

AICC II

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Chapter 1

Introduction

1.1 About this course

In this course, there will be three main topics that will be studied:

- Communication
- Information and Data science
- Cryptography, Secrecy, Privacy

1.2 Cours Grading

- 90% Final exam during exam period
- 10 % Quizzes (online on Moodle)
 - There will be 6 quizzes. BO5
 - On the quizzes, you can update your answer as many times as you want before the deadline
- Quizzes are highly coorelated with homework.

1.2.1 How to be efficient and do well in this course

Before class:

- Browse through the slides to know what to expect
- review the background material as needed

After class:

- read the notes: they are the reference
- do the review questions

Before the exercise session

- are you up to date with the theory?
- Solve what you can ahead of time and finish during the exercise session
- write down **your** solution

February 18, 2025 — **Lecture 1 : Discrete Probability**

Chapter 2

Entropy

2.1 Initial case: Finite Ω : set of all possible outcomes

Definition 1 *Sample space Ω is the set of all possible outcomes*

Definition 2 *Event E : a subset of Ω . Since the outcomes are equally likely:*

$$p(E) = \frac{|E|}{|\Omega|}$$

2.2 Conditional Probability

Conditional probability

Definition 3 *The **conditional probability** $p(E|F)$ is the probability that E occurs, given that F has occurred (hence assuming that $|F| \neq 0$) :*

$$p(E|F) = \frac{|E \cap F|}{|F|}$$

Independent Events

Event E and F are called **independent** if $p(E|F) = p(E)$

*Personal
remark*

this means that even if we know that F has occurred the probability of E is still the same.

**General Case:
Finite Ω , arbitrary $p(\omega)$**

Having equally likely outcomes is pretty rare in real life, just take two dices and do the sum of the result and you will see that all the possible outcome doesn't have the same probability. In order to express those types of distribution we use the probability mass function:

Definition 4 *Sample space* Ω : set of all possible outcomes
Probability distribution (probability mass function) p :
 A function $p : \Omega \rightarrow 1$ such that:

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

If we sum up all the probability it gives us 1.

mass function to a subset Given $E \subset \Omega$ we can define the domain of the probability mass function p is extended to the power set of Ω :

$$p(E) = \sum_{\omega \in E} p(\omega)$$

2.3 Conditional probability and Independent Events

General form The general form for the conditional probability is:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

for F such that $p(F) \neq 0$

Independent events As before E and F are called independent if $p(E|F) = p(E)$, Equivalently, E and F are independent iff $p(E \cap F) = p(E)p(F)$.

Disjoin event if E_1 and E_2 are disjoint event then:

$$p(E_1 \cup E_2) = p(E_1) + p(E_2)$$

Law of total probability For any $F \subseteq \Omega$ and its complement F^c ,

$$p(E) = p(E|F)p(F) + p(E|F^c)p(F^c)$$

which sounds very intuitive because by definition F and F^c are disjoint.

Generally

Theoreme 1 If Ω is the union of disjoint event F_1, F_2, \dots, F_n then:

$$p(E) = p(E|F_1)p(F_1) + p(E|F_2)p(F_2) + \dots + p(E|F_n)p(F_n)$$

Proof

We prove the law of total probability for $\Omega = F \cup F^c$ (the general case follows straightforwardly)

$$\begin{aligned} p(E) &= p(\underbrace{(E \cap F) \cup (E \cap F^c)}_{\text{union of disjoint sets}}) \\ &= p(E \cap F) + p(E \cap F^c) \\ &= \frac{p(E \cap F)}{p(F)}p(F) + \frac{p(E \cap F^c)}{p(F^c)}p(F^c) \\ &= p(E|F)p(F) + p(E|F^c)p(F^c) \end{aligned}$$

Bays' Rule**Theoreme 2**

$$p(F|E) = \frac{p(E|F)p(F)}{p(E)}$$

Proof

We use the definition of conditional probability to write $p(E \cap F)$ two ways and solve for $p(F|E)$:

$$p(F|E)p(E) = p(E \cap F) = p(E|F)p(F)$$

2.4 Random variable**Random variable**

Definition 5 A Random variable is a function X such as $X : \Omega \rightarrow \mathbb{R}$

Probability distribution

p_x , $p_x(X = x)$ or $p_x(x)$ is the probability that $X = x$, i.e, the probability of the event

$$E = \{\omega \in \Omega : X(\omega) = x\}$$

Hence,

$$p_x(x) = \sum_{w \in E} p(\omega)$$

Example

You rolle a dice.
if the outcome is 6, you receive 10CHF. Otherwise, you pay 1 CHF.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$\text{For each } \omega, p(\omega) = \frac{1}{6}$$

Then define:

$$X(\omega) = \begin{cases} 10, & \omega = 6 \\ -1, & \omega \in \{1, 2, 3, 4, 5\} \end{cases}$$

Hence, we have

$$p_x(X) = \begin{cases} \frac{1}{6}, & x = 10 \\ \frac{5}{6}, & x = -1 \end{cases}$$

2.4.1 Two random variables

Two random variables

Definition 6 Let $X : \Omega \rightarrow \mathbb{R}$ and $Y : \Omega \rightarrow \mathbb{R}$ be two random variables. The probability of the event $E_{x,y} = \{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}$ is:

$$p_{x,y}(x, y) = \sum_{\omega \in E_{x,y}} p(\omega)$$

- p_x is called **marginal distribution** (of $p_{x,y}(x, y)$ with respect to x)
- p_y can be computed similarly

2.5 Expected Value

Expected value

Definition 7 The expected value $\mathbb{E}[X]$ of a random variable $X : \Omega \rightarrow \mathbb{R}$ is :

$$\begin{aligned} \mathbb{E}[X] &= \sum_{\omega} X(\omega)p(\omega) \\ &= \sum_x xp_x(x) \end{aligned}$$

linearity

Expectation is a linear operation in the following sense:

Let X_1, X_2, \dots, X_n be random variables and $\alpha_1, \alpha_2, \dots, \alpha_n$ be scalars. Then:

$$\mathbb{E}\left[\sum_{i=1}^n X_i \alpha_i\right] = \sum_{i=1}^n \alpha_i \mathbb{E}[X_i]$$

Random variable and independency

Two random variables X and Y are independent if and only if, for all realizations x and y :

$$p(\{X = x\} \cap \{Y = y\}) = p(\{X = x\})p(\{Y = y\})$$

Or, more concisely, iff

$$p_{x,y}(x, y) = p_x(x)p_y(y)$$

Generalization

Theorem 3 Given n random variables, X_1, \dots, X_n are independent if and only if:

$$p_{x_1, \dots, x_n}(x_1, \dots, x_n) = \prod_{i=1}^n p_{x_i}(x_i)$$

Condition probability

The conditional distribution of Y given X is the function:

$$p_{x,y}(x|y) = \frac{p_{x,y}(x, y)}{p_x(x)}$$

Independent random variables

The following statements are equivalent to the statement that X and Y are two independent random variables:

- $p_{x,y} = p_x p_y$
- $p_{y|x}(y|x) = p_y(y)$
- $p_{y|x}(y|x) = p_y(y)$ is not a function of x
- $p_{x|y}(x|y) = p_x(x)$

- $p_{x|y}(x|y)$ is not a function of y

Summary 1 • *Random Variable*

- *Probability distribution*
 - *Joint distribution of multiple variables*
 - *Marginal distribution*
 - *Conditional distribution*
- *Independence*

February 19, 2025 — **Lecture 2 : Source and entropy**

Expected value and operation The addition works well with Expectation such that

$$\mathbb{E}[X + Y] = \mathbb{E}[x] + \mathbb{E}[Y]$$

However, the product doesn't work well,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

if and only if X and Y are independent random variables.

2.6 Entropy

Introduction We communicate by revealing the value of sequence of variables that we call **(Symbols), Information**

In modern language, Hartley was saying that the value of a symbol provides information if and only if the symbol is a **random variable**.

How much information is carried by a symbol such as S ?

- Suppose that $S \in \mathcal{A}$ is a symbol that can take $|\mathcal{A}|$ possible values
- The amount of information conveyed by n such symbol should be n times the informations conveyed
- there are $|\mathcal{A}|^n$ possible values for n symbols
- This suggests that $\log |\mathcal{A}|^n = n \log |\mathcal{A}|$ is the appropriate measure for information

However, this approach doesn't work:

<i>Example</i>	Imagine having a town where there are 360 days and 5 rainy days, this leads to have only 2 possibilities, $ \mathcal{A} = 2$ which make the quantity of information $\log_2 2 = 1$ bits. Which intuitively sounds kind of false, the forecast doesn't give us that information knowing that it is sunny $\frac{360}{365}$ % of the times, it is kind of expected.
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An article in 1948 from Shannon fixes the problem by defining **Entropy** the **uncertainty** or **entropy** $H(S)$ associated to a discrete random variable S :

Definition

Definition 8

$$H_b(S) = - \sum_{S \in \text{supp}(p_s)} p_s(s) \log_b p_s(s)$$

Where $\text{supp}(s) = \{s : p_s(s) > 0\}$.

Few comments

$$H_b(S) = - \sum_{S \in \text{supp}(p_s)} p_s(s) \log_b p_s(s)$$

- The condition $S \in \text{supp}(p_s)$ is needed because $\log_b p_s(s)$ is not define when $p_s(s) = 0$ this convention allows us to use the notation :

$$H_b(S) = - \sum_{s \in \mathcal{A}} p_s(\log_b p_s(s))$$

- The choice of b determines the unit, $b = 2$ is the **bit**

We also can see this as an "average" of $-\log_b p_s(S)$ which is:

$$H(S) = \mathbb{E}[-\log_b p_s(S)]$$

Example

A sequence of 4 decimal digits, s_1, s_2, s_3, s_4 representing the number to open Anne's lock can be seen as the output of a source S_1, S_2, S_3, S_4 with $S_i = \{0, \dots, 9\}$.

If Anne picks all digits at random and independently, the all outcomes are equally likely:

$$p_{S_1, S_2, S_3, S_4}(S_1, S_2, S_3, S_4) = \frac{1}{10^4}$$

If we search the entropy of this we get:

$$H_2(S) = \log_2 |\mathcal{A}| = \log_2 10^4 \approx 13.3 \text{ bits}$$

2.6.1 Information-Theory Inequality**Lemma (IT-Inequality)**

lemme 1 For a positive real number r ,

$$\log_b r \leq (r - 1) \log_b(e)$$

with equality if and only if $r = 1$

This proof juste using the deriative

**Entropy
Bounds****Theoreme 4** *The entropy of a discrete random variable $S \in \mathcal{A}$ satisfies:*

$$0 \leq H_b(S) \leq \log_b |\mathcal{A}|$$

With equality on the left if and only if $p_s(S) = 1$ and on the right if and only if $p_s(S) = \frac{1}{|\mathcal{A}|}$ for all s .

2.6.2 Random variables and Entropy **n random vari-
able**

the formula for entropy can be expanded to any number of random variables. If X and Y are two discrete random variables, with (joint) probability distribution $p_{x,y}$ then:

$$H(X, Y) = - \sum_{(x,y) \in X \times Y} p_{x,y}(x, y) \log p_{x,y}(x, y)$$

**1.4 of text-
books****Theoreme 5** *Let S_1, \dots, S_n be discrete random variables. Then*

$$H(S_1, S_2, \dots, S_n) \leq H(S_1) + H(S_2) + \dots + H(S_n)$$

With equality if and only if S_1, \dots, S_n are independent.

