parse-numbers = false per-mode = symbol

# Fundamental of digital systems

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2025-02-17 — **Cours 1 : Number Systems**

# Chapitre 1

# Numbser Systems

## 1.1 Digital representations

**Introduction**

- In mathematics, a **tuple** is a finite ordered sequence of elements.
  - An **n-tuple** is a tuple of $n$ elements, where $n$ is a nonnegative integer
- In a **digital representation**, a number is represented by an **ordered n-tuple**
  - Each element of the n-tuple is called a **digit**
  - The n-tuple is called a **digit vector** (or string of digits)
  - The number of digits $n$ is called the **precision** of the representation

### 1.1.1 Representation of nonnegative integers

**Integer Digit-Vector**

- **Digit-vector (string)** representing the integer $x$ is denoted by :

$$X = (X_{n-1}, X_{n-2}, \ldots, X_1, \overbrace{X_0}^{\text{zero-origin}})$$

  We see here that it is a leftward-increasing indexing
- **Least-significant** digit (also called low order digit) : $X_0$
- **Most-significant** digit (also called high-order digit) : $X_{n-1}$

**Elements of a number System**

$$X = (X_{n-1}, X_{n-2}, \ldots, X_1, X_0)$$

- The number system to represent the integer $x$ consists of
  - the number of **digits** n
  - A set of numerical **values** for the digits
    - if a **set of values for a digit** $X_i$ is $D_i$, the cardinality of $D_i$ is $|D_i|$
  - A rule of interpretation
    - Mapping between the set of digit-vector values and the set of integers
  - **Set size**
    - The set of integers is a finite set of at most $K$ elements

$$K = \Pi_{i=0}^{n-1}|D_i|$$

**Example : Decimal number system**

$$X = (X_{n-1}, X_{n-2}, \ldots, X_1, X_0)$$

- Number of digits $n$
  - Can be any, but let us consider $n = 6(e.g., 17, 9899, 676799, \ldots)$
  - Leading zeros are irrelevant
- Digit set in decimal number system
  - $D_i = \{0, 1, 2, \ldots, 9\}$ of cardinality 10
- The correponding set size of $K$ is one million values, from 0 to $K - 1$
  - $K = \prod_{i=0}^{n-1} 10 = 10^6$

**(Non)Redundant Number systems**

- A number system is **nonredundant** if
  - ... each digit-vector represents a **different** integer
  - E.g., the decimal system is nonredundant as every number is unique
- Alternatively, a number system is **redundant** if ...
  - ... there are integers represented by **more than one** digit-vector

**Weighted (Positional number systems**

- Most frequently used number systems are **weighted systems**
- The rule of representation :

$$x = \sum_{i=0}^{n-1} X_i W_i$$

Where $W = (W_{n-1}, W_{n-2}, \ldots, W_1, W_0)$ is the **weight-vector** of size $n$

- Equivalent formulation :

$$x = X_{n-1}W_{n-1} + X_{n-2}W_{n-2} + \cdots + X_1W_1 + X_0W_0$$

**Example Decimal Number system**

- Weights are a power of 10. Example :
  - Digit Vector $X = (8, 5, 4, 6, 0, 3)$
  - Weight vector $W = (10^5, 10^4, 10^3, 10^2.10^1, 10^0)$

$x = 8 \cdot 10^5 + 5 \cdot 10^4 + 4 \cdot 10^3 + 7 \cdot 10^2 + 0 \cdot 10^1 + 3 \cdot 10^0$
$x = 854703_{10}$

- When weights are of the format
  - $W_0 = 1$ and
  - $W_i = W_{i-1}R_{i-1}, \;\; i \leq i \leq n-1$

We have a **radix number system**

**Radix number system**

**Définition 1** *Radix number systems are weighted number system in which the weight vector is related to the* ***radix vector*** $R = (R_{n-1}, R_{n-2}, \ldots, R_1, R_0)$ *as follows :*

$$W_0 = 1; \; W_i = W_{i-1}R_{i-1}, \; 1 \leq i \leq n-1$$

- Equivalent to

$$W_0 = 1;\ W_i = \prod_{j=0}^{i-1} R_j$$

- E.g., in the decimal number system $W_0 = 1; W_i = \prod_{j=0}^{i-1} 10$

*Fixed and Mixed-Radix number systems*

- In a **fixed-radix** system, all elements of the radic-vector have the same value **r (the radix)**
- The weight vector in a fixed-radix system :

$$W = (r^{n-1}, r^{n-2}, \dots, r^2, r^1, 1)$$

and the integer $x$ becomes

$$x = \sum_{i=0}^{n-1} X_i \cdot r^i$$

*Example*

- Characteristics of the decimal number system :
  — Radix $r = 10$
  — **Fixed-radix** system

**Binary/Octal/-Hexadecimal to/from Decimal** I won't really go into the details here but the main thing to know is to convert from a system to one another (with the most famous ones)

