Geometry

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These brief notes are based on lectures given (virtually) by Professor I. Smith in Lent term 2021. Credit is also due to Evan Chen for the style file for these notes[1].

Contents

§1 Topology

This is (mostly!) a pure course, so we build up our object of study from definitions, and then eventually get to prove interesting things about those object. The following definition is the most important and foundational in this course.

Definition 1.1 (Surface)**.** A **surface** is a topological space Σ such that every $p \in \Sigma$ has a neigbourhood homeomorphic to $\mathbb{R}^2$.

In this course, we also impose the conditions that Σ is Hausdorff and second countable.

> **Remark 1.2.** Generalising the above to homeomorphism to $\mathbb{R}^n$ gives rise to a **manifold**, a more general object.

> **Remark 1.3.** Forgotten what Hausdorff means again? Never forget again: a topological space X is Hausdorff iff we can 'house off' every pair of points: that is to say $\forall p \neq q$, there exist *disjoint* open sets $U, V \subset X$ such that $p \in U$ and $q \in V$.

(1.1) is a *local* condition on our topological space. We will want to work with more global properties of our surfaces, so we define **atlases**.

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[1]Available here: https://github.com/vEnhance/dotfiles/blob/master/texmf/tex/latex/evan/evan.sty.

Definition 1.4. An **atlas** for a surface Σ is a set of open sets called **charts** $\{U_i\}$ (indexed by some index set $\mathcal{I}$, say), such that

$$\bigcup_{i\in\mathcal{I}} U_i = \Sigma. \tag{1}$$

Usually, we associate with each U_i a homeomorphism $\phi_i : U_i \to V_i \subset \mathbb{R}^2$ and call (U_i, ϕ_i) a **chart**.

What's the point of this definition? We know that all $p \in \Sigma$ have local ngbds homeomorphic to $\mathbb{R}^2$, so don't we essentially already have a bunch of open sets that cover our surface? The elegance of atlases is that they allow us to describe surfaces we do not have a clean parametrisation for *with a single atlas.*

Example 1.5 (Single charts do not suffice)
In 1A Vector Calculus, we commonly parametrised S^2 by

$$\sigma(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix} \tag{2}$$

where $u \in U := [0, 2\pi]$ and $v \in V := [0,\pi]$. But we can't just choose $U \times V$ as our ngbd to all points on S^2; to flesh this point out: if we *tried* to make a homeomorphism $\phi : S^2 \to U \times V$ we would fail, since $v = 0$ (or π) leads to the u coordinate being arbitrary and injectivity breaking down [a].

Atlasses fix this problem in a clean way. For example can just pick $U = (0, 2\pi)$ and $V = (0,\pi)$, which then only misses out the north and south poles, and half a great circle that joins them. Another open set that's a rotation of $U \times V$ will then allow us to cover all S^2 in two charts.

[a] in addition, there are issues with U not being open

In the following section, we specialise to surfaces that are subspaces of $\mathbb{R}^3$. Note that note this is not possible for all surfaces; the classic example is the Klein bottle, which self-intersects when we try and embed it into $\mathbb{R}^3$, and hence (considering the subspace topology on $\mathbb{R}^3$) at these points of intersection points do not have local neighbourhoods homeomorphic to $\mathbb{R}^3$.

Definition 1.6. **Smooth** means infinitely differentiable.

Definition 1.7. A **diffeomorphism** is a homeomorphism that is smooth, and has a smooth inverse.

Definition 1.8 (Transition map)**.** The **transition maps** between charts are intuitively defined;

$$\phi_\beta \circ \phi_\alpha^{-1} : V_\alpha \to V_\beta \tag{3}$$

maps between the open sets in $\mathbb{R}^3$, and has the suitably restricted domain

$$\phi_\alpha(U_\alpha \cap U_\beta) \tag{4}$$

(the notation gets dense when discussing transition maps, but all we're doing is always making sure things are well-defined).

