

- $X = +13$ ball 13 is heavier
- $X = -1$ balle 1 is lighter
- $\vdots$
- $X = -13$ ball 13 is lighter

Then we know that $| \mathcal{X} | = 27$. This is one way to answers those question.

1. If $X = 0$ or not (then there is or not a different ball)
2. Then $| X |$ gives us the information
3. the sign of X if the ball is heavier or lighter

Observation The number of weighings is equal to the length of the ternary codeword

Then:

Theorem 22

$$\mathbb{E}[number\ of\ weighings] \geq H_3(X)$$

It has to be three by the way the problem is stated. The code is ternary **Therefore** the base for the entropy is 3.
Moreover, our strategy must work **irrespective** of the probability distribution of X.
We can also see:

Theorem 23

$$\mathbb{E}[number\ of\ weighings] \geq max_{p(x)}(H_3(X))$$

Where in our example gives us:

$$\log_3 27 = 3$$

It doesn't need the be an integer it is only the professor that choose on purpose to make it clean

But does there indeed exists such a code

FACT:

Entropy does **not** guarantee the existence of such a strategy

Entropy serves as a lower bound and **not** the best way to do it.

But can what if?
Let us suppose it exists! Then entropy tells us a few basic facts.

Fact 1

if 3 weighings S_1, S_2, S_3 uniquely specify X, Then we **must have**:

$$H_3(X) = H_3(S_1, S_2, S_3)$$

Proof

$$H(X, S_1, S_2, S_3) = H(X) + \overbrace{H(S_1, S_2, S_3 \mid X)}^{=0}$$
$$= H(S_1, S_2, S_3) + \overbrace{H(X \mid S_1, S_2, S_3)}^{=0}$$

It is true because if we know S_1, S_2, S_3 then we know all X then the entropy of 0.
For $H(X \mid S_1, S_2, S_3)$, because S_1, S_2, S_3 uniquely specify X then knowing them implies that this entropy is o.

Fact 2

If 3 weighings S_1, S_2, S_3 uniquely specify X, then we must have:

- S_1, S_2, S_3 uniformly distributed
- S_1, S_2, S_3 independent

Proof

$$H_3(S_1, S_2, S_3) = 3$$

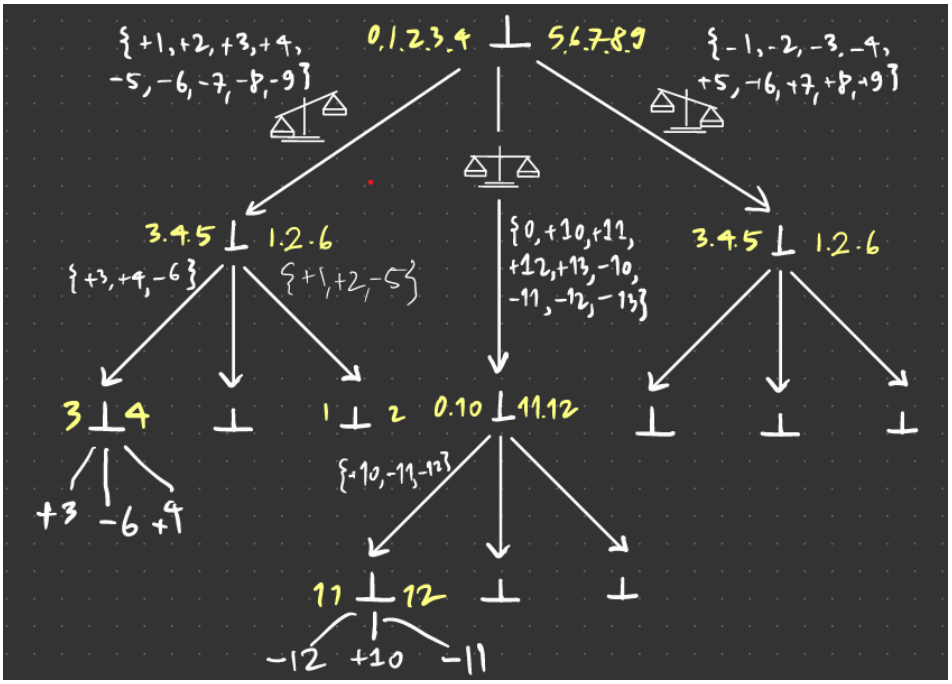
This is a *must*.
But also:

$$H_3(S_1) + H(S_2 \mid S_1) + H(S_3 \mid S_1, S_2) \leq H_3(S_1) + H(S_2) + H(S_3)$$
$$\leq \log_3 3 + \log_3 3 + \log_3 3$$

Where it is an equality if and only if the distribution is uniform and independent.

Example

Let's see how to actually find a way to ask those question:



March 12, 2025 — **Lecture 8 : Prediction, learning, and Cross-Entropy-Loss**

Billard Balls Can we use the 20 questions approach to solve the 14 bullars riddle?

Answer No, because the kind of questions that we can "ask", when wa are weighing, is quite limited.
For instance, the first question cannot be "is 1 or 2 heavy?".

Strategies But is there a strategy that requires only 3 weighings?
From source compression, we can establish the following facts?

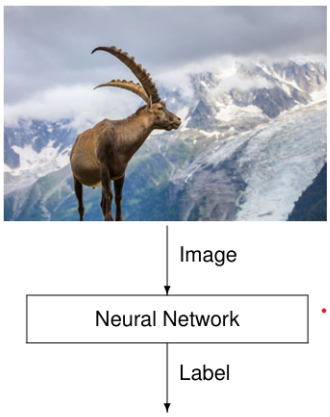
- For each weighings, the three outcomes must be equally likely
- The weighings must be independent of each other

It is because we carefully selected the numbers (alphabet size of 27; each weighing has 3 possible outcomes) that there is a strategy that exactly matches the entropy lower bound 3 weighings. If you change the numbers, it will not generally be true that there is a strategy that *exactly* matches the lower bound.

2.7.6 Prediction, Learning and cross-Entropy Loss

The goal here is to change the way to use entropy, entropy has always be seen as something that *means* something, a lower bound, a quantity of information. Here we will use it to do calculation juste like a *tool*.

Example



Label	*Probability*
Ibex	0.98
Kangaroo	0.005
Lynx	0.002
Wombat	0.002
Dog	0.001
Cat	0.001
Turtle	0.001
Dolphin	0.001
Elephant	0.001
Kookaburra	0.001
Other	0.005

There weren't probability at this time in the slide so imagine without it

The question we want to ask is, *"Is our neural network performing well"*

- Given an image $\mathcal{X}$
- Our machine (Neural network)
- Outputs $Q(x)$
- The label: Label(x)

Zero-one loss

$$L\{Q(x) \neq \text{Label}(x)\} = \begin{cases} 1 \text{ if } Q(x) \neq \text{Label}(x) \\ 0 \text{ if } Q(x) = \text{Label}(x) \end{cases}$$

Given a lot of image, we want to have a **Classification error**:

$$\frac{\sum_{\mathcal{X}} L\{Q(x) \neq \text{Label}(x)\}}{\text{number of images}}$$

Is the function of mis-labeled images.

Pros and Cons

Pros

- Very intuitive
- Interpretable

Cons

- Not differentiable

With probability

Our neural network produces:

$$Q(\text{label} \mid \text{image})$$

The true label distribution is:

$$P_{true}(\text{label} \mid \text{image}) = \begin{cases} 1, & \text{correct label} \\ 0, & \text{wrong label} \end{cases}$$

(We are assuming for simplicity that for each image, there is a single correct label).

- Ideally, we would like:

$$Q(\text{label} \mid \text{image}) = P_{true}(\text{label} \mid \text{image}) \quad \forall \text{pairs}$$

However this is only a dream

- Instead, people like to consider **cross entropy loss**
- that is, we wish ou $Q(\text{label}|\text{image})$ to **minimize**

$$L(P_{true}(\text{label} \mid \text{image}), Q(\text{label} \mid \text{image})) = -\sum_{\text{label}} P_{true}(\text{label} \mid \text{image}) \log_D Q(\text{label} \mid \text{image})$$

- Given training data (image, label), for $i = 1, 2, \ldots, n$ we select $Q(\text{ label} \mid \text{image})$ to minimize the cross entropy loss.

Cross entropy loss

$$L(P, Q) = -\sum_{y} P(y) \log_D Q(y)$$

Where

- P is the true distribution
- Q is our approximation (via neural network)

Why is it popular?

- Good properties for training with "gradient descent" in certain standard architectures.
- Theoretical properties.

A (very) simple neural network

Takes a screen of the blackboard

- it transform the image into a vector
- Then takes is through the weighs w_i all the way to d
- the we take it through the soft max which is two functions:

$$Q(o \mid x) = \frac{e^{z_0}}{e^{z_0} + e^{z_1}}$$
$$Q(1 \mid x) = \frac{e^{z_1}}{e^{z_0} + e^{z_1}}$$

The goal is given a lot of training data, we want to select the w_0, b_0, w_1, b_1 such at to minimize the total cross entropy loss.

