Geometry

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§1 Topology

This is (mostly!) a pure course, so we build up our object of study from definitions, and then eventually get to prove interesting things about those object. The following definition is the most important and foundational in this course.

Definition 1.1 (Surface)**.** A **surface** is a topological space Σ such that every $p \in \Sigma$ has a neigbourhood homeomorphic to $\mathbb{R}^2$.

In this course, we also impose the conditions that Σ is Hausdorff and second countable.

> **Remark 1.2.** Generalising the above to homeomorphism to $\mathbb{R}^n$ gives rise to a **manifold**, a more general object.

> **Remark 1.3.** Forgotten what Hausdorff means again? Never forget again: a topological space X is Hausdorff iff we can 'house off' every pair of points: that is to say $\forall p \neq q$, there exist *disjoint* open sets $U, V \subset X$ such that $p \in U$ and $q \in V$.

(**??**) is a *local* condition on our topological space. We will want to work with more global properties of our surfaces, so we define **atlases**.

Definition 1.4. An **atlas** for a surface Σ is a set of open sets called **charts** $\{U_i\}$ (indexed by some index set $\mathcal{I}$, say), such that

$$\bigcup_{i \in \mathcal{I}} U_i = \Sigma. \tag{1}$$

Usually, we associate with each U_i a homeomorphism $\phi_i : U_i \to V_i \subset \mathbb{R}^2$.

What's the point of this definition? We know that all $p \in \Sigma$ have local ngbds homeomorphic to $\mathbb{R}^3$, so don't we essentially already have a bunch of open sets that cover our surface? The elegance of atlases is that they allow us to describe surfaces we do not have a clean parametrisation for *with a single atlas*.

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[1] Available here: https://github.com/vEnhance/dotfiles/blob/master/texmf/tex/latex/evan/evan.sty.

Example 1.5 (Single charts do not suffice)
In 1A Vector Calculus, we commonly parametrised S^2 by

$$\sigma(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix} \tag{2}$$

where $u \in U := [0, 2\pi]$ and $v \in V := [0, \pi]$. But we can't just choose $U \times V$ as our ngbd to all points on S^2; we don't get a homeomorphism $\phi : S^2 \to U \times V$ since $v = 0$ (or π) leads to the u coordinate

In the following section, we specialise to surfaces that are subspaces of $\mathbb{R}^3$. Note that note this is not possible for all surfaces; the classic example is the Klein bottle, which self-intersects when we try and embed it into $\mathbb{R}^3$, and hence (considering the subspace topology on $\mathbb{R}^3$) at these points of intersection points do not have local neighbourhoods homeomorphic to $\mathbb{R}^3$.

Definition 1.6 (Smooth Surface)**.** A smooth surface in $\mathbb{R}^3$ is

§2 Linear Algebra

The previous section shows

Definition 2.1. The shape operator $\mathbb{S}$ is the negative derivative of the Gauss map

$$\mathbb{S} = -DN|_p. \tag{3}$$

The following theorem shows that this makes sense as a linear map $T_p\Sigma \to T_p\Sigma$.

Theorem 2.2

$$I(v, w) = II(\mathbb{S}v, w) \tag{4}$$

Our bilinear forms I and II are only defined on 2D subspaces of $\mathbb{R}^3$, but in fact $\mathbb{S}$ always lies in the tangent space $T_p\Sigma$:

□

§3 Analysis

§3.1 Geodesic normal form

References

[1] *Prof.* Rajen D. Shah (2021), *Mathematics of Machine Learning*, http://www.statslab.cam.ac.uk/~rds37/teaching/machine_learning/notes.pdf.

[2] Philippe Rigollet, *18.657: Mathematics of Machine Learning*, https://ocw.mit.edu/courses/mathematics/18-657-mathematics-of-machine-learning-fall-2015/lecture-notes/MIT18_657F15_LecNote.pdf.