Some Analysis Things

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In these notes, I make some brief comments on the IB Analysis courses[1] Credit is due to Evan Chen for the style file for these notes[2].

§1 Integration

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[1] strictly, just Analysis and Topology and Complex Analysis, but I hoep that the reader agrees that Analysis and Topology is ... more than one courses worth of material!

[2] Available here: https://github.com/vEnhance/dotfiles/blob/master/texmf/tex/latex/evan/evan.sty.

Theorem 1.1 (Interchanging Differentiation and Integration)
Let $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ be a cts function of θ and t, and suppose $\frac{\partial f}{\partial \theta}$ is also cts.
Then

$$\frac{d}{d\theta}\int_0^1 f(\theta,t)dt = \int_0^1 \frac{\partial f}{\partial \theta}(\theta,t)dt. \tag{1}$$

Proof. The key idea is that differentiability is a local property, so we can force the domain of $\frac{\partial f}{\partial \theta}$ to be compact.

WLOG let $\theta = 0$. Now $\forall \varepsilon > 0$, pick $\delta > 0$ so that $\left|\frac{\partial f}{\partial \theta}(\theta,t) - \frac{\partial f}{\partial \theta}(0,t)\right| < \varepsilon$ for $(\theta,t) \in [-\delta,\delta] \times [0,1]$. Then let

$$F(\theta) = \int_0^1 f(\theta,t)dt. \tag{2}$$

Then we can directly calculate (for $|h| < \delta$)

$$\frac{1}{h}(F(h) - F(0))) = \int_0^1 \frac{f(h,t) - f(0,t)}{h}dt = \int_0^1 \frac{\partial f}{\partial \theta}(\theta_t,t)dt. \tag{3}$$

where the last equality follows from the mean value theorem. $\theta_t \in (-|h|,|h|)$ is a function of t[a].

But our choice of delta means that this differs by at most ε from

$$\int_0^1 \frac{\partial f}{\partial \theta}(0,t)dt, \tag{4}$$

so we're done. □

[a] Extra: can we always make it a *cts* function of t?

Theorem 1.2 (CIF for Derivatives)
Let U be a domain and $f : U \to \mathbb{C}$ holomorphic. Let $D(0,1) \subseteq U$ and $w \in D(0,1)$. Then

$$f^{(n)}(w) = \frac{1}{n!}\oint_{\partial D(0,1)} \frac{f(z)}{(z-w)^{n+1}}dz. \tag{5}$$

Proof. Case $n = 1$ is the ordinary integral formula, and we show how the $n = 2$ arises by applying (**??**). The higher order cases arise similarly.

We can write the $n = 1$ case as

$$\oint_{\partial D(0,1)} \frac{f(z)}{z-w}dz = \int_0^1 \frac{f(\gamma(t))\gamma'(t)}{\gamma(t)-w}dt. \tag{6}$$

where $\gamma(t) = e^{2\pi i t}$. We can now directly apply the previous result, since the integrand's partial w derivative is indeed cts, provided we localise to a ball around w (so that the $\frac{1}{\gamma(t)-w}$ term doesn't get very large). □

§2 Differentiation

I don't think this part of the course needs to be anywhere near as feared as it is currently.

Definition 2.1 (Norm)**.** Let V be a real vector space. A norm on V is a function $||.|| : V \to \mathbb{R}$ satisfying

- $||v|| \geq 0$, with equality iff $v = 0$.
- $||v + w|| \leq ||v|| + ||w||$.
- $||\lambda v|| = |\lambda| ||v||$.

> **Remark 2.2.** This naturally induces a topology on V by turning it into a metric space with distance function $d(v, w) = ||v - w||$.

Theorem 2.3 (Only one norm)
Let V be a finite dimensional vector space. Then all norms on V are Lipschitz equivalent.

Proof. Fix a basis $e_1, ..., e_n$ of V. Then let $||.||_2$ be the Euclidean norm, i.e

$$||\lambda_1 e_1 + .. + \lambda_n e_n||_2 = \sqrt{\lambda_1^2 + ... + \lambda_n^2}. \tag{7}$$

We show that all norms are Lipschitz equivalent to the Euclidean norm, and since Lipschitz equivalence is an equivalence relation, this will suffice.

- $\exists m > 0$ such that $||v|| \leq m||v||_2$ for all $v \in V$:

  Direct application of the triangle inequality. Let

  $$E = \max_{i=1}^{n} ||e_i|| > 0. \tag{8}$$

  Then if $v = \lambda_1 e_1 + ... + \lambda_n e_n$, and

  $$\Lambda = \max_{i=1}^{n} |\lambda_i|, \tag{9}$$

  then

  $$||v|| \leq |\lambda_1|||e_1|| + ... + |\lambda_n|||e_n|| \leq E(|\lambda_1| + |\lambda_2| + ... + |\lambda_n|) \tag{10}$$

  where the first inequality follows from the triangle inequality. So

  $$||v|| \leq nE\Lambda. \tag{11}$$

  Now $||v||_2 \geq \Lambda$ and hence $m = \frac{1}{nE}$ works (note we need Λ independence, but E dependence is fine since the former is a property of the specific v, but the latter a property of the norm).

- $\exists M > 0$ such that $||v|| \geq M||v||_2$ for all $v \in V$:

  Less obvious.

□

Why does this matter? Recall the definition of differentiablility

Definition 2.4. $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at p with derivative the linear map $T(p)(h)$ if

$$\lim_{h:||h||_2 \to 0} \frac{||f(p+h) - f(p) - T(p)(h)||_2}{||h||_2}. \tag{12}$$

But of course there's nothing special about $||.||_2$! Here's an application:

Proposition 2.5 (2019 P1L)
The matrix function $f(M) = M^{-1}$ is differentiable at I.

Proof. @todo, maybe mention the technicality with function being fundamentally local to I, linear map fundamentally local to 0. $\square$

§3 Notation and Glossary

§3.1 Notation

§3.2 Glossary

- Cts: continuous.