Applied Probability

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Part II, Lent Term 2022

These notes are my best attempt at making a course with 'applied' in the title i) exciting and ii) intuitive. Credit due to Evan Chen for the style file for these notes[1].

Proof that $\lambda > p\mu_2$ implies the number of customers is not positive recurrent: Consider the following independent Poisson clocks:

- A, a clock with rate λ.
- D_i (where $i \in \mathbb{N}$), a clock with rate $i\mu_2 p(1-p)^{i-1}$.

These together form a CTMC (X_t) on $\mathbb{Z}$, where initially the state is $+1$, when an A clock rings we transition $+1$, and when a D_i clock rings we transition -1. We now want to perform two steps:

1. Couple (X_t) with $(M_t + N_t)$ (which will be 'greater than' it).
2. Show (X_t) is not positive recurrent.

For 1., consider $X_0 = 1$, and $M_0 = 0, N_0 = 1$.

Exercise 0.1. Find a process $(X_t)_{t\geq 0}$ that has stationary but not independent increments, and a process that has independent but not stationary increments.

- Stationary but not independent: take some Poi(1) random variable, and when it fires, jump some random $U[0,1]$ distance.
- Independent but not stationary: deterministic thing which doesn't have fixed jumps e.g $0, 1, 0, 1, ...$ on intervals of length 1.

Theorem 0.2

The following are equivalent:

- (X_t) is a CTMC with generator Q.
- (X_t) satisfies the limiting transition property that
  - When $y \neq x$, $\mathbb{P}(X_{t+h} = y | X_t = x) = hq_{xy} + O(h^2)$.
  - $\mathbb{P}(X_{t+h} = x | X_t = x) = 1 - h\sum_{y\neq x} q_{xy} + O(h^2)$.

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[1] Available here: https://github.com/vEnhance/dotfiles/blob/master/texmf/tex/latex/evan/evan.sty.

§1 Reversibility

What happens when we run CTMCs in reverse? We will restrict to the case (Q, π) where we have an invariant distribution, much like we did in IB Markov Chains.

Theorem 1.1 (Reversibility)
Let the irreducible, non-explosive CTMC (X_t) with generator Q have invariant distribution π. Then fixing some constant end time $T > 0$, the random process $(\hat{X}_t) = (X_{T-t})$ for $0 \le t \le T$ is a CTMC, with invariant distribution

$$\hat{q}_{xy} = \frac{\pi_y}{\pi_x} q_{yx}. \tag{1}$$

Remark 1.2. This is intuitively the case, since the distribution of X_T will also follow the invariant distribution, and $\pi_x \hat{q}_{xy} = \pi_y q_{yx}$ encodes the fact that the reversed process will have transitions from state x to state y identical to the transitions from state y to state x in the original process.

We can bash this claim out with a lot of algebra, although building from the remark, we can see the result as a case of Bayes' theorem.

Proof. Since π is invariant, $\forall t \in [0, T]$, $\hat{X}_t$ has distribution π. Now fix such a t - we go for the second characterisation of (0.2).

For h small, and $y \neq x$, by Bayes' theorem,

$$\mathbb{P}(\hat{X}_{t+h} = y | \hat{X}_t = x) = \mathbb{P}(X_{T-t-h} = y | X_{T-t} = x) = \frac{\mathbb{P}(X_{T-t} = x | X_{T-t-h} = y)\mathbb{P}(X_{T-t-h} = y)}{\mathbb{P}(X_{T-t} = x)} \tag{2}$$

which equals $\frac{h \pi_y q_{yx}}{\pi_x}$. Therefore indeed $\hat{X}$ is a CTMC. $\square$