For a single image $\mathcal{X}$

because why is juste binary we use:

Total Loss

$$L_{total}(w_o, b_o, w_1, b_1) = -\sum_{i=1}^{k} \log \frac{e^{x_i w_0 + b_0}}{e^{x_i w_0 + b_0} + e^{x_i \cdot w_1 + b_1}} - \sum_{i=k+1}^{n} \log \frac{e^{w_1 k_i + b_1}}{e^{w_0 x_i + b_0} + e^{w_1 x_i + b_1}}$$

Cross entropy loss

Cross entropy loss:

$$L(P, Q) = -\sum_{y} P(y) \log_D Q(y)$$

Theorem 24 *For a fixed probability distribution P, the minimum:*

$$min_Q L(P, Q)$$

Is attained if and only if we selected $Q^* = P$ in this case,

$$L(P, Q^*) = L(P, P) = H(P)$$

Where $H(P)$ is the entropy of the probability distribution P

Proof The proof, which will be done in class, uses once again the "IT inequality".

The theorem is saying this:

$$H(P) \leq L(P, Q)$$

With equality in one case which is $P = Q$.

$$\begin{aligned} H(P) - L(P, Q) &\leq 0 \\ -\sum_y P(y) \log P(y) + \sum_y P(y) \log Q(y) &\leq 0 \\ = sum_y P(y) \log \frac{Q(y)}{P(y)} \leq \sum_y P(y) \left[\frac{Q(y)}{P(y)} - 1 \right] \log(e) & \\ = \sum_y (Q(y) - P(y)) \log(e) & \\ = 0 & \end{aligned}$$

Note

We don't see it in AICC II but let's introduce the notion: **KL-Divergence** (aka KL distance):

$$D_{kl}(p \mid\mid k) = \sum_y p(y) \log \frac{P(y)}{Q(y)}$$

- Fact 1:
  $D_{kl}(P \mid\mid Q) \geq 0$ with equality iff $P = Q$ (this is just the proof seen earlier

2.8 Summary of chapter 1

Entropy

$$H_D(X) = -\sum_x p(x) \log_D p(x)$$

For $D = 2$, we simply write $H(X)$ and we all the units bits.
Entropy has many useful properties, including:

- $0 \leq H_D(X) \leq \log_D \mid \mathcal{X} \mid$
- $H_D(X \mid Y) \leq H_D(X)$ with equality if and only if X and Y are independent
- $H_D(X, Y) = H_D(X) + H_D(Y \mid X)$

Data Compression

- Every uniquely decodable binary code must use at least $H(X)$ bits per symbol on average
- There exists a binary code that uses between $H(X)$ and $H(X) + 1$ bits per symbol on average
- Hence, for a source string of length n:
  - Every uniquely decodable binary code must use at least $H(S_1, S_2,$

Models

Coin Flip The coin flip is not convertible, With a file of result, there is no way to compress the file

Sunny Rainy Here, the entropy, is not 1 then we are able to compress the file here.
This is the first view of mark of model.
Given $S_1, S_2, S_3, \ldots,$ Are S_1, S_3 independent?

$$\begin{aligned} p(S_1, S_3) &= \sum_{S_2} p(S_1, S_2, S_3) \\ &= \sum_{S_2} p(S_1) p(S_2 \mid S_1) p(S_3 \mid S_2) \end{aligned}$$

Entropy and algorithm

We explored examples where entropy can give a lower bound on algorithmic performance.

- Example: in search-type problems, give a lower bound on the minimum number of necessary queries.

Cross-Entropy Loss

- Machine (e.g., Neural Network) outputs a distribution $Q(y)$ over all possible labels
- Cross entropy loss: Select $Q(y)$ to minimize:

$$L(P, Q) = -\sum_{y} P(y) \log_D Q(y)$$

March 18, 2025 — **Lecture 9 : Introduction to cryptography**

Chapter 3

Cryptography

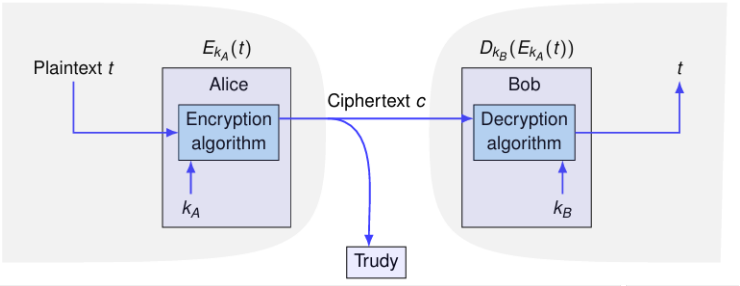
3.1 One-Time pad, Perfect Secrecy, Public-Key

Why cryptography

cryptography gives us the tools to:

- authenticate the sender and the receiver
- verify the integrity of the message
- keep the message confidential

Basic setup for condidentiality



Here Alice want to sent the plaintext t to Bob:

- She encrypts t using her key k_A. Theresult is the ciphertext $c = E_{k_A}(t)$
- She sends c to Bob over a public channel
- Bob decrypts c using his key k_B. The result is $D_{k_B}(E_{k_A}(t)))t$
- For Trudy, it is nearly impossible to recover t from c without knowing k_B

Basic Terminology

- pleintext, ciphertext (also called cryptogram), key, encrypter, decrypter
- cryptography: the art of composing cryptograms
- cryptanalysis the art of breaking cryptograms
- a cryptanalyst has broken the system when he cann quickly determine the plaintext from the cryptogram, no matter what key is used
- attacker: same as cryptanalyst

Ancient cryptography

Caesar's cipher

Suppose that we are using the English alphabet augmented by a few special characters, "space", "comma", and "period". An alphabet of 29 characters, represented by the integers $0, 1, \ldots, 28$

- The key k is an integer between 0 and 28, known to Alice and Bob and to nobody else.
- The encryption algorithm substitues the i-th letter of the alphabet with the $(i+k)th$ letter (mod 29)
- The decryption algorithm substitues the j-th letter with the $(j-k)$-th mod 29

Therefore, here the secrecy of the message rely only on the secrecy of the algorithm.

Various attacks possible

We distinguish between the following attacks:

- **ciphertext-only**: one or more cryptograms available to the cryptanalyst know to have been encrypted with the same key
- **known plaintext**: the cryptanalyst has one or more plaintext and the resulting cryptograms, know to have been encrypted with the same key
- **chosen plaintext** for any plaintext that he requires, the cryptanalyst can obtain the cryptogram under the same key

Ideally, a cryptographic system should be secure against a chosen plaintext attack, At the very least, it should be secure against a cipthertext-only attack.

However with a computer, the key can easily be found using the letter-frequency attack. The question now is how to make the letter frequency attack unfruitful?

Example (Vigenère's cipher with an n-length key

- Chosen plaintext attack: encode the same letter until you have the $n-$length key
- **known plaintext attack** compare input/output until you have the n-length key
- **ciphertext-only attack**
  - brute force approach: try all 29^n keys is you know n
    - for $n = 21$ the number of key is 5.13^{30}
    - for $n = 100$, the number of keys is 1.73^{146}
  - If you know n, you can partition input/output into n parts, each of which is a Caesar cipher with its own key
  - Letter frequency approach: effective if the plaintext-length to key-length ration is sufficiently large

The one-time pad

Preliminary assumptions

- The plaintext t, the key k and the cryptogram c are n-length binary sequences over the alphabet $\mathcal{A} = \{0, 1\}$
- The key k, is produced by selecting each bit independently and with uniform distribution
- Alice and Bob use a private chanel to exchange the key ahead of time

Encryption

$$c = t \oplus k$$

Decryption

$$c \oplus k = (t \oplus k) \oplus k = t \oplus (k \oplus k) = t$$

Perfect secrecy

Definition 19 *A cryptosystem has **perfect secrecy** if the plaintext T and the cryptogram C are statistically independent*

Perfect secrecy is the ultimate kind of security against a ciphertext-only attack: The attacker cannot do better than guessing the plaintext T

Perfect secrecy of the one time pad

- The n-length key k is selected at random (uniform distribution ober $\{0, 1\}^n$)
- The key k and the message t are selected independently
- The ciphertext is $c = t \oplus k$

$$p_{C|T}(c \mid t) = p_{K|T}(c \ominus t \mid t) = p_K(c \ominus t) = \frac{1}{2^n}$$

Hence C and T are independent: knowledge of C is useless in guessing T

Weakness of the one time pad

Example

A cryptanalyst that has the plaintext t and the corresponding cryptogram c immediately gets the key:

$$k = c \ominus t$$

Hence the pad (the key) should be used only once

Pros and Cons of one time pad

Pros

- Very simple algorithm
- as secure as it gets against a ciphertext-only attack and key used one
- of instructional value to prove that the perfect secrecy is possible

Cons

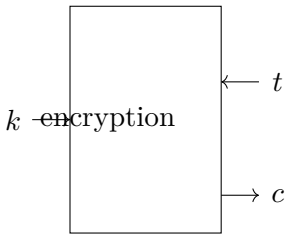
- The key is as long as the plaintext (this is fundamental, see later)

- The key needs to be exchanged ahead of time over a private chanel
- a ciphertext-only attack can break the system if the key is used twice (see homework)
- a known plaintext attack reveals the key

The "one-time pad" has been used extensively in diplomatic and espionage circles

Perfect secrecy requires high entropy keys

The following theorem makes no assumption on the encryption algorithm:



Theorem 25 *Perfect secrecy implies:*

$$H(T) \leq H(K)$$

Proof Perfect secrecy $H(T) = H(T \mid C)$ and decodability ($H(T \mid K, C) = 0$) imply:

$$\begin{aligned} H(T) &= H(T \mid C) \\ &\leq H(T, K \mid C) \\ &= H(K \mid C) + H(T \mid K, C) \\ &= H(K \mid C) \\ &\leq H(K) \end{aligned}$$