 February 25, 2025 — **Lecture 2 : suite**
**Ex hat party
1950**

- n men, all have the same hat
- they throw hats in a corner
- leaving, they randomly take a hat

Solution

$$\text{Let } R_i = \begin{cases} 1, & \text{if person } i \text{ leaves with their own hat} \\ 0, & \text{otherwise} \end{cases}$$

Entropy

$$H_2(S) = \sum_i p(s) \log \frac{1}{2p(s)} \quad (2.1)$$

$$= \frac{1}{8} \log_2 \frac{8}{2} + \frac{1}{8} \log_2 8 \quad (2.2)$$

$$\approx \frac{1}{8} + \frac{1}{8} \cdot 3 \quad (2.3)$$

*personal re-
mark*

We can see it as an average of "surprise".
Where the average is the randomness. (≈ 0.55)

2.6.3 Entropy bounds

Bound

$$0 \leq H_b(S) \leq \log_b \mathcal{A}$$

2.7 Source Coding Purpose

Source coding is often seen as a way to compress the source.

More generally, the foal of source coding is to efficiently describe how much information there is to a *file*

2.7.1 Setup

Setup

The **encoder** is specified by: :

- the input alphabet \mathcal{A} (the same as the source alphabet)
- the output alphabet \mathcal{D} (typically $\mathcal{D} = \{0, 1\}$);
- the codebook \mathcal{C} Which consists of finite sequences over \mathcal{D} ;
- By the one to one encoding map $\Gamma : \mathcal{A}^k \rightarrow \mathcal{C}$ where k is a positive integer.

For now, $k = 1$.

Example

For each code, the encoding map Γ is specified in the following table: A mettre une image.

<i>Example</i>	Code <i>C</i> or <i>B</i> are uniquely decodable : (A mettre une image 106)
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Prefix Free codes

Definition 10 *If no codeword is a prefix of another codeword, the code is said to be prefix free.*

<i>Example</i>	The codeword 01 is a prefix of 011 .
----------------	--

- A prefix free code is always uniquely decodable
- A uniquely decodable code is **not necessarily** prefix free

<i>A prefix code</i>	A prefix free code is also called instantaneous code :
----------------------	--

- Think of phone numbers
- Think about streaming: instantaneous codes minimize the decoding delay (for given codeword length)

Code for one random variable

We start by considering codes that encode **one single random variable** $S \in \mathcal{A}$.

To encode a sequence S_1, S_2, \dots of random variables, we encode one random variable at a time.

Complete tree of a code

Slide 113 screen.

Binary tree

- There is a root (the beginning)
- A vertex (another node)

- A **leaf** is the last vertex
 - Which is like a (arbre généalogique)
- Ternary Tree** The same as a binary tree but with three children.
- With/Without prefix** slide 115.
- Decoding tree** • Obtained from the complete tree by keeping only branches that form a codeword
- Useful to visualize the decoding process
- Slide 116

2.7.2 Codeword length

- The codeword length is defined the obvious way:
- Example: ct \mathcal{A}

Γ_B	
codeword lengths	
a	0
b	1
c	2
d	3
1110	
4 height	

- We would like the average codeword length to be as small as possible.

2.7.3 Kraft McMillan

Part 1. Necessary condition for the code to be uniquely decodable

Theoreme 6 *If a D -ary code is uniquely decodable then its codeword length i_1, \dots, i_M satisfy*

$$D^{-l_1} + \dots + D^{-l_M} \leq 1$$

Kraft's inequality

Example For code O we have :

$$2^{-2} + 2^{-2} + 2^{-2} + 2^{-2} = 1$$

Recall Kraft McMillan

Theoreme 7

Example A For code A we have $2^{-1} + 2^{-2} + 2^{-2} + 2^{-2} = 1.25 > 1$.
 KRaft-McMillan's inequality is not fulfilled.
 There exists no uniquely decodable code with those codeword lengths.

Proof of K-MM Part I We prove a slightly weaker result, namely that the codeword lengths of prefix free codes satisfy K-MM inequality.

Let $L = \max_i l_i$ be the complete tree's depth.

- There are D^L terminal leaves
- There are D^{L-l_i}
- No two codewords share a terminal leaf (The code is prefix free)
- Hence $D^{L-l_1} + D^{L-l_2} + \dots + D^{L-l_M} \leq D^L$

After dividing both sides by D^L we obtain Kraft's inequality:

$$D^{-l_1} + D^{-l_2} + \dots + D^{-l_M} \leq 1$$

Exercise What is the **converse** of Kraft McMillan part 1?
 The **Converse** of Kraft McMillan part 1 is not true (Consider e.g. two codewords: 01 and 0101)
 However, the following statement is almost as good :

Theoreme 8 *If the positive integer I_1, \dots, I_M satisfy Kraft's inequality for some positive integer D , then there exists a D -ary **prefix free code** (hence uniquely decodable) that has codewords*

This says that if the inequality is true, then we **can** find D such that there exists a binary prefix which makes it decodable **and** prefix free!

2.7.4 Important Consequence of Kraft McMillan

Part I

Theoreme 9 *If a **D -ary code is uniquely decodable**, then its codeword length I_1, \dots, I_M satisfy Kraft's inequality :*

$$D^{-l_1} + \dots + D^{-l_M} \leq 1$$

Part II

Theoreme 10 *If the positive integer l_1, \dots, l_M satisfy Kraft's inequality for some positive integer D , then there exists a D -ary **prefix free code** that has those codeword lengths.*

The Kraft McMillan theorem implies that any uniquely decodable code can be substituted by a prefix free code of the same codeword lengths.

Prefix free codes

Our focus will be on prefix free codes. Reasons:

- No loss of optimality: codewords can be as short as for any uniquely decodable code;
- a prefix free codeword is recognized as soon as its last digit is seen:

- important, e.g. a phone number;
- advantageous to limit the decoding delay in, say streaming

**Average Code-
word length**

- The typical use of a code is to encode a sequence of random variables
-

Example

$$\mathcal{A} = \{a, b, c, d\} \quad D = 2$$

Blackboard with table *cct* $s \in \mathcal{A}$

$$\begin{array}{r} \Gamma(s) \\ l(s) \\ p(s) \\ \hline \end{array}$$

$$\begin{array}{r} a \\ 0 \\ 1 \\ 0.05 \\ \hline b \\ 10 \\ 2 \\ 0.05 \\ c \\ 110 \\ 3 \\ 0.1 \\ d \\ 1111 \\ 4 \\ 0.8 \end{array}$$

$$\mathcal{E}[\text{length}] = 0.05 + 1 + 0.05 \cdot 2$$

Definition 11 Let $l(\Gamma(s))$ be the length of the codeword associated to $s \in \mathcal{A}$. The average codeword length is:

$$L(S, R) = \sum_i p_s(s) l(\Gamma(s))$$

Units

The unit of $L(S, \Gamma)$ are **code symbols**

When $D = 2$, the unit of $L(S, \Gamma)$ are bits.

**Average code-
word length:
Lower Bound**

Theoreme 11 Let $\Gamma : \mathcal{A} \rightarrow \mathcal{C}$ be the encoding map of a D -ary

Proof

We want to prove that:

$$\begin{aligned}
H(s) - \sum_s p(s)l(s) &= - \sum_s p(s) \log p(s) - \sum_s p(s)l(s) \\
&= - \sum_s p(s) \log p(s) - \sum_s p(s) \log 2^{l(s)} \\
&= - \sum_s p(s) \log(p(s) \cdot 2^{l(s)}) \leq \dots
\end{aligned}$$

Therefore:

$$\begin{aligned}
&= \sum_s p(s) \log\left(\frac{1}{p(s)} 2^{-l(s)}\right) \\
&\leq \sum_s p(s) \left(\frac{1}{p(s)} 2^{-l(s)} - 1\right) \cdot C \\
&= \left(\sum_s 2^{-l(s)} - \sum_s p(s)\right) \cdot C \\
&\leq 0
\end{aligned}$$

We know that the left side is less or equal to 1 because of the Kraft Inequality, therefore it is bounded.

 February 26, 2025 — **Lecture 4 : Continue**
Key observation

The right hand side of:

$$L(S, \Gamma) = \sum_{s \in \mathcal{A}} p(s)l(\Gamma(s))$$

$$H_D(S) = \sum_{s \in \mathcal{A}} p(s) \log_D \frac{1}{p_S(s)}$$

are identical if $l(\Gamma(s))$

- Unfortunately $l(\Gamma(s)) = \log_D \frac{1}{p_S(s)}$ is often not possible (not an integer)
- How about choosing

Theoreme 12 • For every random variable $S \in \mathcal{A}$

Theorem

Theoreme 13 The average codeword length of a D -ary Shannon-Fano code for the random variable S fulfils:

$$H_D(S) \leq L(S, \Gamma_{SF}) < H_D(S) + 1$$

Proof

it suffices to prove the upper bound (we have already proved the lower bound)

First suppose that we could use $l_i = -\log p_i$. The average

length would be:

$$L(S, \Gamma) = \sum_i p_i l_i = \sum_i p_i (-\log_D p_i) = H_D(S)$$

Instead we use $l_i = \lceil -\log p_i \rceil < -\log p_i + 1$

March 4, 2025 — Lecture 5 : Conditional Entropy

Key Idea

Pack multiple symbols into "supersymbols"

- $(S_1, S_2, S_3, \dots, S_n)$
- Now, apply our Main result to such supersymbols

Theoreme 14 *The average codeword-length of a uniquely decodable code Γ for S must satisfy:*

$$H_D(S_1, S_2, \dots, S_n) \leq L((S_1, S_2, \dots, S_n), \Gamma)$$

And there exists a uniquely decodable code Γ_{SF} satisfying:

$$L((S_1, S_2, \dots, S_n), \Gamma_{SF}) < H_D(S_1, S_2, \dots, S_n) + 1$$

Our Next Nugget

Understand

Example

Audio recording:

- We can easily anticipate the next image in a video, there

KEY(simple) Independent

Definition 12 *The source models a sequence S_1, S_2, \dots, S_n of n coin flips*

So $S_i \in \mathcal{A} = \{H, T\}$ where H stands for heat, T for tails.

$p_{S_i}(H) = p_{S_i}(T) = \frac{1}{2}$ for all $(s_1, S_2, \dots, S_n) \in \mathcal{A}^n$

Not independent

Definition 13 *The source models a sequence S_1, S_2, \dots, S_n of weather conditions.*

So $S_i \in \mathcal{A} = \{S, R\}$, where S stands for sunny and R for rainy

The weather on the first day is uniformly distributed in \mathcal{A} .

For all other days, with probability $q = \frac{6}{7}$ the weather is as for the day before

Conditional Probability

Recall how to determine the conditional probability:

$$p_{X|Y}(x | y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

It gives the probability of the event $X = x$, given that the event $Y = y$ has occurred.

it is defined for all y for which $p_Y(y) > 0$

Remark

There is good slide with good schema in slide 176-179

**Conditional
Expectation of
 X given $Y = y$**

$$p_{X|Y}(\cdot | y)$$

is the probability distribution of the alphabet of X , juste like $p_X(\cdot)$

Definition 14 *The conditional expectation of X given $Y = y$ is defined as:*

$$\mathcal{E}[X | Y = y] = \sum_{x \in \mathcal{X}}$$

**Conditional
Entropy of X
given $Y = y$**

$p_{X|Y}(\cdot | y)$ is a probability distribution on the alphabet of X , juste like $p_X(\cdot)$
Every probability distribution has an entropy associated to it:

- $p_X(\cdot) \rightarrow H(X)$
- $p_{X|Y}(\cdot | y) \rightarrow H(X | Y = y)$

Definition 15 *The conditional entropy of X given $Y = y$ is defined as:*

$$H_D(X | Y = y) = - \sum_{x \in \mathcal{X}} p_{X|Y}(\cdot | y) \log_D p_{X|Y}(\cdot | y)$$

| *Example* A faire

**Entropy
Bounds**

Theoreme 15 *The conditional entropy of a discrete random variable $X \in \mathcal{X}$ conditioned on $Y = y$ satisfies:*

$$0 \leq H_D(X | Y = y) \leq \log_D |\mathcal{X}|$$

With equality on the left iff $p_{X|Y}(x, y) = 1$ for some x , and with equality on the right iff $p_{X|Y}(x | y) = \frac{1}{|\mathcal{X}|}$

Example

The proff is identical to our proof of the basic entropy bounds

Question?

