A Programmer's Introduction to Mathematics: Chapter 8 Exercise solutions

Arthur Allshire

January 4, 2019

8.2.1: Linearity of derivative

Exercise. Prove Theorem 8.9 that the map $f \mapsto f'$ is linear.

Proof. Using the notation in Theorem 8.9, we know that

$$\begin{aligned}
D(f+g)(c) &= \lim_{x \to c} \frac{f(x) + g(x) - (f(c) + g(c))}{x - c} \\
&= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} + \frac{f(x) - f(c)}{x - c} \right) \\
D(f) + D(g) &= \left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \right) + \left(\lim_{x \to c} \frac{g(x) - g(c)}{x - c} \right) \\
D(kf)(c) &= \lim_{x \to c} \frac{kf(x) - kf(c)}{x - c}, \text{ where } k \in \mathbb{R} \\
&= \lim_{x \to c} k \frac{f(x) - f(c)}{x - c}
\end{aligned}$$

The problem thus boils down to showing that the limit of a sum is the same as the sum of limits, and that multiplication by a constant inside and outside a limit is equivalent.

To prove the first part, we go back to the definitions of convergance and limits. Let $x_1, x_2, x_3, ...$ be any sequence converging on $c \in \mathbb{R}$. Let $f(x_n) \to L$ and $g(x_n) \to M$ for any sequence $x_n \to c$. By the definition of convergence, $\forall \delta > 0, \exists k \in \mathbb{N}$ such that both $|f(x_n) - L| < \delta$ and $|g(x_n) - M| < \delta$ for each $n > k$. We now construct the series $f(x_1)+g(x_1), f(x_2)+g(x_2), f(x_3)+g(x_3), ...$ Adding the two previous equations,

$$\begin{gathered}
|f(x_m) - L| + |g(x_m) - M| < 2\delta \\
|(f(x_m) + g(x_m)) - (L + M))| < \epsilon, \text{ where } \epsilon = 2\delta
\end{gathered}$$

Since δ may be any real number greater than 0, so must ϵ, and hence $f(x_n) +$

$g(x_n) \to L + M$. So,

$$\lim_{x \to c} f(x) + g(x) = L + M$$
$$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

Similarly, let x_n be a sequence such that $x_n \to c$, and $f(x)$ be a function such that the series $f(x_n) \to L$, for every series x_n. Thus there is a value $k \in \mathbb{N}$ such that $|f(x_n) - L| < \delta, \forall \delta > 0$ for each $n > k$. Let $af(x_1), af(x_2), ...$ be another sequence. Multiplying the previous expression by some $a \in \mathbb{R}$,

$$a|f(x_n) - L| < a\delta$$
$$|af(x_n) - aL| < \epsilon, \text{ where } \epsilon = a\delta$$

Since δ may be any real number greater than 0, so must ϵ, and hence the limit of $af(x_n)$ is aL. This implies that $\lim_{x \to c} af(x) = a \lim_{x \to c} f(x)$, as desired, thus completing the proof.

□

8.2.2: Product of limits

Exercise. Using the definition of the limit of a function, prove that:

$$\lim_{x \to a} [f(x)g(x)] = (\lim_{x \to a} f(x)(\lim_{x \to a} g(x))$$

Proof. Let $a_1, a_2, ...$ be a series converging on L, and $b_1, b_2, ...$ be a series converging on M. For every threshold $\epsilon > 0$ there is a $k \in \mathbb{N}$ such that all the a_n after a_k, and all of the b_n after b_k are within ϵ of L

We now construct the series $a_1 b_1, a_2 b_2, ...$ To prove it converges on LM we must show that $|a_n b_n - LM| < \delta$ for all $\delta > 0$ Clearly now for all $n > k$, the term $a_n b_n$ is within $(L + \epsilon)(M + \epsilon)$ of LM. Letting $\delta = \epsilon^2 + \epsilon(L + M)$, we then obtain (by the expansion of the previous binomial) $|a_n b_n - LM| < \delta$. Since ϵ may be any real number greater than 0, so must δ and hence the series converges on LM.

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then for all series $x_n \to c$, $f(x_n) \to L$ and $g(x_n) \to M$, and thus using the above $f(x_n)g(x_n) \to LM$, and so

$$\lim_{x \to c} f(x)g(x) = (\lim_{x \to c} f(x))(\lim_{x \to c} g(x))$$

□

8.2.3: Proving divergence

Exercise. Prove that $a_n = \frac{2^{\sqrt{n}}}{n^{10}}$ diverges.

See Discussion of this proof here (that was a link - you may need to download this PDF to click as it doesn't render properly on GH.)

Proof. As $\frac{\mathrm{d}}{\mathrm{d}x}(\log_2 x) = \frac{1}{x} > 0, \forall x > 0$, we know $\log_2 x$ is an increasing function. Hence $b_n = \log_2 a_n$ diverges if and only if a_n diverges.

We know that $b_n = \log_2 \frac{2^{\sqrt{n}}}{n^{10}} = \sqrt{n} - 10 \log_2 n$. Since

$$\frac{\mathrm{d}}{\mathrm{d}n}\sqrt{n} = \frac{1}{2\sqrt{n}} > 10\frac{10}{\ln 2}\frac{1}{n} = \frac{\mathrm{d}}{\mathrm{d}n}10\log_2 n$$

for a sufficiently large n, and since $\lim_{x\to\infty}\sqrt{n} = \infty$ and $\lim_{x\to\infty} 10\log_2 n = \infty$, we conclude that b_n diverges and thus so does a_n. $\square$