Geometry

Arthur Conmy[*]

Part IB, Lent Term 2021

These brief notes are based on lectures given (virtually) by Professor I. Smith in Lent term 2021. Many thanks to Artem, Kene, Daniel and David (and more) for many helpful discussions and clarifications. Credit is also due to Evan Chen for the style file for these notes[1].

I am trying to continually reorder sections, add in more discussions and cover all the course. I hope by the end of the exams this will seem more complete.

Contents

§1 Notation

I think most notation is used in the course, except perhaps the following

- $\subseteq^\circ$: open set in.

- $\mapsto$: always read as 'maps to'.

§2 Geometry from the abstract lens

This is (mostly!) a pure course, so we build up our object of study from definitions, and then eventually get to prove interesting things about those object (think IA Groups and taking these weird axioms, and following where they take us through a bunch of results). The following definition is the most important and foundational in this course.

Definition 2.1 (Surface)**.** A **surface** is a topological space Σ such that every $p \in \Sigma$ has a neigbourhood homeomorphic to $\mathbb{R}^2$.

In this course, we also impose the conditions that Σ is Hausdorff and second countable.

> **Remark 2.2.** Generalising the above to homeomorphism to $\mathbb{R}^n$ gives rise to a **manifold**, a more general object.

---

[1]Available here: https://github.com/vEnhance/dotfiles/blob/master/texmf/tex/latex/evan/evan.sty.

> **Remark 2.3.** Forgotten what Hausdorff means again? Never forget again: a topological space X is Hausdorff iff we can 'house off' every pair of points: that is to say $\forall p \neq q$, there exist *disjoint* open sets $U, V \subset X$ such that $p \in U$ and $q \in V$.

(**??**) is a *local* condition on our topological space. We will want to work with more global properties of our surfaces, so we define **atlases**.

Definition 2.4. An **atlas** for a surface Σ is a set of open sets called **charts** $\{U_i\}$ (indexed by some index set $\mathcal{I}$, say), such that

$$\bigcup_{i \in \mathcal{I}} U_i = \Sigma. \tag{1}$$

Usually, we associate with each U_i a homeomorphism $\phi_i : U_i \to V_i \subset \mathbb{R}^2$ and call (U_i, ϕ_i) a **chart**.

What's the point of this definition? We know that all $p \in \Sigma$ have local ngbds homeomorphic to $\mathbb{R}^2$, so don't we essentially already have a bunch of open sets that cover our surface? The elegance of atlases is that they allow us to describe surfaces we do not have a clean parametrisation for *with a single atlas*.

Example 2.5 (Single charts do not suffice)

In IA Vector Calculus, we commonly parametrised S^2 by

$$\sigma(u, v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix} \tag{2}$$

where $u \in U := [0, 2\pi]$ and $v \in V := [0, \pi]$. But we can't just choose $U \times V$ as our ngbd to all points on S^2; to flesh this point out: if we *tried* to make a homeomorphism $\phi : S^2 \to U \times V$ we would fail, since $v = 0$ (or π) leads to the u coordinate being arbitrary and injectivity breaking down [a].

Atlasses fix this problem in a clean way. For example can just pick $U = (0, 2\pi)$ and $V = (0, \pi)$, which then only misses out the north and south poles, and half a great circle that joins them. Another open set that's a rotation of $U \times V$ will then allow us to cover all S^2 in two charts.

---
[a]in addition, there are issues with U not being open

How might we define a global topology with respect to a bunch of possibly very different charts?

Proposition 2.6 (Global Topology on Surfaces)

Suppose we have a countable collection (U_α, ϕ_α), where each U_α is a set, and

$$\phi_\alpha : U_\alpha \to V_\alpha \subseteq^\circ \mathbb{R} \tag{3}$$

are *bijections* such that $\forall \alpha, \beta$,

$$\phi_\alpha(U_\alpha \cap U_\beta) \subseteq^\circ V_\alpha. \tag{4}$$

Define

$$\Sigma = \bigcup_\alpha U_\alpha. \tag{5}$$

Then

1. There is a topology on Σ with $U \subseteq \Sigma$ open iff $\forall \alpha$,

$$\phi_\alpha(U \cap U_\beta) \subseteq^\circ V_\alpha. \tag{6}$$

2. If we suppose that in addition to the above, $\forall \alpha, \beta$, the transition map

$$\phi_\beta \phi_\alpha^{-1} \tag{7}$$

is a homeomorphism, and that, as a technical condition to enforce the Hausdorff property,

$$\{(x, x) : x \in U_\alpha \cap U_\beta\} \subseteq U_\alpha \times U_\beta \tag{8}$$

is closed in $U_\alpha \times U_\beta$, then with this topology Σ is a topological surface.

This is a long result, so first some discussion.

The first part is clear, since we need to check the three topology axioms and they are all immediate.

The much tricker part is checking that we get a surface given the transition map condition.

What we want to do is prove that the very suggestive-looking (U_α, ϕ_α) are actually charts for the topological surface. Then this will be enough to deduce that Σ is a topological surface (provided that it's Hausdorff and second countable), since $\forall p \in \Sigma$ we can find some chart $\phi : U \to V$ that $p \in U$ and then localise to a small open disk around $f(p) \in V$ which will be homeomorphic to some open set in U (by the continuity of ϕ) and this will exactly be our local neighbourhood of p that's homeomorphic to $\mathbb{R}^2$.

To do this, we need verify the following:

> **Lemma 2.7**
>
> Let $V \subseteq^\circ V_\alpha$. Then $\phi_\alpha^{-1}(V)$ is open in Σ.
>
> *Proof.* To check this we need verify that $\phi_\beta(U_\beta \cap \phi_\alpha^{-1}(V)) \subseteq^\circ V_\beta$ (by definition of this topology).
>
> But this is
>
> $$\phi_\beta(U_\beta) \cap \phi_\beta(\phi_\alpha^{-1}(V)) \tag{9}$$
>
> which is the intersection of two open sets, since that transition map is a homeomorphism. $\qquad\square$

From here, that diagonal condition should remind you of the following Analysis and Topology example sheet question.

> **Example 2.8** (Analysis and Topology Example Sheet 2, Question 8)
>
> The diagonal of a set Y is the set $\Delta_Y = \{(x,y) \in Y \times Y : x = y\}$. Show that a topological space Y is Hausdorff if and only if Δ_Y is closed in $Y \times Y$.

In this case, our condition is subtley different, and in fact we can directly exhibit disjoint, open $U \ni p$, $V \ni q$ by splitting into the two cases of p and q lying in a common U_α (easy) and lying in U_α and U_β and using the closed condition; details skipped[4].