Definition 1.9 (Smooth Surface)**.** A smooth surface in $\mathbb{R}^3$ is a surface Σ given as the union of several U_i such that each transition map is a diffeomorphism.

> **Remark 1.10.** Note that infinite differentiability is a strong condition; we're trying to set things up so we can prove nice things about shapes in space, worrying about wild real analytic pathologies as little as we possibly can.

To study smooth surfaces in $\mathbb{R}^3$, actually we can specialise to certain restricted local parametrisations called *allowable* parametrisations:

Theorem 1.11

The following are equivalent:

- Σ is a smooth surface in $\mathbb{R}^3$.
- Σ is locally the graph of a smooth function over one of the coordinate planes.
- Σ is locally cut out by the vanishing set of a smooth function with nonzero derivative. That is, $\forall p \in \Sigma$, there is an open $U \in \mathbb{R}^3$ such that $\Sigma \cap U = f^{-1}(0)$ where $f : U \to \mathbb{R}$ is smooth and $Df|_p \neq 0$.
- Σ is locally the image of an **allowable** parametrisation, i.e. some $\Sigma : V \to \Sigma$ where $V \subset \mathbb{R}^2$ is open and $D\sigma$ has full rank throughout V.

Proof. Some implications are easy.

The harder implications are those that we'd like to use the inverse function theorem to show, but since our maps are those of the form $\mathbb{R}^m \to \mathbb{R}^n$ where $m \neq n$, this is harder. The submersion theorem and the implicit function theorem are the corollaries of the inverse function theorem needed to show these implications.

@todo; flesh out details in implicit function theorem proof. □

[a]Please send any corrections and/or feedback to asc70@cam.ac.uk.

§2 Linear Algebra

Continuing to work with smooth surfaces in $\mathbb{R}^3$, we can now consider how to measure familiar quantities such as length, area, angle given our σ parametrization. A central tool is the *first fundamental form* ... @todo just write these formulae out and leave as exercises.

The second fundamental form is another bilinear form that is motivated by the divergence of Σ from its tangent space locally near a point: by Taylor's, for h, j small,

$$\sigma(u+h, v+j) \approx \sigma(u,v) + h\sigma_u + j\sigma_v + \frac{1}{2}(\sigma_{uu}h^2 + 2\sigma_{uv}hj + \sigma_{vv}j^2) \tag{5}$$

to second order. Hence the divergence from the tangent plane $T_p\Sigma$ locally at p is (to second order)

$$[\sigma(u+h, v+j) - \sigma(u,v)].n = \frac{1}{2}\begin{pmatrix} h & j \end{pmatrix} \begin{pmatrix} n.\sigma_{uu} & n.\sigma_{uv} \\ n.\sigma_{uv} & n.\sigma_{vv} \end{pmatrix} \begin{pmatrix} h \\ j \end{pmatrix} \tag{6}$$

So we define the second fundamental form as the bilinear form

@todo finish this and other relation for bilinear form.

Theorem 2.1 (Alternative characterisation of SFF)

$$SFF = -(Dn)^T D\sigma \tag{7}$$

□

Since we have an allowable parametrisation everywhere on Σ, we can define a unit normal

$$\frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||} \tag{8}$$

everywhere[3] on Σ. We define the **Gauss map** as exactly this map, $N : \Sigma \to S^2$. It is an intuitive result that this is independent of paramterisation.

> **Remark 2.2.** Important: keep track of dimensions. The Gauss map is a map from (a subset of) $\mathbb{R}^3$ to (a subset of) $\mathbb{R}^3$. In what follows however, we will specialise to two dimensional bilinear forms on the tangent space.

Now also consider n as the map $U \to S^2$ (defined locally on charts) such that

$$n = N \circ \sigma. \tag{9}$$

Now since $n.n = 1$ always, in fact n_u and n_v are perpendicular to n, so $Dn|_p$ (as a map) always sends things to the tangent plane at $n(p)$ on S^2, which of course coincides with the tangent plane at p on Σ (denoted $T_p\Sigma$).

