A Programmer's Introduction to Mathematics: Chapter 2 Exercise solutions

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2.6: Prove Vieta's Formulas

Exercise. Let $f(x) = a_0 + a_1 x + ... + a_n x^n$ be a degree n polynomial, and suppose it has k real roots $r_1, ..., r_n$. Prove Vieta's Formulas, which are

$$\sum_{i=1}^{n} r_i = -\frac{a_{n-1}}{a_n}$$

$$\prod_{i=1}^{n} r_i = (-1)^n \frac{a_0}{a_n}.$$

Proof. Let $g(x) = \frac{f(x)}{a_n} = \frac{a_0}{a_n} + \frac{a_1}{a_n} x + ... + x^n$. Now, the roots of $g(x)$ are the (all real) roots of $f(x)$ - $r_1, ..., r_n$ - and hence, by the Fundamental Theorem of Algebra (see Ex. 2.12), the monic polynomial $g(x)$ may be written as

$$g(x) = \prod_{i=1}^{n} x - r_i = \sum_{i=0}^{n} \frac{a_i}{a_n} x_i$$

Now, equating the coefficients of x^{n-1} an both sides (note the expansion of the product on the left hand side results in the multiplication of $-r_i$ by x $n-1$ times), we get

$$\sum_{i=1}^{n} r_i = -\frac{a_{n-1}}{a_n},$$

as required. Similarly, we may equate the constant term (the "coefficient of x^0") on both sides, observing that on the left hand side this is the product of the roots multiplied by $(-1)^n$, and dividing through by $(-1)^n$ to obtain the second equation.

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