Given T, K, C we search for the entropy:

$$\begin{aligned} H(t,k,c) &= H(c) + \overbrace{H(t \mid c)}^{=H(t)} + H(k \mid t, c) \\ &= H(c) + H(k \mid c) + \underbrace{H(t \mid k, c)}_{=0} \\ &\implies H(t) + H(k, \mid t, c) \leq h(k \mid c) \end{aligned}$$

This implies:

$$H(t) + \underbrace{H(k \mid t, c)}_{\geq 0} \leq H(k)$$

Because if we take out the *"Knowing c"* we cannot have some bigger. Therefore:

$$H(t) \leq H(k)$$

Entropy plays a key role also in cryptography

March 19, 2025 — **Lecture 10 : Encryption?**

Exercice

Determine the minimum average length of the binary key for a cryptosystem that has the following characteristics

- the message is an uncompressible binary string of length n
- the system achieves perfect secrecy

Solution

- $H(T)$ must be essentially n bits (otherwise further compression is possible
- Perfect secrecy requires $H(T) \leq H(K)$
- hence $H(K)$ is at least n
- The average blocklength of the binary key is at least n bits

Symettric-Key crypto Systems: key-distribution problem

A symmetric-key cryptosystem is one for which both ends use the same key ($k_A = k_B = k$). All example considered so fare rely on a symmetric key
There exists fast (ans secure) symmetric key cryptosystems, but:

- Anybody that has the key can encrypt and/or decrypt
- The key cannot be sent over an insecure channel
- In an n user network, each user needs $n-1$ keys to communicate privately with every other user. Key distribution is a problem as it hat to be done over a secure channel. And keys have to be changed frequently
- We have a real problem (the first 6min. and 20 secs of:

  `https://www.youtube.com/watch?v=YEBfamv-_do`

However, is there a way to distribute keys over a pubic channel?

Solution

In 1976, Diffie and Hellman came up with a solution.

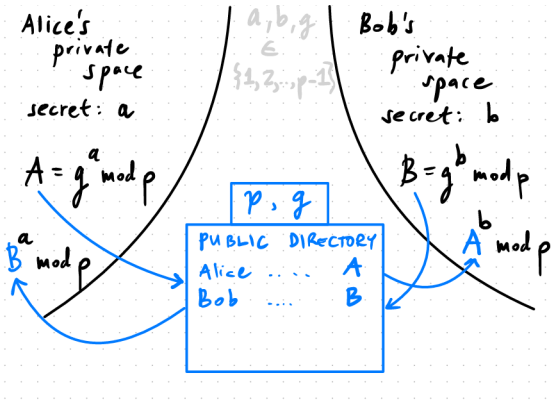
Example

You pick a number $p = 7$ which is a prime $\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6\}$. We then take $g = 3$

i	$g^i \mod 7$
0	
1	3
2	2
3	6
4	4
5	5
6	1

The gives us a permutation. We have to be careful with choosing the g because for example if we take $g = 2$, and take g^1 and g^4 the result is 2 which leads that we don't have a permutations

How does it is seen:



We have a public directory like a phone book where everyone has access. Alice pick a random a. Then she takes a public $A = g^a \mod p$ which we will be written in the public directory. Then Bob do the same thing.
When Alice and Bob want to have an interaction. Alice will look for Bob in the phonebook, take the B in her private space, take the result of $B^a \mod p$ (she is the only one to know a). Bob do the same thing: $A^b \mod p$
At this point

- Alice has $B^a \mod p$

$$B^a = (g^b \mod p)^a \mod p$$

- Bob has $A^b \mod p$

$$A^b = (g^a \mod p)^b \mod p$$

Fact

Theorem 26 *For all $x, y, m \in \mathbb{Z}$:*

$$[(x \mod m) \cdot (y \mod m)] \mod m = xy \mod m$$

We then get $B^a = g^{ab} \mod p$ and $A^b = g^{ab} \mod p$ We juste need to find the inverse of $g^i \mod p$ Which is a discrete logarithm problem. It is very hard to find the discrete logarithm. The gives the secrecy of this encryption. The secrecy is only here because it is hard to compute this inverse

Secrecy

But can't anybody generate this key?
We all know g and we know that $A = g^a \mod p$ we juste has to find

Eve wants to listen

Assuming that the cryptosystem used by Bob and Alise is secure, the best option for Eve is to find the key k.
She know p, g, A and B.
In generale, there seems to be no better way than finding the number a for which $g^a = A$, and the comput $k = B^a$
This is a problem. Let us check out some number. Suppose

p is a 2048 bit number. (it must be prime, but let us neglect this and assume $p = 2^{2048}$ How long does it take to compute:

$$2 \log_2 p = 4096$$

Multiplications to performs $a \to g^a$ (called discrete exponentiation). With a computer that performs 10^{10} multiplications per seconds, the exponentiation is done seamlessly.
It takes roughly:

$$\exp\left(\left(\frac{64}{9}\right)^{\frac{1}{3}} (\ln p)^{\frac{1}{3}} \ln \ln p)^{\frac{2}{3}}\right) \approx 10^{35}$$

Mutliplication to perform $g^a \to a$ (called discrete logarithm to the base g. With the same computer, it takes about 10^{25} seconds, which is about $7 \cdot 10^7$ times the ages of the earth.

Conclusion Diffie and Hellman's public key distribution scheme is clever, efficient, and it seems to be secure.

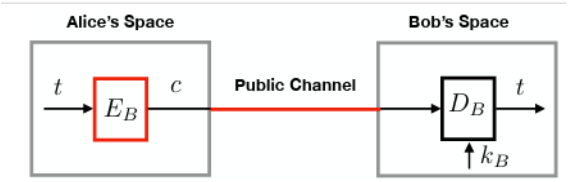
Paradigm shift The issue with the security is the computation. For example in maybe a couple of years the quantum computer will be able to compute this discrete algorithm very fast. The DG system would instanlty become insecure.
To the constrast, perfect secrecy offers provable security even when the enemy has infinite time and computing power. Most cryptographic systems rely on computational security

One way function Discrete exponentiation is an example of a one way function: a function for which a fast algorithm exists one way but not in the other way. The case for the discrete logarithm is that there is a way back but it is very slow. However, the best case would be that there is not way back.

Example IF a computer were to save user's names and passwords, a sstem manager would have access to both.
This is not the case if the operating system stores, along the name a one-way function f of our password.(The password itself is never stored)

Trapdoor One-way function A tropdoor one way function is a one-way function with an extra feature called the trapdoor information: with this information, the hard-to-carry out inverse computation becomes easy.
Diffie and Hellmann realized that with such a tool the key distribution problem would diseappear

Asymetric cryptography Suppose that Allice wants to send private information to Bob
Bob has a trapdoor one-way function, implemented by an algorithm E_B that he publsishes in a open directory
He is the only one who has the trapdoor information k_B. Hence he has the algorithm D_B that implements the inverse function.
Alice and Bob no longer need a shared key:

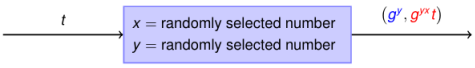


ElGammal's trapdoor one way function

Setup

- Fix a large prime number p. Hereafter all the numbers are in $\{0, 1, \ldots, p1\}$ and arithmetic is modulo p (mor on it later).
- Pick a generator g
- Pick randomly selected numbers x and y. Unlike p and g,x, and y are kept secret.

- Here is a trapdoor one-way function, with **trapdoor information** x.



Given the trapdoor information x, we can invert the function as follows: Compute the inverse of $g(^y)^x = g^{xy}$, multiply the result with $g^{xy}t$. The result is t.

More concretly

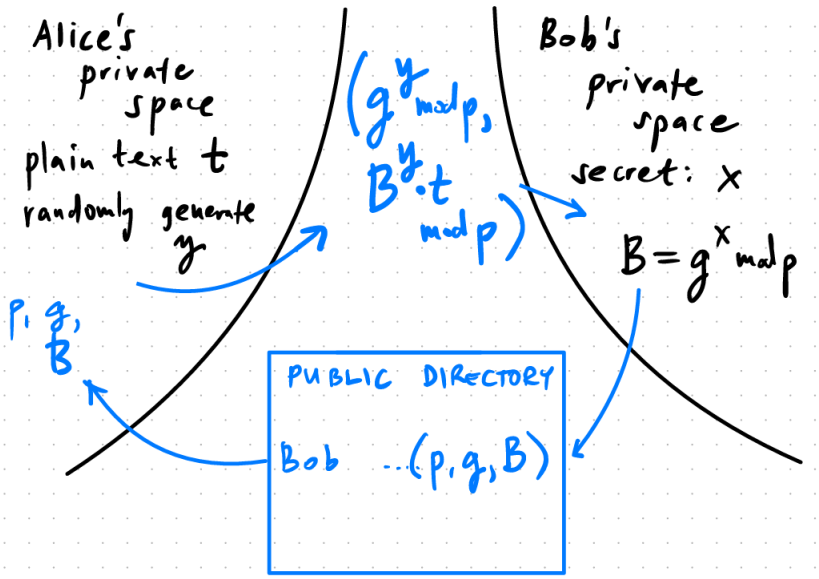
Alice has:

- t = plaintex, t
- y = random number, y

Bob:

- x = random number x.