So, having begun with the Gauss map between (subsets of) $\mathbb{R}^3$, we now have a map between (subsets of) $\mathbb{R}^2$. In fact, since

$$Dn|_p = DN|_{\sigma(p)} \circ D\sigma|_p \tag{10}$$

by the chain rule, defining

Definition 2.3. The shape operator $\mathbb{S}$ is the negative derivative of the Gauss map

$$\mathbb{S} = -DN|_{\sigma(p)}. \tag{11}$$

Considered as a map from the tangent space (in the σ parametrization) at p to itself, we have, in the evaluation of the first fundamental form below $\mathbb{S}v = -DN|_{\sigma(p)} D\sigma|_p v = -Dn|_p v$ so[4]

[3]Subscripts generally denote partial differentiation.

[4]I am not entirely happy with my symbol pushing here. To elaborate, I am interpreting $I(v, w)$ as $(D\sigma v)^T D\sigma w$, i.e v and w contain coordinates (two numbers in $\mathbb{R}$) that $D\sigma$ turn into the familiar first fundamental form expression.

Theorem 2.4

$$I(\mathbb{S}v, w) = II(v, w) \tag{12}$$

Follows immediately from the chain rule and characterisations of I and II given:

$$I(\mathbb{S}v, w) = -v^T(D\sigma)^T(DN)^T(D\sigma)w = -v^T(Dn)^T D\sigma w = II(v, w). \tag{13}$$

□

§3 Analysis

§3.1 Geodesic normal form

§4 Hyperbolic geometry

Given some surface Σ with *constant* curvature, we can rescale all the atlases by a factor of $k \neq 0$, namely $\phi \mapsto k\phi$ and WLOG assume that Σ has curvature $\kappa \in \{-1, 0, 1\}$. We've extensively studied the two non-negative cases here, and while we've seen on the example sheet the *tractoid* which has constant curvature -1, we will now see that there are much easier (2D!) 'models' to work with than that surface.

Proposition 4.1

Let $\Sigma = \mathfrak{h}$ the upper half plane model, equipped with the following abstract Riemannian metric:

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}. \tag{14}$$

Then Σ is hyperbolic (i.e has constant curvature -1 throughout Σ).

Proof. Sadly, despite the disk model being probably the most common hyperbolic model we will work with, this comes down a clever trick:

Under the change of coordinates $x = e^{-u}\tanh(v)$ and $y = e^{-u}\mathrm{sech}(v)$ (check that these indeed parametrise in the way we need) we can explicitly calculate $\mathrm{d}x$ and $\mathrm{d}y$ via the multivariable chain rule from IA[a]. Then the metric is

$$\mathrm{d}u^2 + \cosh^2(u)\mathrm{d}v^2 \tag{15}$$

and hence the Gaussian curvature of $\mathfrak{h}$ is

$$-\frac{\sqrt{G}_{uu}}{\sqrt{G}} = -1 \tag{16}$$

@todo write the geodesic normal form section to reference the formula. □

[a] I don't think that formalising this applied math hand-waving with differential forms (i.e $\mathrm{d}x$ etc.) is actually too far out of reach of this course, and it is a nice application of the multivariable calculus developed in Analysis and Topoligy. See https://venhance.github.io/napkin/Napkin.pdf section XII or even http://pi.math.cornell.edu/~sjamaar/manifolds/manifold.pdf for a complete exposition. However, I currently do not have a good intuition for why squaring these differential forms is a legitimate manipulation, and this is needed to derive the result.

§4.1 Möbius transformations

At this point in the course, we begin working with inversions and Möbius transformations, and while the problems approached are interesting, it's easy to get lost in wondering why we're doing all this. To set the record straight

Definition 4.2. In $\mathbb{H}^2$, a **flag** is a triple consisting of

- A (global) geodesic γ.
- A point $p \in \gamma$.
- A particular side of γ.