Do we also have the following entropy bound:

$$H_D(X | Y = y) \stackrel{???}{\leq} H_D(X)?$$

Answer: no.

| *Example*

(Or "counterexample" if better), Juste for ease of calculation, let us set $\delta = 0$ (but this is not necessary for the example to work). Then, we have:

$$H_D(X | Y = 0) = h_D(\varepsilon) \text{ and } H_D(X | Y = 1) = 0$$

where $h_d(\cdot)$ is the binary entropy function (with $\log_D(\cdot)$). But we have:

$$H_D(X) = h_D\left(\frac{1 - \varepsilon}{2}\right)$$

Conditional entropy can either go up or down (if we give the answer the entropy is 0)

Conditional Entropy of X given Y

The most useful and impactful definition is the *average* conditional entropy of X given $Y = y$, averaged over all values of y under the marginal distribution $p_Y(y)$. Formally, we thus define:

Definition 16 *The conditional entropy X given Y is defined as:*

$$H_D(X | Y) = \sum_{y \in \mathcal{Y}} p_Y(y) \left(- \sum_{x \in \mathcal{X}} p_{X|Y}(x | y) \log_D p_{X|Y}(x | y) \right)$$

Example

For the Bit flipper channel, we have;

$$H_D(X | Y) = p(Y = 0)H_D(X | Y = 0) + p(Y = 1)H_D(X | Y = 1)$$

We search now:

$$H(X | Y) = p(Y \text{ is Head})H(XY \text{ is head}) + p(Y \text{ is Tail})H(X | Y \text{ is tail}) = \frac{1}{2}.$$

Conditional Entropy of X given Y

Theoreme 16 *The conditional entropy of discrete random variable $X \in \mathcal{X}$ conditioned on Y satisfies:*

$$0 \leq H_D(X | Y) \leq \log_D |\mathcal{X}|$$

With equality on the left iff for every y there exists an x such that $p_{X|Y}(x | y) = 1$ and with equality on the right iff $p_{X|Y}(x | y) = \frac{1}{|\mathcal{X}|}$ for all x and all y .

This follows directly from our bounds on $H_D(X | Y = y)$

Having $p_{X|Y}$

We know that $p(X | Y) = \frac{1}{|\mathcal{X}|}$ for all y .

$$\begin{aligned} p(x) &= \sum_{y \in \mathcal{Y}} p(y)p(x | y) \\ &= \sum_y p(y) \frac{1}{|\mathcal{X}|} \\ &= \frac{1}{|\mathcal{X}|} \cdot \sum_y \overbrace{p(y)}^{=1} \end{aligned}$$

Conditioning Reduces Entropy

The following bound is important and impactful (and also intuitively pleasing!)

Theoreme 17 For any two discrete random variables X and Y ,

$$H_D(X | Y) \leq H_D(X)$$

with equality iff X and Y are independent random variables

In words, **On average**, the uncertainty about X can only become smaller if we know Y .

As we have seen, this is not true point-wise: We may have $H_D(X | Y = y) > H_D(X)$ for some values of y .
It works only on average.

Proof

$$\begin{aligned} H(X | Y) - H(X) &= \\ &= \sum_y p(y) \left(- \sum_x p(x | y) \log p(x | y) \right) + \sum_x p(x) \log p(x) \\ &= \sum_{x,y} p(y) p(x | y) \log \frac{1}{p(x | y)} + \sum_{x,y} p(y | x) p(x) \log p(x) \\ &= \sum_{x,y} p(x, y) \log \frac{p(x)}{p(x | y)} \\ &\leq \sum_{x,y} p(x, y) \left(\frac{p(x)}{p(x | y)} - 1 \right) \cdot \log e \\ &= \sum_{x,y} (p(x)p(y) - p(x, y)) \log(e) \\ &= \left(\left(\sum_y p(x)p(y) \right) - \left(\sum_x p(x)p(y) \right) \right) \end{aligned}$$

Conditional Entropy of $f(x)$ Let X be an arbitrary random variable. Let $f(x)$ be a (deterministic) function of x .

$$H(f(x) | X) = 0$$

Proof

To find this conditional entropy:

Let $Y = f(x)$

$$p(y | y) = \begin{cases} 1, & y = f(x) \\ 0, & y \neq f(x) \end{cases}$$

the probability that y is $f(x)$ is only true if $f(x) = y$.
This implies that the entropy is equal to 0:

$$H(y | x) = 0$$

**Conditioning
reduced En-
tropy**

A generalization of the previous bound is also interest to us:

Theoreme 18 For any three discrete random variables X, Y and Z ,

$$H_D(X | Y, Z) \leq H_D(X | Z)$$

With equality iff X and Y are conditionally independent random variables given Z (that is, if and only if $p(x, y | z) = p(x | z)p(y | z)$ for all x, y, z ,

You can see it as make the Z fall which makes it $p(x, y) = p(x)p(y)$

Proof

It is only mathematics:

$$\begin{aligned} H_D(X | Y, Z) - H_D(X | Z) &= \mathbb{E} \left[\log_D \frac{1}{p_{X|Y,Z}(X | Y, Z)} \right] + \mathbb{E} [\log_D p_{X|Z}(X | Z)] \\ &= \mathbb{E} \left[\log_D \frac{p_{X|Z}(X | Z)}{p_{X|Y}(X | Y, Z)} \right] \\ &= \mathbb{E} \left[\log_D \frac{p_{X|Z}(X | Z) p_{Y|Z}(Y | Z) p_Z(Z)}{niquesamere} \right] \end{aligned}$$

March 4, 2025 — **Lecture 6 : Conditional Entropy review**

**Main defini-
tions**

We have here two mains definitions:

The entropy for for a "case" of a random variable:

$$H(X | Y = y) = - \sum_x p(X | y) \log p(X | y)$$

And, the conditional entropy on a random variable:

$$\begin{aligned} H(X | Y) &= \sum_y p(y) H(X | Y = y) \\ &= - \sum_y \sum_x p(x, y) \log p(x | y) \end{aligned}$$

The main thing to understand here is that $H(X | Y)$ is the *Generalization* of the first definition. It is all the possible values of Y together. This is why we sum up all possible value of y . The second way to write $H(X | Y)$ is like taking all the possible pairs together and calculating the entropy of each pairs.

Main Result: The main result behind this is:

$$\begin{aligned} 0 &\leq H(X | Y = y) \leq \log |\mathcal{X}| \\ 0 &\leq H(X | Y) \leq \log |\mathcal{X}| \end{aligned}$$

And the inequality:

$$H(X | Y) \leq H(X)$$

Conditional entropy of $f(x)$ Let X be an arbitrary random variable.
Let $f(x)$ be a (deterministic) function of x :

$$H(f(x) | x) = 0$$

For example:

$$\begin{aligned} X &\in \{0, 1, 2, 3\} \\ f(x) &= X \mod 2 \end{aligned}$$

Which is:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

Then,

$$P(f(x) | X) = \begin{cases} 0, & \text{if } x = 0, 2 \\ 1, & \text{if } x = 1, 3 \end{cases}$$

If we now compute the entropy for $X = 0$ and $X = 1$ etc..., we get:

$$\begin{aligned} H(f(x) | X = 0) &= 0 \\ H(f(x) | X = 1) &= 0 \\ &\vdots \end{aligned}$$

Lisa rolls two dice

Lisa rolls two dice and annoucnes the sum L written as a two digit number. The alphabet of $L = L_1L_2 = \{02, 03, 04, 05, 06, 07, 08, 09, 10, 11, 12\}$ Where the alphabet of $L_1 = \{0, 1\}$ and the alphabet of $L_2 = \{0, 1, 2, \dots, 9\}$. We are looking for the probability that $L_2 = 2$ knowing that $L_1 = 1$:

$$p_{L_2|L_1}(2 | 1)$$

What we are doing here is the joint distribution:

$$p_{L_2|L_1}(2 | 1) = \frac{P_{L_1, L_2}(1, 2)}{P_{L_1}(1)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

After running over all possible values for (i, j) , we obtain:

$L_2 = j \mid L_1 = i$	0	1	$p_{L_2}(j)$
0	0	$\frac{3}{36}$	$\frac{3}{36}$
1	0	$\frac{2}{36}$	$\frac{2}{36}$
2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{2}{36}$
\vdots	\vdots	\vdots	\vdots
$p_{L_i}(i)$	$\frac{5}{6}$	$\frac{1}{6}$	height

now you can do the same with conditional probability and then after computing all those value:

$$H(L_2 | L_1) = \frac{5}{6} \cdot 2.857 + \frac{1}{6} \cdot 1.459 = 2.624 \text{ bits}$$

Now we can observe that:

$$2.624 = H(L_2 | L_1) \leq H(L_2) = 3.22$$

Which says that on average, knowing something takes out some randomness

The chain rule for entropy

Recall that the joint entropy of two random variables X, Y is completely naturally defined as:

$$H_D(X, Y) = - \sum_x \sum_y p_{X,Y}(x, y) \log_D p_{X,Y}(x, y)$$

Or as seen earlier:

$$H_D(X, Y) = H_D(X) + H_D(Y)$$

Using the fact the $p_{X,Y}(x, y) = p_X(x)p_{Y|X}(y | x)$, we can write this as:

$$\begin{aligned} H_D(X, Y) &= - \sum_x p_X(x) \left(\sum_y p_{Y|X}(y | x) \log_D (p_X(x) p_{Y|X}(y | x)) \right) \\ &= - \sum_x p_X(x) \left(\sum_y p_{Y|X}(y | x) (\log_D p_X(x) + \log_D p_{Y|X}(y | x)) \right) \\ &= - \sum_x p_X(x) \left[\left(\sum_y p_{Y|X}(y | x) \log_D p_X(x) \right) + \left(\sum_y p_{Y|X}(y | x) \log_D p_{Y|X}(y | x) \right) \right] \\ &= H(X) + H(Y | X) \end{aligned}$$

Theoreme 19

$$H(X, Y) = H(X) + H(Y | X)$$

*Professor
remark*

Firstly:

$$H(Y, X) = H(X, Y)$$

Which is either proved by:

$$\begin{aligned} H(Y, X) &= H(Y) + H(X | Y) \\ &= H(X, Y) \end{aligned}$$

The relation proved before in words it:

To find the joint entropy of two random variables, we can first calculate the entropy of one of the two, and then add to it the conditional entropy of the second, given the first.

The chain rule entropy

Theoreme 20 Let S_1, \dots, S_n be discrete random variables. Then:

$$H_D(S_1, S_2, \dots, S_n) = H_D(S_1) + H_D(S_2 | S_1) + \dots + H_D(S_n | S_1, \dots, S_{n-1})$$

The above result says that the uncertainty of a collection of random variables (in any order) is the uncertainty of the first, plus the uncertainty of the second when the first is known, plus the uncertainty of the third when the first two are known, etc. . .

