AICC II

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Liste des cours

Chapitre 1

Introduction

1.1 About this course

In this course, there will be three main topics that will be studied :

- Communication
- Information and Data science
- Cryptography, Secrecy, Privacy

1.2 Cours Grading

- 90% Final exam during exam period
- 10 % Quizzes (online on Moodle)
  - There will be 6 quizzes. BO5
  - On the quizzes, you can update your answer as many times as you want before the deadline
- Quizzes are highly coorelated with homework.

1.2.1 How to be efficient and do well in this course

Before class :

- Browse through the slides to know what to expect
- review the background material as needed

After class :

- read the notes : they are the reference
- do the review questions

Before the exercice session

- are you up to date with the theory ?
- Solve what you can ahead of time and finish during the exercice session
- write down **your** solution

Mardi 18 février 2025 — **Cours 1 : Discrete Probability**

Chapitre 2

Entropy

2.1 Initial case : Finite Ω : set of all possiblie outcomes

Definition 1 ***Sample space*** *Ω is the set of all possible outcomes*

Definition 2 ***Event*** *E : a subset of Ω. Since the outcomes are equally likely :*

$$p(E) = \frac{|E|}{|\Omega|}$$

2.2 Conditional Probability

Conditional probability

Definition 3 *The **conditional probability** $p(E|F)$ is the probability that E occurs, given that F has occured (hence assuming that $|F| \neq 0$) :*

$$p(E|F) = \frac{|E \cap F|}{|F|}$$

Independent Events

Event E and F are called **independent** if $p(E|F) = p(E)$

Personal remark

this means that even if we know that F has occured the probability of E is still the same.

General Case : Finite $\boldsymbol{\Omega}$, arbitary $\boldsymbol{p(\omega)}$

Having equally likely outcomes is pretty rare in real life, juste take two dices and do the sum of the result and you will se that all the possible outcome doesn't have the same probability. In order to express those types of distribution we use the probability mass function :

Definition 4 ***Sample space*** Ω *: set of all possiblie outcomes*
Probability distribution (probability mass function) p *:*
A function $p : \Omega \to 1$ *such that :*

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

If we sum up all the probablity it gives us 1.

muss function to a subset Given $E \subset \Omega$ we can define the domain of the probability mass function p is extended to the power set of Ω :

$$p(E) = \sum_{\omega \in E} p(\omega)$$

2.3 Conditional probability and Independent Events

General form The general form for the conditional probability is :

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

for F such that $p(F) \neq 0$

Independet events As before E and F are called independent if $p(E|F) = p(E)$, Equivalently, E and F are independent iff $p(E \cap F) = p(E)p(F)$.

Disjoin event if E_1 and E_2 are disjoint event then :

$$p(E_1 \cup E_2) = p(E_1) + p(E_2)$$

Law of total probability For any $F \subseteq \Omega$ and its complement F^c,

$$p(E) = p(E|F)p(F) + p(E|F^c)p(F^c)$$

which sounds very intuitive because by definition F and F^c are disjoint.

Generally

Theoreme 1 *If* Ω *is the union of disjoint event* $F_1, F_2, \ldots, F_n$ *then :*

$$p(E) = p(E|F_1)p(F_2) + p(E|F_2)p(F_2) + \cdots + p(E|F_n)p(F_n)$$

Proof We prove the law of total probability for $\Omega = F \cup F^c$ (the general case follows straighforwardly)

$$\begin{aligned}
p(E) &= p(\underbrace{(E \cap F) \cup (E \cap F^c)}_{\text{union of disjoint sets}}) \\
&= p(E \cap F) + p(E \cap F^c) \\
&= \frac{p(E \cap F)}{p(F)} p(F) + \frac{p(E \cap F^c)}{p(F^c)} p(F^c) \\
&= p(E|F)p(F) + p(E|F^c)p(F^c)
\end{aligned}$$

Bays' Rule

Theoreme 2
$$p(F|E) = \frac{p(E|F)p(F)}{p(E)}$$

Proof We use the definition of conditional probability to write $p(E \cap F)$ two ways and solve for $p(F|E)$:

$$p(F|E)p(E) = p(E \cap F) = p(E|F)p(F)$$

2.4 Random variable

Random variable

Definition 5 *A Random variable is a function X such as $X : \Omega \to \mathbb{R}$*

Probability distribution

p_x, $p_x(X = x)$ or $p_x(x)$ is the probability that $X = x$, i.e, the probability of the event

$$E = \{\omega \in \Omega : X(\omega) = x\}$$

Hence,

$$p_x(x) = \sum_{w \in E} p(\omega)$$

Example You rolle a dice.
if the outcome is 6, you receive 10CHF. Otherwise, you pay 1 CHF.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$\text{For each } \omega, p(\omega) = \frac{1}{6}$$

Then define :

$$X(\omega) = \begin{cases} 10, & \omega = 6 \\ -1, & \omega \in \{1, 2, 3, 4, 5\} \end{cases}$$

Hence, we have

$$p_x(X) = \begin{cases} \frac{1}{6}, & x = 10 \\ \frac{5}{6}, & x = -1 \end{cases}$$

2.4.1 Two random variables

Two random variables

Definition 6 *Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be two random variables. The probability of the event $E_{x,y} = \{w \in \Omega : X(\omega) = x$ and $Y(\omega) = y\}$ is :*

$$p_{x,y}(x, y) = \sum_{w \in E_{x,y}} p(\omega)$$

- p_x is called **marginal distribution** (of $p_{x,y}(x, y)$ with respect to x)
- p_y can be computed similarly

2.5 Expected Value

Expected value

Definition 7 *The expected value $\mathbb{E}[X]$ of a random variable $X : \Omega \to \mathbb{R}$ is :*

$$\mathbb{E}[X] = \sum_{\omega} X(\omega)p(\omega)$$
$$= \sum_{x} xp_x(x)$$

linearity

Expectation is a linear operation in the folowwing sence :
Let $X_1, X_2, \ldots, X_n$ be random variables and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be scalars. Then :

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i \alpha_i\right] = \sum_{i=1}^{n} \alpha \mathbb{E}[X_i]$$

Random variable and independecy

Two random variable X and Y are independent if and only if, for all realizations x and y :

$$p(\{X = x\} \cap \{Y = y\}) = p(\{X = x\})p(\{Y = y\})$$

Or, more concisely, iff

$$p_{x,y}(x, y) = p_x(x)p_y(y)$$

Generalization

Theoreme 3 *Given n random variables, $X_1, \ldots, X_n$ are independent if and only if :*

$$p_{x_1,\ldots,x_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_{x_i}(x_i)$$

Condition probability

The conditional distrivution of Y given X is the function :

$$p_{x,y}(x|y) = \frac{p_{x,y}(x, y)}{p_x(x)}$$

Independent random variables

The folowig statement are equivalent to the statemant that X and Y are two indepedent random variables :

- $p_{x,y} = p_x p_y$
- $p_{y|x}(y|x) = p_y(y)$
- $p_{y|x}(y|x) = p_y(y)$ is not a function of x
- $p_{x|y}(x|y) = p_x(x)$

- $p_{x|y}(x|y)$ is not a function of y

Summary 1
- *Random Variable*
- *Probability distribution*
  - *Joint distribution of multiple variables*
  - *Marginal distribution*
  - *Conditional distribution*
- *Independence*

Mercredi 19 février 2025 — **Cours 2 : Source and entropy**

Expected value and operation The addition works well with Expectation such that

$$\mathbb{E}[X+Y] = \mathbb{E}[x] + \mathbb{E}[Y]$$

However, the product doesn't work well,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

if and only if X and Y are independent random variables.

