A Programmer's Introduction to Mathematics: Chapter 8 Exercise solutions

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8.2.1: Linearity of derivative

Exercise. Prove Theorem 8.9 that the map $f \mapsto f'$ is linear.

Proof. Using the notation in Theorem 8.9, we know that

$$\begin{aligned}
D(f+g)(c) &= \lim_{x \to c} \frac{f(x) + g(x) - (f(c) + g(c))}{x - c} \\
&= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} + \frac{f(x) - f(c)}{x - c}\right) \\
D(f) + D(g) &= \left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right) + \left(\lim_{x \to c} \frac{g(x) - g(c)}{x - c}\right) \\
D(kf)(c) &= \lim_{x \to c} \frac{kf(x) - kf(c)}{x - c}, \text{ where } k \in \mathbb{R} \\
&= \lim_{x \to c} k \frac{f(x) - f(c)}{x - c}
\end{aligned}$$

The problem thus boils down to showing that the limit of a sum is the same as the sum of limits, and that multiplication by a constant inside and outside a limit is equivalent.

To prove the first part, we go back to the definitions of convergance and limits. Let $x_1, x_2, x_3, ...$ be any sequence converging on $c \in \mathbb{R}$. Let $f(x_n) \to L$ and $g(x_n) \to M$ for any sequence $x_n \to c$. By the definition of convergence, $\forall \delta > 0, \exists k \in \mathbb{N}$ such that both $|f(x_n) - L| < \delta$ and $|g(x_n) - M| < \delta$ for each $n > k$. We now construct the series $f(x_1)+g(x_1), f(x_2)+g(x_2), f(x_3)+g(x_3), ...$ Adding the two previous equations,

$$\begin{gathered}
|f(x_m) - L| + |g(x_m) - M| < 2\delta \\
|(f(x_m) + g(x_m)) - (L + M))| < \epsilon, \text{ where } \epsilon = 2\delta
\end{gathered}$$

Since δ may be any real number greater than 0, so must ϵ, and hence $f(x_n) +$

$g(x_n) \to L + M$. So,

$$\begin{aligned}\lim_{x \to c} f(x) + g(x) &= L + M \\ &= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)\end{aligned}$$

Similarly, let x_n be a sequence such that $x_n \to c$, and $f(x)$ be a function such that the series $f(x_n) \to L$, for every series x_n. Thus there is a value $k \in \mathbb{N}$ such that $|f(x_n) - L| < \delta, \forall \delta > 0$ for each $n > k$. Let $af(x_1), af(x_2), ...$ be another sequence. Multiplying the previous expression by some $a \in \mathbb{R}$,

$$\begin{aligned}a|f(x_n) - L| &< a\delta \\ |af(x_n) - aL| &< \epsilon, \text{ where } \epsilon = a\delta\end{aligned}$$

Since δ may be any real number greater than 0, so must ϵ, and hence the limit of $af(x_n)$ is aL. This implies that $\lim_{x \to c} af(x) = a \lim_{x \to c} f(x)$, as desired, thus completing the proof.

$\square$