Proposition 4.3

Let $\mathcal{F}$ be the set of isometries that act transitively on the set of flags in $\mathbb{H}^2$. Then $\mathcal{F}$ is generated by the Möbius transformations that preserve the particular model of $\mathbb{H}^2$ that we're working with, and a reflection in a geodesic.

Remark 4.4. We only need one particular reflection (e.g $z \mapsto -\bar{z}$, reflection in the imaginary axis[a]) to generate $\mathcal{F}$ due to being able to obtain all such geodesics with MTs.

[a]this works in either model.

Remark 4.5. If we insist such isometries preserve orientation, we in fact only get Möbius transformations, without the reflections.

[b]Please send any corrections and/or feedback to asc70@cam.ac.uk.

Remark 4.6. In fact we have a uniqueness statement too: we have a unique hyperbolic isometry that maps a flag to another flag.

Proof. One direction here is not too difficult: to show Möbius transformations are isometries, it is not too difficult to check that their generators are isometries.

The proof that all orientation-preserving isometries are Möbius transformations is more difficult, and the claims made should be verified! Work in the disk model. Given some orientation-preserving isometry T, we can find a Möbius transformation M such that $M \circ T$ fixes 0 and 1. It's clear that isometries must also preserve geodesics, so in fact $M \circ T$ preserves $\mathbb{R} \cap D$. In addition, isometries in this model must preserve angles, because if locally an isometry sheared space, then curve lengths would not be preserved[7]. So, once again applying geodesic preservation, we have that $i\mathbb{R} \cap D$ is preserved by $M \circ T$ (it can't be reflected since we're assuming we're in the orientation-preserving case). From this point, the isometry must be the identity since points are determined uniquely by their distance (in hyperbolic, NOT Euclidean metric) to the real and imaginary axis, along with whether they are closest to the positive or negative parts of both of these axes. I made a picture!

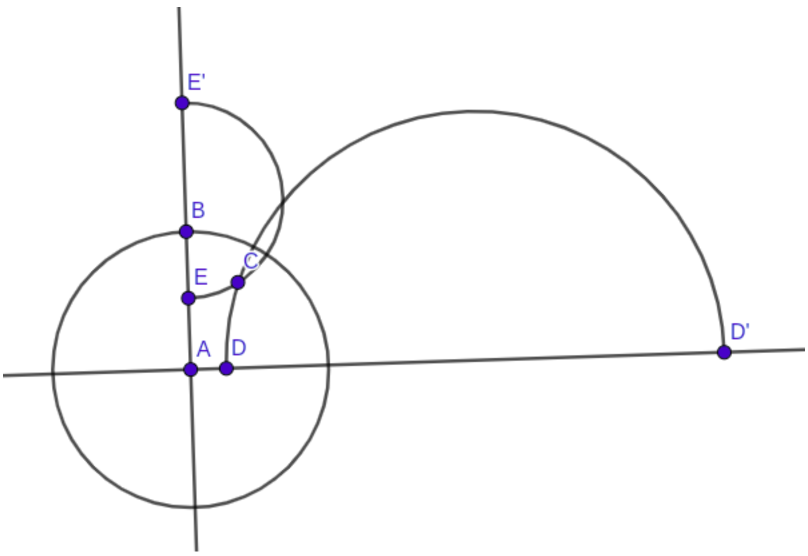

Figure 1: Given some point C in the disk, the distance of C from the real and imaginary axes are the lengths of the geodesic segments CE and CD shown. This gives rise to the fact that an isometry that fixes $\mathbb{R}$ and $i\mathbb{R}$ is infact the identity.

@todo: write orientation preserving section. □

[7]I would appreciate a clearer reason for this statement!

References

[1] Rajen D. Shah (2021), *Mathematics of Machine Learning*, http://www.statslab.cam.ac.uk/~rds37/teaching/machine_learning/notes.pdf.

[2] Philippe Rigollet, *18.657: Mathematics of Machine Learning*, https://ocw.mit.edu/courses/mathematics/18-657-mathematics-of-machine-learning-fall-2015/lecture-notes/MIT18_657F15_LecNote.pdf.