### Representation of signed Integers

**Sign-and-Magnitude (SM)**

- A signed integer $x$ is represented by a pair

$$(x_s, x_m)$$

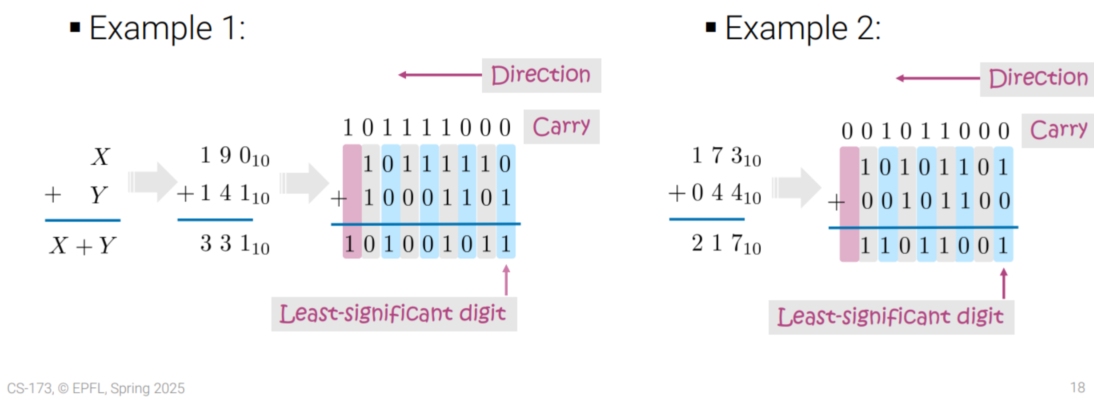
where $x_s$ is the **sign** and $x_m$ is the **magnitude** (positive integer)
- Sign (positive, negative) is represented by a binary bariables
  — $0 \implies$ positive ; $1 \implies$ negative
- Magnitude can be represented as any positive integer
  — In a conventional radix-r system, the range of n-digit magnitude is :

$$0 \leq x_m \leq r^n - 1$$

2025-02-20 — **Cours 2 : Arithmetic Operations with integers**

## 1.2 Addition of unsigned Integers

**By hand** We use here the same principle as a classical addition by hand of decimal numbers :




**How many Bits are needed** To represent the **sum of two $n$-bit unsigned numbers** we use $n+1$
For exemple the minimum space is when there are $0+0$ which leads to :

$$s_{min} = 0 + 0 = 0$$

and for the maximum :

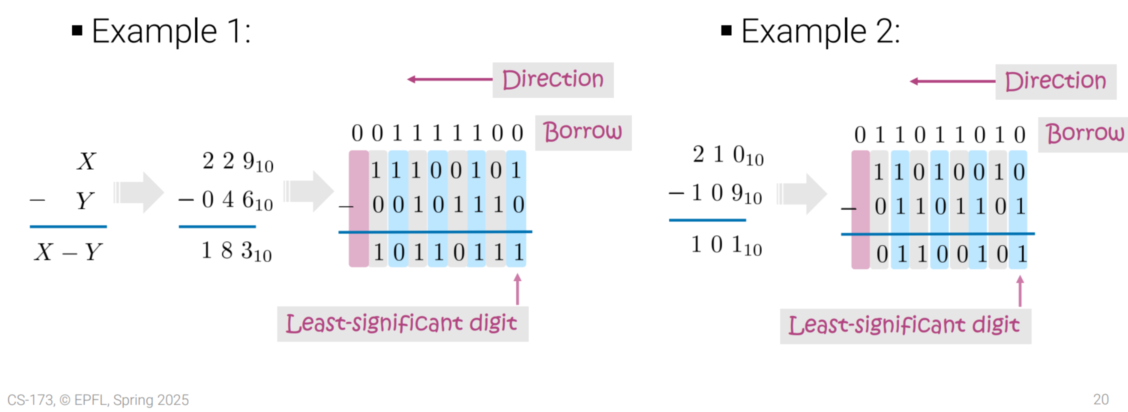
$$s_{max} = (2^n - 1) + (2^n - 1) = 2 \cdot 2^n - 2 = 2^{n+1} - 2$$

which makes it $n+1$ bits for the sum.

- But we do not always have the extra bit in hardware
- When the magnitude of the result exceeds the largest representable value, we say an **overflow** occurs and the result is incorrect.

### 1.2.1 Substraction of Unsigned Integers

- We use here the same idea as for decimal numbers :




**Negative result**

- Negative results cannot be represented using an unsigned system
- When trying to represent a value smaller than the minimu representable by the given number of bits $n$, an integer **underflow** occurs, and the result is incorrect.