§2.1 Surfaces in R^3

In the following subsection, we specialise to surfaces that are subspaces of $\mathbb{R}^3$. Note that note this is not possible for all surfaces; the classic example is the Klein bottle, which self-intersects when we try and embed it into $\mathbb{R}^3$, and hence (considering the subspace topology on $\mathbb{R}^3$) at these points of intersection points do not have local neighbourhoods homeomorphic to $\mathbb{R}^3$.

Definition 2.9. **Smooth** means infinitely differentiable.

Definition 2.10. A **diffeomorphism** is a homeomorphism that is smooth, and has a smooth inverse.

Definition 2.11 (Transition map)**.** The **transition maps** between charts are intuitively defined (from a restricted domain that's a subset of V_α, $\tilde{V}$ say)

$$\phi_\beta \circ \phi_\alpha^{-1} : \tilde{V} \to V_\beta \tag{10}$$

this maps between open sets in $\mathbb{R}^2$, and has the explicit, suitably restricted domain

$$\tilde{V} = \phi_\alpha(U_\alpha \cap U_\beta) \tag{11}$$

(the notation gets dense when discussing transition maps, but all we're doing is always making sure things are well-defined).

Definition 2.12 (Smooth Surface)**.** A smooth surface in $\mathbb{R}^3$ is a surface Σ given as the union of several U_i such that each transition map is a diffeomorphism.

---

[4]hint: take the complement of that diagonal set that's defined in the condition.

> **Remark 2.13.** Note that infinite differentiability is a strong condition; we're trying to set things up so we can prove nice things about shapes in space, worrying about wild real analytic pathologies as little as we possibly can.

To study smooth surfaces in $\mathbb{R}^3$, actually we can specialise to certain restricted local parametrisations called *allowable* parametrisations:

Theorem 2.14

The following are equivalent:

- Σ is a smooth surface in $\mathbb{R}^3$.

- Σ is locally the graph of a smooth function over one of the coordinate planes.

- Σ is locally cut out by the vanishing set of a smooth function with nonzero derivative. That is, $\forall p \in \Sigma$, there is an open $U \in \mathbb{R}^3$ such that $\Sigma \cap U = f^{-1}(0)$ where $f : U \to \mathbb{R}$ is smooth and $Df|_p \neq 0$.

- Σ is locally the image of an **allowable** parametrisation, i.e. some $\Sigma : V \to \Sigma$ where $V \subset \mathbb{R}^2$ is open and $D\sigma$ has full rank throughout V.

Proof. This is a particularly heavy part of the course, so I'll only discuss the important ideas here.

Note that the third statement concerns inverting f, and the fourth statement involves a full rank statement, which *if* we were working over maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. Both of these statements should strongly suggest we should use the Inverse Function Theorem from analysis. However, our maps are those of the form $\mathbb{R}^m \to \mathbb{R}^n$ where $m \neq n$, this is harder. The submersion theorem and the implicit function theorem are the corollaries of the inverse function theorem needed to show these implications. $\qquad\square$

---

§2.2 Some tools to help us out

Theorem 2.15 (Inverse Function Theorem)

Let $f : U \to \mathbb{R}^n$ be a C^1 function on U, where $U \subseteq^{\circ} \mathbb{R}^n$. Then there are open subsets $V, W \in \mathbb{R}^n$ so that $f|_V : V \to W$ is a bijection, and $f\upharpoonright_V^{-1}$ is differentiable with derivative

$$(Df|_{f^{-1}(x)})^{-1}. \tag{12}$$

Proof. Ultimate result of IB Analysis and Topology. $\qquad\square$

> **Remark 2.16.** The derivative expression can be memorised since we can write
>
> $$f(f^{-1}(x)) = x \tag{13}$$
>
> and differentiate, using the chain rule.

To begin working with different dimensions,

Theorem 2.17 (Submersion Theorem)

Let $f : U \to V$ where $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ are open be locally differentiable at p and have surjective derivative. Then there exist coordinate systems (i.e diffeomorphisms with open sets U' and V') such that in these coordinate systems, f is the *submersion*[a]

$$F(x_1, ..., x_n) = (x_1, ..., x_m). \tag{14}$$

Proof. As with many analysis results, make convenience of life assumptions that $p = 0$ and it is the first n columns of Df that are linearly independent.

Now we basically force ourselves into the equal dimension setting: define

$$\Phi(x_1, ..., x_n) = (f(x_1, ..., x_n), x_{m+1}, ..., x_n) \tag{15}$$

which, when thought about for a while, is a locally differentiable function around 0 with derivative invertible (we can explicitly write out the derivative matrix). So applying inverse function theorem, Φ^{-1} exists. Now after thinking for a while, in fact

$$f \circ \Phi^{-1} \tag{16}$$

is exactly the submersion requested. $\qquad\square$

---
[a](ignore this waffle) the italics here are because I can't resist saying *submersion* as if I'm a death metal singer. Maybe I need to stop listening to Code Orange while revising.

Theorem 2.18 (Implicit Function Theorem)

If we have the same setup as in the submersion theorem, but $n = n' + m$ and we break vectors in $\mathbb{R}^n$ into their n'-dimensional component x and their m-dimensional component y, and the last m columns of Df are invertible locally then there exists g such that

$$f(x, y) = 0 \iff g(x) = y \tag{17}$$

Proof. We essentially already have the convenience of life assumption, meaning that applying submersion, in some coordinates $f(x, y) = \tilde{f}(y)$, where the form of $\tilde{f}$ depends on x. This means that $f(x, y) = 0$ is equivalent to $y = g(x)$, i.e given some x, locally evaluate the invertible $\tilde{f}$ and check which y vector is the zero vector in the changed coordinates. $\qquad\square$

Remark 2.19. As far as I can tell, we are mostly using these results as tools to do geometry, not as analysis facts in their own right. For that reason, the proofs have been brief, and indeed we are not careful in specifying 'smooth' versus 'C^1' and so forth. From the lecturers errata page:

> When dealing with topological surfaces, all maps are continuous. When dealing with smooth surfaces, all maps are smooth, even if I sometimes forget to say so explicitly.

We will later similarly see such lax checking that conditions hold when we use Picard-Lindelof to deduce curve and geodesic existence.

§3 Getting our hands on some $\mathbb{R}^3$ surfaces

Continuing to work with smooth surfaces in $\mathbb{R}^3$, we can now consider how to measure familiar quantities such as length, area, angle given our σ parametrization. A central tool is the *first fundamental form* ...

The first fundamental form gives a notion of dot product (equivalently, a bilinear form) of our parametrisation $\sigma(u, v)$. Specifically, given some allowable σ, the bilinear form on the tangent plane to Σ at p is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle \sigma_u, \sigma_u \rangle & \langle \sigma_u, \sigma_v \rangle \\ \langle \sigma_u, \sigma_v \rangle & \langle \sigma_v, \sigma_v \rangle \end{pmatrix}. \tag{18}$$

We more commonly write

$$E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2. \tag{19}$$

We've seen similar uses of differentials forms (i.e isolated $\mathrm{d}x$ etc. terms) before in, for example, IA Vector Calculus. Where do they come from?

