Lecture 12: First Order Linear Differential Equations, Intro to First Order Circuits

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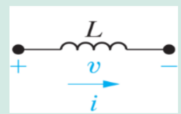
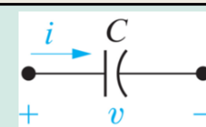
ECEN 214 – Electrical Circuit Theory (Spring 2020)

Outline

- Solution Methods to First Order Linear Differential Equations
- Forming the Differential Equation for a Sample First Order Circuit

Highlights from Last Lecture (Inductor and Capacitor Summary/Comparison)

	Inductor	Capacitor
Symbol		
Units	Henries [H]	Farads [F]
Describing equation	$v(t) = L\frac{di(t)}{dt}$	$i(t) = C\frac{dv(t)}{dt}$
Other equation	$i(t) = \frac{1}{L}\int_{t_o}^{t} v(\tau)d\tau + i(t_o)$	$v(t) = \frac{1}{C}\int_{t_o}^{t} i(\tau)d\tau + v(t_o)$
Initial condition	$i(t_o)$	$v(t_o)$
Behavior with const. source	If $i(t) = I$, $v(t) = 0$ $\rightarrow$ short circuit	If $v(t) = V$, $i(t) = 0$ $\rightarrow$ open circuit
Continuity requirement	$i(t)$ is continuous so $v(t)$ is finite	$v(t)$ is continuous so $i(t)$ is finite

Highlights from Last Lecture (Inductor and Capacitor Summary/Comparison Cont’d)

	Inductor	Capacitor
Power	$p(t) = v(t)i(t) = Li(t)\frac{di(t)}{dt}$	$p(t) = v(t)i(t) = Cv(t)\frac{dv(t)}{dt}$
Energy	$w(t) = \frac{1}{2}Li(t)^2$	$w(t) = \frac{1}{2}Cv(t)^2$
Initial energy	$w_o(t) = \frac{1}{2}Li(t_o)^2$	$w_o(t) = \frac{1}{2}Cv(t_o)^2$
Trapped energy	$w(\infty) = \frac{1}{2}Li(\infty)^2$	$w(\infty) = \frac{1}{2}Cv(\infty)^2$
Series-connected	$L_{eq} = L_1 + L_2 + L_2$ $i_{eq}(t_o) = i(t_o)$	$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$ $v_{eq}(t_o)$ $= v_1(t_o) + v_2(t_o) + v_3(t_o)$
Parallel-connected	$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$ $i_{eq}(t_o)$ $= i_1(t_o) + i_2(t_o) + i_3(t_o)$	$C_{eq} = C_1 + C_2 + C_2$ $v_{eq}(t_o) = v(t_o)$

Order of a Differential Equation

- A differential equation is a relationship between an independent variable, say t, and a dependent variable, say y, and one or more derivatives of y with respect with t. Examples:

$$\frac{dy}{dt} + t^2 y = 4t, t^2 y \frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$$

(Note that $y \equiv y(t)$, except otherwise specified, wherever we use y in an equation, it is defined as a function dependent on t)

- The order of a differential is the highest derivative present in the equation. For instance, $\frac{dy}{dt} + t^2 y = 4t$ is a first order differential equation and $t^2 y \frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$ is a second order differential equation

First Order Linear Differential Equation

- A general first order **linear** differential equation can be written in the form:

$$\frac{dy}{dt} + P(t)y = x(t)$$

Linear in the above form indicates that $\frac{dy}{dt}$ and y are raised to powers of 1. $P(t)$ and $x(t)$ are not necessarily linear functions of t

- A general first order **linear, constant coefficient,** differential equation can be written by replacing $P(t) = k$, where k is a constant:

$$\frac{dy}{dt} + ky = x(t)$$

First Order Linear Differential Equation

- A general first order **linear, constant coefficient,** differential equation (k is a constant): $\frac{dy}{dt} + ky = x(t)$
- The first order circuits we will analyze can be expressed in the above linear, constant coefficient form
- The above equation can be viewed as describing the input/output relationship for a linear dynamical system

Solution Methods (Case 1): By Direct Integration

- Consider the general form of interest:

$$\frac{dy}{dt} + ky = x(t)$$

- If $k = 0$,

$$\frac{dy}{dt} = x(t)$$

- Then the equation can be solved by direct integration

$$y = \int x(t)dt + C$$

Where C is the arbitrary constant of integration

To find the value of the arbitrary constant, we need an initial condition in the form of $y(0) = y_0$ or $\frac{dy(0)}{dt} = y_1$, where y_0 and y_1 are given constants

Example 1 (By Direct Integration)

Find the solution of the differential equation, $\frac{dy}{dt} = \sin(t)$ given that $y(0) = 2$

$$y = \int \sin(t)\, dt$$

$$y = -\cos(t) + C$$

Apply initial condition: $y(0) = 2$

$$y(0) = -\cos(0) + C, 2 = -\cos(0) + C, \Rightarrow C = 2 + 1 = 3$$

Solution: $y = \cos(t) + 3$

Solution Methods (Case 2): By Separating the Variables

- Consider the general form of interest:

$$\frac{dy}{dt} + ky = x(t)$$

- If $x(t) = 0$, the resulting is called a first order **linear, constant coefficient,** "**homogenous**" differential equation

$$\frac{dy}{dt} + ky = 0$$

- The solution can be obtained by separating y and t as follows:

