Resolução da 2ª Prova de Álgebra Linear

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Q1. $[v_1, \dots, v_m] = \{ \alpha_1 v_1 + \dots + \alpha_m v_m \mid \alpha_i \in K \}$.

• Como $\underbrace{0}_{\text{vetor}} = \underbrace{0}_{\text{zero}} . v_1 + \dots + \underbrace{0}_{\text{zero}} . v_m$, temos $\underbrace{0}_{\text{vetor}} \in [v_1, \dots, v_m]$

•• Dados $u, v \in [v_1, \dots, v_m]$, temos

$$u = \alpha_1 v_1 + \dots + \alpha_m v_m \quad e \quad v = \beta_1 v_1 + \dots + \beta_m v_m \quad , \text{onde } \alpha_i, \beta_i \in K.$$

Então,

$$u + v = \underbrace{(\alpha_1 + \beta_1)}_{\in K} . v_1 + \dots + \underbrace{(\alpha_m + \beta_m)}_{\in K} . v_m \in [v_1, \dots, v_m].$$

••• Dados $\lambda \in K$ e $u \in [v_1, \dots, v_m]$, temos

$$u = \alpha_1 . v_1 + \dots + \alpha_m v_m \quad , \text{onde } \alpha_i \in K.$$

Então,

$$\lambda u = \underbrace{(\lambda \alpha_1)}_{\in K} . v_1 + \dots + \underbrace{(\lambda \alpha_m)}_{\in K} . v_m \in [v_1, \dots, v_m].$$

Portanto, $[v_1, \dots, v_m]$ é um subespaço vetorial de V. //

Q2. (a) Como $\dim_{\mathbb{R}} \mathbb{R}^4 = 4 = \# \mathcal{B}$, basta provarmos que $\mathcal{B}$ é LI. Suponha que

$$\alpha_1 . (0,0,1,1) + \alpha_2 . (-1,1,1,2) + \alpha_3 . (1,1,0,0) + \alpha_4 . (2,1,2,1) = (0,0,0,0)$$

Então

$$(-\alpha_2 + \alpha_3 + 2\alpha_4 , \alpha_2 + \alpha_3 + \alpha_4 , \alpha_1 + \alpha_2 + 2\alpha_4 , \alpha_1 + 2\alpha_2 + \alpha_4) = (0,0,0,0)$$

2 Segue que

$$\begin{cases} -\alpha_2+\alpha_3+2\alpha_4=0 \\ \alpha_2+\alpha_3+\alpha_4=0 \\ \alpha_1+\alpha_2+2\alpha_4=0 \\ \alpha_1+2\alpha_2+\alpha_4=0 \end{cases} \sim \begin{cases} \alpha_1+\alpha_2+2\alpha_4=0 & \leftarrow L_3 \\ \alpha_2-\alpha_4=0 & \leftarrow L_4-L_3 \\ \alpha_2+\alpha_3+\alpha_4=0 & \leftarrow L_2 \\ -\alpha_2+\alpha_3+2\alpha_4=0 & \leftarrow L_1 \end{cases}$$

$$\sim \begin{cases} \alpha_1+\alpha_2+2\alpha_4=0 \\ \alpha_2-\alpha_4=0 \\ \alpha_3+2\alpha_4=0 & \leftarrow L_3-L_2 \\ \alpha_3+\alpha_4=0 & \leftarrow L_4+L_2 \end{cases} \sim \begin{cases} \alpha_1+\alpha_2+2\alpha_4=0 \\ \alpha_2-\alpha_4=0 \\ \alpha_3+2\alpha_4=0 \\ -\alpha_4=0 & \leftarrow L_4-L_3 \end{cases}$$

Resolvendo, encontramos $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$. Portanto, $\mathcal{B}$ é LI. //

b) $(a,b,c,d)=\alpha_1.(0,0,1,1)+\alpha_2.(-1,1,1,2)+\alpha_3.(1,1,0,0)+\alpha_4.(2,1,2,1)$

$(a,b,c,d)=(-\alpha_2+\alpha_3+2\alpha_4,\ \alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+2\alpha_4,\ \alpha_1+2\alpha_2+\alpha_4)$

