A Programmer's Introduction to Mathematics: Chapter 10 Exercise solutions

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Theorem 10.14 Proof

Exercise. Let U, V, W be three vector spaces. Let $f : U \to V$ and $g : V \to W$ be linear maps. Then

$$M(g \circ f) = M(g)M(f)$$

where $g \circ f$ denotes the function composition $x \mapsto g(f(x))$, and $M(g)M(f)$ denotes matrix multiplication.

Let $n, m, l \in \mathbb{N}$ be the respective dimensions of U, V, W, and $x \in U$ be an arbitrary vector. First, a lemma to isolate all of the messy index stuff.

Lemma. Where $x \in U$, we have $M(g)(M(f)x) = (M(g)M(f))x$.

Proof. Let

$$x = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, M(f) = \begin{bmatrix} \beta_{1,1} & \dots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{m,1} & \dots & \beta_{m,n} \end{bmatrix}, \text{ and } M(g) = \begin{bmatrix} \delta_{1,1} & \dots & \delta_{1,m} \\ \vdots & \ddots & \vdots \\ \delta_{l,1} & \dots & \delta_{l,m} \end{bmatrix}. \text{ So,}$$

$$\begin{aligned}
M(f)x &= \begin{bmatrix} \sum_{i=1}^n \alpha_i \beta_{i,1} \\ \vdots \\ \sum_{i=1}^n \alpha_i \beta_{i,m} \end{bmatrix} \\
M(g)(M(f)x) &= \begin{bmatrix} \delta_{1,1} & \dots & \delta_{1,m} \\ \vdots & \ddots & \vdots \\ \delta_{l,1} & \dots & \delta_{l,m} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n \alpha_i \beta_{i,1} \\ \vdots \\ \sum_{i=1}^n \alpha_i \beta_{i,m} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^m \delta_{1,j} \sum_{i=1}^n \alpha_i \beta_{i,j} \\ \vdots \\ \sum_{j=1}^m \delta_{l,j} \sum_{i=1}^n \alpha_i \beta_{i,j} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^m \sum_{i=1}^n \delta_{1,j} \alpha_i \beta_{i,j} \\ \vdots \\ \sum_{j=1}^m \sum_{i=1}^n \delta_{l,j} \alpha_i \beta_{i,j} \end{bmatrix}. \text{ Also,} \\
M(g)M(f) &= \begin{bmatrix} \delta_{1,1} & \dots & \delta_{1,m} \\ \vdots & \ddots & \vdots \\ \delta_{l,1} & \dots & \delta_{l,m} \end{bmatrix} \begin{bmatrix} \beta_{1,1} & \dots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{m,1} & \dots & \beta_{m,n} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^m \delta_{1,j} \beta_{j,1} & \dots & \sum_{j=1}^m \delta_{1,j} \beta_{j,n} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^m \delta_{l,j} \beta_{j,1} & \dots & \sum_{j=1}^m \delta_{l,j} \beta_{j,n} \end{bmatrix} \\
(M(g)M(f))x &= \begin{bmatrix} \sum_{j=1}^m \delta_{1,j} \beta_{j,1} & \dots & \sum_{j=1}^m \delta_{1,j} \beta_{j,n} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^m \delta_{l,j} \beta_{j,1} & \dots & \sum_{j=1}^m \delta_{l,j} \beta_{j,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^n \alpha_i \sum_{j=1}^m \delta_{1,j} \beta_{j,i} \\ \vdots \\ \sum_{i=1}^n \alpha_i \sum_{j=1}^m \delta_{l,j} \beta_{j,i} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^m \sum_{i=1}^n \delta_{1,j} \alpha_i \beta_{i,j} \\ \vdots \\ \sum_{j=1}^m \sum_{i=1}^n \delta_{l,j} \alpha_i \beta_{i,j} \end{bmatrix} \text{(order doesn't matter)} \\
&= M(g)(M(f)x), \text{ as required.}
\end{aligned}$$

□

Using this, we may fairly simply complete the proof.

Proof. By definition, $g(f(x)) = M(g \circ f)x$. We also know that $f(x) = M(f)x$ Thus, $g(f(x)) = g(M(f)x) = M(g)(M(f)x) = (M(g)M(f))x$ (using the lemma) and hence the function composition in matrix form $M(g \circ f) = M(g)M(f)$. □

Map to matrix preserves inverses

Exercise. From page 156: Prove that if a linear map is a bijection, then its inverse is also a linear map, and the linear-map-to-matrix correspondence preserves inverses.

First we prove the linearity of the inverse

Proof. Let V, W be two vector spaces, and $f : V \mapsto W$ be a linear bijective map. Since f is a bijection, there exists an inverse $f^{-1} : W \mapsto V$. Let $x, y \in V$, and $f(x) = x',\ f(y) = y'$. Let c be a scalar. Now,

$$\begin{aligned}
f(x+y) &= f(x) + f(y), \text{ by linearity of f,} \\
x + y &= f^{-1}(f(x) + f(y)), \text{ taking the inverse,} \\
f^{-1}(x') + f^{-1}(y') &= f^{-1}(x' + y') \\
f(cx) &= cf(x), \text{ by linearity of f} \\
cx &= f^{-1}(cf(x)), \text{ taking the inverse, and hence} \\
cf^{-1}(x') &= f^{-1}(cx')
\end{aligned}$$

And hence f^{-1} is linear, as required. $\square$