$\frac{dy}{dt} = -ky, \Rightarrow \frac{1}{y}\frac{dy}{dt} = -k, \Rightarrow \frac{1}{y}dy = -kdt$, Integrate both sides

$\int \frac{1}{y} dy = \int -kdt \,, \Rightarrow \ln(y) = -kt + C_1$

$y = C_2 e^{-kt}$, where C_1 and C_2 are arbitrary constants

$y = C_2 e^{-kt}$ is called the **homogenous solution or natural response** of the linear system. This is the system's response when the input signal, $x(t)$, is absent or set to zero and is due to energy stored in the system

Example 2 (By Separating Variables)

Find the solution of the differential equation, $2\frac{dy}{dt} + 5y = 0$

Write into the general form by dividing by through by 2 to get,
$\frac{dy}{dt} + 2.5y = 0$, which gives that $k = 2.5$.
The general solution we got for this form from previous slide is $y = Ce^{-kt}$.
Therefore, the general solution is $y = Ce^{-2.5t}$ where C is an arbitrary constant which can be resolved if we are given the initial condition of the dependent variable, y

Solution Methods (General Case): By Integrating Factor

- Consider the general form of interest:

$$\frac{dy}{dt} + ky = x(t)$$

- Multiply both side of the above equation by $e^{\int kdt}$, that is, e^{kt}

$$e^{kt}\frac{dy}{dt} + ke^{kt}y = e^{kt}x(t)$$

- The left hand side of the resulting equation, $e^{kt}\frac{dy}{dt} + ke^{kt}y$, can be written as $\frac{d}{dt}(e^{kt}y)$

$$\frac{d}{dt}\left(e^{kt}y\right) = e^{kt}x(t)$$

$e^{\int kdt}$, that is, e^{kt}, is the multiplying factor that simplified the task and is called an integrating factor

- Integrate both side with respect to t

$$e^{kt}y = \int e^{kt}x(t)dt \ + C_1$$

Solution Methods: By Integrating Factor

- Expression from previous slide

$$e^{kt}y = \int e^{kt}x(t)dt \; + C_1$$

Where C_1 is an arbitrary constant of integration

$$y = e^{-kt}\int e^{kt}x(t)dt \; + C_1 e^{-kt}$$

- The first term of the above equation is called the particular solution or forced response, $y_P(t) = e^{-kt}\int e^{kt}x(t)dt$ and is the response of the system due to input, $x(t)$
- The second term is called homogenous solution or natural response, $y_H(t) = C_1 e^{-kt}$, which is the same term we got through separation of variable with $x(t)$ set to 0.

$$\therefore y = y_P(t) + y_H(t)$$

Example 3 (By Integrating Factor)

Find the solution of the differential equation, $2\frac{dy}{dt} + 5y = 2e^{-2t}$

Write into the general form by dividing by through by 2 to get, $\frac{dy}{dt} + 2.5y = e^{-2t}$, which gives that $k = 2.5$, and $x(t) = e^{-2t}$
You can solve it multiplying both side by the integrating factor, $e^{\int k dt} = e^{2.5t}$, to get $e^{2.5t}\frac{dy}{dt} + 2.5e^{2.5t}y = e^{2.5t}e^{-2t}$, $\Rightarrow \frac{d}{dx}(e^{2.5t}y) = e^{2.5t}e^{-2t}$ and then integrate both side. Or just substitute $k = 2.5$, and $x(t) = e^{-2t}$ into the equation we derived in the previous slide.

The general solution we got for this form from previous slide is
$y = \underbrace{e^{-kt}\int e^{kt}x(t)dt}_{y_P(t)} + \underbrace{C_1 e^{-kt}}_{y_H(t)}$.

Example 3 Cont'd (By Integrating Factor)

Find the solution of the differential equation, $2\frac{dy}{dt} + 5y = 2e^{-2t}$

$\frac{dy}{dt} + 2.5y = e^{-2t}$, which gives that $k = 2.5$, and $x(t) = e^{-2t}$

The form of the general solution we derived previously is $y = \underbrace{e^{-kt}\int e^{kt}x(t)dt}_{y_P(t)} + \underbrace{C_1 e^{-kt}}_{y_H(t)}$.

$$y_P(t) = e^{-2.5t}\int e^{2.5t}e^{-2t}dt = e^{-2.5t}\int e^{0.5t}dt = \frac{e^{-2.5t}e^{0.5t}}{0.5} = 2e^{-2t}$$

$$\therefore y = y_P(t) + y_H(t) = 2e^{-2t} + C_1 e^{-2.5t}$$

Example 3 Cont'd (By Integrating Factor)

Find the solution of the differential equation, $2\frac{dy}{dt} + 5y = 2e^{-2t}$

($\frac{dy}{dt} + 2.5y = e^{-2t}$, which gives that $k = 2.5$, and $x(t) = e^{-2t}$)

Alternative Approach to Find Particular Solution by Method of Undertermined Coefficient

Notice that the particular solution gave us $2e^{-2t}$. In general, the particular solution for an exponential input, $x(t) = e^{-bt}$ is $y_P(t) = a_0 e^{-bt}$ where a_0 is a coefficient to be determined.