§3.1 Differential Forms

This section is largely pulled from [**?**], however hopefully synthesised for an audience who has finished IB Analysis and Topology.

Recall from that course the following results:

> **Theorem 3.1** (Projecting derivatives)
>
> Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ has components $f_i : \mathbb{R}^m \to \mathbb{R}$ for $1 \leq i \leq n$. The the following are equivalent
>
> - f is differentiable at p.
>
> - Each f_i is differentiable at p,
>
> *Proof.* Good exercise (I think).   □

This is important since it basically means that to study differentiability of functions $f : \mathbb{R}^m \to \mathbb{R}^n$, we can essentially WLOG let $n = 1$.

Theorem 3.2 (Computing derivatives in n dimensions.)

Suppose that $U \subseteq^\circ \mathbb{R}^m$ and $f : U \to \mathbb{R}$ has continuous partial derivatives at $p \in U$. Then f is differentiable at p, with derivative

$$Df(p)(h) = \frac{\partial f}{\partial x_1} h_1 + \cdots + \frac{\partial f}{\partial x_m} h_m. \tag{20}$$

Proof. Decompose as a telescoping sum

$$f(x_1 + h_1, \cdots, x_m + h_m) - f(x_1, \cdots, x_m) \tag{21}$$
$$= f(x_1 + h_1, \cdots, x_m + h_m) - f(x_1 + h_1, \cdots, x_{m-1} + h_{m-1}, x_m) \tag{22}$$
$$+ f(x_1 + h_1, \cdots, x_{m-1} + h_{m-1}, x_m) - f(x_1 + h_1, \cdots, x_{m-1}, x_m) \tag{23}$$
$$+ \cdots \tag{24}$$
$$+ f(x_1 + h_1, x_2, \cdots, x_m) - f(x_1, \cdots, x_m). \tag{25}$$

And now by an 'm-ε' proof (i.e m applications of the triangle inequality), we use continuity of the m partial derivatives to deduce the desired derivative expression.

However, actually doing this is somewhat more subtle than it may seem at first (for example, to me!), since the limit $||h||_2 \to 0$ need be independent of the relative 'speeds' that each h component approach 0.

We illustrate how we do this in the case $m = 2$; the argument is easily generalised to the higher order cases.

Example 3.3

WLOG show differentiability at 0.

The quantity Q that we need show approaches 0 as $h_1 \to 0$ and $h_2 \to 0$ is

$$Q = \frac{f(h_1, h_2) - f(0, 0) - h_1 \frac{\partial f}{\partial x}|_0 - h_2 \frac{\partial f}{\partial y}|_0}{\sqrt{h_1^2 + h_2^2}} \tag{26}$$

Now by the triangle inequality, and that $\sqrt{h_1^2 + h_2^2} \geq |h_1|, |h_2|$,

$$|Q| \leq \left| \frac{f(h_1, h_2) - f(0, h_2) - h_1 \frac{\partial f}{\partial x}|_0}{h_1} \right| + \left| \frac{f(0, h_2) - f(0, 0) - h_2 \frac{\partial f}{\partial y}|_0}{h_2} \right| \tag{27}$$

as $h_2 \to 0$, clearly the latter term approaches 0 (this is easy due to the remark (**??**) made below). However, for the first term the 'different speeds' issue immediately arises. To remedy this, what we can do is localise to an open set $U \subseteq^\circ \mathbb{R}^2$ around 0 so that when $(x, y) \in U$,

$$\left| \frac{\partial f}{\partial x}|_{(x,y)} - \frac{\partial f}{\partial x}|_0 \right| < \varepsilon. \tag{28}$$

Then this means that if $(h_1, h_2) \in U$,

$$\frac{f(h_1, h_2) - f(0, h_2)}{h_1} \tag{29}$$

differs from $\frac{\partial f}{\partial x}|_0$ by at most ε, since otherwise by the *mean value theorem,* somwhere in U we would have $\frac{\partial f}{\partial x}$ differing by more than ε from $\frac{\partial f}{\partial x}|_0$.

□

We're doing geometry here, so everything is smooth and has nice derivative expressions like above (see (**??**)).

Note that we can rewrite (**??**) as

$$Df(p)(h) = \left(\frac{\partial f}{\partial x_1} e_1^* + \cdots + \frac{\partial f}{\partial x_m} e_m^* \right)(h) \tag{30}$$

where $e_1^*, \cdots, e_m^*$ is the dual basis to $e_1, \cdots, e_m$, since the derivative result is saying exactly the derivative decomposes into this projection onto all the coordinate directions.

This is the motivation for the $\mathrm{d}f$ and $\mathrm{d}x$ notation. Specifically, in the case $n = 1$, define $\mathrm{d}x = e_1^*$. Then

$$\mathrm{d}f = Df = \frac{\partial f}{\partial x} \mathrm{d}x \tag{31}$$

So for example when $f = \sin x$, $\mathrm{d}f = \cos x \mathrm{d}x$ and we give in to the temptation and write $\frac{\mathrm{d}f}{\mathrm{d}x} = \cos x$.

What does $\mathrm{d}x^2$ mean? This is simply the bilinear form that maps a pair of $\mathbb{R}^2$ vectors to the product of the projection onto the x coordinates. This motivates the notation for the fundamental forms.

§3.2 The Second Fundamental Form

The second fundamental form is another bilinear form that is motivated by the divergence of Σ from its tangent space locally near a point: by Taylor's, for h, j small,

$$\sigma(u + h, v + j) \approx \sigma(u, v) + h\sigma_u + j\sigma_v + \frac{1}{2}(\sigma_{uu}h^2 + 2\sigma_{uv}hj + \sigma_{vv}j^2) \tag{32}$$

to second order. Hence the divergence from the tangent plane $T_p\Sigma$ locally at p is (to second order)

$$[\sigma(u + h, v + j) - \sigma(u, v)].n = \frac{1}{2} \begin{pmatrix} h & j \end{pmatrix} \begin{pmatrix} n.\sigma_{uu} & n.\sigma_{uv} \\ n.\sigma_{uv} & n.\sigma_{vv} \end{pmatrix} \begin{pmatrix} h \\ j \end{pmatrix} \tag{33}$$

Definition 3.4. The second fundamental form II is defined as the bilinear form appearing above;

$$\mathrm{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} n.\sigma_{uu} & n.\sigma_{uv} \\ n.\sigma_{uv} & n.\sigma_{vv} \end{pmatrix}. \tag{34}$$

Remark 3.5. This definition of the second fundamental form is only well motivated for $II(v, v)$. What's the interpretation for $II(v, w)$?