2.6 Entropy

Introduction We communicate be revealing the value of sequence of variables that we call (**Symbols**), **Information**
In modern language, Hartley was saying that the value of a symbole provides information if and only if the symbol is a **random variable**.
How much information is carried by a symbol such as S ?

- Suppose that $S \in \mathcal{A}$ is a symbol that can take $|\mathcal{A}|$ possible values
- The amount of information conveyed by n such symbol should be n times the informations conveyed
- there are $|\mathcal{A}|^n$ possible values for n symbols
- This suggests that $\log |\mathcal{A}|^n = n \log |\mathcal{A}|$ is the appropriate mesure for information

However, this approach doesn't works :

Example Imagine having a town where there are 360 days and 5 rainy days, this leads to have only to possibilities, $|\mathcal{A}| = 2$ which make the quantity of information $\log_2 2 = 1$ bits. Which intuitively sounds kind of false, the forecast doesn't give us that information knowing that it is sunny $\frac{360}{365}$ % of the times, it is kind of excpected.

An article in 1948 from Shannon fixes the problem by defining **Entropy**

Definition the **uncertainty** or **entropy** $H(S)$ associated to a discrete random variable S :

Definition 8

$$H_b(S) = -\sum_{S \in supp(p_s)} p_s(s) \log_b p_s(s)$$

Where $supp(s) = \{s : p_s(s) > 0\}$.

Few comments

$$H_b(S) = -\sum_{S \in supp(p_s)} p_s(s) \log_b p_s(s)$$

- The condition $S \in supp(p_s)$ is needed because $\log_b p_s(s)$ is not define when $p_s(s) = 0$ this convention allows us to use the notation :

$$H_b(S) = -\sum_{s \in \mathcal{A}} p_s(\log_b p_s(s))$$

- The choice of b determines the unit, $b = 2$ is the **bit**

We also can see this as an "*average*" of $-\log_b p_s(S)$ which is :

$$H(S) = \mathbb{E}[-\log_b p_s(S)]$$

Example

A sequence of 4 decimal digits, s_1, s_2, s_3, s_4 representing the number to open Anne's lock can be senn as the output of a source S_1, S_2, S_3, S_4 with $S_i = \{0, \ldots, 9\}$.
If Anne picks all digits at randm and indepedently, the all outcomes are equally likely :

$$p_{S_1,S_2,S_3,S_4}(S_1, S_2, S_3, S_4) = \frac{1}{10^4}$$

If we search the entropy of this we get :

$$H_2(S) = \log_2 |\mathcal{A}| = \log_2 10^4 \approx 13.3 \; bits$$

2.6.1 Information-Theory Inequality

Lemma (IT-Inequality)

lemme 1 *For a positive real number* r,

$$\log_b r \leq (r-1) \log_b(e)$$

with equality if and only if $r = 1$

This proof juste using the deriative

Entropy Bounds

Theoreme 4 *The entropy of a discrete random variable $S \in \mathcal{A}$ satisfies :*

$$0 \leq H_b(S) \leq \log_b |\mathcal{A}|$$

With equality on the left if and only if $p_s(S) = 1$ and on the right if and only if $p_s(S) = \frac{1}{|\mathcal{A}|}$ for all s.

2.6.2 Random variables and Entropy

n random variable

the formula for entropy can be expanded to any number of random variables. If X and Y are two discrete random variables, with (joint) probability distribution $p_{x,y}$ then :

$$H(X,Y) = -\sum_{(x,y)\in X\times Y} p_{x,y}(x,y)\log p_{x,y}(x,y)$$

1.4 of textbooks

Theoreme 5 *Let $S_1, \ldots, S_n$ be discrete random variables. Then*

$$H(S_1, S_2, \ldots, S_n) \leq H(S_1) + H(S_2) + \cdots + H(S_n)$$

With equality if and only if $S_1, \ldots, S_n$ are indepedent.

Mardi 25 février 2025 — **Cours 2 : suite**

Ex hat party 1950

- n men, all have the same hat
- they throw hats in a corner
- leaving, they randomly take a hat

Solution

$$\text{Let } R_i = \begin{cases} 1, & \text{if person} i \text{leaves with their own hat} \\ 0, & \text{otherwise} \end{cases}$$

Entropy

$$H_2(S) = \sum_i p(s)\log\frac{1}{2p(s)} \tag{2.1}$$

$$= \frac{1}{8}\log_2\frac{8}{2} + \frac{1}{8}\log_2 8 \tag{2.2}$$

$$\approx \frac{1}{8} + \frac{1}{8}\cdot 3 \tag{2.3}$$

personal remark

We can see it as an average of ”surprise”.
Where the average is the randomness. (≈ 0.55)

2.6.3 Entropy bounds

Bound

$$0 \leq H_b(S) \leq \log_b \mathcal{A}$$

2.7 Source Coding Purpose

Source coding is often seen as a way to compress the source.
More generally, the foal of source coding is to efficiently describe how much information there is to a *file*

2.7.1 Setup

Setup

The **encoder** is specified by : :

- the input alphabet$\mathcal{A}$ (the same as the source alphabet)
- the output alphabet $\mathcal{D}$(typically $\mathcal{D} = \{0, 1\}$) ;
- the codebook $\mathcal{C}$ Which consists of finite sequences over $\mathcal{D}$;
- By the one to one encoding map $\Gamma : \mathcal{A}^k \to \mathcal{C}$ where k is a positive integer.

For now, $k = 1$.

Example

For each code, the encoding map Γ is specified in the following table : A mettre une image.

Example Code C or B are uniquely decodable : (A mettre une image 106)

Prefix Free codes

Definition 10 *If no codeword is a prefix of another codeword, the code is said to be prefix free.*

Example The codeword **01** is a prefix of **01**1.

- A prefix free code is always uniquely decodable
- A uniquely decodable code is **not necessarily** prefix free

A prefix code A prefix free code is also called instantaneous code :

- Think of phone numbers
- Think about streaming : instantaneous codes minimize the decoding delay (for given codeword length)

Code for one random variable

We start by considering codes that encode **one single random variable** $S \in \mathcal{A}$.
To encode a sequence $S_1, S_2, \ldots$ of random variables, we encode one random variable at a time.

Complete tree of a code

Slide 113 screen.

Binary tree

- There is a root (the beginning)
- A vertex (another node)

- A **leaf** is the last vertex
- Which is like a (arbre généalogique)

Ternary Tree The same as a binary tree but with three children.

With/Without prefix slide 115.

Decoding tree

- Obtained from the complete tree by keeping only branches that form a codeword
- Useful to visualize the decoding process

Slide 116

2.7.2 Codeword length

- The codeword length is defined the obvious way :
- Example : *ct*

$\mathcal{A}$ Γ_B codeword lengths
a 0 1
b 10 2
c 110 3
d 1110 4 height

- We would like the average codeword length to be as small as possible.

2.7.3 Kraft McMillan

Part 1. Necessary condition for the code to be uniquely decodable

> **Theoreme 6** *If a D-ary code is uniquely decodable then its codeword length $i_1, \ldots, i_M$ satisfy*
>
> $$D^{-l_1} + \cdots + D^{-l_M} \leq i$$
>
> *Kraft's inequality*

Example For code O we have :

$$2^{-2} + 2^{-2} + 2^{-2} + 2^{-2} = 1$$

Recall Kraft McMillan

> **Theoreme 7**

Example A For code A we have $2^{-1} + 2^{-2} + 2^{-2} + 2^{-2} = 1.25 > 1$.
KRaft-McMillan's inequality is not fulfilled.
There exists no uniquely decodable code with those codeword lengths.

