- **Part 1: Definition of Differential Equations and Solutions**

Definition 1. A **differential equation** is an equation involving the derivative of a function. If the function depends on a single variable, then only ordinary derivatives appear, and the equation is called an **ordinary differential equation (ODE)**. If the function depends on several variables and partial derivatives appear, then the equation is called a **partial differential equation (PDE)**. A function that satisfies a differential equation is called a **solution** of the differential equation.

Example 2. Verify that $y_1(t) = e^{-3t}$ and $y_2(t) = e^t$ are solutions of the ODE: $y'' + 2y' - 3y = 0$.

Example 3. Verify that $u(x,t) = e^{-4t}\sin(x)$ is a solution of the PDE: $4u_{xx} - u_t = 0$.

Example 4. Determine the values of r for which the ODE $y''' - 3y'' + 2y' = 0$ has a solution of the form $y = e^{rt}$.

Example 5. Determine the values of r for which the ODE $t^2y'' - 4ty' + 4y = 0$ has a solution of the form $y = t^r$ for $t > 0$.

Part 2: Classification of Differential Equations

Definition 6. The **order** of a differential equation is the order of the highest derivative that appears in the equation. If a differential equation can be written as a linear combination of the unknown function and its derivatives, it is called **linear**. The general form of a linear ordinary differential equation is:

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t).$$

A differential equation that is not linear is called **nonlinear**.

Example 7. Classify each differential equation as an ordinary or partial differential equation. Determine the order of each differential equation and whether the equation is linear or nonlinear.

a) $(1+y)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$

b) $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (\cos^2 t)y = t^3$

c) $u_{xx} + u_{yy} + u_x + 2u = 0$

d) $u_t + uu_x = 1 + u_{xx}$

- **Part 3: Mathematical Models**

A question you have asked yourself at this point is why are differential equations so important? It turns out that they are fundamental to the study of many physical processes. Let's see this with an example.

Example 8. Newton's second law says that the forces that act upon an object is equal to mass times acceleration. In math terms, if F is the acting force, m is the mass and a is the acceleration then:

This is actually a differential equation! Note that

where v is the velocity of the object and u is its displacement.

Now let's assume that the object is falling in the atmosphere. What forces are acting on it?

Putting everything together, we have

$$F = mg - \gamma v \quad \text{and} \quad F = m\frac{dv}{dt} \quad \text{so that } m\frac{dv}{dt} = mg - \gamma v.$$

As you can see, differential equations help us understand things in nature, and the above example is only a very small taste of their power. They are mathematical models that describe relationships of change in the physical world, for example, population dynamics, heat transfer, objects in motion, chemical reactions, fluid flow, and much, much more!

Example 9. Suppose that $m = 2$ kg and $\gamma = 2$ kg/s in the equation

$$m\frac{dv}{dt} = mg - \gamma v.$$

Find the velocity of the object at any time t.

Because of the arbitrary constant C, we call the solution $v(t) = Ce^{-t} + g$ the **general solution** to the equation $dv/dt = g - v$. Even though we say "the," it is actually a whole family of solutions indexed by the parameter C. The graph corresponding to any particular value of C is called an **integral curve**.

What if we also knew that the velocity of the object was 15.8 m/s at time $t = 0$?

Knowing the value at a particular time is called an **initial condition**. A differential equation along with an initial condition is called an **initial value problem (IVP)**. The solution to an IVP is any function that satisfies the differential equation AND the initial condition.

- **Part 4: Direction Fields and Equilibrium Solutions**

Consider a differential equation of the form

$$\frac{dy}{dt} = f(t, y).$$

Recall that the derivative, y', represents the slope of the tangent line to the graph of $y(t)$. By choosing a lattice of points (t, y) in the ty-plane and sketching line segments with slopes y', we obtain a **direction field** describing the overall behavior of solutions of the equation.

Definition 10. An **equilibrium solution** of a differential equation is a solution from which there is no change. That is, a solution $y(t)$ for which

$$\frac{dy}{dt} = y' = 0.$$

Solutions may approach or move away from an equilibrium solution over time.

Example 11. Draw a direction field and sketch some integral curves for the differential equation

$$y' = y - 2.$$

Based on the direction field, determine the behavior of y as $t \to \infty$.

- **Part 5: Using a Computer to Plot Direction Fields**

There are many resources and tools available to do much of what we will cover in this class numerically. Some are extremely powerful, such as Mathematica, MATLAB, Python, Octave, Sage, etc. For direction fields, I will use a simple Java applet created by John C. Polking at Rice University. You can click on the link: https://aeb019.hosted.uark.edu/dfield.html or find the link under "Computer Resources" inside eCampus.

Example 12. Use the Java applet to create a direction field and some integral curves for the ODE:

$$y' = y(y-5).$$

Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of $t = 0$, describe this dependency.

Example 13. Use the Java applet to create a direction field and some integral curves for the ODE:

$$y' = y^2 \sin(t) + e^{-t} y.$$

Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of $t = 0$, describe this dependency.