The scheme look like this:



Bob sends g^x to Alice, then Alice sends the cryptogram $(g^y, g^{xy}t)$ to Bob

Note: x and y are transaction specific

March 25, 2025 — **Lecture 11 : Number Theory**

3.2 Number theory

Introduction

In the digital world, the information is represented by the elements of a finite set, and we should be able to do math with them. Which means that the finite set should be a finite field. Our bigger goal of the next few lectures is to develop the tools to understand when and how we can turn a finite set into a finite field.

Operation with integers

Within $\mathbb{Z}$ (the set of integers) we can

- add, subtract, multiply
- but not divide $\frac{7}{2}$ is not an integer
- What comes closest to the (regular) division is the euclidean division

Euclidiean division

The division algorithm

Given integers a (the dividend) and m the divisor:

$$a = mq + r, \quad 0 \leq r < \mid m \mid$$

The computation of q and r as above is called euclidean division

euclidean division in mainstream programming languages

In C/C++/Java/ Python we use the operator % to compute r as follows:
if a and m are both positive, then $r = \%m$
If one or the other or both are negative, different languages behave differently, but the general rule is:

- if $a\%m$ is nonnegative, then $r = a\%m$
- if $a\%m$ is **negative**, then $r = a\%m + m$

Congruence is an equivalence relation

A binary relation $\sim$ on a set is an **equivalence relation** if and only if the following three axioms are satisfied:

- $a \sim a$ (reflexivity)
- if $a \sim b$ then $b \sim a$ (symmetry)
- if $a \sim b$ and $b \sim c$ then $a \sim c$ (transitivity)

Substriture $a \sim a$ with $a \equiv a \mod m$ etc..., to see that congruene is an equivalence relation
One of the consequences is that we can form equivalence classes and we can work with one representative of each class (this will become useful later.)

Modulo

Theorem 27 *If:*

$$\begin{aligned} a &\equiv a' \mod m \\ b &\equiv b' \mod m \end{aligned}$$

then:

$$\begin{aligned} a + b &\equiv a' + b' \mod m \\ ab &\equiv a'b' \mod m \\ a^n &\equiv (a')^n \mod m \end{aligned}$$

In particular, if $a' = (a \mod m)$ and $b' = b \mod m)$, then we obtain the following facts (useful is mon calculation):

- $(a + b) \equiv ((a \mod m) + (b \mod m)) \mod m$
- Hence:

$$a + b \mod = (a \mod m) + (b \mod m)) \mod m$$

- $ab \equiv ((a \mod m)(b \mod m) \mod m$
-

$$ab \mod m = ((a \mod m)(b \mod m) \mod m$$

- $a^n \equiv (a \mod m)^n \mod m$

$$a^n \mod m = (a \mod m)^n \mod m$$

Example is $9^{1000} + 9^{10^6}$ divisible by 5?
We compute first:

$$9 \equiv -1 \mod 5$$

Which gives us:

$$9^{1000} + 9^{10^6} \equiv (-1)^{1000} + (-1)^{10^6} \equiv 1 + 1 \equiv 2 \mod 5$$

Hence, $9^{1000} + 9^{10^6}$ is not divisible by 5

Check digits mod 97

Write down an integer in decimal notation *e.g*:

$$021\ 235\ 1234$$

Compute its remainder after division by 97:

$$021\ 235\ 1234 \mod 97 = 95$$

Appends the remainder to the number, as check digit:

$$021\ 235\ 123495$$

A common mistake consists in transposing two digits:

$$021\ 253\ 1234\ 95$$

The check digits are no longer consistent:

$$021\ 253\ 1234 \mod 97 = 63$$

Procedure mod 97 − 10

It's a variant of the previous one:
Append 00

April 1, 2025 — **Lecture 13 : Multiplicative Inververse**

Why modular arithmetic

Modular arithmetic is the foundation of number theory, therefore we need number theory for cryptography and for channel coding.

Introduction to $\mathbb{Z}/m\mathbb{Z}$

Instead of considering integers and congruences (mod m) and write equation like:

$$a + b \equiv c(\mod m)$$

We would like to make our life easier like this:

$$a + b = c$$

This can be done, if we give a new meaning to a, b and c. namely we make them the congruence classe $[a]_m, [b]_m [c]_m$ and $[c]_m$.

Definition 20 *Let $m > 1$ be an integer, called the modulus.*
The set of all integers congruent to $a(\mod m)$ is called the congruence class of a modulo m.
it is denoted by $[a]_m$.

It is the same foundation as the one of finite field (corps fini) in linear algebra. (With prof. Sherer).

Other definition

Definition 21 *The set of all congruence classes modulo m is denoted by $\mathbb{Z}/m\mathbb{Z}$ (which is read $\mathbb{Z} \mod m$.*

Note Some authors use the notation $\mathbb{Z}_m$

Example

Some more example to see how this works:
if we have the class $[a]_m$ and:

$$a = mq + r, \text{ with } 0 \leq r \leq m - 1$$

Then we have that:

$$[a]_m = [r]_m$$

If we take for example $[-13]_9$ and $[5]_9$ we can see here that there are equal such that

$$[-13]_9 = [5]_9$$

Sum

In $\mathbb{Z}/m\mathbb{Z}$ we define the sum and the product as follows:

- $[a]_m + [b]_m = [a + b]_m$
- $[a]_m[b]_m = [ab]_m$

The result is the same regardless the choice of representative. In fact:

- if we choose $[a + km]_m$ instead of $[a]_m$
- and $[b + lm]_m$ instead of $[b]_m$
- Then we obtain $[a + km]_m + [b + lm]_m = [a + km + b + lm]_m$ which is equal to $[a + b]_m$

Properties of + in $\mathbb{Z}/m\mathbb{Z}$

The sum has the following properties:

- $[a]_m + ([b]_m + [c]_m) = ([a]_m + [b]_m) + [c]_m$
- There exists an additive identity, namely $[0]_m$:

$$[a]_m + [0]_m = [0]_m + [a]_m = [a]_m$$

- There exists an inverse with respect to addition: every $[a]_m$ has an inverse, denoted $[-a]_m$ such that:

$$[a]_m + [-a]_m = [-a]_m + [a]_m = [0]_m$$

- Commutativity

$$[a]_m + [b]_m = [b]_m + [a]_m$$

Properties of $\times$ in $\mathbb{Z}/m\mathbb{Z}$

The multiplication has the following properties:

- associativity

$$[a]_m([b]_m[c]_m) = ([a]_m[b]_m)[c]_m$$

- multiplicative identity, namely $[1]_m$:

$$[a]_m[1]_m = [1]_m[a]_m = [a]_m$$

- commutativity

$$[a]_m[b]_m = [b]_m[a]_m$$

Mixed properties

- Distributivity:

$$[a]_m([b]_m + [c]_m) = [a]_m[b]_m + [a]_m[c]_m$$

The notation $k[a]_m$ in $\mathbb{Z}/m\mathbb{Z}$

For an arbitrary positive integer k, $k[a]_m$ is a short hand for

$$\underbrace{[a]_m + [a]_m + \cdots + [a]_m}_{\text{k times}}$$

We can easily verify that:

$$k[a]_m = [ka]_m = [k]_m[a]_m$$

Multiplicative Inverse

Some elements of $\mathbb{Z}/m\mathbb{Z}$ have the **multiplicative inverse**.
The multiplicative inverse of $[a]_m$, if it exists is an element $[b]_m$ such that:

$$[a]_m[b]_m = [1]_m$$

Theorem 28 *The multiplicative inverse if it exits it is unique, and it is denoted by* $([a]_m^{-1})$.

Furthermore $(([a]_m)^{-1})^{-1} = [a]_m$

Proof

Suppose first that $ab = 1$ and $ac = 1$ (a has two inverse).
The we now that $ab = ac$. If we multiply both side by b:

$$bab = bac$$

However we know that $ab = 1 = ba$:

$$b \cdot 1 = c \cdot 1$$
$$b = c$$

April 2, 2025 — **Lecture 14 : Read Slide**

Powers in $\mathbb{Z}/m\mathbb{Z}$

For an positive integer k,

- $([a]_m)^k$ is a short hand for $\underbrace{[a]m[a]m\dots[a]_m}_{\text{k times}}$
- $([a]_m)^0 = [1]_m$

Note that we do not consider negative power because it is problematic in general except -1 which is juste the multiplicative inverse

Exercise

Suppose $[a]_m \in \mathbb{Z}/m\mathbb{Z}$ has a multiplicative inverse.
Does there exist k such that:

$$([a]_m)^k = [0]_m$$

We denote first $[b]_m = ([a]_m)^{-1}$. The first statement implies that:

$$([b]_m)^k \cdot ([a]_m)^k = [0]_m$$

Which is:

$$\begin{aligned}[0]_m &= ([b]_m)^{k-1}\overbrace{[b]_m[a]_m}^{=[1]_m}([a]_m)^{k-1} \\ &= ([b]_m)^{k-1}([a]_m)^{k-1} \\ &= \cdots = [1]_m\end{aligned}$$

Which is false. Therefore, The answers is no.