Proof of K-MM Part I We prove a slightly weaker result, namely that the codeword lengths of prefix free codes satisfy K-MM inequality.
Let $L = \max_i l_i$ be the complete tree's depth.

- There are D^L terminal leaves
- There are D^{L-l_i}
- No two codewords share a terminal leaf (The code is prefix free)
- Hence $D^{L-l_i} + D^{L-l_2} + \cdots + D^{L-l_m} \leq D^L$

After dividing both sides by D^L we obtain Kraft's inequality :

$$D^{-l_1} + D^{-l_2} + \cdots + D^{-l_M} \leq 1$$

Exercice What is the **converse** of Kraft McMillan part 1 ?
The **Converse** of Kraft McMillan part 1 is not true (Consider e.g. two codewords : 01 and 0101)
However, the following statement is almost as good :

> **Theoreme 8** *If the positive integer $I_1, \ldots, I_M$ satisfy Kraft's inequality for some positive integer D,then there exists a D-ary* ***prefix free code*** *(hence uniquely decodable) that has codewords*

This says that if the inequality is true, then we **can** find D such that there exists a binary prefix which makes it decodable **and** prefix free !

2.7.4 Important Consequence of Kraft McMillan

Part I

> **Theoreme 9** *If a* ***D-ary code is uniquely decodable****, then its codeword length $I_1, \ldots I_M$ satisfy Kraft's inequality :*
>
> $$D^{-l_1} + \cdots + D^{-l_M} \leq 1$$

Part II

> **Theoreme 10** *If the positive integer $l_1, \ldots, l_M$ satisfy Kraft's inequality for some positive integer D, then there exists a D-ary* ***prefix free code*** *that has those codeword lengths.*

The Kraft McMillan theorem implies that any uniquely decodable code can be substituted by a prefix free code of the same codeword lengths.

Prefix free codes Our focus will be on prefix free codes. Reasons :

- No loss of optimality : codewords can be as short as for any uniquely decodable code ;
- a prefix free codeword is recognized as soon as its last digit is seen :

— important, e.g. a phone number ;
— advantageous to limit the decoding delay in, say streaming

Average Codeword length

- The typical use of a code is to encode a sequence of random variables
-

Example

$$\mathcal{A} = \{a, b, c, d\} \quad D = 2$$

Blackboard with table *cct* s ∈A
$\Gamma(s)$
$l(s)$
$p(s)$

a
0
1
0.05
b
10
2
0.05
c
110
3
0.1
d
1111
4
0.8

$$\mathcal{E}[\text{length}] = 0.05 + 1 + 0.05 \cdot 2$$

Definition 11 *Let $l(\Gamma(s))$ be the length of the codeword assiociated to $s \in \mathcal{A}$ The average codeword length is :*

$$L(S, R) = \sum_i p_s(s) i(\Gamma(s))$$

Units The unit of $L(S, \Gamma)$ are **code symbols**
When $D = 2$, the unit of $L(S, \Gamma)$ are bits.

Average codeword length : Lower Bound

Theoreme 11 *Let $\Gamma : \mathcal{A} \to \mathcal{C}$ be the encoding map of a D-ary*

Proof We want to prove that :

$$
\begin{aligned}
H(s) - \sum_s p(s)l(s) \\
&= -\sum_s p(s)\log p(s) - \sum_s p(s)l(s) \\
&= -\sum_s p(s)\log p(s) - \sum_s p(s)\log 2^{l(s)} \\
&= -\sum_s p(s)\log(p(s)\cdot 2^{l(s)}) \leq \dots
\end{aligned}
$$

Therefore :

$$
\begin{aligned}
&= \sum_s p(s)\log(\frac{1}{p(s)}2^{-l(s)}) \\
&\leq \sum_s p(s)\left(\frac{1}{p(s)}2^{-l(s)} - 1\right)\cdot C \\
&= (\sum_s 2^{-l(s)} - \sum_s p(s))\cdot C \\
&\leq 0
\end{aligned}
$$

We know that the left side is less or equal to 1 because of the Kraft Inequality, therefore it is bounded.

Mercredi 26 février 2025 — **Cours 4 : Continue**

Key observation The right hand side of :

$$L(S,\Gamma) = \sum_{s\in\mathcal{A}} p(s)l(\Gamma(s))$$

$$H_D(S) = \sum_{s\in\mathcal{A}} p(s)\log_D \frac{1}{p_S(s)}$$

are identical if $l(\Gamma(s))$

- Unfortunately $l(\Gamma(s)) = \log_D \frac{1}{p_S(s)}$ is often not possible (not an integer)
- How about choosing

Theoreme 12 • *For every random variable $S \in \mathcal{A}$*

Theorem

Theoreme 13 *The average codeword length of a D-ary Shannon-Fano code for the random variable S fulfils :*

$$H_D(S) \leq L(S,\Gamma_{SF}) < H_D(S) + 1$$

Proof it suffices to prove the upper bound (we have already proved the lower bound)
First suppose that we could use $l_i = -\log p_i$. The average

length would be :

$$L(S,\Gamma) = \sum_i p_i l_i = \sum_i p_i(-\log_D p_i) = H_D(S)$$

Instead we use $l_i = \lceil -\log p_i \rceil < -\log p_1 + 1$

Mardi 4 mars 2025 — **Cours 5 : Conditional Entropy**

Key Idea Pack multiple symbols into " *supersymbols*"

- $(S_1, S_2, S_3, \ldots, S_n)$
- Now, apply our Main result to such supersymbols

Theoreme 14 *The average codeword-length of a uniquely decodable code Γ for S must satisfy :*

$$H_D(S_1, S_2, \ldots, S_n) \leq L((S_1, S_2, \ldots, S_n), \Gamma)$$

And there exists a uniquely decodable code Γ_{SF} satisfying :

$$L((S_1, S_2, \ldots, S_n), \Gamma_{SF}) < H_D(S_1, S_2, \ldots, S_n) + 1$$

Our Next Nugget Understand

Example Audio recording :

- We can easily anticipate the next image in a video, there

KEy(simple) Independent

Definition 12 *The source models a seuquence $S_1, S_2, \ldots, S_n$ of n coin flips*
So $S_i \in \mathcal{A} = \{H, T\}$ where H stands for heat, T for tails.
$p_{S_i}(H) = p_{S_i}(T) = \frac{1}{2}$ *for all* $(s_1, S_2, \ldots, S_n) \in \mathcal{A}^n$

Not independent

Definition 13 *The source models a sequence $S_1, S_2, \ldots, S_n$ of weather conditions.*
So $S_i \in \mathcal{A} = \{S, R\}$, where S stands for sunny and R for rainy
The weather on the first day is uniformly distributed in $\mathcal{A}$.
For all other days, with probability $q = \frac{6}{7}$ the weather is as for the day before

Conditional Probability Recall how to determine the conditional probability :

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

It gives the probability of the event $X = x$, given that the event $Y = y$ has occured.
it is defined for all y for which $p_Y(y) > 0$

Remark There is good slide with good schema in slide 176-179

Conditional Expectation of X given $Y = y$

$$p_{X|Y}(\cdot \mid y)$$

is the probability distribution of the alphabet of X, juste like $p_x(\cdot)$

Definition 14 *The conditional expectation of X given $Y = y$ is defined as :*

$$\mathcal{E}[X \mid Y = y] = \sum_{x \in \mathcal{X}}$$

Conditional Entropy of X given $Y = y$

$p_{X|Y}(\cdot \mid y)$ is a probability distribution on the alphabet of X, juste like $p_X(\cdot)$
Every probability distribution has an entropy associated to it :

- $p_x(\cdot) \to H(X)$
- $p_{X|Y}(\cdot \mid y) \to H(X \mid Y = y)$

Definition 15 *The conditional entropy of X given $Y = y$ is defined as :*

$$H_D(X \mid Y = y) = -\sum_{x \in \mathcal{X}} p_{X|Y}($$

Example A faire

Entropy Bounds

Theoreme 15 *The conditional entropy of a discrete random variable $X \in \mathcal{X}$ conditioned on $Y = y$ satisfies :*

$$0 \leq H_D(X \mid Y = y) \leq \log_D \mid \mathcal{X} \mid$$

With equality on the left iff $p_{X|Y}(x, y) = 1$ for some x, and with equality on the right iff $p_{X|Y}(x \mid y) = \frac{1}{|\mathcal{X}|}$

The proff is identical to our proof of the basic entropy bounds

Example

Question ?
Do we also have the following entropy bound :

$$H_D(X \mid >= y) \overbrace{\leq}^{???} H_D(X)?$$

Answer : no.