Então,

$$\begin{cases} -\alpha_2+\alpha_3+2\alpha_4=a \\ \alpha_2+\alpha_3+\alpha_4=b \\ \alpha_1+\alpha_2+2\alpha_4=c \\ \alpha_1+2\alpha_2+\alpha_4=d \end{cases} \sim \begin{cases} \alpha_1+\alpha_2+2\alpha_4=c & \leftarrow L_3 \\ \alpha_2-\alpha_4=d-c & \leftarrow L_4-L_3 \\ \alpha_2+\alpha_3+\alpha_4=b & \leftarrow L_2 \\ -\alpha_2+\alpha_3+2\alpha_4=a & \leftarrow L_1 \end{cases}$$

$$\sim \begin{cases} \alpha_1+\alpha_2+2\alpha_4=c \\ \alpha_2-\alpha_4=d-c \\ \alpha_3+2\alpha_4=b-d+c & \leftarrow L_3-L_2 \\ \alpha_3+\alpha_4=a+d-c & \leftarrow L_4+L_2 \end{cases} \sim \begin{cases} \alpha_1+\alpha_2+2\alpha_4=c \\ \alpha_2-\alpha_4=d-c \\ \alpha_3+2\alpha_4=b-d+c \\ -\alpha_4=a-b-2c+2d & \uparrow L_4-L_3 \end{cases}$$

Resolvendo, encontramos $\alpha_4 = -a+b+2c-2d$,

$\alpha_3 = 2a-b-3c+3d$, $\alpha_2 = -a+b+c-d$ e

$\alpha_1 = 3a-3b-4c+5d$. //

(c) Para $v=(-6,-2,0,3)$, substituímos $a=-6$, $b=-2$, $c=0$ e $d=3$ no item (b) e encontramos

$\alpha_1 = 3$, $\alpha_2 = 1$, $\alpha_3 = -1$ e $\alpha_4 = -2$. Então,

$$(-6,-2,0,3) = 3.(0,0,1,1)+1.(-1,1,1,2)-1.(1,1,0,0)-2.(2,1,2,1).$$

$$=(3,1,-1,-2)_{\mathcal{B}}. \text{//}$$

Q3. (a)
$$\begin{aligned}
Nuc(T) &= \{(x,y,z) \mid T(x,y,z)=(0,0)\} \\
&= \{(x,y,z) \mid (x-2y,3y+z)=(0,0)\} \\
&= \{(x,y,z) \mid \underbrace{x-2y=0}_{x=2y} \text{ e } \underbrace{3y+z=0}_{z=-3y}\} \\
&= \{(2y,y,-3y) \mid y \in \mathbb{R}\} \\
&= \{y\cdot(2,1,-3) \mid y \in \mathbb{R}\} \\
&= [(2,1,-3)],
\end{aligned}$$

sendo $\mathcal{B}=\{(2,1,-3)\}$ uma base para $Nuc(T)$. //

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(b) $Im(T) = \{T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\}$

$= \{(x-2y, 3y+z) \mid (x,y,z) \in \mathbb{R}^3\}$

$= \{x.(1,0) + y.(-2,3) + z.(0,1) \mid x,y,z \in \mathbb{R}\}$

$= [(1,0), (-2,3), (0,1)]$

$= [(1,0), (0,1)]$

pois $(-2,3)$ é comb. lin dos demais.

$= \mathbb{R}^2$,

sendo $\mathcal{B} = \{(1,0), (0,1)\}$ uma base para $Im(T)$. //

Q4. $T(x,y,z) = T(xe_1 + y.e_2 + ze_3)$

$= x.T(e_1) + y.T(e_2) + z.T(e_3) \quad (*)$

Devemos ter $T(1,-1,1) = (0,0,0,0) = 0$

vetor nulo

ie., $1.T(e_1) - 1.T(e_2) + 1.T(e_3) = 0$

ie., $T(e_3) = -T(e_1) + T(e_2)$.

Substituindo em $(*)$,

$T(x,y,z) = x.T(e_1) + y.T(e_2) + z.(-T(e_1) + T(e_2))$

$= (x-z).T(e_1) + (y+z).T(e_2)$

Agora podemos definir

$$T(e_1) = (1,0,0,1) \quad e \quad T(e_2) = (0,1,-1,0)$$

Então,

$$T(x,y,z) = (x-z).(1,0,0,1) + (y-z).(0,1,-1,0)$$

$$= (x-z, y-z, -y+z, x-z).$$