### 1.2.2 Two's Complement Addition/substraction

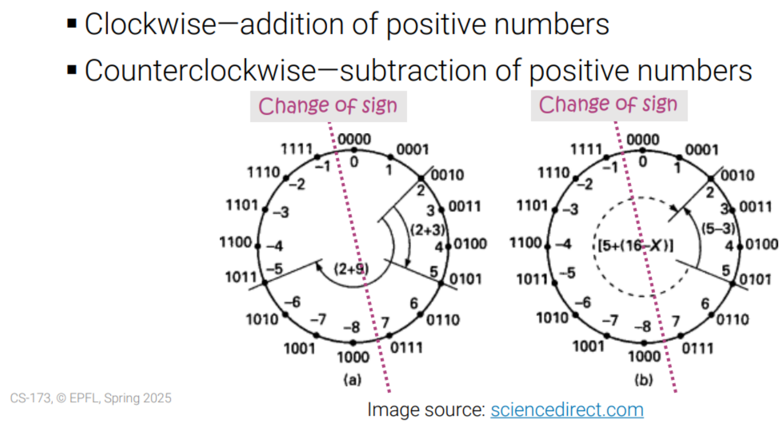
**Addition** We use here the same algorithm as for the unsigned numbers, and if the result exceeds the range, **overflow** occurs.
To refresh how signes numbers works, for example $1000_2 = -8_{10}$ which is the "*most negative*" number with 4 bits. Then we add the right side of the number as positive integers like this :

$$\underbrace{1}_{-8_{10}}\overbrace{010_2}^{2} = -6_{10}$$

If we want to sum up $-5$ and 7 for exemple :

$$1011 + 0111$$
$$\overbrace{1+1}^{1}0$$
$$\overbrace{1}^{1}\ 10$$
$$\overbrace{1}^{1}\ 010$$
$$0010 = 2_{10}$$

We use it as a clock :




**The preferred representation in digital Systems**

- If we start with the smallest (most negative) number $1000_2 = -8_{10}$ and count up all succesive numbers up to $0111_2 = 7_{10}$ can be obtained by adding 1 to the previous one :
  - The result will always be correct as long as the range is not exceeded
  - Simple operation
  - Not as simple for sign and magnitude
  - Good for hardware implementation
    - **Win-win** : the same hardware can perform the addition of unsigned numbers

**Overflow Detection rules**

- Same algorithm as for the unsigned numbers
- If the result exceeds the range, **overflow** occurs
- **Overflow detection rules**
  - If the signes of the two numbers are the same but different from the sign of the sum, the overflow occured
  - Alternative formulation : if $c_{in}$ into $c_{out}$ out of the sign position are different, the overflow occured
  - Adding two numbers of different signs never produces an overflow

### 1.2.3 Binary multiplication

**How** We use the same "*algorithm*" that the one we use by hand. For a binary representation :

$$\begin{aligned} X \cdot Y &= X \cdot \sum_{i=0}^{n-1} Y_i \cdot 2^i \\ &= \sum_{i=0}^{n-1} X \cdot Y_i \cdot 2^i \\ &= Y_{n-1} \cdot \underbrace{X \cdot 2^{n-1}}_{\text{Mulit Left-shifted by } n-1} + \cdots + Y_2 \overbrace{X \cdot 2^2}^{\text{Mult Left shifted by 2}} + Y_1 \cdot X \cdot 2^1 + Y_0 \cdot X \cdot 2^0 \end{aligned}$$

**How many bits**

**Théorème 1** *Given a $n$-bits intger and a $m$-bits intgers, there product can at most require $n+m$ bits.*

We can see the multiplication as a sequence of $m$ additions with an $n$-bit number.

**Two's Complement multiplication**

Recall of a value in two's complement (signed byte) :

$$x = -X_{n-1}2^{n-1} + \sum_{i=0}^{n-2} X_i 2^i$$

- Inspired by the previous algorithm :

$$\begin{aligned}
X \cdot Y &= X \cdot (-Y_{n-1} \cdot 2^{n-1}) + X \sum_{i=0}^{n-2} Y_i \cdot 2^i \\
&= -X \cdot Y_{n-1} \cdot 2^{n-1} + \sum_{i=0}^{n-2} X \cdot Y_i \cdot 2^i \\
&= -Y_{n-1} \cdot X \cdot 2^{n-1} + Y_{n-2} \cdot X \cdot 2^{n-2} + \cdots + Y_2 \cdot X \cdot 2^2 + Y_1 \cdot X \cdot 2^1 + Y_0 \cdot X \cdot 2^0
\end{aligned}$$

Let us not forget the sign-extend the partial result

For this only $n+m$ bits are kept ; any higher-order bits are discarded. (that the "*reason*" how $-5 \cdot -3 = 15$

**Sign-Magnitude and Two's Complement**

I juste want to underline the difference between those two representation.

*Sign Magnitude*

In the Sign magnitude representation with $n$ bits, we use the **most significant bit** (MSB) to use it as a sign :

- 0 for positive numbers
- 1 for negative numbers

The remaining bits represent the absolute magnitude of the number :
To write 5 in a 4-bits number, we use $0101_2 = +5_{10}$
To write $-5$ in a 4-bits number, we use $1101_2 = -5_{10}$
We see here that it is very intuitive and mirrors human notation with a sign.

*Two's Complement Representation*

Here, there is two point of view the one introduce in the course is to see it as a clock, in a clockwise (le sens des aiguilles d'une montre) it is positive, and unclockwise (dans le sens contraire à celui d'une montre) it is negative and begin. The negative also start at $-1$ but the bit to represent $-1$ is $1111_2$ which is just on the left.