Example (Or " *counterexample*" if better), Juste for ease of calculation, let us set $\delta = 0$ (but this is not necessary for the example to work). Then, we have :

$$H_D(X \mid Y = 0) h_D(\varepsilon) \text{ and } H_D(X \mid Y = 1) = 0$$

where $h_d(\cdot)$ is the binary entropy function (with $\log_D(\cdot)$). But we have :

$$H_D(X) = h_D(\frac{1-\varepsilon}{2})$$

Conditional entropy can either go up or down (if we give the answer the entropy is 0)

Conditional Entropy of X given Y

The most useful and impactful definition is the *average* conditional entropy of X given $Y = y$, averaged over all values of y under the marginal distribution $p_Y(y)$. Formally, we thus define :

Definition 16 *The conditional entropy X given Y is defined as :*

$$H_D(X \mid Y) = \sum_{y \in \mathcal{Y}} p_Y(y) \left(- \sum_{x \in \mathcal{X}} p_{X|Y}(x \mid y) \log_D p_{X|Y}(x \mid y) \right)$$

Example For the Bit flipper channel, we have ;

$$H_D(X \mid Y) = p(Y = 0) H_D(X \mid Y = 0) + p(Y = 1) H_D(X \mid Y = 1)$$

We search now :

$$H(X \mid Y) = p(Y \text{ is Head}) H(XY \text{ is head}) + p(Y \text{ is Tail}) H(X \mid Y \text{ is tail}) = \frac{1}{2} \cdot$$

Conditional Entropy of X given Y

Theoreme 16 *The conditional entropy of discrete random variable $X \in \mathcal{X}$ conditioned on Y satisfies :*

$$o \leq H_D(X \mid Y) \leq \log_D \mid \mathcal{X} \mid$$

With equality on the left iff for every y there exists and y such that $p_{X|Y}(x \mid y) = 1$ and with equality on the right iff $p_{X|Y}(x \mid y) = \frac{1}{|\mathcal{X}|}$ for all x and all y.

This follows directly from our bounds on $H_D(X \mid Y = y)$

Having $p_{X|Y}$

We know that $p(X \mid Y) = \frac{1}{|\mathcal{X}|}$ for all y.

$$\begin{aligned} p(x) &= \sum_{y \in \mathcal{Y}} p(y) p(x \mid y) \\ &= \sum_{y} p(y) \frac{1}{\mid \mathcal{X} \mid} \\ &= \frac{1}{\mid \mathcal{X} \mid} \cdot \sum_{y} \overbrace{p(y)}^{=1} \end{aligned}$$

Conditioning Reduces Entropy

The following bound is important and impactful (and also intuitively pleasing !)

Theoreme 17 *For any two discrete random variables X and Y,*

$$H_D(X \mid Y) \leq H_D(X)$$

with equality iff X and Y are independent random variables

In words, **On average**, the uncertainty about X can only become smaller if we know Y.

As we have seen, this is not true point-wise : We may have $H_D(X \mid Y = y) > H_D(X)$ for some values of y.
It works only on average.

Proof

$$\begin{aligned}
H(X \mid Y) - H(X) = & \\
&= \sum_y p(y)\left(-\sum_x p(x \mid y)\log p(x \mid y)\right) + \sum_x p(x)\log p(x) \\
&= \sum_{x,y} p(y)p(x \mid y)\log\frac{1}{p(x \mid y)} + \sum_{x,y} p(y \mid x)p(x)\log p(x) \\
&= \sum_{x,y} p(x,y)\log\frac{p(x)}{p(x \mid y)} \\
&\leq \sum_{x,y} p(x,y)\left(\frac{p(x)}{p(x \mid y} - 1\right)\cdot\log e \\
&= \sum_{x,y}(p(x)p(y) - p(x,y))\log(e) \\
&= \left(\left(\sum_y p(x)p(y)\right) - \left(\sum_x p(x)p(y)\right)\right)
\end{aligned}$$

Conditional Entropy of $f(x)$

Let X be an arbitrary random variable. Let $f(x)$ be a (deterministic) function of x.

$$H(f(x) \mid X) = 0$$

Proof

To find this conditional entropy :
Let $Y = f(x)$

$$p(y \mid y) = \begin{cases} 1, & y = f(x) \\ 0, & y \neq f(x) \end{cases}$$

the probability that y is $f(x)$ is only true if $f(x) = y$.
This implies that the entropy is equal to 0 :

$$H(y \mid x) = 0$$

Conditioning reduced Entropy

A generalization of the previous bound is also interest to us :

Theoreme 18 *For any three discrete random variables X, Y and Z,*

$$H_D(X \mid Y, Z) \leq H_D(X \mid Z)$$

With equality iff X and Y are conditionally independent random variables given Z (that is, if and only if $p(x, y \mid z) = p(x \mid z)p(y \mid z)$ for all x, y, z,

You can see it as make the Z fall which makes it $p(x, y) = p(x)p(y)$

Proof It is only mathematics :

$$\begin{aligned} H_D(X \mid Y, Z) - H_D(X \mid Z) &= \mathbb{E}\left[\log_D \frac{1}{p_{X|Y,Z}(X \mid Y, Z)}\right] + \mathbb{E}[\log_D p_{X|Z}(X \mid Z) \\ &= \mathbb{E}\left[\log_D \frac{p_{X|Z}(X \mid Z)}{p_{X|Y}(X \mid Y, Z)}\right] \\ &= \mathbb{E}\left[\log_D \frac{p_{X|Z}(X \mid Z)p_{Y|Z}(Y \mid Z)p_Z(Z)}{niquesamere}\right. \end{aligned}$$

Mardi 4 mars 2025 — **Cours 6 : Conditional Entropy review**

Main definitions

We have here two mains definitions :
The entropy for for a "*case*" of a random variable :

$$H(X \mid Y = y) = -\sum_x p(X \mid y) \log p(X \mid y)$$

And, the conditional entropy on a random variable :

$$\begin{aligned} H(X \mid Y) &= \sum_y p(y) H(X \mid Y = y) \\ &= -\sum_y \sum_x p(x, y) \log p(x \mid y) \end{aligned}$$

The main thing to understand here is that $H(X \mid Y)$ is the *Generalization* of the first definition. It is all the possible values of Y together. This is why we sum up all possible value of y. The second way to write $H(X \mid Y)$ is like taking all the possible pairs together and calculating the entropy of each pairs.
Main Result : The main result behind this is :

$$\begin{aligned} 0 \leq H(X \mid Y = y) \leq \log \mid \mathcal{X} \mid \\ 0 \leq H(X \mid Y) \leq \log \mid \mathcal{X} \mid \end{aligned}$$