Function with multiplicative inverse

Theorem 29 *In $\mathbb{Z}/m\mathbb{Z}$ the following statement are equivalent:*

- *$[a]_m$ has an inverse*
- *For all $[b]_m$, $[a]_m x = [b]_m$ has a unique solution*
- *There exists $a[b]_m$, such that $[a]_m x = [b]_m$ has a unique solution*

Proof $1 \to 2$ We multiply both side of $[a]_m x = [b]_m$ by $[a]_m^{-1}$ then we have $x = [a]_m^{-1}[b]_m$ showing that there is a solution and the solution is unique.

Proof $2 \to 1$ For $[b]_m = [1]_m$ we obtain $[a]_m x = [1]_m$ which is a solution by assumption. The solution is the inverse of a.

Proof $2 \to 1$ True since (3) is a weaker statement than (2). (one says for all and the other says there exists)

Proof 3 → 2

Here the claim is: $\exists b$ it has a unique solution $\implies$ $\forall b$ there is a unique solution. The negation of this claim is:
$\exists b$ either there is no solution or multiple solution $\implies$ $\nexists b$ there is a unique solution.
It says that if there is a b with **multiple** solution implies that there is no b with a unique solution.

Proof Let $[a]_m[x_1]_m = [b^*]_m$ and $[a]_m[x_2]_m = [b^*]_m$. Now we define:

$$[x_3]_m = [x_1]_m - [x_2]_m[a]_m[x_3]_m = [0]_m \tag{3.1}$$

Now we select **any** $\bar{b}_m$, we then suppose that $[a]_m[x_4]_m = \left[\bar{b}\right]_m$, Because of the fact that we have the solution x_3 before, we can juste add them up:

$$[a]_m([x_3]_m + [x_3]_m) = \left[\bar{b}\right]_m$$

Proof 3 $\implies$ 2 We prove this using the contrapositive i.e, we assume that there is a $[b]_m$ such that $[a]_m x = [b]_m$ has either no solution or multiple solutions, and we prove that for no $[b]_m$, $[a]_m x = [b]_m$ has a unique solution.

- So suppose that $[a]_m x = [b]_m$ has no solution or multiple solutions
- By the pigeonhole principle, the map $x \to ax$ is neither **injective** nor **surjective**.
- We can find $a[b^*]_m$ $[a]_m x = [b^*]_m$ has multiple solutions say, x_1, x_2. We define then $x_3 = x_1 - x_2 \neq [0]_m$
- Hence, $[a]_m x_3 [a]_m x_1 - [a]_m x_2 = [b^*]_m - [b^*]_m = [0]_m$
- So the solution $[a]_m x = [0]_m$ has at least two solution x_3 and $[0]_m$
- If $[a]_m x = [b]_m$ has a solution, say x_4, then $x_3 + x_4$ is also a solution.
- We conclude that for no $[b]_m$, $[a]_m x = [b]_m$ has a unique solution.

Example If it exists, find the solution of $[2]_7 x + [3]_7 = [1]_7$

$$\begin{aligned}
[2]_7 x &= [1]_7 + (-[3]_7) \\
&= [2]_7 x = [-2]_7 \\
&= x \qquad\qquad\qquad\qquad = ([-2]_7)^{-1}[5]_7 \\
&= [4]_7[5]_7 \\
&= [20]_7 \\
&= [6]_7
\end{aligned}$$

$\mathbb{Z}/p\mathbb{Z}$

$\mathbb{Z}/p\mathbb{Z}$ with p prime if p is prime, all elements of $\mathbb{Z}$ $p\mathbb{Z}$ except $[0]_p$ have a multiplicative inverse The proof is only taking the fact that the gcd between every number in the classe and p is 1, therefore has an inverse. (except 0 which have p).

Euclidiean algorithm For all this part I think the best is the slide in the pdf Slides 2025 week7 between 100 and 135. or a youtube video.

April 8, 2025 — Lecture 15 : Commutative Groups

What's next After $\mathbb{Z}$ $m\mathbb{Z}$ we could proceed in two directions:

Finite groups Focus on finite groups, which are finite sets with one operation, like $(\mathbb{Z}/m\mathbb{Z}, +)$ we do so now because we need them for cryptography.

Finite field Focus on finite field, which are finite sets with two operations, like $(\mathbb{Z}/m\mathbb{Z}, +, \cdot)$ With the extra property that every non-zero

element gas a multiplicative inverse. We do so later as we need finites fields for channel coding

3.2.1 Commutative Group

Definition 22 *A **commutative group** (also called Abelian group) is a set G endowed with a binary operation * that combines any two elements a and b to form another element denoted $a * b$. The groupe operation * must satisfy the following five axioms:*

- *(Closure) For all $a, b \in G$, $a * (b * c) = (a * b) * c$*
- *(Associativity) For all $a, b \in G$ $a *(b * c) = (a * b) * c$*
- *(Identity element): There exists an element $e \in G$ such that for all $a \in G$, **$a * e = e * a = e$***
- *(Inverse element) For all $a \in G$ there exists a $b \in G$ such that $a * b = b * a = e$*
- *(Commutativity) For all $a, b \in G$ $a * b = b * a$*

$\mathbb{Z}/m\mathbb{Z}*$

To obtain a commutative group with modulo multiplication, we take only the elements of $\mathbb{Z}/m\mathbb{Z}$ that have multiplicative inverse. The resulting set is denoted $\mathbb{Z}/m\mathbb{Z}*$ For every integer $m > 1$ ($\mathbb{Z}/m\mathbb{Z}^*m\cdot$) is a commutative group.

Euler function

Definition 23 *Euler's function $\varphi(n)$ (also called Euler's totient function) is the number of positive integers in $\{1, \ldots, n\}$ that are relatively prime to n.*

Observation

Here we can see two main thing :

- $\varphi(m)$ is the cardinality of $\mathbb{Z}/m\mathbb{Z}^*$
- if p is prime, $\varphi(p) = p - 1$

The cartesian product of a commutative groupe is a commutative group

Recall of the axioms of a commutative group:

- (Closure) For all $a, b \in G$, a * (b * c) = (a * b) * c
- (Associativity) For all $a, b \in G$ a *(b * c) = (a * b) * c
- (Identity element): There exists an element $e \in G$ such that for all $a \in G$, **a * e = e * a = e**
- (Inverse element) For all $a \in G$ there exists a $b \in G$ such that a * b = b * a = e
- (Commutativity) For all $a, b \in G$ a * b = b * a

We can see the $(a_1, a_2) \in (G_1, op_1) \times (G_2, op_2)$ we see that this **is** a commutative group.

Isomorphism

Some sets endowed with an operation might look different, buit they are actually the same once their elements are re-labeled.

Definition 24 *Let (G, *) and (H, ⊕) be sets, each endowed with an arbitrary binary operation.*
an ***isomorphism*** *from (G, *) to (H, ⊕) is a bijction $\psi : G \to H$ such that*

$$\psi(a*) = \psi(a) \oplus \psi(b)$$

holds for all $a, b \in G$
*We say that (G, *) and (H ⊕) are* ***isomorphic*** *if there exists an isomorphism between them.*

Je me suis arreeter vers les slide 50

April 15, 2025 — Lecture 17 : public key cryptography

Proof that $m1$ and m_2 coprime $\implies$ φ is bijective

First we prove that ψ is one-to-one:

- $\iff$ m_1 and m_2 divide $(k - k')$
- because m_1 and m_2 have no common facots, $m_1 m_2$ divides $(k - k')$
- Hence $[k]_{m_1 m_2}$

April 16, 2025 — Lecture 18 : Last lecture on crypto

with m a prime

First we select m a prime, we then take $m = p$.
Then, we select e such that gcd $(e, p-1) = 1$
We select d such that $[ed]_{p-1} = 1$
Works but is not secret.

Select $m = pq$

select e such that $\gcd(e, p-1) = 1$
select d such that $[ed]_{p-1} = 1$

Difference

Here that thing that is hard for people who want to crack the code, is to find p and q from m.

Chinese remainder

If we take as before $\mathbb{Z}$ $m_1, m_2, \mathbb{Z}$, with m_1, m_2 coprime.

$$[k]_{m_1,m_2} \to \left([k]_{m_1}, [k]_{m_2}\right)$$

$$[0]_{m_1,m_2} \to \left([0]_{m_1}, [0]_{m_2}\right)$$

We see here that **this** doesn't have a multiplicative inverse, the reason is that from the Chinese remainder, we will get 0:

$$[m_1]_{m_1,m_2} \to \left([0]_{m_1}, [m_1]_{m_2}\right)$$

Now We do the Chinese remainder theorem with p and q (two distinct prime): $\mathbb{Z}$ $pq\mathbb{Z}$: the question is what are **all** the elements that do not have a multiplicative inverse?. The answer is all the multiple of q or p, which you can do a mapping with the Chinese remainder:

$$[k]_{pq} \to \left([0]_p, [k]_q\right)$$

Or:

$$\to \left([k]_p, [0]_q\right)$$

The issue is from does who have a 0 is one of their "multiplicative".

Remark We only use the Chinese remainder **ONLY** for proving, we won't use it to compute in RSA. (if Alice know p and q she can then find every cipher text).