Theorem 3.6 (Alternative characterisation of SFF)

$$SFF = -(Dn)^T D\sigma \tag{35}$$

Note that $n.\sigma_u = n.\sigma_v = 0$ so we get this from partially differentiating these with respect to u and v. $\quad\square$

Since we have an allowable parametrisation everywhere on Σ, we can define a unit normal

$$\frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||} \tag{36}$$

everywhere on Σ.

We define the **Gauss map** as exactly this map, $N : \Sigma \to S^2$. It is an intuitive result that this is independent of paramterisation.

We will now need to differentiate the Gauss map. Doing this directly leads to a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$. However, since the Gauss map has both domain and range restricted to smooth surfaces which 'feel' like 2D objects, we might hope that we can make sense of such a 'derivative', with restricted domain and range, in terms of linear maps between copies of $\mathbb{R}^2$.

In what follows I use the notation D_N rather than DN, to emphasise the subtlety of the switching between the $\mathbb{R}^2$ and $\mathbb{R}^3$ viewpoints.

Define D_N by

$$D_N : T_p\Sigma \to T_{N(p)}S^2 \tag{37}$$

such that $D\sigma(v) \mapsto D(N \circ \sigma)(v)$.

This is a linear map since $D(N \circ \sigma) = DN \circ D\sigma$.

> **Remark 3.7.** This is essentially restricting the *domain* of the Gauss map to a 2D surface.

Also consider n as the map $U \to S^2$ (defined locally on charts) such that

$$n = N \circ \sigma. \tag{38}$$

Now since $n.n = 1$ always, in fact n_u and n_v are perpendicular to n, so $Dn|_p$ (as a map) always sends things to the tangent plane at $n(p)$ on S^2, which of course coincides with the tangent plane at p on Σ.

So, having begun with the Gauss map between (subsets of) $\mathbb{R}^3$, we now have a map between (subsets of) $\mathbb{R}^2$. Also note that

$$Dn|_p = DN|_{\sigma(p)} \circ D\sigma|_p \tag{39}$$

by the chain rule.

Definition 3.8. The shape operator $\mathbb{S}$ is the linear map

$$\mathbb{S} : T_p\Sigma \to T_p\Sigma \tag{40}$$

defined by

$$D\sigma(v) \mapsto -D_N D\sigma(v). \tag{41}$$

This is linear since it inherits linearity from D_N.

Considered as a map from the tangent space (in the σ parametrization) at p to itself, we have, in the evaluation of the first fundamental form below $\mathbb{S}v = -DN|_{\sigma(p)}D\sigma|_p v = -Dn|_p v$ so[10]

---

[10]I am not entirely happy with my symbol pushing here. To elaborate, while interpreting $\mathrm{I}(v, w)$ as $(D\sigma v)^T D\sigma w$, i.e v and w contain coordinates (two numbers in $\mathbb{R}$) that $D\sigma$ turn into the familiar first fundamental form expression seems to work, but I feel the careful work we did to ensure get a 2D endomorphism is sort of lost when we do derivations like this.

Theorem 3.9

$$\mathrm{I}(\mathbb{S}v, w) = \mathrm{II}(v, w) \tag{42}$$

Follows immediately from the chain rule and characterisations of I and II given:

$$\mathrm{I}(\mathbb{S}v, w) = -v^T (D\sigma)^T (DN)^T (D\sigma) w = -v^T (Dn)^T D\sigma w = \mathrm{II}(v, w). \tag{43}$$

$\square$

We can now use these tools to build further upon the abstract surface formalisms began earlier.

Definition 3.10 (Riemannian metrics on $\mathbb{R}^2$ subsets.)**.** We can define a **Riemannian metric** on a subset $V \subset \mathbb{R}^2$ by a smooth map

$$z \mapsto \begin{pmatrix} E(z) & F(z) \\ F(z) & G(z) \end{pmatrix} \tag{44}$$

where the map is to positive definite bilinear forms (since a map to matrices would be too general).

The next definition is nothing surprising

Definition 3.11 (Riemmanian metrics on abstract smooth surfaces)**.** A **Riemannian metric** on a surface given as an atlas is a map defined for each such atlas as in (**??**) with the following compatibility condition on the transition maps $f_{\alpha\beta} := \phi_\beta \circ \phi_\alpha^{-1}$: we unsurprisingly want the first fundamental forms to agree, which amounts to needing the following to hold, and so pretending for a moment we have embedded surfaces in $\mathbb{R}^3$ with associated parametrizations σ_α and σ_β, we would want

$$(D\sigma_\alpha)^T D\sigma_\alpha = (D(\sigma_\beta \circ f))^T D(\sigma_\beta \circ f) \tag{45}$$

but of course we're being abstract, so actually we expand via the chain rule and insist

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}_\alpha = (Df)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix}_\beta Df. \tag{46}$$

Proposition 3.12 (Riemannian metric topology)

Given a Riemannian metric g, it induces a topology via the metric

$$d(p, q) = \inf_{\gamma} L(\gamma) \tag{47}$$

where the infimum is taken over the set of smooth curves with $\gamma(0) = p$, $\gamma(1) = q$. Moreover, this topology is equivalent to the natural surface topology (**??**).

Lemma 3.13 (Coarseness of Topology)

Let τ_1 and τ_2 be topologies on a set X. Then τ_2 is at least as fine as τ_1 if $\forall x \in X$, and sets $U \subseteq^{\circ} X$ containing x, there exists $U' \in \tau_2$ with $x \in U' \subseteq U$.

Proof. Just algorithmically find U' for all $x \in U$ and union these all up to find U in τ_2. $\qquad\square$

Of course, if two toplogies are as fine as each other, they're the same ('$y \leq x \leq y \implies x = y$').

So how do we show this? Sketch: the key idea is that whenever we're computing lengths, really we're integrating a bunch of $|\sigma_u|$ and $|\sigma_v|$ terms. So localise, and use compactness to deduce lower and upper bounds on the size of these quantities, which in turn leads to bounds for the Riemannian metric topology. $\qquad\square$

Remark 3.14. We've consistently 'dropped down' to $\mathbb{R}^2$ in order to work with tangent spaces in this section, which we've always had tied to an $\mathbb{R}^3$ picture of an embedded smooth surface in this course.

To understand tangent spaces for abstract surfaces, chapter 45 of [**?**] develops intuition for how this is done (and I imagine this is covered in higher math as ever).

§3.3 Gaussian Curvature

With the linear algebraic tools setup, we can now move onto to begin to appreciate that random interlude about pizza from IA Vector Calculus (but still no full proofs).