And the inequality :

$$\boxed{H(X \mid Y) \leq H(X)}$$

Conditional entropy of $f(x)$

Let X be an arbitrary random variable.
Let $f(x)$ be a (deterministic) function of x :

$$H(f(x) \mid x) = 0$$

For example :

$$X \in \{0, 1, 2, 3\}$$
$$f(x) = X \mod 2$$

Which is :

$$f(x) = \begin{cases} 0 \text{ if x is even} \\ 1 \text{ if x is odd} \end{cases}$$

Then,

$$P(f(x) \mid X) = \begin{cases} 0, & \text{if } x = 0, 2 \\ 1, & \text{if } x = 1, 3 \end{cases}$$

If we now compute the entropy for $X = 0$ and $X = 1$ etc..., we get :

$$H(f(x) \mid X = 0) = 0$$
$$H(f(x) \mid X = 1) = 0$$
$$\vdots$$

Lisa rolls two dice

Lisa rolls two dice and annoucnes the sum L written as a two digit number. The alphabet of $L = L_1L_2 = \{02, 03, 04, 05, 06, 07, 08, 09, 10, 11, 12\}$ Where the alphabet of $L_1 = \{0, 1\}$ and the alphabet of $L_2 = \{0, 1, 2, \ldots, 9\}$.
We are looking for the probability that $L_2 = 2$ knowing that $L_1 = 1$:

$$p_{L_2|L_2}(2 \mid 1)$$

What we are doing here is the joint distribution :

$$p_{L_2|L_1}(2 \mid 1) = \frac{P_{L_1,L_2}(1, 2)}{P_{L_1}(1)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

After running over all possible values for (i, j), we obtain :

$L_2 = j$ $L_1 = i$	0	1	$p_{L_2}(j)$
0	0	$\frac{3}{36}$	$\frac{3}{36}$
1	0	$\frac{2}{36}$	$\frac{2}{36}$
2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{2}{36}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$p_{L_i}(i)$	$\frac{5}{6}$	$\frac{1}{6}$	height

now you can do the same with conditional probability and then after computing all those value :

$$H(L_2 \mid L_1) = \frac{5}{6} \cdot 2.857 + \frac{1}{6} \cdot 1.459 = 2.624 \text{ bits}$$

Now we can observe that :

$$2.624 = H(L_2 \mid L_1) \leq H(L_2) = 3.22$$

Which says that on average, knowing something takes out some randomness

The chain rule for entropy

Recall that the joint entropy of two random variables X, Y is completely naturally defined as :

$$H_D(X,Y) = -\sum_x \sum_y p_{X,Y}(x,y) \log_D p_{X,Y}(x,y)$$

Or as seen earlier :

$$H_D(X,Y) = H_D(X) + H_D(Y)$$

Using the fact the $p_{X,Y}(x,y) = p_X(x) p_{Y|X}(y \mid x)$, we can write this as :

$$\begin{aligned}
H_D(X,Y) &= -\sum_x p_X(x) \left(\sum_Y p_{Y|X}(y \mid x) \log_D (p_X(x) p_{Y|X}(y \mid x)) \right) \\
&= -\sum_x p_X(x) \left(\sum_y p_{Y|X}(y \mid x) (\log_D p_X(x) + \log_D p_{Y|X}(y \mid x)) \right) \\
&= -\sum_x p_X(x) \left[\left(\sum_y p_{Y|X}(y \mid x) \log_D p_X(x) \right) + \left(\sum_y p_{Y|X}(y \mid x) \log_D p_{Y|X}(y \mid x) \right) \right] \\
&= H(X) + H(Y \mid X)
\end{aligned}$$

Theoreme 19

$$H(X,Y) = H(X) + H(Y \mid X)$$

Professor remark

Firstly :

$$H(Y,X) = H(X,Y)$$

Which is either proved by :

$$\begin{aligned}
H(Y,X) &= H(Y) + H(X \mid Y) \\
&= H(X,Y)
\end{aligned}$$

The relation proved before in words it :
To find the joint entropy of two random variables, we can first calculate the entropy of one of the two, and then add to it the conditional entropy of the second, given the first.

The chain rule entropy

Theoreme 20 *Let $S_1, \ldots, S_n$ be discrete random variables. Then :*

$$H_D(S_1, S_2, \ldots, S_n) = H_D(S_1) + H_D(S_2 \mid S_1) + \cdots + H_D(S_n \mid S_1, \ldots, S_{n-1})$$

The above result says that the uncertainty of a collection of random variables (in any order) is the uncertainty of the first, plus the uncertainty of the second when the first is known, plus the uncertainty of the third when the first two are know, etc...
Let us see how :

$$\begin{aligned}
&H(\underbrace{S_1, S_2, \ldots, S_{n-1}}_{=Z}, S_n) \\
&= H(Z) + H(S_n \mid Z) \\
&= H(\underbrace{S_1, S_2, \ldots, S_{n-2}}_{=Z'}, S_{n-1}) + H(S_n \mid S_1 \ldots, S_{n-1}) \\
&= H(Z') + H(S_{n-1} \mid Z') + H(S_n \mid S_{1,n-1}
\end{aligned}$$

Until we get $Z^{'\cdots'} = S_1$.

Example Let X, Y, Z be discrete random variables. We have :

$$\begin{aligned}
H(X,Y,Z) &= H(X) + H(Y \mid X) + H(Z \mid X, Y) \\
&= H(X) + H(Z \mid X) + H(Y \mid X, Z) \\
&= H(Y) + H(X \mid Y) + H(Z \mid X, Y) \\
&= H(Y) + H(Z \mid Y) + H(X \mid Y, Z) \\
&= H(Z) + H(X \mid Z) + H(Y \mid X, Z) \\
&= H(Z) + H(Y \mid Z) + H(X \mid Y, Z)
\end{aligned}$$

Theoreme 21 *Let $S_1, \ldots, S_n$ be discrete random variables. Then :*

$$H(S_1, S_2, \ldots, S_n) \leq H(S_1) + H(S_2) + \cdots + H(S_n)$$

With equality iff, $S_1, \ldots, S_n$ are independent

Proof

$$\begin{aligned}
H(S_1, S_2, S_3) &= H(S_1) + H(S_2 \mid S_1) + H(S_3 \mid S_1, S_2) \\
&\leq H(S_1) + H(S_2) + H(S_3)
\end{aligned}$$

Another way around

Sometimes it is convenient to compute the conditional entropy using the chain rule for entropies. For instance :

$$\boxed{H(X \mid Y) = H(X, Y) - H(Y)}$$

It can be useful to make it easier to compute $H(X \mid Y)$ because on the right side, it is only marginal entropies with $p \log p$ which are *"easy to compute"*

corollaire 1

$$H(X,Y) \geq H(X)$$
$$H(X,Y) \geq H(Y)$$

The above inequalities follow from the chain rule for entropies and the fact that entropy (condition or not) is nonnegative.