The other way is to see it as the most significant bit (MSB) as negative, $1000_2 = -8$ and the rest of the bits being positive. To write $-5$ you have to write it as $-8 + 3 = -5$ which goes to $1011_2 = -5_{10}$

The pros for this notation is that there is only one representation for 0 where there is two for the other ($1000 = 0000 = 0$), The arithmetic operation are easier because we don't have to carry a sign everywhere and it is mor efficient in hardware implementation.

FIGURE 1.1 – Comparison table

| Feature | Sign-Magnitude | Two's complement |
|---|---|---|
| $-5_{10}$ | 1101 | 1011 |
| Zero representation | 0000(+0) and 1000(−0) | 0000 |
| Range (4-bit) | $[-7, 7]$ | $[-8, +7]$ |

2025-02-24 — **Cours 3 : Fractional (Nointeger) Number**

## 1.3 Fractional number

### 1.3.1 Fixed-Point Representation

**General Format**

**Définition 2** *Fixed-Point Numbers are :*

- *Integers*

$$I = -N, \ldots, N$$

- *Rational numbers ("binary" rationals) of the form :*

$$x = \frac{a}{2^f}$$

*where $a \in I$ and $f$ positive integer*

The fixed-point representation of a number $x$ consists of integer $x_{int}$ and fraction $x_{fr}$ components represented by $m$ and $f$ digits, respectively :

$$x = x_{int} + x_{fr}$$

**Définition 3** *Digit-vector representation :*

$$X = (X_{m-1}X_{m-2}\dots X_1X_0 \underbrace{.}_{\textit{Radix point}} X_{-1}X_{-2}\dots X_{-f})$$

- *For* ***unsigned*** *numbers :*

$$x = \sum_{i=-f}^{m-1} X_i 2^i$$

- *For* ***signed*** *number in two's complement :*

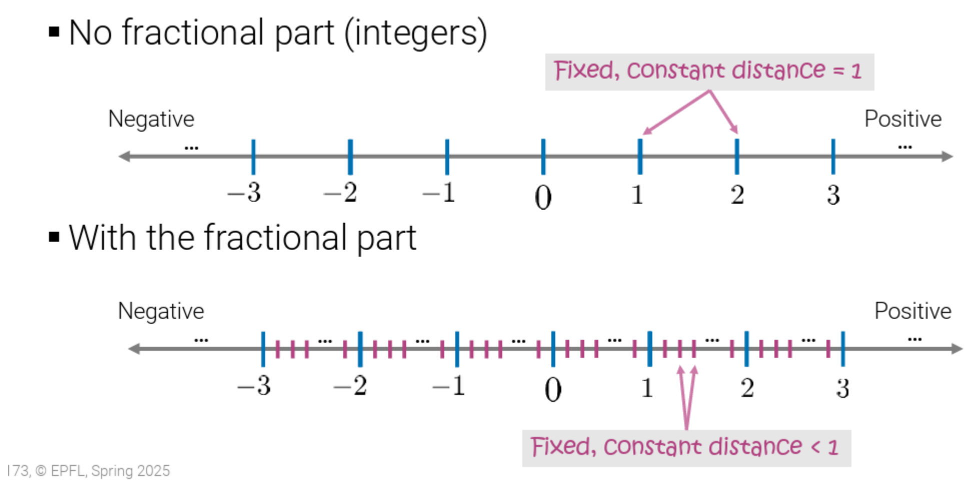
$$x = -X_{m-1}2^{m-1} + \sum_{i=-f}^{m-2} X_i 2^i$$

### 1.3.2 Radix point

**Separator between the integer and fractional parts**

$$X = (X_{m-1}X_{m-2}\dots X_1X_0 \underbrace{.}_{\text{Radix point}} X_{-1}X_{-2}\dots X_{-f})$$

- The position of the radix-point is assumed to be fixed
  — Hence the name fixed-point
- If the radix point is not shown, it is assumed to be to the right of the least significant digit (i.e, no fractional part)
  — In that case, the number is an integer
- Also known as decimal point, binary point, etc. . .




**Example**

- Decimal number system and $m = 5, f = 5$
- Example decimal digit vector :
  — $X = (10077.01690)$
  — $x = 1 \cdot 10^4 + 7 \cdot 10^1 \cdots + 0,009$

- Most negative (min) :

$$x_{min} = -99999.99999 = -99999\frac{99999}{10^5}$$

- Largest number (max, positive) :

$$x_{max} = +99999.99999 = +99999\frac{99999}{10^5}$$

**Fixed-point Representation**

- Given an unsigned fixed-point binary format $m = 3, f = 4$
  — and an example binary digit vector :

$$X = (101.0111)$$

- Q : Find the equivalent decimal number :

$$X = (101.0111); x = 2^2 + 2^0 + 2^{-2} + 2^{-3} + 2^{-4} = 5.4375$$

*Example* With sign-and-magnitude and $m = 5, f = 3$, Example of a binary digit vector :

$$X = (10101.110);$$
$$x = -(4 + 1 + 0.5 + 0.25) = -5.75$$

Therefore, The most negative number can be

$$x_{min} = 11111.111_2 = -15\frac{7}{8}$$

On the other hand, the largest number :

$$x_{max} = 01111.111 = 15\frac{7}{8}$$

*Two's complement* With two's complement the work is the same as usual (the first digit is negative) :

$$X = (1010.1101);$$
$$x = -8 + 2 + 0.5 + 0.0625 = -5.1875$$

Here the most negative number is $x_{min} = 1000.0000_2 = -8$ and the largest one is $x_{max} = 0111.1111_2 = 7\frac{15}{16}$