Definition 3.15 (Gaussian Curvature). Recall that the the shape operator $\mathbb{S}$ (**??**) can be considered as a linear map from the tangent space $T_p\Sigma$ to itself. Therefore we can make sense of its determinant:

The Gaussian curvature κ of a smooth surface in $\mathbb{R}^3$ is the quantity defined by

$$\kappa(p) = \det(\mathbb{S})_p. \tag{48}$$

In practice, we are really going to compute this a bit differently:

Proposition 3.16

$$\kappa = \frac{LN - M^2}{EG - F^2} \tag{49}$$

Recall from (**??**) that

$$\mathrm{I}(\mathbb{S}v, w) = \mathrm{II}(v, w). \tag{50}$$

So in fact, $\mathbb{S}\mathrm{I} = \mathrm{II}$ which implies the formula after taking determinants. $\qquad\square$

However, there's another characterisation of Gaussian curvature less tied to particular parametrizations.

Proposition 3.17

Let $(A_n)_{n \geq 1}$ be a sequence of nested open neighbourhoods of p that shrink arbitrarily small[a]. Then

$$|\kappa| = \lim_{n \to \infty} \frac{\mathrm{Area}_{S^2}(N(A_i))}{\mathrm{Area}_\Sigma(A_i)} \tag{51}$$

Proof. The key idea here is to calculate areas via cross products. We can do this since to calculate areas from σ parametrisations we integrate $\sqrt{EG - F^2}$, and this is precisely $\sigma_u \times \sigma_v$.

So (stare at it) if $\sigma(U_i) = A_i$,

$$\mathrm{Area}_\Sigma(A_i) = \int_{U_i} ||\sigma_u \times \sigma_v|| dudv. \tag{52}$$

To deal with the Gauss map area, note that $n = N \circ \sigma$ is itself a paramterisation of the surrounding region of the unit sphere we need. So apply the same formula, extracting the u and v partial derivatives via the total derivative decomposition (**??**),

$$\mathrm{Area}_{S^2}(N(A_i)) = \int_{U_i} ||n_u \times n_v|| dudv = \int_{U_i} ||DN(\sigma_u) \times DN(\sigma_v)|| dudv. \tag{53}$$

Now note that since everything here is linear (and in particular we're in 2D so even direct expansion will suffice), if $(v_1 v_2)$ is the matrix that sends e_1 to v_1 and e_2 to v_2, and $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map, then

$$\det(Lv_1 Lv_2) = \det(L)\det(v_1 v_2). \tag{54}$$

So

$$\mathrm{Area}_{S^2}(N(A_i)) = \int_{U_i} |\kappa(u, v)| ||\sigma_u \times \sigma_v|| dudv \tag{55}$$

Now by continuity, we can factor out $|\kappa|$ and get the result. $\qquad\square$

---
[a]formally, eventually they all lie in $B(p, \varepsilon)$ for all $\varepsilon > 0$.

> **Remark 3.18.** We can get the sign of kappa, too, by figuring out whether A_i and $N(A_i)$ are similarly orientated.

§3.4 Flavours of surfaces

[14]

Definition 3.19 (Global Isometry). $f : \Sigma_1 \to \Sigma_2$ is an **isometry** if it preserves the lengths of all curves.

We won't, however use this definition much.

Definition 3.20 (Local Isometry). Surfaces Σ_1 and Σ_2 are locally isometric at $p \in \Sigma_1$ and $q \in \Sigma_2$ if there exists open neighbourhoods U_1 and U_2 around the two points such that there exists an isometry between U_1 and U_2.

This is more useful, because of the following result.

Proposition 3.21 (Isometries determined by FFF)

Regions Σ_1 and Σ_2 are locally isometric at $p \in \Sigma_1$ and $q \in \Sigma_2$ iff we can parametrise them such that the parametrisations have the same FFF.

Proof. The if direction is obvious since we compute the lengths of curves from the FFF.

For the only if direction, it suffices to show that **given the lengths of curves only, we can extract the FFF from this**.

Indeed, given σ a parametrisation around p with $\sigma(0,0) = p$, then let $\gamma_\varepsilon(t) = \sigma(t,0)$ for $0 \le t \le \varepsilon$. Then

$$\frac{d}{d\varepsilon} L(\gamma_\varepsilon) = \frac{d}{d\varepsilon} \int_0^\varepsilon E(\sigma(t,0))^{1/2} dt \to E(p) \tag{56}$$

in the limit $\varepsilon \to 0$. We can use the curves $\gamma = \sigma(0,t)$ and $\gamma = \sigma(t,t)$ in a similar manner in order to extract F and G. So the isometry between the two open neighbourhoods will simply 'carry over' the FFF since it preserves the lengths of curves, and we can use the above computation (on the image surface!) in order to extract the same E, F and G on this surface. $\qquad \square$

---

[a]Please send any corrections and/or feedback to asc70@cam.ac.uk.

Definition 3.22. An *ruled* surface is a surface of the form

$$\sigma(u,v) = \gamma(u) + v\delta(u) \tag{57}$$

where σ is a smooth curve in the plane and δ is always orthogonal to γ.

Definition 3.23. A *developable* surface is a surface of the form

$$\sigma(u,v) = \gamma(u) + v\gamma'(v). \tag{58}$$

---

[14]Hmm, we need Gaussian curvature to do this section, but ideally this would have been covered as soon as FFF had been covered. Harrumph.

Theorem 3.24

Developable surfaces are locally isometric to the plane.

Proof. This is equivalent to having the same FFFs, and we can compute for a developable surface parametrised by arc length its first fundamental form

$$(1 + v^2\kappa^2)dx^2 + 2dxdy + dy^2. \tag{59}$$

This is ugly, but actually it shows that all we need to do is cookup some *plane* curve $\hat{\gamma}(t) = (x(t), y(t), 0)$ that has curvature agreeing with γ's curvature.

To satisfy this requirement, we need to find a solution to the system

$$\begin{cases} x''(t) = \kappa(t)y'(t) \\ y''(t) = -\kappa(t)x'(t) \end{cases} \tag{60}$$

that is also parametrised by arc lengthI am a bit uncertain as to what to do when locally the curvature's sign is at an inflection point, and whether the solution generated will really agree with both the positive and negative parts of the curvature.

To do this, setup the differential equation for $\begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}. \tag{61}$$

Now since locally κ will be differentiable, we can localise so that this system is in the form $y' = f(t, y)$ where f is Lipschitz in the second variable, i.e a solution exists via Picard-Lindelof.

Then we can choose arbitrary $x(0)$ and $y(0)$, rescale so that $x'(0)^2 + y'(0)^2 = 1$ and note that this curve will indeed be parametrised by arc length, since

$$\frac{d}{dt}(x'(t)^2 + y'(t)^2) = 2x'(t)x''(t) + 2y'(t)y''(t) = 0 \tag{62}$$

from the original system of equations.

$\square$

.