Example

From lisa rolls two dice :

$$\begin{aligned} H(L_1, L_2) &= 3.2744 \\ H(L_1) &= 0.6500 \\ H(L_2) &= 3.2188 \end{aligned}$$

We compute :

$$\begin{aligned} H(L_2 \mid L_1) &= H(L_1, L_2) - H(L_1) = 3.2744 - 0.6500 = 2.6254 \\ H(L_1 \mid L_2) &= H(L_1, L_2)_H(L_2) = 3.2744 - 3.2188 = 0.056 \end{aligned}$$

And verify that indeed :

$$\begin{aligned} H(L_1 \mid L_2) &\leq H(L-1) \leq H(L_1, L_2) \\ H(L_2 \mid L_1) &\leq H(L_2) \leq H(L_1, L_2) \end{aligned}$$

2.7.5 Random Processes

A.K.A Source models

Definition 17 *The source models a sequence $S_1, S_2, \ldots, S_n$ of n coin flips*
So $S_i \in \mathcal{A} = \{H, T\}$, where H stands for heads, T for tails, $i = 1, 2, \ldots, n$
$p_{S_i}(H) = p_{S_i}(T) = \frac{1}{2}$ for all i, and coin flips are independent.
Hence,

$$p_{S_1,S_2,\ldots,S_n}(S_1, S_2, \ldots, S_n) = \frac{1}{2^n}, \quad \forall (S_1, S_2, \ldots, S_n) \in \mathcal{A}^n$$

Definition 18 *The source models a sequence $S_1, S_2, \ldots, S_n$ of weather conditions.*
So $S_i \in \mathcal{A} = \{S, R\}$, where S stands for sunny and R for rainy, $i = 1, 2, \ldots, n$.
The weather on the first day is uniformly distributed in $\mathcal{A}$.
For all other days, with probability $q = \frac{6}{7}$ the weather is as for the day before

What we can see here that is the conditional probability, for example :

$$\begin{aligned} p(S_2 = \text{ sun} \mid S_1 = \text{ sun}) &= q \\ p(S_2 = \text{ rain} \mid S_1 = \text{ sun}) &= 1 - q \end{aligned}$$

However :

$$p(S_3 = \text{sun} \mid S_1 = \text{sun}, S_2 = \text{sun})$$
$$= p(S_3 = \text{ sun} \mid S_2 = \text{ sun}) = q$$

More generally :

$$P(S_n \mid S_1, S_2, \ldots, S_{n-1}) = p(S_n \mid S_{n-1})$$

Mardi 11 mars 2025 — **Cours 7 : Entropy and algorithm**

Experience little play

- Think of something
- Ask yes or no question
- Find the answer

the game was called twenty questions in old U.S tv. We want to use entropy to understand this game.

Last Week

$$H_D(X) = H_D(P) - -\sum_x p(x) \log_D p(x)$$

We also saw those two bounds :

$$0 \leq H_D(X) \leq \log_D \mid \mathcal{A} \mid$$

Information is always about option, more options you have, more information (the first way to introduce *"entropy"*)
We also saw :

$$H(X \mid Y = y) = -\sum_x p(x \mid y) \log_D p(x \mid y)$$
$$H(X \mid Y = y) = -\sum_y \ldots$$

And we also saw that on average :

$$H(X \mid Y) \leq H(X)$$
$$H(X \mid Y, Z) \leq H(X \mid Y) \leq H(X)$$

We also saw the chain rule :

$$H(S_1, S_2, S_3, S_4)$$
$$= H(S_2, S_4, S_1, S_3)$$

The order in entropy doesn't matter,

$$= H(S_1) + H(S_2 \mid S_1) + H(S_3 \mid S_1, S_2) + H(S_4 \mid S_1, S_2, S_3)$$

An intresting way to use this, is if we combine the inequalities and the chain rule. The equality on the right sight is true if and only if X and Y are independent. there fore :

$$H(S_1, S_2, S_3, S_4) = H(S_1) + H(S_2) + H(S_3) + H(S_4)$$

this equality is true if and only if S_1, S_2, S_3, S_4 are independent.

The 20 question problem Let X be a random variable. What is the minimum number of "yes/no" question needed to identify X?, which question should be asked.

Solution Let us consider a binary code Γ for $X \in \mathcal{X}$
Once Γ is fixed, we know $x \in \mathcal{X}$ if and only if we know the codeword $\Gamma(x)$.
The strategy consists in asking the ith question so as to obtein the ith bit of the codword $\Gamma(x)$.
The expected number of question $L(X, \Gamma)$, which is minimized if Γ is the encoding map of Huffman code

Example Suppose that we know that $\mathcal{X} = \{$ cat, dog, pony$\}$, with :

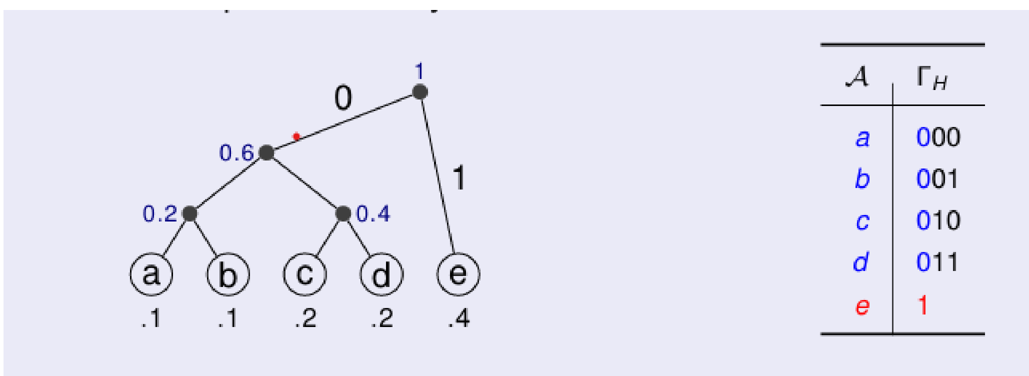
$$p(cat) = \frac{1}{2}$$
$$p(dog) = \frac{1}{4}$$
$$p(pony) = \frac{1}{4}$$

We want to make it the best way, the question we should ask is :

- is the animal a cat ?

X	a	b	c	d	e height

We then do a Huffman tree :



We know here that this, will be optimal

Optimality We have seen that a prefix free code for $X \in \mathcal{X}$ leads to a querying strategy to find the realization of X.
Similarly, a deterministic querying strategy leads to a binary prefix-free code for X. Here is why :

- Before the first question we know that $x \in \mathcal{X}$
- Without loss of generality, the first question can be formulated in terms of "is $x \in \mathcal{A}$"? for some $\mathcal{A} \subset \mathcal{X}$, (The choice of $\mathcal{A}$ is determined from the strategy, that we fix once and for all)
- Is the answer is YES, the we know that $x \in \mathcal{A} \subset \mathcal{X}$. Otherwise $x \in A^c \subset \mathcal{X}$. Either way we have reduced the size of the set that contains x.

- We continue asking similar questions until the value of x is fully determined, the we stop.

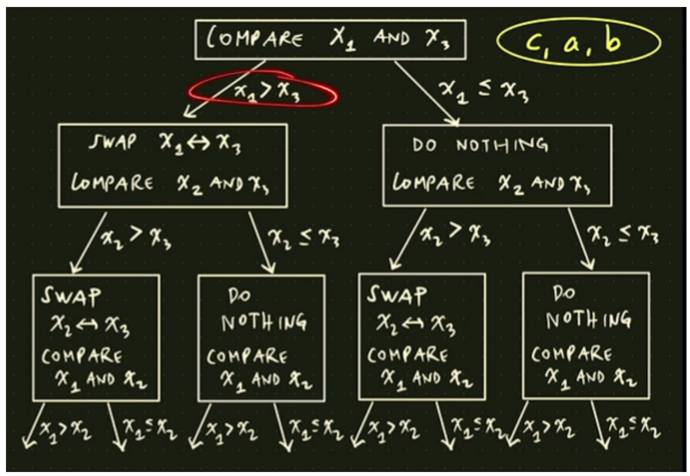
Here, the sequence of Yes or no answers is a binary codeword associated to x. The code obtained when we consider all possible values of x is a binary prefix-free code. Since the tree is prefix free, its averag codeword-length cannot be smaller than that of a Huffman code.