## 1.4 Concepts of finite precision math

**Precision**

**Définition 4** *The precision is the maximum number of non-zero bits*

For example if we have $X = (X_{m-1}X_{m-2} \dots X_1 X_1.X_{-1}X_{-2} \dots X_{-f}$ then, the precisions is the sum of $f$ and $m$ :

$$\text{Precisions}(x) = m + f$$

**Resolution**

**Définition 5** *The resolution is the smallest possible difference between two consecutive numbers*

For example if a number as $f = 5$ (5 digits for is fractional part) then we know that the smallest possible difference between the number is $\frac{1}{2^5} = \frac{1}{32}$ , for integer ($f = 0$) the resolution is $\frac{1}{2^0} = 1$
However, in the general case :

$$\text{Resolution(x)} = 2^{-f}$$

**Rang**

**Définition 6** *The range is the difference between the most positive and the most negative number representable.*

For example with **two's complement**
If we take, $m = 5, f = 3$, we compute $x_{max} = \sum_{i=-f}^{m-2} 2^i = 15\frac{7}{8}$, $x_{min} = -2^{m-1} = -16$. Then, the range is equal to $x_{max} - x_{min} = 31\frac{7}{8}$
In the general case, for fixed point and two's complement :

$$\text{Range}(x) = x_{max} - x_{min} = \sum_{i-=f}^{m-2} 2^i - (-2^{m-1})$$

### 1.4.1 Accuracy

**definition**

**Définition 7** *The accuracy is the magnitude of the maximum difference between a* ***real*** *value and its representation.*

The worst case (max difference) occurs for a real value exactly in the middle between two subsequent representable numbers (the real value lays between two equaly distant representation).
In the general case =

$$\text{Accuracy}(x) = \frac{\text{Resolution}(x)}{2}$$

**Dynamic Range**

**Définition 8** *The dynamic range is the* ***ratio*** *of, the maximum* ***absolute*** *value representable and the minimum positive value absolute (i.e nonzero) value representable.*

If we take the two's complement, with $m = 5, f = 3$ then the maximum absolute value is $- - 2^4 = 16$. For the minimum positive value we have $2^{-3} = \frac{1}{8}$.
The dynamic range is said $= \frac{x_{max}}{x_{min}} = 128$ In the general case, for fixed-point and two's complement :

$$\text{Dynamic Range}(x) = \frac{2^{m-1}}{2^{-f}} = 2^{m-1+f}$$

*Personal remark* You can see as the *size* of all the representable value divided by 2, 128. We have here 8 bits which means that we have $2^8$ possible value which goes exactly to 256.

### 1.4.2 Floating-Point Number representation

**Floating-Point (FP) Representation** As with any other number representation in a digital system, $FP$ representation is encoded in a finite number of bits. It represents only a **finite subset** of the **infinite set** of real numbers.
A real number that is **exactly** represented is called a **floating-point (FP) number**. All other real number either fall out of range (overflow or underflow) or are represented by $FP$ numbers that approximate their value. The process of approximation is called **roundoff** and produces a **roundoff error**.

**Significand, Exponent, Base** $FP$ representation consists of two components :

- the signed **significand** (also called **mantissa**) $M^*$
- the signed **exponent** $E$

$$x = M^* \times b^E$$

where $b$ is a constant called the **base**

Reminds us of the usual scientific notation, base 10 :

$$+35200 = \underbrace{3.52}_{\text{Coefficient}} \cdot 10^{+4} \quad -0.099 = -9.9 \cdot \overbrace{10^{-2}}^{\text{Exponent}}$$

**Benefits of Floating-Point** Consider 32 bit two's complement signed integers :

$$\text{Dynamic Range}_1(x) = \frac{x_{max}}{x_{min}} = \frac{2^{32-1}}{2^0} = 2^31 \approx 2 \cdot 10^9$$

New, let's consider alors a 32 bit but floating-point number, with 24-significand in sign and magnitude and 8-bits exponent in two's complement.

$$\text{Dynamic Range}_2(x) = \frac{x_{max}}{x_{min}} = \frac{(2^{23}-1) \cdot 2^{2^{(8-1)}-1}}{2^0 \cdot 2^{-2^{8-1}}} = (2^{23}-1) \cdot 2^{255} \approx 5 \cdot 10^{83}$$

We can see here that the dynamic range increase a lot by a factor of $\approx 10^{74}$

**Benefit** We can also see the benefits the resolution which also reduces of for example when taking a 32-bits with 8 fractional bits (fixed-point) and on the other side, 24 bits significand in sign and magnitude and 8 bit exponent in two's complement. If we compute each resolutions :

$$\text{Resolution}_1(x) = 2^{-8} = 0.00390625$$
$$\text{Resolution}_2(x) = 2^0 \cdot 2^{-2^{(8-1)}} = 2^{-2^7} = 2^{-128}$$

If we compute the ratio :

$$\frac{\text{Resolution}_2(x)}{\text{Resolution}_1(x)} = \frac{2^{-128}}{2^{-8}} = 2^{-120} \approx 7.523 \cdot 10^{-37}$$

### 1.4.3 Significand : Sign-and-Magnitude

**Floating-Point Representation** Today, the most used representation for significand is sign and magnitude because it simplifies multiplication in hardware.
The floating-point representation becomes :

$$x = (-1)^S \times M \times b^E$$

Where $S \in \{0, 1\}$ is the **sign** and $M$ is the **magnitude** of the signed significant

> In the rest of the lecture, we assume significand is always represented in sign-and-magnitude.