Fermat + Chinese Remainders

- Let p q be distinct primes
- Let k be a multiple of both $(p-1)$ and $(q-1)$
- for all non-negative l

$$\left([a]_p\right)^{lk+1} = [a]_p$$

$$\left([a]_q\right)^{lk+1} = [a]_q$$

- Using the Chinese remainders theorem, we combine into:

$$\left([a]_{pq}\right)^{lk+1} = [a]_{pq}$$

We have proved the following result (TextBook Thm 10.3):

Theorem 30 *Let p and q be disctinct prime number and let k be a multiple*

Proof, idea of the proof Because $(\mathbb{Z}/pq\mathbb{Z}, \cdot)$ is isomorhpic to (by the mapping of the Chinese remainder) $(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}, \cdot)$ The relation is equivalent to:

$$\left([n]_p\right)^{1+m} = [n]_o$$

And:

$$\left([n]_q\right)^{1+m} = [n]_q$$

If $[n]_p = 0$ then the equation is true, else, $[n]_p$ is inversible because p is prime, therefore, by Euleur theorem $\left([n]_p\right)^{p-1} = [1]_p$. Furthermore, m is a multiple of $p-1$, this means, there exists an integer $l \geq 0$ such that $m = l(p-1)$:

$$\left([n]_p m = \left([n]_p\right)^{(p-1)l} = \left([1]_p\right)^e ll = [1]_p\right.$$

3.2.2 RSA, Rivest

High level

Suppose that we can find:

- Integer m (modulus)
- Integer e (encoding exponent)
- integer d (decoding exponent)

Such that, for all integers $t \in \mathbb{Z}/m\mathbb{Z}$ (plaintext)

$$\left[(t^e)^d\right]_m = [t]_m$$

Then:

- The receiver generates m, e, d (we sill se later on how)

- (m, e) is the public encoding key –Announced in a phone like public directory
- (m, d) is the priate decoding key – d never leaves the receiver
- To send the plaintext $t \in \mathbb{Z}/m\mathbb{Z}$
- The encoder forms the cryptogram $c = t^e \bmod m$ Exponentiation is easy
- The intended decoder performs $c^d \bmod m$ and obtains the plaintext t.

Example

suppose we have $m = 33, e = 7, d = 3$
suppose that the plaintext is $t = 2$
The encryption is $c = t^e \bmod m = 2^7 \bmod 33 = 128 \bmod 33 = 29$
The decryption is:

$$c^d \bmod m = 29^3 \bmod 33 = \ldots = 2$$

s expected.

RSA keys generation

- generate large prime p and q at random (which is approximately $\frac{\log n}{n}$), we want to use a prime that has never been use before, imagine using a m that as already be used, we can just check if another key match our and if it is, it will be cracked.
- $m = pq$ is the modulus used for encoding and decoding
- let k be a multiple of $(p-1)$ and $(q-1)$ to be kept secret
- For instance $k = \varphi(pq)$ or $k = lcm(p-1, q-1)$
- Produce the public (encoding) exponent e such that gcd(e, k) = 1
- (a common choice is $e = 65537 = 2^{16} + 1$ which is a prime number. No need for e to be distinct for each recipient)
- the public key is (m, e)

The choice for $e = 2^{16} + 1$ is here because it is very easy to compute big power like this: $\left(\left((t^2)^2\right)^2 \ldots\right)^2 \cdot t = t^{2^{16}+1}$ which makes it easy to comput the cipher text to the power of e.

How decoding workds

$[t]_m \in \mathbb{Z}/m\mathbb{Z}$ with $m = pq$ Hence:

$$\begin{aligned}\left([t]_m^2\right)^d &= [t]_m^{ed} \\ &= [t]_{pq}^{1-kl} \\ &= [t]_{pq} \text{ Fermat + CRs} \\ &= [t]_m\end{aligned}$$

Example (toy-key generation)

taking $p = 3$, $q = 11$, $m = 33$, $k = \varphi(pq) = (2-1)(11-1) = 10$.

- $e = 7$ which is relatively prime with k
- $d = 3$ (check that $ed \bmod k = 1$)
- the public key is $(m, e) = (33, 7)$
- The private key is $(m, d) = (33, 3)$

Now let us encrypt. Each letter of the alphabet is converted into a number in $\{1, 2, \ldots, m - 1 = 32\}$ (we avoid 1 to avoir $c = 0 = t$)

- We use the natural order, $a \to 1, b \to 2$ etc...

Attack

How to decrypt not knowing d? here the possibilites (that we know of):

- factor m to find p and q. Very hard to do if m is large (say $\approx 2^{500}$)
- in $\mathbb{Z}$ $m\mathbb{Z}$ solve $c = x^e$ for x Which is very hard to do if m is large.
- guess k
- guess t
- guess d

The trapDoor one-way function Behind RSA

The trapdoor one-way function is:

$$t \to x = t^e \bmod m$$

Where e is called the encoding exponent.
Instead of publishing the function, it suffices to publish (m, e). This is called the public key.
Someone that knows (m, d) can perform

$$c \to t = c^d \bmod m$$

where d is called the decoding exponent
Hence the trapdoor informations is (m, d). It is called the private key.

3.3 Digital signature

We have used trapdoor one-way functions for privacy. In conjunction with hash function, they are equally suited for authenticity.
The goal here to "prove" that this is actually us that written the message.

Issue

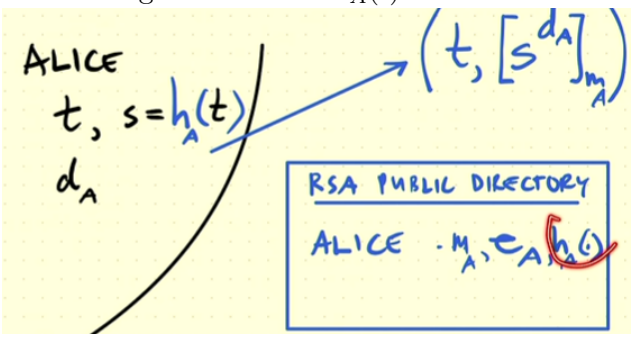
Public receives a message from Alice?
But is the message really from Alice?

RSA public directory

With RSA, Alice sends the message t but she also sends $\left[t^{da}\right]_{m_A}$. The question is how do we know with this additional information that this is Alice that sent this message.

Following Questsion

Alice has a message **and** a hashing function $s = h_A(t)$ which sends then in the public: $\left(t, \left[s^{d_A}\right]_m\right)$



How does the public check?

Step 1		$\left[\left(\left[s^{d_A}\right]_m\right)^{e_a}\right]_m = s^a$
Step 2		$t \to h_A(t) = s$
Step 3	if $s =$	

Hash function

A hash function is a many-to one function, used to map a sequence of arbitrary length to a fixed length bit sequence of, say, 200 bits
what we expect from a hash function, is that even the smallest change in the input produces a different output
Ideally it should be so that one has to try about 2^{200} alternative inputs to hope to find a sequence that produces a given output.

Digital Signature

to sign a document, we appends to the document a hash function of the docmunent in such a way that only the signee could have done it

Trusted Agency

How do we know that the directory storing all the public keys has not been tampered with?

Example Alies queries the public directory for bob's directory

oui The directory information signed by a trusted agency, Say Symantex. Here is how:

Symantec's public key is distributed one and for all via a channel that cannot be tapered with (e.g., hard coded into the crypto hardware)

- Each directory entry is digitally signed by symantec. We call the result a certificate
- Anybody that has Symantec's public key can verify that the information areceived from the directory is authentic

3.3.1 Summary of chapter 2

Perfect secrecy is possible but requires long keys:

One-time pad In onte time pad, the cryptogram is compute like this:

$$\text{Cryptogram} = \text{Plaintext} \oplus \text{SharedKey}$$

- If the sharedKey is perfectly (uniformly) random and shared between encrypter and decrypter ahead of time
- and the sharedKey is kept secret from anyone else
- Then the One-time pad offers perfect secrecy
- Hence: it is expensive to implement. Only worth it for spies and such.

Practical cryptography is based on algorithmic/computational complexity
pPublic key cryptography. Most public key cryptographic algorithm fall into one of the following two categories:

- those that are based on the belied that diescrete exponentiation (in mulitplicative cyclic group) is a one way function (e.g. Diffie-Hellamn and ElGamal)
- Those that are based on the difficulty of factoring (e.g. RSA)

To understand RSA and Diffie-Hellman, we need Number Theory and algebra.

3.3.2 Number theory and algebra

First with modulo Operation and the euclid Algorithm (we can find the multiplicative inverse gcd etc....

Group

- $\mathbb{Z}/m\mathbb{Z}$ with addition is always a group
- $\mathbb{Z}/m\mathbb{Z}$ with mulitplication: need to retain only those elements that have a multiplicative inverse: $\mathbb{Z}/m\mathbb{Z}^*$
- Finding multiplicative inverses in $\mathbb{Z}/m\mathbb{Z}$: Bézout identity; Extended Euclid algorithm
- How many element in $\mathbb{Z}/m\mathbb{Z}$ have a multiplicative inverse? the answer is Euler's function (torient function) $\varphi(n)$
- Group isomorphism
- Order of group elements Lagrance's theorem: Order of any group element must divide the cardinality of the group

Product group

Theoreme 31 *Main theorem: Cartesian product of groups is again a group*

The special Isomorphism

The special isomorphism between $\mathbb{Z}/m_1m_2\mathbb{Z}$ and $\mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$ when m_1 and m_2 are coprime:

- Holds for both addition and multiplication, including for elements that do not have multiplicative inverse,
- Hence, this is more than juste a group ismorphism

Computationally hard problem

Discrete logarithm

leads to Diffie Hellman(and, by slight extension, El Gamal)

- Encryption $A = g^a, B = g^b$
- Leads to a shered key: $C = A^b = B^a$
- To understand that it workds, we need cyclic groups.