Sorting via pairwise comparisons

Given an **unsorted** List with n elements.
For example $l = [c, a, b]$ with $n = 3$
Repeat :

1. Select two position $1 \leq i \leq j \leq n$
2. Compare and swap :
   - If $x_i > x_j$
   - Then swap elements $x_i \iff x_j$
   - Else do nothing

One way to understands how it works :



The first observation :
The sequence of pairwise comparisons must identify the exact order of the unsorted list.
The second observation :
The sequence of pairwise comparisons in a uniquely decodable (actually, prefix-free) binary code for x.
There fore, we must have :

$$\mathbb{E}[\text{number of comparisons}] \geq H_2(X)$$

However what is the X ? We see it as a random variable because we don't really know what the unsorted list is.
For example $n = 3$ we have $\mathcal{X} = \{abc, acb, bac, bca, cab, cba\}$ where $\mathcal{X}$ is the set of all permutations.
However what is $p(x)$? Here, we want to talk about our algorithm working for all $p(x)$.

$$E \geq max_{p(x)} H_2(X) = \log_2 \mid \mathcal{X} \mid$$
$$= \log_2 n!$$

We already know a bounds on factorial :

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}}$$

Therefore :

$$H_2(x) \approx \log_2 \frac{n^n}{e^{n-1}}$$
$$= n \log_2 n - (n-1) \log_2 e$$

Which is *"dominated"* by $n \log_2 n$

Billard Balls

There are 14 billards balls numbered as shown :

Among balls $1 - 13$, at most one **could** be heavier/lighter than the others. What is the minimum number of weightings to simultaneously determine :

- If one ball is different
- if there is such a ball which one,
- And whether the different ball is heavier/lighter

Here we want to use entropy to solve this problem. The goal here is to associated the number of weightings to code. The goal is to see it as a tree.

The steps of picking two sets is *"mandatory"* we have to pick two sets in order to compare something, and in order to compare something, you have to compare something...
From this comparisons, there will be three possibilities. with three possibilites, We are specifying a Ternary code. The issue here is that we are losing information, yes we only get a binary tree however we wouldn't be able to have the same amount of information as with a ternary tree.
What we are saying here is, with any strategy to solve this problem **can** be written in this way. Hence we can read this tree as a ternary code.

*But a code for **What**?*

What are we finding with this code ?
A code for X :

- $X = 0$: all balls are equals
- $X = +1$: ball 1 is heavier
- $\vdots$
- $X = +13$ ball 13 is heavier
- $X = -1$ balle 1 is lighter

- ⋮
- $X = -13$ ball 13 is lighter

Then we know that $\mid \mathcal{X} \mid = 27$. This is one way to answers those question.

1. If $X = 0$ or not (then there is or not a different ball)
2. Then $\mid X \mid$ gives us the information
3. the sign of X if the ball is heavier or lighter

Observation The number of weighings is equal to the length of the ternary codeword

Then :

Theoreme 22

$$\mathbb{E}[number\ of\ weighings] \geq H_3(X)$$

It has to be three by the way the problem is stated. The code is ternary **Therefore** the base for the entropy is 3.
Moreover, our strategy must work **irrespective** of the probability distribution of X.
We can also see :

Theoreme 23

$$\mathbb{E}[number\ of\ weighings] \geq max_{p(x)}(H_3(X))$$

Where in our example gives us :

$$\log_3 27 = 3$$

It doesn't need the be an integer it is only the professor that choose on purpose to make it clean

But does there indeed exists such a code

FACT :

Entropy does **not** guarantee the existence of such a strategy

Entropy serves as a lower bound and **not** the best way to do it.

But can what if ?
Let us suppose it exists ! Then entropy tells us a few basic facts.

Fact 1 **if** 3 weighings S_1, S_2, S_3 uniquely specify X, Then we **must have** :

$$H_3(X) = H_3(S_1, S_2, S_3)$$

Proof

$$H(X, S_1, S_2, S_3) = H(X) + \overbrace{H(S_1, S_2, S_3 \mid X)}^{=0}$$
$$= H(S_1, S_2, S_3) + \overbrace{H(X \mid S_1, S_2, S_3)}^{=0}$$

It is true because if we know S_1, S_2, S_3 then we know all X then the entropy of 0.
For $H(X \mid S_1, S_2, S_3)$, because S_1, S_2, S_3 uniquely specify X then knowing them implies that this entropy is o.

Fact 2

If 3 weighings S_1, S_2, S_3 uniquely specify X, then we must have :

- S_1, S_2, S_3 uniformly distributed
- S_1, S_2, S_3 independent

Proof

$$H_3(S_1, S_2, S_3) = 3$$

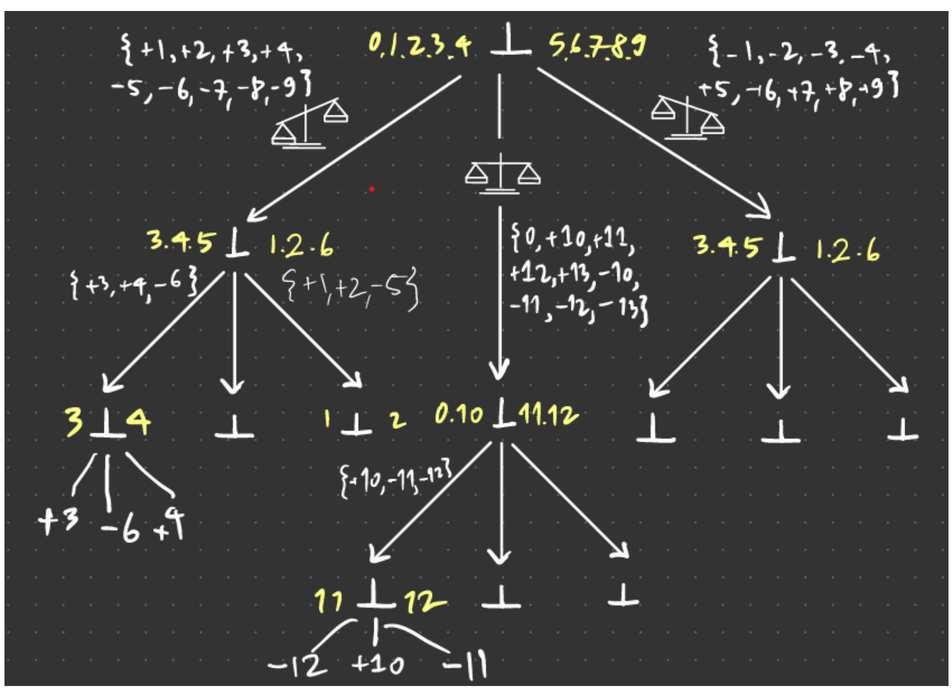
This is a *must*.
But also :

$$H_3(S_1) + H(S_2 \mid S_1) + H(S_3 \mid S_1, S_2) \leq H_3(S_1) + H(S_2) + H(S_3)$$
$$\leq \log_3 3 + \log_3 3 + \log_3 3$$

Where it is an equality if and only if the distribution is uniform and independent.

Example

Let's see how to actually find a way to ask those question :



Mercredi 12 mars 2025 — **Cours 8 : Prediction, learning, and Cross-Entropy-Loss**

Billard Balls Can we use the 20 questions approach to solve the 14 bullars riddle ?

> *Answer* No, because the kind of questions that we can "ask", when wa are weighing, is quite limited.
> For instance, the first question cannot be "is 1 or 2 heavy ?".

Strategies But is there a strategy that requires only 3 weighings ?
From source compression, we can establish the following facts ?