**Digit vector** Many digit vectors are conceivable, but we focus on the following :

$$X = (\underbrace{SE_{m-1}}_{\text{Sign}E_{m-2}...E_1E_0M_{n-1}M_{n-2}...M_0}$$

Where $E_i$ is the exponent and $M_i$ is the magnitude.
There is $(n+1)$ bit significand in **sign and magnitude** and $m$ bit exponent.

**Redundant** In the most general case, the representation :

$$x = (-1)^S \times M \times b^E$$

is redundant. Sign and magnitude is redundant, Multiple magnitude and exponent combinations can give the same number.

> *Example* If we take for example :
>
> $$(1010)_2 \times 2^{-2} = 10 \times 2^{-2} = 2.5$$
> $$(0101)_2 \times 2^{-1} = 5 \times 2^{-1} = 2.5$$
> $$(1.01)_2 \times 2^1 = 1.25 \times 2^1 = 2.5$$
>
> Floating-point representation is **redundant unless it is normalized**!
> If we take a magnitude that is **normalized** :
>
> $$1 \leq M < 2$$

Then :

$$1010.1000_2 = 1.0101_2 \times 2^3 = 10.5$$
$$-(0.00000011)_2 = -1.1_2 \times 2^{-7} = -0.01171875$$

Juste to be clearer, the normalized one here, is $1.0101_2 \times 2^3$ and $-1.1_2 \times 2^{-7}$.
For example let put $20_{10}$ normalized.
First, $20_{10} = 10100_2$, however $1 \leq M < 2$, which leads us to : $1.0100_2 \times 2^4$. The $M$ being between 1 and 2 doesn't mean that the decimal number has a $1, \ldots$.

**Hidden Bit and Fraction** As the significand is normalized, the first digit of the magnitude is **always** binary 1. If something is always the same, it can be omitted (saving precious bits)
The first digit of the significand is omitted and called **hidden bit**.
The binary point is assumed to the right of the hidden bit. The represented part of the significand is called **fraction F**.

*Example*

$$\overbrace{101.001_2}^{\text{unormalized significand}} \times 2^{-4} = \underbrace{1.01001_2}_{\text{Normalized significand}} \times 2^{-2} = \overbrace{.01001_2}^{\text{hidden is not represented}} \times 2^{-}$$

**Summary**

- Common significand representation is the following :
  - Sign-and-magnitude
  - Normalized
  - One hidden bit
- Corresponding significand value becomes :

$$(-1)^S \times (1 + \sum_{i=1}^{n} M_{n-1} 2^{-1})$$

## 1.5 Exponent

**Exponent** Exponent needs to be signed

- **Positive** for representing very large numbers ( **large absolute** value)
- **Negative** for representing very small numbers ( **small absolute** value)

**Biased representation** Exponent can take any signed representation we know but there is one particular representation, called **biased**, which simplifies comparing two $FP$ numbers in hardware.
Biased representation of a digit vector $X = (X_{n-1} \ldots X_1 X_0)$

$$x = \sum_{i=0}^{n-1} X_i 2^i - B$$

Typically, the bias equals $B = 2^{n-1} - 1$

**Biased representation, Cntd.** Where's the catch ?

- Resulting number are sorted just like unsigned integers but cover both the positive and negative numbers
- efficient hardware (superior to two's complement)
- Min exponent is represented as all zeros
  — $FP$ zero can be represented as all zeros (significand and exponent)

**Summary**

**Exponent**

- Common representation of an -$m$ bit exponent is biased with base $B = 2^{m-1} - 1$
- For the binary digit vector :

$$X = (SE_{m-1}E_{m-2} \ldots E_1E_0.M_{n-1}M_{n-2} \ldots M_0)$$

this biased exponent **value** becomes :

$$e = \sum_{j=0}^{m-1} E_j 2^j - (2^{m-1} - 1)$$

**Floating point format** There could be many floating point formats, but we will often assume :

- $(n+1)$-bit significand
- Sign and magnitude
- Normalized, one hidden bit
- $m$-bit exponent
  — Biased, $B = 2^{m-1} - 1$

$$X = (SE_{m-1}E_{m-2} \ldots E_1E_0.M_{n-1}M_{n-2} \ldots M_0)$$

$$x = (-1)^S \times (1 + \sum_{i=1}^{n} M_{n-i} 2^{-i}) \times 2^{\sum_{j=0}^{m-1} E_j 2^j - (2^{m-1}-1)}$$