§3.5 A first peek at Möbius transformations

You probably remember the following from IA Groups:

Definition 3.25. A Möbius transformation $\mathcal{M}$ is a bijection from $\mathbb{C}_\infty$ to itself given by
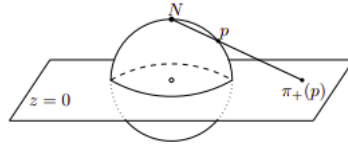
$$z \mapsto \frac{az + b}{cz + d} \tag{63}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

These turn out to have special significance to **stereographic projection**:

Definition 3.26. Stereographic projection is the map from the unit sphere S^2 to the plane $\mathbb{R}^3 \cap \{z = 0\}$ given by

$$(x, y, z) \mapsto \left(\frac{x}{1 - z}, \frac{y}{1 - z}, 0 \right) \tag{64}$$

which is interpreted as projection from the north pole N:

Figure 1: Stereographic projection (given by the map π_+).

Exercise 3.27. Check the formula works!

> **Remark 3.28.** Where possible, avoid using the formula, and use nice geometric arguments (brevity is great for clarity, and exams!). This isn't always possible, however (see next claim).

The important link between Möbius transformations and stereographic projection is the following:

Definition 3.29. The subgroup $\mathrm{PSU}(2) \leq \mathrm{PSL}(2,\mathbb{C})$ is given by matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \tag{65}$$

where $a\bar{a} + b\bar{b} = 1$.

Proposition 3.30

Rotations of S^2 (namely, the action of $\mathrm{SO}(3)$ on it) correspond 1-to-1 with Möbius transformations (of the plane) in the group $\mathrm{PSU}(2)$ of unitary matrices in $\mathrm{PSL}(2,\mathbb{C})$.

---

§4 Geodesics

This part is mostly applying our tool of the first fundamental form in the Euler-Lagrange setting.

Proposition 4.1

Lengths of curves are given by

$$L(\gamma) := \int_\gamma ||\gamma'(t)|| \mathrm{d}t = \int_\gamma \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} \mathrm{d}t. \tag{66}$$

Exercise 4.2. Show this (hint[17]).

We don't actually want to apply Euler-Lagrange to $L(\gamma)$ directly, since the square root is messy and it turns out we can avoid it, and still get equations that tell us when curves are length minimisers.

Definition 4.3 (Energy)**.** The energy functional is given by

$$E(\gamma) := \int_\gamma ||\gamma'(t)||^2 \mathrm{d}t \tag{67}$$

---

[17]write $||\gamma'(t)|| = \sqrt{\gamma'(t).\gamma'(t)}$

> **Remark 4.4.** Unsurprisingly, L is parametrisation independent (exercise ...). However, WARNING: E is not.

From here, we can apply Euler-Lagrange to study energy minimisers over curves between two points, since E is an expression in the form

$$\int f(t, u(t), u'(t), v(t), v'(t)) \mathrm{d}t. \tag{68}$$

Proposition 4.5

Doing this, we get the fairly gross geodesic equations

$$\frac{d}{dt}\left(E\dot{u} + F\dot{v}\right) = \frac{1}{2}\left(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2\right) \tag{69}$$

and

$$\frac{d}{dt}\left(F\dot{u} + G\dot{v}\right) = \frac{1}{2}\left(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2\right). \tag{70}$$

---

[a]Please send any corrections and/or feedback to asc70@cam.ac.uk.

> **Remark 4.6.** As with most Euler-Lagrange things, these need not be committed to memory; the Euler-Lagrange idea of considering small perturbations $f \mapsto \delta f$ (see any VP notes) is quick to rederive even in timed conditions.

As promised, we now show that studying geodesics, i.e length minimisers, amounts to studying E minimisers:

Proposition 4.7 (Integral Cauchy-Schwarz)

Define an inner product on continuous functions $f : [a, b] \to \mathbb{R}_{ge0}$ as follows:

$$\langle f, g \rangle = \int_a^b f(t)g(t)\mathrm{d}t. \tag{71}$$

Then

$$\langle f, g \rangle^2 \leq \langle f, f \rangle \langle g, g \rangle \tag{72}$$

With equality iff f and g are scalar multiples.

Proof. Any Cauchy-Schwarz proof generalises to this infinite dimensional setting with no issues. $\square$

Applying C-S with $||\gamma'(t)||$ and the identically 1 function,

$$L(\gamma)^2 \leq kE(\gamma) \tag{73}$$

where k is the length of the interval used to parametrise γ.

> **Proposition 4.8**
>
> Energy minimisers are length minimisers.
>
> *Proof.* Suppose that for contradiction's sake γ minimised energy but wasn't a length minimiser. Then pick some γ' that was a length minimiser, parametrised with constant speed. Then (**??**) gives us that
>
> $$E(\gamma) > \frac{1}{k}L(\gamma)^2 \geq \frac{1}{k}L(\gamma')^2 = E(\gamma') \tag{74}$$
>
> i.e γ isn't in fact an energy minimiser, contradiction. $\square$

To avoid using the messy geodesic equations, we might be able to use the following alternative characterisation:

> **Proposition 4.9**
>
> Let Σ be a smooth surface in $\mathbb{R}^3$ and $\gamma : [a, b] \to \Sigma$ be a smooth curve on Σ. Then γ is a geodesic if and only if the derivative of the tangent vector to γ is everywhere normal to Σ.
>
> *Proof.* Exercise! Hint[a]. $\square$
>
> ───────────
> [a]There are two geodesic equations, and normal to the tangent plane is equivalent to perpendicular to σ_u and σ_v ... you guess what you need to do!

§4.1 Geodesic normal form

I assume that the rigour won't be covered in exams (and it is pretty technical analysis anyway), so the following will be handwaved:

> **Proposition 4.10** (Geodesic existence)
>
> $\forall p \in \Sigma$ embedded in $\mathbb{R}^3$ and all tangent vectors v at p, there's a geodesic $\gamma : [0, \varepsilon) \to \Sigma$ locally, with $\gamma(0) = p$ and $\gamma'(0) = v$.
>
> *Proof.* Working with the gross geodesic equations, this is a question about existence of solutions to differential equations, specifically some of the form
>
> $$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} u'' \\ v'' \end{bmatrix} = ... \tag{75}$$
>
> Where the RHS is some gross (but smooth!) expression in terms of u, v, u' and v'.
> Now the first fundamental form is invertible (wth smooth inverse), so after inverting it we're indeed in the Picard-Lindelöf setting. Strengthening what we proved in Analysis and Topology, it turns out that this smooth setup means solutions are also smooth (proof omitted). Hand wave over. $\square$

Given this existence result, we now can make a coordinate system by travelling along the (locally existent) geodesic $\gamma(v)$ and then constructing for every $\gamma(v)$ a geodesic γ_v that is perpendicular to γ and is tangent to γ. This can be thought of as a bunch of orthogonal 'offshoot' branches from a common stem.
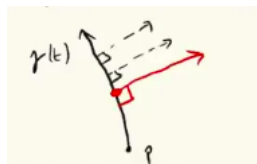   Here's the picture to have in mind:

Figure 2: γ is parametrised by t, not v, but the 'offshoot' intuition is shown.