Factorization of large integers

leads to Cocks and RSA.

- Encryption: $t^e \bmod m$ where t is the plaintext and $m = pq$, where p and q are primes
- Decryption $(t^e)^d \bmod m$
- To understant that it works (meaning that $(t^e)^d \bmod m = t$ for all plaintexts t), we need to understand $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$

Authencity: Digital Signatures

In practice, so called symetric key cryptoSystems are important. The common secret key is typically only a few hundred bits, distributed e.g., via Diffie-Hellman. Encryption/decryption can be implemented more efficiently (faster algorithm, smaller hardware). Think: one-time pad, but with an imperfect key. There is no proof that the resulting algorithm is secure.

April 29, 2025 — **Lecture 19 : Error Correction**

Point to point communication system

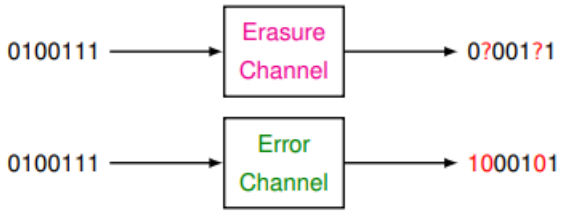
Now we have compressed the source coding, we encrypted it. What we want now is to actually transfer the message to the receiver side. However we have to protect ourself from the mean world. For example with the communication over the internet it is our application which does the compression and encryption.

Motivation Channel Model

The question is how the bad thing happens:

- The internet often drops packets due to congestion
- Not all the bits on a storage device can be retrieved
- **Wireless signals are very noisy**

We consider two types of channel models:



If we model the world has being a **Erasure** channel mode, then the following happens:
So your model replace some of your bit with question mark. (those bit are no longer readable). So for example you store your holiday picture into your hdd but some water fell on it, then there will be some error and those error is the question mark.
The model can **only** replace the bit by a question mark.
The question is "how we decide which bit has to be replace with a question mark?" For example like this:
So for the first 0 let us say we throw a dice, if the outcome is 6, we will replace the outcome by a question mark or we won't. Therefore, here we can say that the erosion probability is $\frac{1}{6}$.
However maybe the eraser is not random, maybe someone who know what we're gonna do (the algorithm to find the erasure) will find the one that will be a question mark.
Here the channel input alphabet is not necessarily binary.
We can take for the alphabet $\mathcal{A} = \{0, 1, 2, 3, 4\}$ and works with this.
For the second model (Error Channel), The channel can completely change entire symbol. The output is in the same alphabet as the input (which make the thing "easier"). What we can do is the substitute (0 to 1, 1 to 0 and then 0 to 0).
This makes the channel way harder to know which of these bits have been flipped.

The channel input alphabet doesn't have to be binary

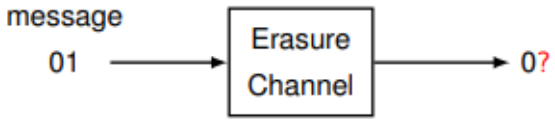
Erasure Weight

Definition 25 *We define the **erasure weigh** p (resp. **error weight p**) as the total number of erasures (resp. errors).*

For example if we take the erasure channel from before, erasure weight: $p = 2$. (error weight: $p = 3$ for the error channel).

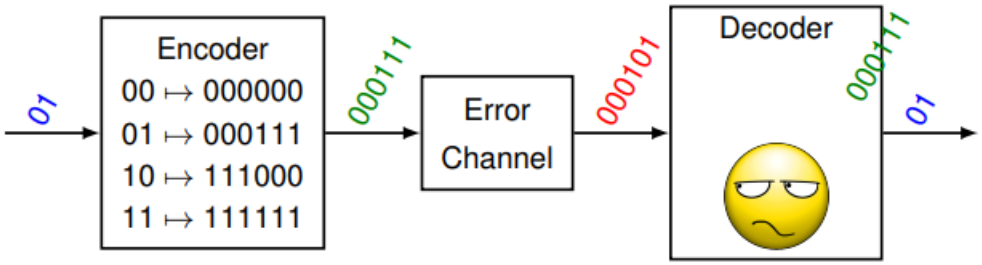
Channel coding to deal with erasures

Here this is the beginning of every algorithm for channel coding which goes:
Suppose that the source outputs 2 bits, and we store them as is (no channel coding):



f any bit is erased, there is no way to determine the original message (all hypothesis are equally possible).
So the goal here is the put redundancy, the goal is the create a redundancy that match exactly the message. The redundancy serves to deal with the back eyes (i don't rembeber).
So first we got rid of redundancy, we encrypt, and the finaly we add back in redundancy (which is not the same as the one we got rid off).
So first we got rid of redundancy, we encrypt, and the finaly we add back in

redundancy (which is not the same as the one we got rid off).
Now suppose that we do channel coding like this:



The channel output is not a valid codeword. The decoders recognizes it, and assumes that the transmitted codeword is the one that agrees in most position with the observed channel output.
We add redundancy like some backups bits to make it stronger.

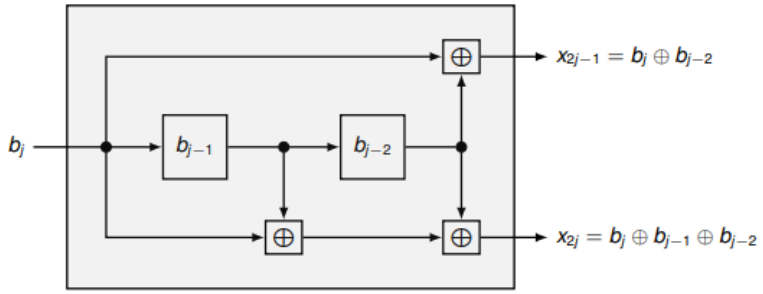
We study only block codes

The above is an (n, k) **block code** with $n = 6$ and $k = 2$: each k source symbols are substituted by n channel symbols over the same alphabet.
Since the alphabet is $\{0, 1\}$, we call it a **binary** (n, k) block code.

> We consider only bock codes

Example

The following is a convolutional encoder: every encoder input symbol produces two encoder output symbols.
The output pair produced at any given time is a linear function of the corresponding encoder input and encoder state (the previous two inputs).



So here we put the bit (from the encryption or whatever) we stores the previous bit. We then compute the two output.
Here this is not a really good code (too slow for our smartphone but it is okay for satellite communications).

Terminology

- The code $\mathcal{C}$ is the set of codewords
- A codeword c is an element of $\mathcal{A}^n$. (the alphabet $\mathcal{A}$ is $\{0,1\}$ in our example)
- The block length is n (which is the length after the $\rightarrow$)
- Each codeword carries $k = \log_2|\mathcal{C}|$ bits of information (however we don't have to use $\log_2$ we can also use $\log_{|\mathcal{A}|}$ which won't have bit as a unit).
- The rate is $\frac{k}{n}$ bits per symbol

Here $|\mathcal{C}|$ doesn't have to be equal to $\mathcal{A}^n$.

Hamming distance

Definition 26 *The **Hamming distance $d(x, y)$** between two n-tuples x and y is the number of positions in which they differ.*

Example

- $x = (101110)$, $y = (100110)$, $d(x, y) = 1$
- $x = (0427222)$, $y = (1227986)$, $d(x, y) = 5$

The Hamming distance is indeed a distance

In math, a function of two variables is a distance if it satisfies the following axioms:

Definition 27 *Distance axioms:*

- ***non-negativity**: $d(x, y) \geq 0$ with equality if and only if $x = y$*
- ***Symmetry**: $d(x, y) = d(y, x)$*
- ***triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$*

Proof

We can first rewrite the distance like this:

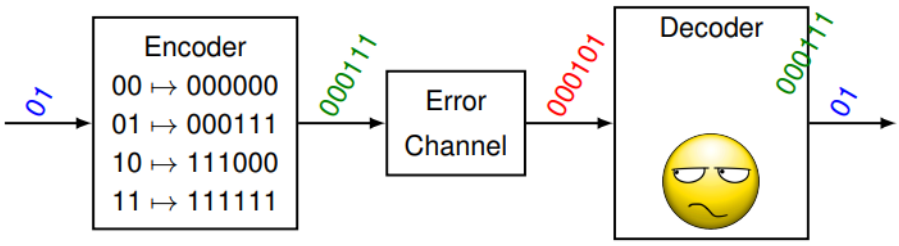
$$d(x, y) = \sum_{i=1}^{n} d(x_i, y_i)$$

Where the x_i and y_i have a distance of 1.
The claim is: $d(x_i, z_i) \leq d(x_i, y_i) + d(y_i, z_i)$.
To prove this we do a proof by cases:

- **Case 1**: $x_i = z_i \iff d(x_i, z_i) = 0$ there fore the inequality hold ($0 \leq d(x_i, y_i) + d(y_i + z_i)$
- **Case 2**: $x_i \neq z_i \iff d(x_i, z_i) = 1$
  Therefore in this case, there is at least one of the $d(x_i, y_i) = 1$ or $d(y_i, z_i)1$ (or both) which is greater or equal to 1. (here we use one because we said before that we were using "vecteurs unitaires".

Minimum Distance decoder

- The decoder guesses the encoder input based on the channel output
- Here we consider only **minimum distance decoders**.