Let us see how:

$$\begin{aligned} & H(S_1, S_2, \dots, S_{n-1}, S_n) \\ & \quad \underbrace{\hspace{1.5cm}}_{=Z} \\ & = H(Z) + H(S_n | Z) \\ & = H(\underbrace{S_1, S_2, \dots, S_{n-2}, S_{n-1}}_{=Z'}) + H(S_n | S_1, \dots, S_{n-1}) \\ & = H(Z') + H(S_{n-1} | Z') + H(S_n | S_1, \dots, S_{n-1}) \end{aligned}$$

Until we get $Z' \dots' = S_1$.

Example

Let X, Y, Z be discrete random variables. We have:

$$\begin{aligned} H(X, Y, Z) &= H(X) + H(Y | X) + H(Z | X, Y) \\ &= H(X) + H(Z | X) + H(Y | X, Z) \\ &= H(Y) + H(X | Y) + H(Z | X, Y) \\ &= H(Y) + H(Z | Y) + H(X | Y, Z) \\ &= H(Z) + H(X | Z) + H(Y | X, Z) \\ &= H(Z) + H(Y | Z) + H(X | Y, Z) \end{aligned}$$

Theoreme 21 Let S_1, \dots, S_n be discrete random variables. Then:

$$H(S_1, S_2, \dots, S_n) \leq H(S_1) + H(S_2) + \dots + H(S_n)$$

With equality iff, S_1, \dots, S_n are independent

Proof

$$\begin{aligned} H(S_1, S_2, S_3) &= H(S_1) + H(S_2 | S_1) + H(S_3 | S_1, S_2) \\ &\leq H(S_1) + H(S_2) + H(S_3) \end{aligned}$$

Another way around

Sometimes it is convenient to compute the conditional entropy using the chain rule for entropies. For instance:

$$H(X | Y) = H(X, Y) - H(Y)$$

It can be useful to make it easier to compute $H(X | Y)$ because on the right side, it is only marginal entropies with $p \log p$ which are "easy to compute"

corollaire 1

$$H(X, Y) \geq H(X)$$

$$H(X, Y) \geq H(Y)$$

The above inequalities follow from the chain rule for entropies and the fact that entropy (condition or not) is nonnegative.

Example

From lisa rolls two dice:

$$H(L_1, L_2) = 3.2744$$

$$H(L_1) = 0.6500$$

$$H(L_2) = 3.2188$$

We compute:

$$H(L_2 \mid L_1) = H(L_1, L_2) - H(L_1) = 3.2744 - 0.6500 = 2.6254$$

$$H(L_1 \mid L_2) = H(L_1, L_2) - H(L_2) = 3.2744 - 3.2188 = 0.056$$

And verify that indeed:

$$H(L_1 \mid L_2) \leq H(L_1) \leq H(L_1, L_2)$$

$$H(L_2 \mid L_1) \leq H(L_2) \leq H(L_1, L_2)$$

2.7.5 Random Processes**A.K.A Source models**

Definition 17 *The source models a sequence S_1, S_2, \dots, S_n of n coin flips*

So $S_i \in \mathcal{A} = \{H, T\}$, where H stands for heads, T for tails, $i = 1, 2, \dots, n$ $p_{S_i}(H) = p_{S_i}(T) = \frac{1}{2}$ for all i , and coin flips are independent.

Hence,

$$p_{S_1, S_2, \dots, S_n}(S_1, S_2, \dots, S_n) = \frac{1}{2^n}, \quad \forall (S_1, S_2, \dots, S_n) \in \mathcal{A}^n$$

Definition 18 *The source models a sequence S_1, S_2, \dots, S_n of weather conditions.*

So $S_i \in \mathcal{A} = \{S, R\}$, where S stands for sunny and R for rainy, $i = 1, 2, \dots, n$.

The weather on the first day is uniformly distributed in \mathcal{A} .

For all other days, with probability $q = \frac{6}{7}$ the weather is as for the day before

What we can see here that is the conditional probability, for example:

$$p(S_2 = \text{sun} \mid S_1 = \text{sun}) = q$$

$$p(S_2 = \text{rain} \mid S_1 = \text{sun}) = 1 - q$$

However:

$$\begin{aligned} p(S_3 = \text{sun} \mid S_1 = \text{sun}, S_2 = \text{sun}) \\ = p(S_3 = \text{sun} \mid S_2 = \text{sun}) = q \end{aligned}$$

More generally:

$$P(S_n \mid S_1, S_2, \dots, S_{n-1}) = p(S_n \mid S_{n-1})$$

March 11, 2025 — **Lecture 7 : Entropy and algorithm**

Experience little play

- Think of something
- Ask yes or no question
- Find the answer

the game was called twenty questions in old U.S tv. We want to use entropy to understand this game.

Last Week

$$H_D(X) = H_D(P) - - \sum_x p(x) \log_D p(x)$$

We also saw those two bounds:

$$0 \leq H_D(X) \leq \log_D |\mathcal{A}|$$

Information is always about option, more options you have, more information (the first way to introduce "entropy")

We also saw:

$$\begin{aligned} H(X \mid Y = y) &= - \sum_x p(x \mid y) \log_D p(x \mid y) \\ H(X \mid Y = y) &= - \sum_y \dots \end{aligned}$$

And we also saw that on average:

$$\begin{aligned} H(X \mid Y) &\leq H(X) \\ H(X \mid Y, Z) &\leq H(X \mid Y) \leq H(X) \end{aligned}$$

We also saw the chain rule:

$$\begin{aligned} H(S_1, S_2, S_3, S_4) \\ = H(S_2, S_4, S_1, S_3) \end{aligned}$$

The order in entropy doesn't matter,

$$= H(S_1) + H(S_2 \mid S_1) + H(S_3 \mid S_1, S_2) + H(S_4 \mid S_1, S_2, S_3)$$

An interesting way to use this, is if we combine the inequalities and the chain rule. The equality on the right side is true if and only if X and Y are independent. therefore:

$$H(S_1, S_2, S_3, S_4) = H(S_1) + H(S_2) + H(S_3) + H(S_4)$$

this equality is true if and only if S_1, S_2, S_3, S_4 are independent.

The 20 question problem

Let X be a random variable. What is the minimum number of "yes/no" question needed to identify X ?, which question should be asked.

Solution

Let us consider a binary code Γ for $X \in \mathcal{X}$

Once Γ is fixed, we know $x \in \mathcal{X}$ if and only if we know the codeword $\Gamma(x)$. The strategy consists in asking the i th question so as to obtain the i th bit of the codeword $\Gamma(x)$.

The expected number of question $L(X, \Gamma)$, which is minimized if Γ is the encoding map of Huffman code

Example

Suppose that we know that $\mathcal{X} = \{\text{cat, dog, pony}\}$, with:

$$p(\text{cat}) = \frac{1}{2}$$

$$p(\text{dog}) = \frac{1}{4}$$

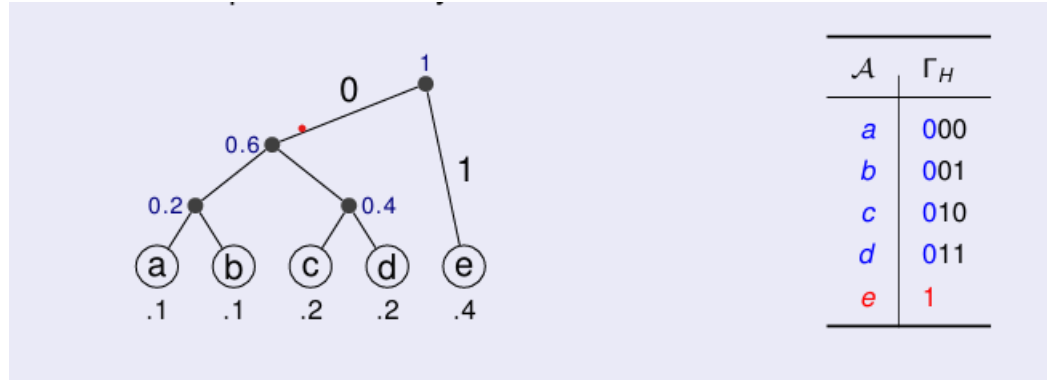
$$p(\text{pony}) = \frac{1}{4}$$

We want to make it the best way, the question we should ask is:

- is the animal a cat?

X a b c d e height

We then do a Huffman tree:



We know here that this, will be optimal

Optimality

We have seen that a prefix free code for $X \in \mathcal{X}$ leads to a querying strategy to find the realization of X .

Similarly, a deterministic querying strategy leads to a binary prefix-free code for X . Here is why:

- Before the first question we know that $x \in \mathcal{X}$
- Without loss of generality, the first question can be formulated in terms of "is $x \in \mathcal{A}$ "? for some $\mathcal{A} \subset \mathcal{X}$, (The choice of \mathcal{A} is determined from the strategy, that we fix once and for all)

- Is the answer is YES, the we know that $x \in \mathcal{A} \subset \mathcal{X}$. Otherwise $x \in \mathcal{A}^c \subset \mathcal{X}$. Either way we have reduced the size of the set that contains x .
- We continue asking similar questions until the value of x is fully determined, the we stop.

Here, the sequence of Yes or no answers is a binary codeword associated to x . The code obtained when we consider all possible values of x is a binary prefix-free code. Since the tree is prefix free, its averag codeword-length cannot be smaller than that of a Huffman code.

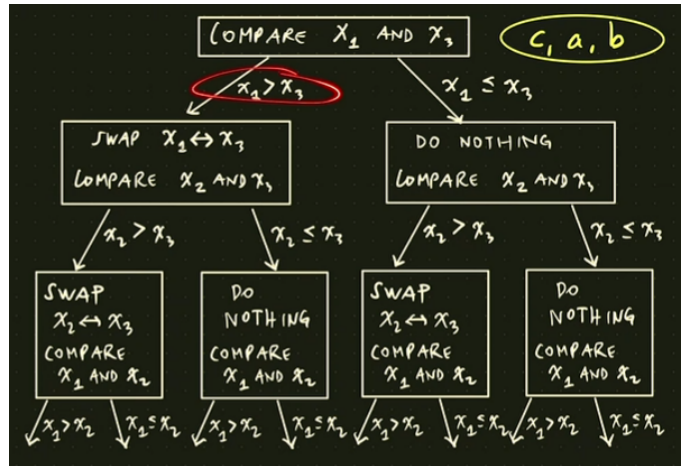
Given an **unsorted** List with n elements.

For example $l = [c, a, b]$ with $n = 3$

Repeat:

1. Select two position $1 \leq i \leq j \leq n$
2. Compare and swap:
 - If $x_i > x_j$
 - Then swap elements $x_i \iff x_j$
 - Else do nothing

One way to understands how it works:



The first observation:

The sequence of pairwise comparisons must identify the exact order of the unsorted list.

The second observation:

The sequence of pairwise comparisons in a uniquely decodable (actually, prefix-free) binary code for x .

There fore, we must have:

$$\mathbb{E}[\text{number of comparisons}] \geq H_2(X)$$

However what is the X ? We see it as a random variable because we don't really know what the unsorted list is.

For example $n = 3$ we have $\mathcal{X} = \{abc, acb, bac, bca, cab, cba\}$ where \mathcal{X} is the set of all permutations.

