Some Analysis Things

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Part IB, Lent Term 2021

In these notes, I make some brief comments on the IB Analysis courses[1] Credit is due to Evan Chen for the style file for these notes[2].

§1 Integration

Theorem 1.1 (Interchanging Differentiation and Integration)
Let $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ be a cts function of θ and t, and suppose $\frac{\partial f}{\partial \theta}$ is also cts.
Then

$$\frac{d}{d\theta}\int_0^1 f(\theta,t)dt = \int_0^1 \frac{\partial f}{\partial \theta}(\theta,t)dt. \tag{1}$$

Proof. The key idea is that differentiability is a local property, so we can force the domain of $\frac{\partial f}{\partial \theta}$ to be compact.

$\forall \varepsilon > 0$, pick $\delta > 0$ so that $\delta \left|\frac{\partial f}{\partial \theta}\right| < \varepsilon$ on $[-\delta,\delta] \times [0,1]$. Then let

$$F(\theta) = \int_0^1 f(\theta,t)dt. \tag{2}$$

Then we can directly calculate (for $|h| < \delta$)

$$\frac{1}{h}(F(h) - F(0))) = \int_0^1 \frac{f(t,h) - f(t,0)}{h}dt = \int_0^1 \frac{\partial f}{\partial \theta}(t,\theta_t)dt. \tag{3}$$

where the last equality follows from the mean value theorem. θ_t is a *cts* function of t, by the continuity of $\frac{\partial f}{\partial \theta}$.

But our choice of delta means that this differs by at most ε from

$$\int_0^1 \frac{\partial f}{\partial \theta}(\theta,t)dt, \tag{4}$$

so we're done. □

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[1]strictly, just Analysis and Topology and Complex Analysis, but I hoep that the reader agrees that Analysis and Topology is ... more than one courses worth of material!

[2]Available here: https://github.com/vEnhance/dotfiles/blob/master/texmf/tex/latex/evan/evan.sty.

Theorem 1.2 (CIF for Derivatives)
Let U be a domain and $f : U \to \mathbb{C}$ holomorphic. Let $0 \in U$ and $w \in D(0,r)$. Then

$$f^{(n)}(w) = \frac{1}{n!} \oint_{\partial D(0,r)} \frac{f(z)}{(z-w)^{n+1}} dz. \tag{5}$$

Proof. Case $n = 1$ is the ordinary integral formula, and we show how the $n = 2$ arises by applying (**??**). The higher order cases arise similarly.

We can write the $n = 1$ case as

$$\oint_{\partial D(0,r)} \frac{f(z)}{z-w} dz = \int_0^1 \frac{f(\gamma(t))\gamma'(t)}{\gamma(t)-w} dt. \tag{6}$$

where $\gamma(t) = e^{2\pi i t}$. We can now directly apply the previous result, since the integrand's partial w derivative is indeed cts, provided we localise to a ball around w.

□

§2 Notation and Glossary

§2.1 Notation

§2.2 Glossary

- Cts: continuous.