## 1.6 Rounding

The result of a floating-point operation is a real number that, to be represented exactly might require a significand with an infinite number of digits.
To obtain a representation close to the exact result, we perform what is called **rounding**

**Rounding modes** Various rounding modes exist

- Round to **nearest**, to **even** when **tie**
- Round towards **zero** (truncate)
- Round towards plus or towards minus **infinity**

Consider the real number $x_{real}$ and the consecutive floating-point number $F_1$ and $F_2$ such that $F_1 \leq x_{real} \leq F_2$, we round it like always (normal definition)

### 1.6.1 IEE Standard 754

***FP* format in IEEE 754**

Exactly what we described

- $(n+1)-$bit significand
- Sign and magnitude, Normalized, one hidden bit
- $m$-bit exponent
  — Biased $B = 2^{m-1} - 1$

$$X = (SE_{m-1}E_{m-2}\dots E_1E_0.M_{n-1}M_{n-2}\dots M_0)$$

There is two types of formats : Basic and extended format :

*Basic formats*

- Sign $S$ 1 bit
- Exponent $E$ : 8 bits
- Fraction $F$ : 23 bits

The default rounding mode is to the nearest, to even when there is a tie.

*Double precision (64 bits)*

- Sign $S$ : 1 bit
- Exponent $E$ : 11 bits
- Fraction $F$ : 52 bits

**Special Values**

- Floating-point **zero** : $E = 0, F = 0$
  — The sign $S$ differentiates between positive and negative zero, Value $1.0 \times 2^{-B}$ is not represented.
- Positive and negative **infinity**
  — Biased exponents all ones, $F = 0$
- **NaN** (not a number)
  — To represent results of invalid operations(for example, the square root of a negative number)
  — Sign = 0 or 1 biased exponents all ones, $F \neq 0$

**Exceptions : Handling of special situations**

The following five exceptions set a flag (i.e " *activate an alarm*") and the computation continues.

- **Overflow**, when the rounded value is **to large** to be represented
  — Result is set to infinity
- **Underflow**, when the rounded value is to small to be represented s to small to be represented
- Division by zero
- Inexact result, when the result is not an exact floating-point number
- invalid result, When $NaN$ is produce by zero
- Inexact result, when the result is not an exact floating-point number
- invalid result, When $NaN$ is produced

2025-02-28 — **Cours 4 : Arithmetic operation**

**Difference between Fixed and Floating point representation** In a fixed point representation, the " *distance*" between each point is fixed, on the other and When using Floating point representation, this distance isn't fixed it is **floating**. The reason for this is the definition of the floating point representation :

$$X = (SE_{m-1}E_{m-2}\dots E_1E_0M_{n-1}M_{n-2}\dots M_0)$$
$$x = (-1)^S \times M \times b^E$$

As you can see, the number can be much more precise as it goes near zero.

### 1.6.2 Arithmetic operations

**Fixed-Point arithmetic** Performing $+$ or $-$ on two binary numbers $x(m, f)$ and $y(m, f)$ is done **the same way** as if the operands were integers.

- Overflow can happen

*Example* (Slide 11) If I forgot to put the screenshot here is the example :

$$X = 000101.110_2 = 5.75_{10}$$
$$Y = 001100.011_2 = 12.375_{10}$$

And we want to sum up these two number,

$$\begin{array}{r} 00101.110 \\ +001100.011 \end{array}$$

We begin at the right, $0 + 1 = 1$ , $X + Y$ =??????.??1 then $1 + 1 = 10$ there for we put a 0 and carry it over, $X + X$ = ??????.?01, then carry$+1 + 0 = 10$ same method at the next index so $X + Y$ =?????0.001 then we get the carry alone, ... and we end up with $010010.001 = 18.125$

*Personal remark* It is the same way because we are always adding power of the 2 event when we are in the " *fractional world*" it is still power of two. We also do the same with decimal number in base 10.

**Two's Complement** For the two's complement the formula is :

$$x \pm y = \left(-X_{(m_x-1)}2^{(m_x-2)} + \sum_{i=-f_x}^{m_x-2} X_i 2^i\right) \pm \left(-Y_{(m_y-1)}2^{(m_y-1)} + \sum_{i=-f_y}^{m_y-2} Y_i 2^i\right)$$

The largest integer-part exponent : $\max(m_x - 1, m_y - 1)$ Consequently $m_{x\pm y} = \max(m_x, m_y) + 1$

The smallest fractional part exponent : $\min(-f_x, -f_y)$ Consequently $f_{x\pm y} = \max(f_x, f_y)$
$m_{x\pm y}$ is the number of bits for the integer component that is needed (usual addition), same thing for the $f_{x\pm y}$

### Fixed Point arithmetic Multiplication

**Introduction** For the multiplication on two binary numbers $x(m, f)$ and $y(m, f)$, we use the same algorithm as if the operands were integers but, the **binary point location changes**.
In two's complement :

$$x \cdot y = \left(-X_{m-1}2^{m-1} + \sum_{i=-f}^{m-2} X_i 2^i\right) \cdot \left(-Y_{m-1}2^{m-1} + \sum_{i=-f}^{m-2} Y_i 2^i\right)$$

