A Programmer's Introduction to Mathematics: Chapter 4 Exercise solutions

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4.2: Prove De Morgan's Law

Exercise. Prove De Morgan's law for sets, which for $A, B \subset X$ states that $(A \cap B)^C = A^C \cup B^C$, and $(A \cup B)^C = A^C \cap B^C$.

Proof.

$$\begin{aligned}
(A \cap B)^C &= \{x : x \in A \text{ and } x \in B\}^C \\
&= \{x : x \notin A \text{ or } x \notin B\} \\
&= \{x : x \notin A\} \cup \{x : x \notin B\} \\
&= \{x : x \in A\}^C \cup \{x : x \in B\}^C \\
&= A^C \cup B^C
\end{aligned}$$

$$\begin{aligned}
(A \cup B)^C &= \{x : x \in A \text{ or } x \in B\}^C \\
&= \{x : x \notin A \text{ and } x \notin B\} \\
&= \{x : x \notin A\} \cap \{x : x \notin B\} \\
&= A^C \cap B^C
\end{aligned}$$

□

4.5: Prove that $\mathbb{N} \times \mathbb{N}$ is countable

Solution I had to look this solution up. This solution is a fairly straightforward method to approach this problem.

4.6: Union of Countable Sets

Exercise. Suppose for each $n \in \mathbb{N}$ we picked a countable set A_n. Prove that the union of all the A_n is countable. Hint: use the previous problem and write the elements of all the A_n in a grid.

Proof. Let $X = A_1 \cup A_2 \cup A_3 \cup ....$ To prove that X is countable we show that there must exist a surjection $\mathbb{N} \to X$. Define a surjective map $g : \mathbb{N} \to \mathbb{N}^2$. From Ex. 4.5, we know such a map must exist. Define $X_{(i,j)}$ as the j-th element from the i-th set which made up X. Now, define $f : \mathbb{N} \to X$ as $f(x) = X_{g(x)}$, which is clearly a surjection (as $g(x)$ is). Thus, since the surjection $f : \mathbb{N} \to X$ exists, X is countable. $\square$