However what is $p(x)$? Here, we want to talk about our algorithm working

**Sorting via
pairwise com-
parisons**

for all $p(x)$.

$$\begin{aligned} E &\geq \max_{p(x)} H_2(X) = \log_2 |\mathcal{X}| \\ &= \log_2 n! \end{aligned}$$

We already know a bounds on factorial:

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}}$$

Therefore:

$$\begin{aligned} H_2(x) &\approx \log_2 \frac{n^n}{e^{n-1}} \\ &= n \log_2 n - (n-1) \log_2 e \end{aligned}$$

Which is "dominated" by $n \log_2 n$

Billard Balls

There are 14 billards balls numbered as shown:



Among balls 1 – 13, at most one **could** be heavier/lighter than the others. What is the minimum number of weightings to simultaneously determine:

- If one ball is different
- if there is such a ball which one,
- And whether the different ball is heavier/lighter

Here we want to use entropy to solve this problem. The goal here is to associated the number of weightings to code. The goal is to see it as a tree.



The steps of picking two sets is "mandatory" we have to pick two sets in order to compare something, and in order to compare something, you have to compare something...

From this comparisons, there will be three possibilities. with three possibilites, We are specifying a Ternary code. The issue here is that we are losing information, yes we only get a binary tree however we wouldn't be able to have the same amount of information as with a ternary tree.

What we are saying here is, with any strategy to solve this problem **can** be written in this way. Hence we can read this tree as a ternary code.

<i>But a code for What?</i>	What are we finding with this code? A code for X : <ul style="list-style-type: none"> • $X = 0$: all balls are equals • $X = +1$: ball 1 is heavier • \vdots
--	---

- $X = +13$ ball 13 is heavier
- $X = -1$ ball 1 is lighter
- \vdots
- $X = -13$ ball 13 is lighter

Then we know that $|\mathcal{X}| = 27$. This is one way to answer those questions.

1. If $X = 0$ or not (then there is or not a different ball)
2. Then $|X|$ gives us the information
3. the sign of X if the ball is heavier or lighter

Observation The number of weighings is equal to the length of the ternary codeword

Then:

Theoreme 22

$$\mathbb{E}[\text{number of weighings}] \geq H_3(X)$$

It has to be three by the way the problem is stated. The code is ternary **Therefore** the base for the entropy is 3.

Moreover, our strategy must work **irrespective** of the probability distribution of X .

We can also see:

Theoreme 23

$$\mathbb{E}[\text{number of weighings}] \geq \max_{p(x)}(H_3(X))$$

Where in our example gives us:

$$\log_3 27 = 3$$

It doesn't need to be an integer it is only the professor that chooses on purpose to make it clean

*But does
there indeed
exist such a
code*

FACT:

Entropy does **not** guarantee the existence of such a strategy

Entropy serves as a lower bound and **not** the best way to do it.

But can what if?

Let us suppose it exists! Then entropy tells us a few basic facts.

if 3 weighings S_1, S_2, S_3 uniquely specify X , Then we **must have**:

$$H_3(X) = H_3(S_1, S_2, S_3)$$

Fact 1

Proof

$$\begin{aligned}
 H(X, S_1, S_2, S_3) &= H(X) + \overbrace{H(S_1, S_2, S_3 | X)}^{=0} \\
 &= H(S_1, S_2, S_3) + \overbrace{H(X | S_1, S_2, S_3)}^{=0}
 \end{aligned}$$

It is true because if we know S_1, S_2, S_3 then we know all X then the entropy of 0.

For $H(X | S_1, S_2, S_3)$, because S_1, S_2, S_3 uniquely specify X then knowing them implies that this entropy is 0.

Fact 2

If 3 weighings S_1, S_2, S_3 uniquely specify X , then we must have:

- S_1, S_2, S_3 uniformly distributed
- S_1, S_2, S_3 independent

Proof

$$H_3(S_1, S_2, S_3) = 3$$

This is a *must*.

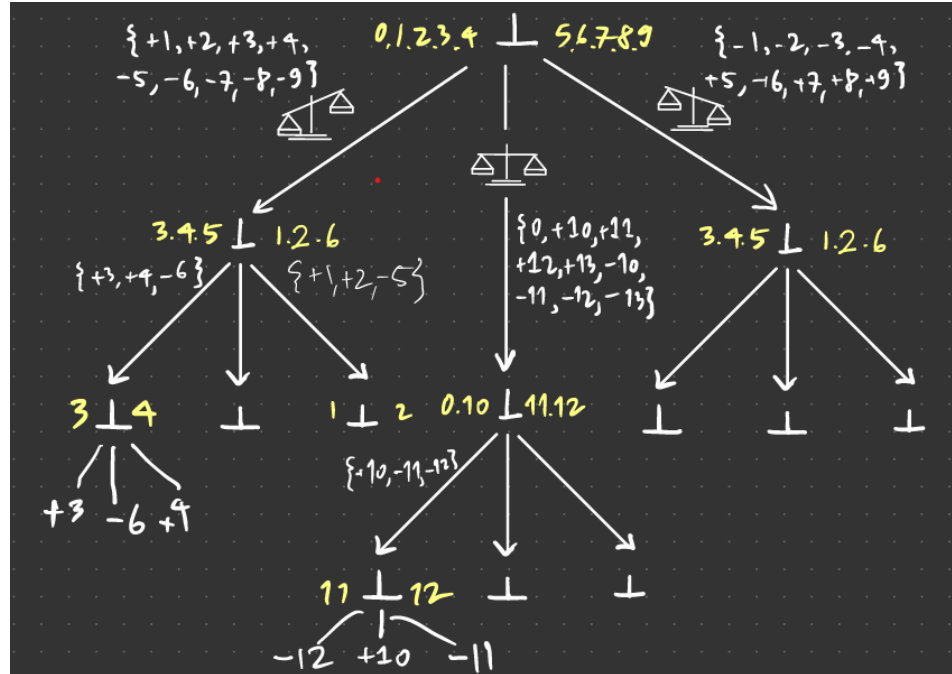
But also:

$$\begin{aligned}
 H_3(S_1) + H(S_2 | S_1) + H(S_3 | S_1, S_2) &\leq H_3(S_1) + H(S_2) + H(S_3) \\
 &\leq \log_3 3 + \log_3 3 + \log_3 3
 \end{aligned}$$

Where it is an equality if and only if the distribution is uniform and independent.

Example

Let's see how to actually find a way to ask those question:



March 12, 2025 — **Lecture 8 : Prediction, learning, and Cross-Entropy-Loss**

Billard Balls Can we use the 20 questions approach to solve the 14 bullars riddle?

Answer No, because the kind of questions that we can "ask", when wa are weighing, is quite limited.
For instance, the first question cannot be "is 1 or 2 heavy?".

Strategies But is there a strategy that requires only 3 weighings?
From source compression, we can establish the following facts?

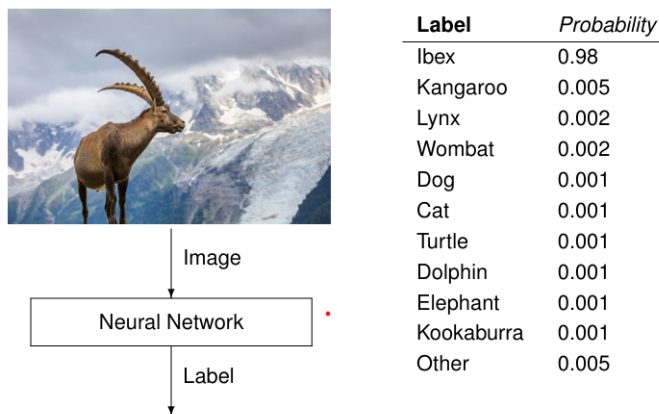
- For each weighings, the three outcomes must be equally likely
- The weighings must be independent of each other

It is because we carefully selected the numbers (alphabet size of 27; each weighing has 3 possible outcomes) that there is a strategy that exactly matches the entropy lower bound 3 weighings. If you change the numbers, it will not generally be true that there is a strategy that *exactly* matches the lower bound.

2.7.6 Prediction, Learning and cross-Entropy Loss

The goal here is to change the way to use entropy, entropy has always be seen as something that *means* something, a lower bound, a quantity of information. Here we will use it to do calculation juste like a *tool*.

Example



There weren't probability at this time in the slide so imagine without it

The question we want to ask is, "*Is our neural network performing well*"

- Given an image \mathcal{X}
- Our machine (Neural network)
- Outputs $Q(x)$
- The label: $\text{Label}(x)$

Zero-one loss

$$L\{Q(x) \neq \text{Label}(x)\} \\ = \begin{cases} 1 & \text{if } Q(x) \neq \text{Label}(x) \\ 0 & \text{if } Q(x) = \text{Label}(x) \end{cases}$$

Given a lot of image, we want to have a **Classification error**:

$$\frac{\sum_{\mathcal{X}} L\{Q(x) \neq \text{Label}(x)\}}{\text{number of images}}$$

Is the function of mis-labeled images.

Pros and
Cons

Pros

- Very intuitive
- Interpretable

Cons

- Not differentiable

With probability

Our neural network produces:

$$Q(\text{label} \mid \text{image})$$

The true label distribution is:

$$P_{true}(\text{label} \mid \text{image}) = \begin{cases} 1, & \text{correct label} \\ 0, & \text{wrong label} \end{cases}$$

(We are assuming for simplicity that for each image, there is a single correct label).

- Ideally, we would like:

$$Q(\text{label} \mid \text{image}) = P_{true}(\text{label} \mid \text{image}) \quad \forall \text{pairs}$$

However this is only a dream

- Instead, people like to consider **cross entropy loss**
- that is, we wish ou $Q(\text{label}|\text{image})$ to **minimize**

$$L(P_{true}(\text{label} \mid \text{image}), Q(\text{label} \mid \text{image})) \\ = - \sum_{\text{label}} P_{true}(\text{label} \mid \text{image}) \log_D Q(\text{label} \mid \text{image})$$

- Given training data (image, label), for $i = 1, 2, \dots, n$ we select $Q(\text{label} \mid \text{image})$ to minimize the cross entropy loss.

**Cross entropy
loss**

$$L(P, Q) = - \sum_y P(y) \log_D Q(y)$$

Where

- P is the true distribution
- Q is our approximation (via neural network)

Why is it popular?

- Good properties for training with "gradient descent" in certain standard architectures.
- Theoretical properties.

A (very) simple neural network Takes a screen of the blackboard

- it transform the image into a vector
- Then takes is through the weighs w_i all the way to d
- the we take it through the soft max which is two functions:

$$Q(o | x) = \frac{e^{z_0}}{e^{z_0} + e^{z_1}}$$

$$Q(1 | x) = \frac{e^{z_1}}{e^{z_0} + e^{z_1}}$$

The goal is given a lot of training data, we want to select the w_0, b_0, w_1, b_1 such at to minimize the total cross entropy loss.

For a single image \mathcal{X}

because why is juste binary we use:

Total Loss

$$L_{total}(w_o, b_o, w_1, b_1) = - \sum_{i=1}^k \log \frac{e^{x_i w_0 + b_0}}{e^{x_i w_0 + b_0} + e^{x_i w_1 + b_1}} - \sum_{i=k+1}^n \log \frac{e^{w_1 k_i + b_1}}{e^{w_0 x_i + b_0} + e^{w_1 x_i + b_1}}$$

Cross entropy loss

Cross entropy loss:

$$L(P, Q) = - \sum_y P(y) \log_D Q(y)$$

Theoreme 24 For a fixed probability distribution P , the minimum:

$$\min_Q L(P, Q)$$

Is attained if and only if we selected $Q^* = P$ in this case,

$$L(P, Q^*) = L(P, P) = H(P)$$

Where $H(P)$ is the entropy of the probability distribution P

Proof

The proof, which will be done in class, uses once again the "IT inequality".