Since the particular solution is the solution due to the input, it has to satisfy the differential equation. For input, $x(t) = e^{-2t}$, the particular solution will be of the form, $y_P(t) = a_0 e^{-2t}$. Substitute into the differential equation to find a_0

$$\frac{dy_P}{dt} + 2.5y_P = e^{-2t}, \Rightarrow -2a_0 e^{-2t} + 2.5a_0 e^{-2t} = e^{-2t}, \Rightarrow 0.5a_0 = 1, \Rightarrow a_0 = 2$$

$\therefore y_P(t) = 2e^{-2t}$ as before

Some Common Forms of Particular Solution

Input Function Form	Particular Solution Form
$x(t) = b$, b is a constant	$y_P = a_0$
$x(t) = e^{-bt}$, b is a constant	$y_P = a_0 e^{-bt}$
$\cos(wt + \varphi)$ or $\sin(wt + \varphi)$, w and φ are constants	$y_P = \mathrm{a}_0 \cos(wt + \varphi) + \mathrm{a}_1 \sin(wt + \varphi)$
Polynomial input, $x(t)$, of degree n	$y_P = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$

Where a_0, a_1, a_2, to a_n are coefficients to be determine. To find the coefficients, substitute the particular solution form into the differential equation, and solve for the coefficient by matching the left and right hand side of the resulting equation

Practice Example 4

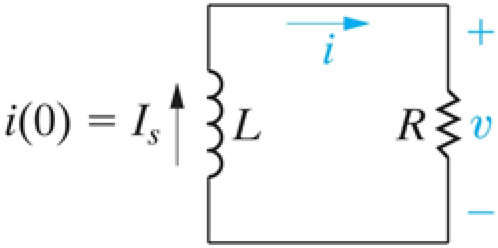
Find the solution of the differential equation, $\frac{dy}{dt} + 2y = t + 3$, with initial condition $y(0) = 2$

Practice Example 4 Cont'd

Find the solution of the differential equation, $\frac{dy}{dt} + 2y = t + 3$, with initial condition $y(0) = 2$

Introduction to First Order Circuit: Natural Response of RL Circuit

- The problem: Given an initial current $i(0) = I_s$ through inductor, L, at $t = 0$, find $i(t)$ for t ≥ 0. For now let's not worry about how the initial current got into the inductor, we will come to that later.
- Note: this is the "natural" response of this circuit because for $t \geq 0$ there is no independent source in the circuit – the response is due only to the "nature" of the components and their interconnection

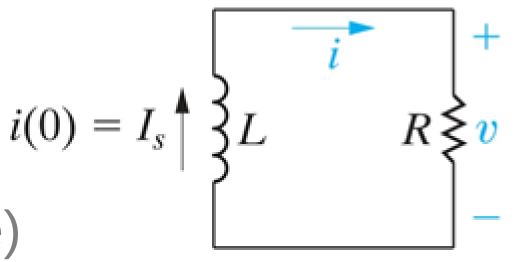


Introduction to First Order Circuit: Natural Response of RL Circuit

- Write the KVL equation for the circuit:

$$L\frac{di(t)}{dt} + Ri(t) = 0$$

$$\Rightarrow \frac{di(t)}{dt} + \frac{R}{L}i(t) = 0$$



This equation is a

- First-order (the highest derivative is a first derivative)
- Homogeneous (the right-hand side is 0)
- Ordinary differential equation
- With constant coefficients

What is the solution based on our previous derivation for first order **linear, constant coefficient,** "**homogenous**" differential equation?

Introduction to First Order Circuit: Natural Response of RL Circuit

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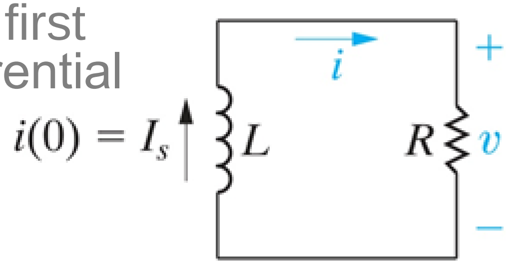

A. $i(t) = C_1 e^{-kt}$, where $k = R/L$ and C_1 is a constant
B. $i(t) = C_1 e^{kt}$, where $k = R/L$ and C_1 is a constant
C. $i(t) = C_1 e^{-kt}$, where $k = L/R$ and C_1 is a constant

What is the value of C_1? (recall that $i(0) = I_s$)
Can you try plotting $i(t)$ with MATLAB or Python for different values of I_s, R, and L? What did you notice from the plot