Remark 4.11. The curves $u \mapsto \gamma_v(u)$ for fixed v are of course geodesics, but WARNING there's no guarentee that $v \mapsto \gamma_v(u)$ is a geodesic for fixed u, except when $u = 0$. This is the reason the geodesic normal form doesn't just collapse to something isometric to the plane below!

Proposition 4.12

Embedded surfaces $\Sigma \subseteq \mathbb{R}^3$ have parametrisations of the form

$$du^2 + G(u,v)dv^2 \tag{76}$$

locally.

Proof. Use (via the handwaving...) the geodesic normal coordinates above:

$$\sigma(u,v) = \gamma_v(u). \tag{77}$$

This is an allowable parametrization because at $u = 0$, $\sigma_u \perp \sigma_v$ (and both are definitely not vanishing), and hence $D\sigma$ is locally invertible, so by the inverse function theorem locally the parametrisation is invertible, so this is an allowable parametrization indeed (σ and $D\sigma$ both injective).

$E = 1$ follows from the geodesics being parametrised by arc length, but it's not obvious that F is identically zero: we have to use the geodesic equation (**??**) on a curve with v constant, i.e an offshooting geodesic. Then the geodesic equation gives that

$$\dot{F} = E_v = 0 \tag{78}$$

since we saw $E = 1$. But on the other hand by the MV chain rule

$$\dot{F} = F_u \dot{u} + F_v \dot{v} = F_u = 0 \tag{79}$$

so in fact F is u independent. When $u = 0$, $F = 0$ so $F \equiv 0$ everywhere. $\square$

---
[a]Please send any corrections and/or feedback to asc70@cam.ac.uk.

An immediate application of this is another, even simpler characterisation of Gaussian curvature

---

Proposition 4.13

Suppose Σ is parametrised by geodesic normal form. Then

$$\kappa = \frac{\sqrt{G}_{uu}}{\sqrt{G}} \tag{80}$$

Proof. Just use (**??**). $\square$

---

§5 Hyperbolic geometry

To motivate this section (and it really needs motivation!), we need some heavy machinery:

Definition 5.1 (Triangles on surfaces)**.** A **triangle** on a surface is a set of three points on that surface joined pairwise by geodesics.

Definition 5.2 (Triangulation)**.** A **triangulation** of a surface is a graph on the surface where edges are geodesics and where the only faces are triangles[20].

Definition 5.3 (Euler Characteristic)**.** The **Euler characteristic** $\chi(\Sigma)$ of a surface Σ is the invariant value

$$V - E + F \tag{81}$$

for a triangulation on that surface.

The proof of invariance in the plane can be found here: `https://en.wikipedia.org/wiki/Euler_characteristic#Proof_of_Euler's_formula`. I don't think that the fact locally surfaces are $\cong \mathbb{R}^2$ is enough to deduce this fact more generally. See II Algebraic Topology (apparently) for a better definition of χ as a topological invariant (i.e quantity preserved by homeomorphism).

Theorem 5.4 (Global Gauss-Bonnet)

Let Σ be an abstract compact smooth surface with abstract Riemannian metric g, and Gaussian curvature κ. Then

$$\int_{\Sigma} \kappa \, \mathrm{d}A = 2\pi \chi(\Sigma). \tag{82}$$

Proof. (Black-boxed). $\square$

---

[20]For the pedantic, more formal definitions can probably be found elsewhere

---

Proposition 5.5 (Donuts!)

Let Σ be a surface with *genus*[a] g embedded in $\mathbb{R}^3$. Then Σ has Euler characteristic

$$\chi(\Sigma) = 2 - 2g. \tag{83}$$

Proof. Sketch: make a triangulation for a torus as a base case, then remove two disk within triangles of two copies of the torus, stitch these together while adding three edges to keep a triangulation and note that the Euler characteristic decreases by 2.   □

---

[a]for now, work with the informal defintion 'number of holes in surface'.

What do we get with all this machinery? Well, suppose we wished to consider metrics on these surfaces that were as symmetrical as possible. Then a natural choice would be to wish for surfaces of constant curvature. By rescaling all the atlases by a factor of $k \neq 0$, namely $\phi \mapsto k\phi$ we would now be assuming Σ has curvature $\kappa \in \{-1, 0, 1\}$.

We've extensively studied the two non-negative cases here, which correspond to the plane and the sphere). However, (**??**) suggests that whenever $g > 1$, we will need to consider the $\kappa = -1$ case.

While we've seen on the example sheet the *tractoid* which has constant curvature -1, we will now see that there are much easier (2D!) so-called 'models' to work with than that surface.

Eventually, we will actually get to build some $g > 1$ surfaces, and bizarrely this will involve trousers. Stay tuned!

Proposition 5.6

Let $\Sigma = \mathfrak{h}$ the upper half plane model, namely the subset of $\mathbb{C}$ $\{z : \operatorname{Im} z > 0\}$ equipped with the following abstract Riemannian metric:

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}. \tag{84}$$

Then Σ is hyperbolic (i.e has constant curvature -1 throughout Σ).

Proof. Sadly, despite the disk model being probably the most common hyperbolic model we will work with, this comes down a clever trick:

Under the change of coordinates $x = e^{-u}\tanh(v)$ and $y = e^{-u}\operatorname{sech}(v)$ (check that these indeed parametrise in the way we need) we can explicitly calculate $\mathrm{d}x$ and $\mathrm{d}y$ via the multivariable chain rule from IA[a]. Then the metric is

$$\mathrm{d}u^2 + \cosh^2(u)\mathrm{d}v^2 \tag{85}$$

and hence the Gaussian curvature of $\mathfrak{h}$ is

$$-\frac{\sqrt{G}_{uu}}{\sqrt{G}} = -1 \tag{86}$$

@todo write the geodesic normal form section to reference the formula.

$\square$

_____

[a]I don't think that formalising this applied math hand-waving with differential forms (i.e $\mathrm{d}x$ etc.) is actually too far out of reach of this course, and it is a nice application of the multivariable calculus developed in Analysis and Topoligy. See `https://venhance.github.io/napkin/Napkin.pdf` section XII or even `http://pi.math.cornell.edu/~sjamaar/manifolds/manifold.pdf` for a complete exposition. However, I currently do not have a good intuition for why squaring these differential forms is a legitimate manipulation, and this is needed to derive the result.

Important intuition: if we're given a point to work with in a hyperbolic question, use the disk model with this point as the origin. Alternatively, if we're given a line, use the half-plane model with the line as the (positive) imaginary axis.