The theorem is saying this:

$$H(P) \leq L(P, Q)$$

With equality in one case which is $P = Q$.

$$\begin{aligned} H(P) - L(P, Q) &\leq 0 \\ - \sum_y P(y) \log P(y) + \sum_y P(y) \log Q(y) &\leq 0 \\ = \sum_y P(y) \log \frac{Q(y)}{P(y)} &\leq \sum_y P(y) \left[\frac{Q(y)}{P(y)} - 1 \right] \log(e) \\ &= \sum_y (Q(y) - P(y)) \log(e) \\ &= 0 \end{aligned}$$

Note

We don't see it in AICC II but let's introduce the notion:
KL-Divergence (aka KL distance):

$$D_{kl}(p \parallel k) = \sum_y p(y) \log \frac{P(y)}{Q(y)}$$

- Fact 1:
 $D_{kl}(P \parallel Q) \geq 0$ with equality iff $P = Q$ (this is just the proof seen earlier)

2.8 Summary of chapter 1

Entropy

$$H_D(X) = - \sum_x p(x) \log_D p(x)$$

For $D = 2$, we simply write $H(X)$ and we all the units bits.

Entropy has many useful properties, including:

- $0 \leq H_D(X) \leq \log_D |\mathcal{X}|$
- $H_D(X | Y) \leq H_D(X)$ with equality if and only if X and Y are independent
- $H_D(X, Y) = H_D(X) + H_D(Y | X)$
- Every uniquely decodable binary code must use at least $H(X)$ bits per symbol on average
- There exists a binary code that uses between $H(X)$ and $H(X) + 1$ bits per symbol on average
- Hence, for a source string of length n :
 - Every uniquely decodable binary code must use at least $H(S_1, S_2,$

Data Compression

Models

Coin Flip The coin flip is not convertible, With a file of result, there is no way to compress the file

Sunny Rainy Here, the entropy, is not 1 then we are able to compress the file here.
 This is the first view of mark of model.
 Given S_1, S_2, S_3, \dots , Are S_1, S_3 independent?

$$\begin{aligned} p(S_1, S_3) &= \sum_{S_2} p(S_1, S_2, S_3) \\ &= \sum_{S_2} p(S_1)p(S_2 | S_1)p(S_3 | S_2) \end{aligned}$$

Entropy and algorithm

We explored examples where entropy can give a lower bound on algorithmic performance.

Cross-Entropy Loss

- Example: in search-type problems, give a lower bound on the minimum number of necessary queries.
- Machine (e.g., Neural Network) outputs a distribution $Q(y)$ over all possible labels
- Cross entropy loss: Select $Q(y)$ to minimize:

$$L(P, Q) = - \sum_y P(y) \log_D Q(y)$$

Chapter 3

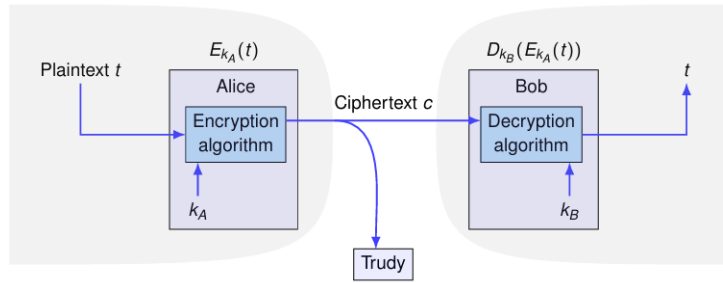
Cryptography

3.1 One-Time pad, Perfect Secrecy, Public-Key

Why cryptography - cryptography gives us the tools to:

- authenticate the sender and the receiver
- verify the integrity of the message
- keep the message confidential

Basic setup for confidentiality



Here Alice want to sent the plaintext t to Bob:

- She encrypts t using her key k_A . The result is the ciphertext $c = E_{k_A}(t)$
- She sends c to Bob over a public channel
- Bob decrypts c using his key k_B . The result is $D_{k_B}(E_{k_A}(t))t$
- For Trudy, it is nearly impossible to recover t from c without knowing k_B

Basic Terminology

- plaintext, ciphertext (also called cryptogram), key, encrypter, decrypter
- cryptography: the art of composing cryptograms
- cryptanalysis the art of breaking cryptograms
- a cryptanalyst has broken the system when he can quickly determine the plaintext from the cryptogram, no matter what key is used
- attacker: same as cryptanalyst

Ancient cryptography

Caesar's cipher

Suppose that we are using the English alphabet augmented by a few special characters, "space", "comma", and "period". An alphabet of 29 characters, represented by the integers $0, 1, \dots, 28$

- The key k is an integer between 0 and 28, known to Alice and Bob and to nobody else.
- The encryption algorithm substitutes the i -th letter of the alphabet with the $(i + k)$ -th letter ($\text{mod } 29$)
- The decryption algorithm substitutes the j -th letter with the $(j - k)$ -th $\text{mod } 29$

Therefore, here the secrecy of the message rely only on the secrecy of the algorithm.

Various attacks possible

We distinguish between the following attacks:

- **ciphertext-only**: one or more cryptograms available to the cryptanalyst know to have been encrypted with the same key
- **known plaintext**: the cryptanalyst has one or more plaintext and the resulting cryptograms, know to have been encrypted with the same key
- **chosen plaintext** for any plaintext that he requires, the cryptanalyst can obtain the cryptogram under the same key

Ideally, a cryptographic system should be secure against a chosen plaintext attack, At the very least, it should be secure against a ciphertext-only attack.

However with a computer, the key can easily be found using the letter-frequency attack. The question now is how to make the letter frequency attack unfruitful?

Example (Vigenère's cipher with an n -length key)

- Chosen plaintext attack: encode the same letter until you have the n -length key
- **known plaintext attack** compare input/output until you have the n -length key
- **ciphertext-only attack**
 - brute force approach: try all 29^n keys is you know n
 - * for $n = 21$ the number of key is 5.13^{30}
 - * for $n = 100$, the number of keys is 1.73^{146}
 - If you know n , you can partition input/output into n parts, each of which is a Caesar cipher with its own key
 - Letter frequency approach: effective if the plaintext-length to key-length ration is sufficiently large

The one-time pad

Preliminary assumptions

- The plaintext t , the key k and the cryptogram c are n -length binary sequences over the alphabet $\mathcal{A} = \{0, 1\}$
- The key k , is produced by selecting each bit independently and with uniform distribution
- Alice and Bob use a private channel to exchange the key ahead of time

Encryption

$$c = t \oplus k$$

Decryption

$$c \oplus k = (t \oplus k) \oplus k = t \oplus (k \oplus k) = t$$

Perfect secrecy

Definition 19 A cryptosystem has *perfect secrecy* if the plaintext T and the cryptogram C are statistically independent

Perfect secrecy is the ultimate kind of security against a ciphertext-only attack: The attacker cannot do better than guessing the plaintext T

Perfect secrecy of the one time pad

- The n -length key k is selected at random (uniform distribution over $\{0, 1\}^n$)
- The key k and the message t are selected independently
- The ciphertext is $c = t \oplus k$

$$p_{C|T}(c | t) = p_{K|T}(c \ominus t | t) = p_K(c \ominus t) = \frac{1}{2^n}$$

Hence C and T are independent: knowledge of C is useless in guessing T

Weakness of the one time pad

Example

A cryptanalyst that has the plaintext t and the corresponding cryptogram c immediately gets the key:

$$k = c \ominus t$$

Hence the pad (the key) should be used only once

Pros and

Cons of one time pad

Pros

- Very simple algorithm
- as secure as it gets against a ciphertext-only attack and key used one
- of instructional value to prove that the perfect secrecy is possible

Cons

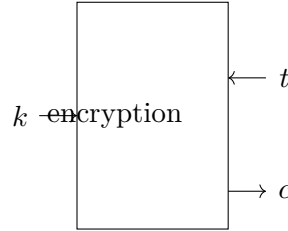
- The key is as long as the plaintext (this is fundamental, see later)

- The key needs to be exchanged ahead of time over a private channel
- a ciphertext-only attack can break the system if the key is used twice (see homework)
- a known plaintext attack reveals the key

The "one-time pad" has been used extensively in diplomatic and espionage circles

Perfect secrecy requires high entropy keys

The following theorem makes no assumption on the encryption algorithm:



Theoreme 25 *Perfect secrecy implies:*

$$H(T) \leq H(K)$$

Proof

Perfect secrecy $H(T) = H(T | C)$ and decodability ($H(T | K, C) = 0$) imply:

$$\begin{aligned} H(T) &= H(T | C) \\ &\leq H(T, K | C) \\ &= H(K | C) + H(T | K, C) \\ &= H(K | C) \\ &\leq H(K) \end{aligned}$$

Given T, K, C we search for the entropy:

$$\begin{aligned} H(t, k, c) &= H(c) + \overbrace{H(t | c)}^{=H(t)} + H(k | t, c) \\ &= H(c) + H(k | c) + \underbrace{H(t | k, c)}_{=0} \\ &\implies H(t) + H(k, | t, c) \leq h(k | c) \end{aligned}$$

This implies:

$$H(t) + \underbrace{H(k | t, c)}_{\geq 0} \leq H(k)$$

Because if we take out the "Knowing c " we cannot have some bigger. Therefore:

$$H(t) \leq H(k)$$

Entropy plays a key role also in cryptography

 March 19, 2025 — **Lecture 10 : Encryption?****Exercise**

Determine the minimum average length of the binary key for a cryptosystem that has the following characteristics

- the message is an uncompressible binary string of length n
- the system achieves perfect secrecy

Solution

- $H(T)$ must be essentially n bits (otherwise further compression is possible)
- Perfect secrecy requires $H(T) \leq H(K)$
- hence $H(K)$ is at least n
- The average blocklength of the binary key is at least n bits

**Symetric-
Key crypto
Systems: key-
distribution
problem**

A symmetric-key cryptosystem is one for which both ends use the same key ($k_A = k_B = k$). All example considered so fare rely on a symmetric key

There exists fast (ans secure) symmetric key cryptosystems, but:

- Anybody that has the key can encrypt and/or decrypt
- The key cannot be sent over an insecure channel
- In an n user network, each user needs $n-1$ keys to communicate privately with every other user. Key distribution is a problem as it hat to be done over a secure channel. And keys have to be changed frequently
- We have a real problem (the first 6min. and 20 secs of:

https://www.youtube.com/watch?v=YEBfamv-_do

However, is there a way to distribute keys over a pubic channel?

Solution

In 1976, Diffie and Hellman came up with a solution.

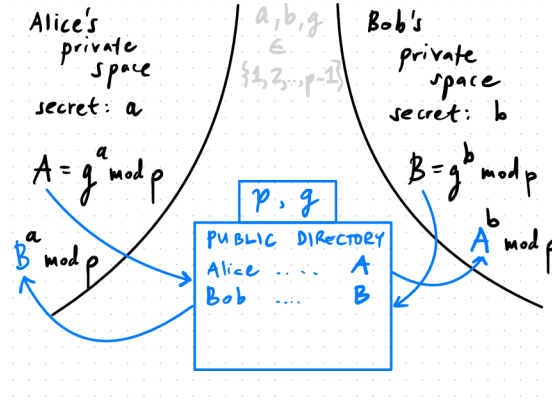
Example

You pick a number $p = 7$ which is a prime $\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6\}$. We then take $g = 3$

i	$g^i \bmod 7$
0	
1	3
2	2
3	6
4	4
5	5
6	1

The gives us a permutation. We have to be careful with choosing the g because for example if we take $g = 2$, and take g^1 and g^4 the result is 2 which leads that we don't have a permutations

How does it is seen:



We have a public directory like a phone book where everyone has access. Alice pick a random a . Then she takes a public $A = g^a \mod p$ which we will be written in the public directory. Then Bob do the same thing.