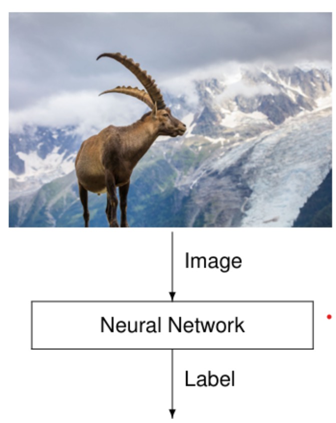
- For each weighings, the three outcomes must be equally likely
- The weighings must be independent of each other

> It is because we carefully selected the numbers (alphabet size of 27 ; each weighing has 3 possible outcomes) that there is a strategy that exactly matches the entropy lower bound 3 weighings. If you change the numbers, it will not generally be true that there is a strategy that *exactly* matches the lower bound.

2.7.6 Prediction, Learning and cross-Entropy Loss

The goal here is to change the way to use entropy, entropy has always be seen as something that *means* something, a lower bound, a quantity of information. Here we will use it to do calculation juste like a *tool*.

Example



Label	*Probability*
Ibex	0.98
Kangaroo	0.005
Lynx	0.002
Wombat	0.002
Dog	0.001
Cat	0.001
Turtle	0.001
Dolphin	0.001
Elephant	0.001
Kookaburra	0.001
Other	0.005

> There weren't probability at this time in the slide so imagine without it

The question we want to ask is, *"Is our neural network performing well"*

- Given an image $\mathcal{X}$
- Our machine (Neural network)
- Outputs $Q(x)$
- The label : Label(x)

Zero-one loss

$$L\{Q(x) \neq \text{Label}(x)\} = \begin{cases} 1 \text{ if } Q(x) \neq \text{Label}(x) \\ 0 \text{ if } Q(x) = \text{Label}(x) \end{cases}$$

Given a lot of image, we want to have a **Classification error** :

$$\frac{\sum_{\mathcal{X}} L\{Q(x) \neq \text{Label}(x)\}}{\text{number of images}}$$

Is the function of mis-labeled images.

Pros and Cons

Pros

- Very intuitive
- Interpretable

Cons

- Not differentiable

With probability

Our neural network produces :

$$Q(\text{label} \mid \text{image})$$

The true label distribution is :

$$P_{true}(\text{label} \mid \text{image}) = \begin{cases} 1, & \text{correct label} \\ 0, & \text{wrong label} \end{cases}$$

(We are assuming for simplicity that for each image, there is a single correct label).

- Ideally, we would like :

$$Q(\text{label} \mid \text{image}) = P_{true}(\text{label} \mid \text{image}) \;\; \forall \text{pairs}$$

However this is only a dream

- Instead, people like to consider **cross entropy loss**
- that is, we wish ou $Q(\text{label}|\text{image})$ to **minimize**

$$L(P_{true}(\text{label} \mid \text{image}), Q(\text{label} \mid \text{image}) = -\sum_{\text{label}} P_{true}(\text{label} \mid \text{image}) \log_D Q(\text{label} \mid \text{image})$$

- Given training data (image, label), for $i = 1, 2, \ldots, n$ we select $Q($ label $\mid$ image$)$ to minimize the cross entropy loss.

Cross entropy loss

$$L(P, Q) = -\sum_{y} P(y) \log_D Q(y)$$

Where

- P is the true distribution
- Q is our approximation (via neural network)

Why is it popular ?

- Good properties for training with "gradient descent" in certain standard architectures.
- Theoretical properties.

A (very) simple neural network

Takes a screen of the blackboard

- it transform the image into a vector
- Then takes is through the weighs w_i all the way to d
- the we take it through the soft max which is two functions :

$$Q(o \mid x) = \frac{e^{z_0}}{e^{z_0} + e^{z_1}}$$
$$Q(1 \mid x) = \frac{e^{z_1}}{e^{z_0} + e^{z_1}}$$

The goal is given a lot of training data, we want to select the w_0, b_0, w_1, b_1 such at to minimize the total cross entropy loss.

For a single image $\mathcal{X}$

because why is juste binary we use :

Total Loss

$$L_{total}(w_o, b_o, w_1, b_1) = -\sum_{i=1}^{k} \log \frac{e^{x_i w_0 + b_0}}{e^{x_i w_0 + b_0} + e^{x_i \cdot w_1 + b_1}} - \sum_{i=k+1}^{n} \log \frac{e^{w_1 k_i + b_1}}{e^{w_0 x_i + b_0} + e^{w_1 x_i + b_1}}$$

Cross entropy loss

Cross entropy loss :

$$L(P, Q) = -\sum_{y} P(y) \log_D Q(y)$$

Theoreme 24 *For a fixed probability distribution P, the minimum :*

$$min_Q L(P, Q)$$

Is attained if and only if we selected $Q^* = P$ in this case,

$$L(P, Q^*) = L(P, P) = H(P)$$

Where $H(P)$ is the entropy of the probability distribution P

Proof The proof, which will be done in class, uses once again the "IT inequality".

The theorem is saying this :

$$H(P) \leq L(P, Q)$$

With equality in one case which is $P = Q$.

$$\begin{aligned} H(P) - L(P, Q) &\leq 0 \\ -\sum_y P(y) \log P(y) + \sum_y P(y) \log Q(y) &\leq 0 \end{aligned}$$

$$= sum_y P(y) \log \frac{Q(y)}{P(y)} \leq \sum_y P(y) \left[\frac{Q(y)}{P(y)} - 1 \right] \log(e)$$
$$= \sum_y (Q(y) - P(y)) \log(e)$$
$$= 0$$

Note

We don't see it in AICC II but let's introduce the notion : **KL-Divergence** (aka KL distance) :

$$D_{kl}(p \mid\mid k) = \sum_y p(y) \log \frac{P(y)}{Q(y)}$$

- Fact 1 :
  $D_{kl}(P \mid\mid Q) \geq 0$ with equality iff $P = Q$ (this is just the proof seen earlier

2.8 Summary of chapter 1

Entropy

$$H_D(X) = -\sum_x p(x) \log_D p(x)$$

For $D = 2$, we simply write $H(X)$ and we all the units bits.
Entropy has many useful properties, including :

- $0 \leq H_D(X) \leq \log_D \mid \mathcal{X} \mid$
- $H_D(X \mid Y) \leq H_D(X)$ with equality if and only if X and Y are independent
- $H_D(X, Y) = H_D(X) + H_D(Y \mid X)$

Data Compression

- Every uniquely decodable binary code must use at least $H(X)$ bits per symbol on average
- There exists a binary code that uses between $H(X)$ and $H(X) + 1$ bits per symbol on average
- Hence, for a source string of length n :
  — Every uniquely decodable binary code must use at least $H(S_1, S_2,$

Models

Coin Flip The coin flip is not convertible, With a file of result, there is no way to compress the file

Sunny Rainy Here, the entropy, is not 1 then we are able to compress the file here.
This is the first view of mark of model.
Given $S_1, S_2, S_3, \ldots$, Are S_1, S_3 independent ?

$$
\begin{aligned}
p(S_1, S_3) &= \sum_{S_2} p(S_1, S_2, S_3) \\
&= \sum_{S_2} p(S_1) p(S_2 \mid S_1) p(S_3 \mid S_2)
\end{aligned}
$$

Entropy and algorithm We explored examples where entropy can give a lower bound on algorithmic performance.

- Example : in search-type problems, give a lower bound on the minimum number of necessary queries.

Cross-Entropy Loss

- Machine (e.g., Neural Network) outputs a distribution $Q(y)$ over all possible labels
- Cross entropy loss : Select $Q(y)$ to minimize :

$$L(P, Q) = -\sum_{y} P(y) \log_D Q(y)$$