§5.1 Möbius transformations

At this point in the course, we return to working with inversions and Möbius transformations, this time in the hyperbolic models. It's easy to get lost in wondering why we're doing all this. To set the record straight

Definition 5.7. In $\mathbb{H}^2$, a **flag** is a triple consisting of

- A (global) geodesic γ.

- A point $p \in \gamma$.

- A particular side of γ.

We care about flags because of the following result, essentially the KEY theorem of the disk models!

> **Proposition 5.8**
>
> Let $\mathcal{F}$ be the set of isometries that act transitively on the set of flags in $\mathbb{H}^2$. Then $\mathcal{F}$ is generated by the Möbius transformations that preserve the particular model of $\mathbb{H}^2$ that we're working with, and a reflection in a geodesic.

Remark 5.9. We only need one particular reflection (e.g $z \mapsto -\bar{z}$, reflection in the imaginary axis[a]) to generate $\mathcal{F}$ due to being able to obtain all such geodesics with MTs.

---
[a]this works in either model.

Remark 5.10. If we insist such isometries preserve orientation, we in fact only get Möbius transformations, without the reflections.

Remark 5.11. In fact we have a uniqueuness statement too: we have a unique hyperbolic isometry that maps a flag to another flag.

Proof. One direction here is not too difficult: to show Möbius transformations are isometries, it is not too difficult to check that their generators are isometries.

The proof that all orientation-preserving isometries are Möbius transformations is more difficult, and the claims made should be verified! Work in the disk model. Given some orientation-preserving isometry T, we can find a Möbius transformation M such that $M \circ T$ fixes 0 and 1. It's clear that isometries must also preserve geodesics, so in fact $M \circ T$ preserves $\mathbb{R} \cap D$. In addition, isometries in this model must preserve angles, because if locally an isometry sheared space, then curve lengths would not be preserved[24]. So, once again applying geodesic preservation, we have that $i\mathbb{R} \cap D$ is preserved by $M \circ T$ (it can't be reflected since we're assuming we're in the orientation-preserving case). From this point, the isometry must be the identity since points are determined uniquely by their distance (in hyperbolic, NOT Euclidean metric) to the real and imaginary axis, along with whether they are closest to the positive or negative parts of both of these axes. I made a picture!

@todo: write orientation preserving section.   □

So what actually are the Möbius transformations that preserve our various models?

> **Proposition 5.12**
>
> In the disk model, the Möbius transformations that preserve the disk (for the last time, equivalently the orientation preserving isometries!) are those of the form
>
> $$z \mapsto \lambda \frac{z - a}{\bar{a}z - 1} \tag{87}$$
>
> where $|\lambda| = 1$ and $|a| < 1$.
>
> *Proof.* Complex analysis example sheet 1.   □

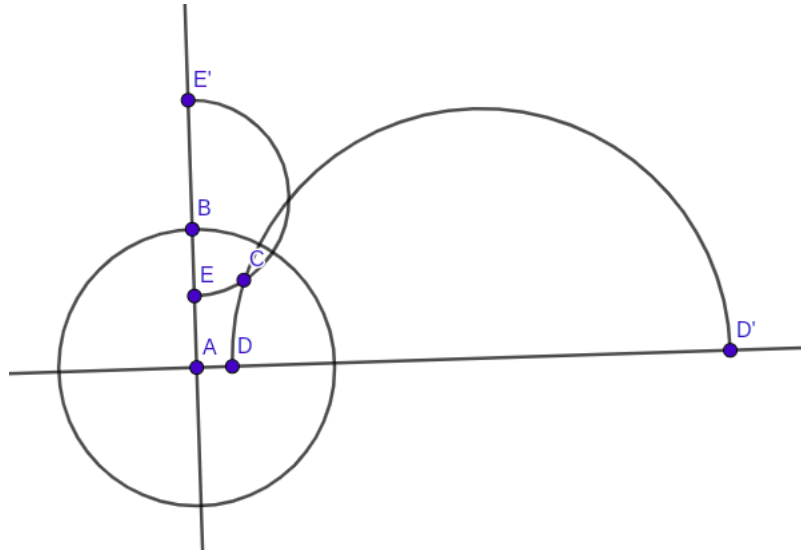[24]I would appreciate a clearer reason for this statement!

Figure 3: Given some point C in the disk, the distance of C from the real and imaginary axes are the lengths of the geodesic segments CE and CD shown. This gives rise to the fact that an isometry that fixes $\mathbb{R}$ and $i\mathbb{R}$ is in fact the identity.

Proposition 5.13

In the half-plane model, the relevant Möbius transformations are those of the form

$$z \mapsto \frac{az + b}{cz + d} \tag{88}$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$.

Proof. Sketch: this isn't actually too hard: we need to fix the real line, which quickly yields that all the coefficients need be real, and then we can just isolate the cases $c = 0$ and $c \neq 0$. In the former the result falls out quickly, and in the latter case, WLOG $c = 1$ so write out

$$\frac{az + b}{z + d} = \frac{-1}{z + d} + a \tag{89}$$

which is the composition of a bunch of two translations and also $z \mapsto -\frac{1}{z}$. This final map is an isometry since we have metric

$$\frac{|\mathrm{d}z|^2}{4|\mathrm{Im}z|^2} \tag{90}$$

and changing variables to $w = -\frac{1}{z}$ preserves this[a]. $\qquad \square$

---
[a]Note that again the differential form manipulation should make you a little uneasy...

Example 5.14 (The hyperbolic plane is *weird*)

Discuss the example sheet question I think.
    This can definitely be skipped for exams.

> **Proposition 5.15** (Pairs of pants (!))
>
> Important: the right angles lead to smooth gluing of geodesics.

§6 That crazy hard section at the end of the course.

For the last section of the course, we return to the torus as it arose from the identification space of the square in Analysis and Topology. What happens when we tweak the square?

If we 'fix' the torus we're working with, and instead consider exactly what's going on when we change the metric but not the underlying surface

Considering parallelograms with area 1, we can still identify opposite sides and get a pseudo-torus.

Surprisingly, we get *all* flat metrics on the torus this way (no proof).

Somewhat like Gauss-Bonnet, in which we tie a topological invariant (Euler characteristic) to something seemingly unrelated (global curvature) and geometrical, this ties together hyperbolic geometry and a space of metric, which apparently is something studied in higher maths than where we are at now.

References

[1] Evan Chen (2021), *An Infinitely Large Napkin*, https://venhance.github.io/napkin/Napkin.pdf.

[2] Daniel Bassett (2021), *IB Geometry*, http://db808.user.srcf.net/Geometry.pdf.

[3] David Bai (2021), *Geometry*, http://zb260.user.srcf.net/notes/IB/geom.pdf.