When Alice and Bob want to have an interaction. Alice will look for Bob in the phonebook, take the B in her private space, take the result of $B^a \mod p$ (she is the only one to know a). Bob do the same thing: $A^b \mod p$

At this point

- Alice has $B^a \mod p$

$$B^a = (g^b \mod p)^a \mod p$$

- Bob has $A^b \mod p$

$$A^b = (g^a \mod p)^b \mod p$$

Fact

Theoreme 26 For all $x, y, m \in \mathbb{Z}$:

$$[(x \mod m) \cdot (y \mod m)] \mod m = xy \mod m$$

We then get $B^a = g^{ab} \mod p$ and $A^b = g^{ab} \mod p$ We juste need to find the inverse of $g^i \mod p$ Which is a discrete logarithm problem. It is very hard to find the discrete logarithm. The gives the secrecy of this encryption. The secrecy is only here because it is hard to compute this inverse

Secrecy

But can't anybody generate this key?

We all know g and we know that $A = g^a \mod p$ we juste has to find

Eve wants to listen

Assuming that the cryptosystem used by Bob and Alise is secure, the best option for Eve is to find the key k .

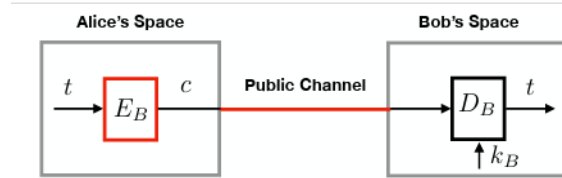
She know p, g, A and B .

In generale, there seems to be no better way than finding the number a for which $g^a = A$, and the comput $k = B^a$

This is a problem. Let us check out some number. Suppose

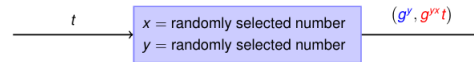
	<p>p is a 2048 bit number. (it must be prime, but let us neglect this and assume $p = 2^{2048}$ How long does it take to compute:</p> $2 \log_2 p = 4096$ <p>Multiplications to performs $a \rightarrow g^a$ (called discrete exponentiation). With a computer that performs 10^{10} multiplications per seconds, the exponentiation is done seamlessly. It takes roughly:</p> $\exp\left(\left(\frac{64}{9}\right)^{\frac{1}{3}} (\ln p)^{\frac{1}{3}} \ln \ln p)^{\frac{2}{3}}\right) \approx 10^{35}$ <p>Multiplication to perform $g^a \rightarrow a$ (called discrete logarithm to the base g. With the same computer, it takes about 10^{25} seconds, which is about $7 \cdot 10^7$ times the ages of the earth.</p>
	<p><i>Conclusion</i> Diffie and Hellman's public key distribution scheme is clever, efficient, and it seems to be secure.</p>
Paradigm shift	<p>The issue with the security is the computation. For example in maybe a couple of years the quantum computer will be able to compute this discrete algorithm very fast. The DG system would instantly become insecure. To the contrast, perfect secrecy offers provable security even when the enemy has infinite time and computing power. Most cryptographic systems rely on computational security</p>
One way function	<p>Discrete exponentiation is an example of a one way function: a function for which a fast algorithm exists one way but not in the other way. The case for the discrete logarithm is that there is a way back but it is very slow. However, the best case would be that there is not way back.</p> <p><i>Example</i> IF a computer were to save user's names and passwords, a sstem manager would have access to both. This is not the case if the operating system stores, along the name a one-way function f of our password.(The password itself is never stored)</p>
Trapdoor One-way function	<p>A tropdoor one way function is a one-way function with an extra feature called the trapdoor information: with this information, the hard-to-carry out inverse computation becomes easy. Diffie and Hellmann realized that with such a tool the key distribution problem would disappear</p>
Asymetric cryptography	<p>Suppose that Allice wants to send private information to Bob Bob has a trapdoor one-way function, implemented by an algorithm E_B that he publishes in a open directory He is the only one who has the trapdoor information k_B. Hence he has the algorithm D_B that implements the inverse function. Alice and Bob no longer need a shared key:</p>

ElGamal's trapdoor one way function



Setup

- Fix a large prime number p . Hereafter all the numbers are in $\{0, 1, \dots, p-1\}$ and arithmetic is modulo p (more on it later).
- Pick a generator g
- Pick randomly selected numbers x and y . Unlike p and g , x and y are kept secret.
- Here is a trapdoor one-way function, with **trapdoor information** x .



Given the trapdoor information x , we can invert the function as follows: Compute the inverse of $g^{(y)x} = g^{xy}$, multiply the result with $g^{xy}t$. The result is t .

More concretely

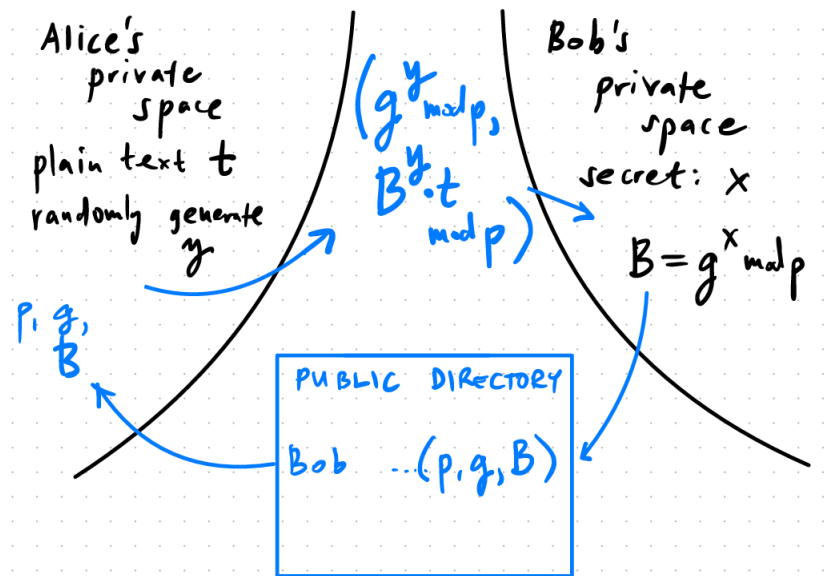
Alice has:

- t = plaintext, t
- y = random number, y

Bob:

- x = random number x .

The scheme looks like this:



Bob sends g^x to Alice, then Alice sends the cryptogram $(g^y, g^{xy}t)$ to Bob

Note: x and y are transaction specific

March 25, 2025 — Lecture 11 : Number Theory

3.2 Number theory

Introduction

In the digital world, the information is represented by the elements of a finite set, and we should be able to do math with them. Which means that the finite set should be a finite field. Our bigger goal of the next few lectures is to develop the tools to understand when and how we can turn a finite set into a finite field.

Operation with integers

Within \mathbb{Z} (the set of integers) we can

- add, subtract, multiply
- but not divide $\frac{7}{2}$ is not an integer
- What comes closest to the (regular) division is the euclidean division

Euclidean division

The division algorithm

Given integers a (the dividend) and m the divisor:

$$a = mq + r, \quad 0 \leq r < |m|$$

The computation of q and r as above is called euclidean division

<i>euclidean</i>	In C/C++/Java/ Python we use the operator % to compute
<i>division in</i>	r as follows:
<i>mainstream</i>	if a and m are both positive, then $r = a \% m$
<i>programming</i>	If one or the other or both are negative, different languages
<i>languages</i>	behave differently, but the general rule is:
	<ul style="list-style-type: none"> • if $a \% m$ is nonnegative, then $r = a \% m$ • if $a \% m$ is negative, then $r = a \% m + m$

Congruence is an equivalence relation

A binary relation \sim on a set is an **equivalence relation** if and only if the following three axioms are satisfied:

- $a \sim a$ (reflexivity)
- if $a \sim b$ then $b \sim a$ (symmetry)
- if $a \sim b$ and $b \sim c$ then $a \sim c$ (transitivity)

Substitute $a \sim a$ with $a \equiv a \pmod{m}$ etc..., to see that congruence is an equivalence relation

One of the consequences is that we can form equivalence classes and we can work with one representative of each class (this will become useful later.)

Modulo

Theoreme 27 *If:*

$$a \equiv a' \pmod{m}$$

$$b \equiv b' \pmod{m}$$

then:

$$a + b \equiv a' + b' \pmod{m}$$

$$ab \equiv a'b' \pmod{m}$$

$$a^n \equiv (a')^n \pmod{m}$$

In particular, if $a' = (a \pmod{m})$ and $b' = (b \pmod{m})$, then we obtain the following facts (useful in calculation):

- $(a + b) \equiv ((a \pmod{m}) + (b \pmod{m})) \pmod{m}$
- Hence:

$$a + b \pmod{m} = ((a \pmod{m}) + (b \pmod{m})) \pmod{m}$$

- $ab \equiv ((a \pmod{m})(b \pmod{m})) \pmod{m}$
-

$$ab \pmod{m} = ((a \pmod{m})(b \pmod{m})) \pmod{m}$$

- $a^n \equiv (a \pmod{m})^n \pmod{m}$

$$a^n \pmod{m} = (a \pmod{m})^n \pmod{m}$$

	<i>Example</i>	is $9^{1000} + 9^{10^6}$ divisible by 5? We compute first:
		$9 \equiv -1 \pmod{5}$
		Which gives us:
		$9^{1000} + 9^{10^6} \equiv (-1)^{1000} + (-1)^{10^6} \equiv 1 + 1 \equiv 2 \pmod{5}$
		Hence, $9^{1000} + 9^{10^6}$ is not divisible by 5

**Check digits
mod 97**

Write down an integer in decimal notation *e.g.*:

021 235 1234

Compute its remainder after division by 97:

$$021\ 235\ 1234 \pmod{97} = 95$$

Appends the remainder to the number, as check digit:

021 235 1234**95**

A common mistake consists in transposing two digits:

021 253 1234 95

The check digits are no longer consistent:

$$021\ 253\ 1234 \pmod{97} = 63$$

**Procedure
mod 97 – 10**

It's a variant of the previous one:
Append 00

April 1, 2025 — **Lecture 13 : Multiplicative Inverse**

**Why modular
arithmetic**

Modular arithmetic is the foundation of number theory, therefore we need number theory for cryptography and for channel coding.

**Introduction to
 $\mathbb{Z}/m\mathbb{Z}$**

Instead of considering integers and congruences (\pmod{m}) and write equation like:

$$a + b \equiv c \pmod{m}$$

We would like to make our life easier like this:

$$a + b = c$$

This can be done, if we give a new meaning to a , b and c . namely we make them the congruence classe $[a]_m$, $[b]_m$ $[c]_m$ and $[c]_m$.

Definition 20 Let $m > 1$ be an integer, called the modulus.
The set of all integers congruent to $a \pmod{m}$ is called the congruence class of a modulo m .
it is denoted by $[a]_m$.

It is the same foundation as the one of finite field (corps fini) in linear algebra. (With prof. Sherer).