How it works is pretty simple. It take the output codeword and compute the hamming distance with every element of the alphabet $\mathcal{C}$. And chose the one with the smallest hamming distance.
For our example let us compute the hamming distance:

$$d_H(000101, 00000) = 2$$
$$d_H(000101, 000111) = 1$$
$$d_H(000101, 111000) = 5$$

We called here 000111 the winner! (we don't know for sure if it is **but** it is a good answer, a good starting point).

Minimum distance decoder

Definition 28 *Let y be the channel output observed by the decoder. A minimum-distance decoder decides that the channel input is (one of) the $\hat{c} \in \mathcal{C}$ for which $d(y, \hat{c})$ is minimized.*

$$\hat{c} \in arg\ min_{x \in \mathcal{C}} d(y, x)$$

We pick one in the minimum if there are more than one with small Hamming distance.
The justification is that a small error weight is more likely than a large one.

Minimum distance

Definition 29 *The minimum distance of a code $\mathcal{C}$ is*

$$d_{min}(\mathcal{C}) = min_{x,y \in \mathcal{C}: x \neq y} d(x, y)$$

Example Given the set $\mathcal{C} = \{000000, 100110, 011001, 111111\}$, we get that

$$d_{min}(\mathcal{C}) = 3$$

What to expect from a decoder

For an error channel:

- Channel error correction: The best is if the decoder recognize and corrects the channel errors. In this case, the encoder input is recovered error-free.
- Channel-error detection: in some circumstances, the encoder is able to detect the presence channel errors but it is unable to correct them. The receiver may or may not ask for retransmission
- Decoding error: the worst is if the decoder tries to do as in (1) and makes the wrong decision.

For an erasure channel:

erasure correction: the best is if the decoder is capable of filling in the erased positions. In this case, the encoder input is recovered error.free

(erasure detection: unlike errors, erasure are always detected)

decoding error: the worse is if the decoder fills in one or more erased position with incorrect symbols.

Error detection

The question now is how does error detection is related to $d_{min}(\mathcal{C})$

Theorem 32 *1. Channel errors of weight $p < d_{min}(\mathcal{C})$ do not lead to a codeword. Hence they are detected.*

2. Some channel errors of weight $p \geq d_{min}(\mathcal{C})$ do lead to another codeword. Hence they cannot be detected by a minimum-distance decoder.

Note Erasures are always detected (by definition)

Proof Firstly, let $c \in \mathcal{C}$ be transmitted and y be received. We know that if $p = d(c, y) < d_{min}(\mathcal{C})$, y cannot be a codeword, therefore the error is detected.
We construct an example in which a channel error of weight $p = d_{min}(\mathcal{C})$ cannot be detected.
Let c and c' be a codeword at distance $d_{min}(\mathcal{C})$.
Suppose that c is the channel input and the channel output is $y = c'$.
y is a codeword. A minimum distance decoder will decide that no channel error has occured.

Example (Error detection)

Let the encoding map be the MOD 97-10 procedure:

$$u \to v = (100 \cdot u) + (98 - [100 \cdot u]_{97})$$

Recall that v is considered as valid if $[v]_{97} = 1$
For example $u = 0216936631 \to v = 021693663165$ Suppose v is transmitted and v' is received, $d(v, v') = 1$.
We can always write $v' = v + a10^k$ with $a \in \{-9, \ldots, -1, 1, \ldots, 9\}$.
The only way for v' to be a valid codeword is if $\left[a10^k\right]_{97} = 0$.
Since $[10]_{97}$ is invertible, so is $\left[10^k\right]_{97}$, hence $a = 0$.
Therefore all weight 1 errors are detected, implying that the minimum distance is at least 2.

Code $\mathcal{C}$

Let use have the code $\mathcal{C}$:

$$\begin{matrix}0000\\0011\\1100\\1111\end{matrix}$$

We want to find n, k, rate and d_{min}.

- $n = 4$
- $k = \log_2|\mathcal{C}| = \log_2 4 = 2$
- rate: $\frac{2}{4} = \frac{1}{2}$
- $d_{min} = 2$

If we take the code:

$$\begin{matrix}0000\\1111\end{matrix}$$

- $n = 4$
- $k = \log_2|\mathcal{C}| = \log_2 2 = 1$
- rate: $\frac{1}{4} = \frac{1}{4}$
- $d_{min} = 4$

For example without binary code:

$$\begin{matrix}000\\001\\002\\\vdots\\222\end{matrix}$$

- $n = 3$
- $k = \log_2|\mathcal{C}| = \log_2 27$
- rate: $\frac{\log_2 27}{3} = \frac{\text{bits}}{\text{channel use}}$ which is the thing you get when you ask swisscom how much 4g you have. If we take for example a better universe:

$$k = \log_3|\mathcal{C}| = \log_3 27 = 3$$

$$\text{rate} = \frac{k}{n} = \frac{3}{3} = 1$$

- $d_{min} = 1$

But we can take the alphabet in a smart way: 0.5

000
011
022
110
220
121
102
201
212

- $n = 3$
- $k = \log_3|\mathcal{C}| = \log_3 9 = 2$
- rate: $\frac{k}{n} = \frac{2}{3}$
- $d_{min} = 2$

Erasure correction: how it relates to $d_{min}(\mathcal{C})$

Theorem 33 *A minimum-distance decoder for a code $\mathcal{C}$* ***corrects*** *(fill in)* ***all the erasures*** *of weight p if and only if $p < d_{min}(\mathcal{C})$*

Let us do it for the previous code (the one right before).
If we take for example 220 → ERASURE → 2?2 the claim now is the we actually can recover from this. Let us check the alphabet which one has a 2 one the first side and a 0 on the other side. We check and we see that there is only one in the alphabet that check all the ticks: 220. We can also say we chose the one that has the smallest hamming distance with the erased codeword.

Proof ⟸

Suppose that $p < d_{min}(\mathcal{C})$.
Let c and y be the input and the output of an erasure channel, respectively, with $d(c, y) = p$.
We show that there is only one way to fille in the erased positions.
let $c \in \mathcal{C}$ and $\overline{c} \in \mathcal{C}$ be two codewords that agree with y in the non-erased positions.
Clearly, $d(c, \overline{c}) \leq p < d_{min}(\mathcal{C})$ This is possible only if $c = \overline{c}$.

Proof ⟹

We use contraposition.
Suppose that $p = d_{min}(\mathcal{C})$
We construct an example where the decoder will not always decode correctly.
Let c and c' be codewords at distance $d_{min}(\mathcal{C})$.
Let c be the channel input, and suppose

Illustrative example

given the alphabet

0000
0011
1100
1111

You can see that if you take the erased 00?? you cannot know which one is it is. we took here the $p = d_{min}(\mathcal{C})$ and as you can see we cannot guess.

Error correction

Theorem 34 *A minimum-distance decoder for a code $\mathcal{C}$* ***corrects all channel errors*** *of weight p if and only if $p < \frac{d_{min}(\mathcal{C})}{2}$.*

Proof

Let have the true codeword C, noisy channel output y. Given the assumption, $d(c, y) = p$. Let $\overline{c}$ be the output of the minimum distance decoder. Therefore if we want to comput the hamming distance between $\overline{c}$ and y we get:

$$d(\overline{c}, y) \leq p$$

Then by the triangle inequality:

$$\begin{aligned} d(c, \overline{c}) &\leq d(c, y) + d(y, \overline{c}) \\ &\leq 2p < d_{min}(\mathcal{C}) \end{aligned}$$

Hence, $c = \overline{c}$
Hence, the minimum distance decoder output the correct answer.
A finir la preuve

Summary

	detection guaranteed if	correction guaranteed if
erasure channel	(not applicable)	$p < d_{min}$
error channel	$p < d_{min}$	p < $\frac{d_{min}}{2}$

Upper bound to $d_{min}(\mathcal{C})$

Recall the important parameters of a block code $\mathcal{C}$ over a D-ary alphabet:

- n, the block length
- $k = \log_D |\mathcal{C}|$ the number of information symbols carried by a codeword. (Equivalently, $\mathcal{C} = D^k$)

Singleton's bound

Theorem 35 *Refardless of the alphabet size, the minimum distance of a block code satisfies:*

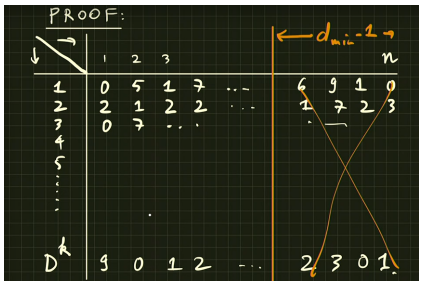
$$d_{min} - 1 \leq n - k$$

Block codes that satisfy the singleton bound with equality are called **Maximum distance separable** codes. (**MDS** codes).

Proof

	1	2	3	...	n
1	0	5	1	...	0
2	2	1	2	...	3
⋮					
D^k	9	0	2	...	2

You can then get rid of the $d_{min} - 1$ from the right like this:



The next question after this is: how many different strings of length $n - (d_{min} - 1)$ can we have?
We can you combination with the alphabet cardinilality being D we get:

$$nbr_{comb} = D^{n-(d_{min}-1)}$$

Hence we must have:

$$D^k \leq D^{n-(d_{min}-1)}$$

Remark

here k doesn't have to be a integer, only D^k **has to** be an integer (by assumption)

Pigeonhole principle

a lire mais pas vu en cours donc voila ("it is a bit overkill")

Exercise