The largest integer-part exponent $(m-1)+(m-1)$ Consequently $m_{xy} = 2m$
The smallest fractional-part exponent : $(-f)+(-f)$ Consequently $f_{xy} = 2f$

**Generalization** Multiple on two binary numbers $x(m_x, f_x)$ and $y(m_y, f_y)$

$$x \cdot y = (x_{int} + x_{fr}) \cdot (y_{int} + y_{fr})$$

In two's complement :

$$x \cdot y = \left(-X_{m_x}2^{m_x-1} + \sum_{i=-f_x}^{m_x-2} X_i 2^i\right) \cdot \left(-Y_{m_y-1}2^{m_y-1} + \sum_{i=-f_y}^{m_y-2} Y_i 2^i\right)$$

- $m_{xy} = m_x + m_y$
- $f_{xy} = f_x + f_y$

*Example : Analogy with Decimal numbers* let us take for example

- 9.99 $m_x = 1, f_x = 2$
- 999.9999, $m_y = 3,\ f_x = 4$

If we take the multiplication :

$$9989.999001 \quad m_{xy} = 1+3 = 4; f_{xy} = 2+4 = 6$$

*Example* For example if we take two number with the format,

- $m_x = m_y = 3$
- $f_x = f_y = 2$

and $X = 010.11$,$Y = 011.01$. (screenshot slide 17)
To explain it in spoken English we do it as a loop of addition without the format (like it is integer) and then with the result, we convert it to fixed-point.

> We have to be careful here to not forget to change the format $(m_{xy} = m_x + m_y \ldots)$.

**Pros and cons of fixed Point representation**

**Pros**

- Arithmetic operations on integers can be applied to fixed-point numbers without modifications
  - Portable : we can reuse the same inetger processing hardware
  - Like with intgers, arithmetic operations are performed efficiently (fast)
  - Used in image and signal processing and communication

**Cons**

- Complex data and algorithm analysis
  - Where to put the binary point to maximize accuracy
- There are other number formats, namely floating-point, that provide more extensive dynamic range and better precision

### Floating-Point Arithmetic

**Addition/Subtraction**

Let $x$ and $y$ be represented as $(S_x, M_x, E_x)$ and $(S_y, M_y, E_y)$

- The significands $M^* = (-1)^S M$ are normalized

Addition/subtraction result is $z$, also represented as $(S_z, M_z, E_z)$ :

$$z = x \pm y = M_x^* \times 2^{E_x} \pm M_y^* \times 2^{E_y}$$

The significand of the result is also normalized :

$$z = M_z^* \times 2^{E_z}$$

**Steps**

Four main steps to compute and produce the result $+/-$

- Add/substract significand and set exponent
  The significand of the number with the **smaller** exponent has to be multiplied by two to the power of the difference between the exponents (this operation is called **alignment**) and the added/subtracted to the other significand

$$M_z^* \begin{cases} (M_x^* \pm (M_y^* \times 2^{(E_y - E_x)})) \times 2^{E_x} & \text{if } E_x \geq E_y \\ ((M_x^* \times 2^{(E_x - E_y)}) \pm M_y^*) \times 2^{E_y} & \text{if } E_x < E_y \end{cases}$$
$$E_z = \max(E_x, E_y)$$

- Normalize the result and update the exponent, if required
- Round the result, normalize, and adjust exponent, if required
- Set flags for special values, if required

**Recap**

- Recal Step 1 : Add/substract significand and set exponent
- Algorithm
  - Substract exponents $d = E_x - E_y$
  - Align significands
    - Compare the exponents of the two operands
    - shift right $d$ positions the significand of the operand with the smallest exponent

  — Select as the exponent of the result the largest exponent
- — Add/subtract signed significands and produce the sign of the result

### 1.6.3 Floating Point +/-

**Normalization** Various situations may occur

- Scenario 2 : When the effective operation is an **addition**, the significand might **overflow**. Steps to perform normalization :
  - — Shift right the significand one position
  - — Increment the exponent by one
- Example :

$$\begin{aligned} &1.1001111 \\ +\ &0.0110110 \\ =\ &10.0000101 \end{aligned}$$

Normalization

Shift right >> 1

Increment the exponent $E = E + 1$

**Rounding** The intermediate result may not be representable with the given format, in this case we perform a rounding.

- Towards zero : truncate the lsb
- Twords $\pm\infty$ : requires addition
- To nearest : require addition

**Tie to even** The $FP$ result is as close as possible to the exact value :

- Minimized rounfoff error (default rounding mode in $IEEE$ 754)
- Tie to even is preferred because it leads to smaller error when the result is divided by two -a frequent operation

Assuming as significand of infinite precision and radix $r$, round to the nearest can be obtained by **adding** $(\frac{r^{-f}}{2})$ to the infinite precision significandd and keeping the resulting $f$ fractional digits

- In case of overflow : normalization and the exponent update are needed

**Max round-off Error** Rounding to nearest. $f$ fractional digits. What is the maximumm difference between the exact value and its $FP$ representation ?

$$d_{max} = \frac{2^{-f}}{2} \times 2^{E_{max}}$$