Other definition

Definition 21 The set of all congruence classes modulo m is denoted by $\mathbb{Z}/m\mathbb{Z}$ (which is read $\mathbb{Z} \pmod{m}$).

| *Note* Some authors use the notation \mathbb{Z}_m

Example

Some more example to see how this works:
if we have the class $[a]_m$ and:

$$a = mq + r, \text{ with } 0 \leq r \leq m - 1$$

Then we have that:

$$[a]_m = [r]_m$$

If we take for example $[-13]_9$ and $[5]_9$ we can see here that there are equal such that

$$[-13]_9 = [5]_9$$

Sum

In $\mathbb{Z}/m\mathbb{Z}$ we define the sum and the product as follows:

- $[a]_m + [b]_m = [a + b]_m$
- $[a]_m [b]_m = [ab]_m$

The result is the same regardless the choice of representative. In fact:

- if we choose $[a + km]_m$ instead of $[a]_m$
- and $[b + lm]_m$ instead of $[b]_m$
- Then we obtain $[a + km]_m + [b + lm]_m = [a + km + b + lm]_m$ which is equal to $[a + b]_m$

Properties of + in $\mathbb{Z}/m\mathbb{Z}$

The sum has the following properties:

- $[a]_m + ([b]_m + [c]_m) = ([a]_m + [b]_m) + [c]_m$
- There exists an additive identity, namely $[0]_m$:

$$[a]_m + [0]_m = [0]_m + [a]_m = [a]_m$$

- There exists an inverse with respect to addition: every $[a]_m$ has an inverse, denoted $[-a]_m$ such that:

$$[a]_m + [-a]_m = [-a]_m + [a]_m = [0]_m$$

- Commutativity

$$[a]_m + [b]_m = [b]_m + [a]_m$$

Properties of \times in $\mathbb{Z}/m\mathbb{Z}$ The multiplication has the following properties:

- associativity

$$[a]_m([b]_m[c]_m) = ([a]_m[b]_m)[c]_m$$

- multiplicative identity, namely $[1]_m$:

$$[a]_m[1]_m = [1]_m[a]_m = [a]_m$$

- commutativity

$$[a]_m[b]_m = [b]_m[a]_m$$

Mixed properties

- Distributivity:

$$[a]_m([b]_m + [c]_m) = [a]_m[b]_m + [a]_m[c]_m$$

The notation $k[a]_m$ in $\mathbb{Z}/m\mathbb{Z}$

For an arbitrary positive integer k , $k[a]_m$ is a short hand for

$$\underbrace{[a]_m + [a]_m + \cdots + [a]_m}_{k \text{ times}}$$

We can easily verify that:

$$k[a]_m = [ka]_m = [k]_m[a]_m$$

Multiplicative Inverse

Some elements of $\mathbb{Z}/m\mathbb{Z}$ have the **multiplicative inverse**. The multiplicative inverse of $[a]_m$, if it exists is an element $[b]_m$ such that:

$$[a]_m[b]_m = [1]_m$$

Theoreme 28 *The multiplicative inverse if it exists it is unique, and it is denoted by $([a]_m^{-1})$.*

Furthermore $(([a]_m)^{-1})^{-1} = [a]_m$

Proof

Suppose first that $ab = 1$ and $ac = 1$ (a has two inverse).
Then we now that $ab = ac$. If we multiply both side by b :

$$bab = bac$$

However we know that $ab = 1 = ba$:

$$b \cdot 1 = c \cdot 1$$

$$b = c$$

April 2, 2025 — **Lecture 14 : Read Slide**

Powers in $\mathbb{Z}/m\mathbb{Z}$

For an positive integer k ,

- $([a]_m)^k$ is a short hand for $\underbrace{[a]_m [a]_m \dots [a]_m}_{k \text{ times}}$
- $([a]_m)^0 = [1]_m$

Note that we do not consider negative power because it is problematic in general except -1 which is just the multiplicative inverse

Exercise

Suppose $[a]_m \in \mathbb{Z}/m\mathbb{Z}$ has a multiplicative inverse.
Does there exist k such that:

$$([a]_m)^k = [0]_m$$

We denote first $[b]_m = ([a]_m)^{-1}$. The first statement implies that:

$$([b]_m)^k \cdot ([a]_m)^k = [0]_m$$

Which is:

$$\begin{aligned} [0]_m &= ([b]_m)^{k-1} \overbrace{[b]_m [a]_m}^{=[1]_m} ([a]_m)^{k-1} \\ &= ([b]_m)^{k-1} ([a]_m)^{k-1} \\ &= \dots = [1]_m \end{aligned}$$

Which is false. Therefore, The answers is no.

Function with multiplicative inverse

Theoreme 29 In $\mathbb{Z}/m\mathbb{Z}$ the following statement are equivalent:

- $[a]_m$ has an inverse
- For all $[b]_m$, $[a]_m x = [b]_m$ has a unique solution
- There exists $a[b]_m$, such that $[a]_m x = [b]_m$ has a unique solution

Proof $1 \rightarrow 2$ We multiply both side of $[a]_m x = [b]_m$ by $[a]_m^{-1}$ then we have $x = [a]_m^{-1} [b]_m$ showing that there is a solution and the solution is unique.

Proof $2 \rightarrow 1$ For $[b]_m = [1]_m$ we obtain $[a]_m x = [1]_m$ which is a solution by assumption. The solution is the inverse of a .

Proof $2 \rightarrow 1$ True since (3) is a weaker statement than (2). (one says for all and the other says there exists)

Proof 3 \rightarrow 2

Here the claim is: $\exists b$ it has a unique solution $\implies \forall b$ there is a unique solution. The negation of this claim is:

$\exists b$ either there is no solution or multiple solution $\implies \nexists b$ there is a unique solution.

It says that if there is a b with **multiple** solution implies that there is no b with a unique solution.

Proof Let $[a]_m [x_1]_m = [b^*]_m$ and $[a]_m [x_2]_m = [b^*]_m$. Now we define:

$$[x_3]_m = [x_1]_m - [x_2]_m [a]_m [x_3]_m = [0]_m \quad (3.1)$$

Now we select **any** \bar{b}_m , we then suppose that $[a]_m[x_4]_m = [\bar{b}]_m$,
Because of the fact that we have the solution x_3 before, we
can just add them up:

$$[a]_m([x_3]_m + [x_3]_m) = [\bar{b}]_m$$

Proof 3 \implies 2 We prove this using the contrapositive i.e, we assume that there is a $[b]_m$ such that $[a]_m x = [b]_m$ has either no solution or multiple solutions, and we prove that for no $[b]_m$, $[a]_m x = [b]_m$ has a unique solution.

- So suppose that $[a]_m x = [b]_m$ has no solution or multiple solutions
- By the pigeonhole principle, the map $x \rightarrow ax$ is neither **injective** nor **surjective**.
- We can find $a[b^*]_m$ $[a]_m x = [b^*]_m$ has multiple solutions say, x_1, x_2 . We define then $x_3 = x_1 - x_2 \neq [0]_m$
- Hence, $[a]_m x_3 [a]_m x_1 - [a]_m x_2 = [b^*]_m - [b^*]_m = [0]_m$
- So the solution $[a]_m x = [0]_m$ has at least two solution x_3 and $[0]_m$
- If $[a]_m x = [b]_m$ has a solution, say x_4 , then $x_3 + x_4$ is also a solution.
- We conclude that for no $[b]_m$, $[a]_m x = [b]_m$ has a unique solution.

Example If it exists, find the solution of $[2]_7 x + [3]_7 = [1]_7$

$$\begin{aligned} [2]_7 x &= [1]_7 + (-[3]_7) \\ &= [2]_7 x = [-2]_7 \\ &= x &= ([-2]_7)^{-1} [5]_7 \\ &= [4]_7 [5]_7 \\ &= [20]_7 \\ &= [6]_7 \end{aligned}$$

$\mathbb{Z}/p\mathbb{Z}$

$\mathbb{Z}/p\mathbb{Z}$ with p prime if p is prime, all elements of $\mathbb{Z}/p\mathbb{Z}$ except $[0]_p$ have a multiplicative inverse
The proof is only taking the fact that the gcd between every number in the classe and p is 1, therefore has an inverse. (except 0 which have p).

Euclidean algorithm For all this part I think the best is the slide in the pdf Slides 2025 week7 between 100 and 135. or a youtube video.

April 8, 2025 — **Lecture 15 : Commutative Groups**

What's next After $\mathbb{Z}/m\mathbb{Z}$ we could proceed in two directions:

- | | |
|---------------|--|
| Finite groups | Focus on finite groups, which are finite sets with one operation, like $(\mathbb{Z}/m\mathbb{Z}, +)$ we do so now because we need them for cryptography. |
| Finite field | Focus on finite field, which are finite sets with two operations, like $(\mathbb{Z}/m\mathbb{Z}, +, \cdot)$ With the extra property that every non-zero |

element has a multiplicative inverse. We do so later as we need finite fields for channel coding

3.2.1 Commutative Group

Definition 22 A **commutative group** (also called Abelian group) is a set G endowed with a binary operation $*$ that combines any two elements a and b to form another element denoted $a * b$. The group operation $*$ must satisfy the following five axioms:

- (Closure) For all $a, b \in G$, $a * (b * c) = (a * b) * c$
- (Associativity) For all $a, b \in G$, $a * (b * c) = (a * b) * c$
- (Identity element): There exists an element $e \in G$ such that for all $a \in G$, $a * e = e * a = a$
- (Inverse element) For all $a \in G$ there exists a $b \in G$ such that $a * b = b * a = e$
- (Commutativity) For all $a, b \in G$, $a * b = b * a$

$\mathbb{Z}/m\mathbb{Z}^*$

To obtain a commutative group with modulo multiplication, we take only the elements of $\mathbb{Z}/m\mathbb{Z}$ that have multiplicative inverse. The resulting set is denoted $\mathbb{Z}/m\mathbb{Z}^*$. For every integer $m > 1$, $(\mathbb{Z}/m\mathbb{Z}^*, \cdot)$ is a commutative group.

Euler function

Definition 23 Euler's function $\varphi(n)$ (also called Euler's totient function) is the number of positive integers in $\{1, \dots, n\}$ that are relatively prime to n .

Observation

Here we can see two main things :

- $\varphi(m)$ is the cardinality of $\mathbb{Z}/m\mathbb{Z}^*$
- if p is prime, $\varphi(p) = p - 1$

The cartesian product of a commutative group is a commutative group

Recall of the axioms of a commutative group:

- (Closure) For all $a, b \in G$, $a * (b * c) = (a * b) * c$
- (Associativity) For all $a, b \in G$, $a * (b * c) = (a * b) * c$
- (Identity element): There exists an element $e \in G$ such that for all $a \in G$, $a * e = e * a = a$
- (Inverse element) For all $a \in G$ there exists a $b \in G$ such that $a * b = b * a = e$
- (Commutativity) For all $a, b \in G$, $a * b = b * a$

We can see the $(a_1, a_2) \in (G_1, op_1) \times (G_2, op_2)$ we see that this is a commutative group.

Isomorphism

Some sets endowed with an operation might look different, but they are actually the same once their elements are re-labeled.

Definition 24 Let $(G, *)$ and (H, \oplus) be sets, each endowed with an arbitrary binary operation.
 an **isomorphism** from $(G, *)$ to (H, \oplus) is a bijection $\psi : G \rightarrow H$ such that

$$\psi(a*) = \psi(a) \oplus \psi(b)$$

holds for all $a, b \in G$

We say that $(G, *)$ and (H, \oplus) are **isomorphic** if there exists an isomorphism between them.

Je me suis arreeter vers les slide 50

April 15, 2025 — **Lecture 17 : public key cryptography**

**Proof that m_1
 and m_2 co-
 prime $\implies \varphi$
 is bijective**

First we prove that ψ is one-to-one:

- $\iff m_1$ and m_2 divide $(k - k')$
- because m_1 and m_2 have no common facots, $m_1 m_2$ divides $(k - k')$
- Hence $[k]_{m_1 m_2}$