Student Information

Full Name : Onur Can TIRTIR
Id Number : 2099380

Answer 1

$P(i)$ is to be the truth value of the equation given in the question, when $n = i$:

1. Basis Step

   When $n = 1$,

   $$\sum_{j=1}^{1} j.(j+1)(j+2)\ldots(j+k-1) = 1.2.3\ldots k = k\,! = \frac{1.2.3\ldots k.(k-1)}{(k-1)}.$$

   Then $P(1)$ is true.

2. Inductive Step

   Say $f(j) = j.(j+1)(j+2)\ldots(j+k-1)$.

   I will try to prove $P(t+1)$ is true, assuming that $P(t)$ is true.

   $P(t)$ is true then

   $$\sum_{j=1}^{t} f(j) = \frac{t.(t+1).(t+2)\ldots(t+k)}{(k+1)}.$$

   Hence

   $$\begin{aligned}\sum_{j=1}^{t+1} f(j) &= \sum_{j=1}^{t} f(j) + (t+1)(t+2)\ldots(t+k)\\ &\overset{\text{IH}}{=} \frac{t.(t+1)(t+2)\ldots(t+k)}{(k+1)} + (t+1)(t+2)\ldots(t+k)\\ &= (t+1)(t+2)\ldots(t+k)(\frac{t}{k+1}+1)\\ &= \frac{(t+1)(t+2)\ldots(t+k)(t+k+1)}{(k+1)}\end{aligned}$$

   Also if we substitute $t+1$ in our assumption, we get again same result as below.

   $$\sum_{j=1}^{t+1} f(j) = \frac{(t+1)(t+2)\ldots(t+k)(t+k+1)}{(k+1)}. \tag{1}$$

   These two equations show that $P(t+1)$ is true under the assumption that $P(t)$ is true. Inductive step is done.

So by mathematical induction, we know that $P(t)$ is true for all positive integers t. Proof is done.

Answer 2

Let $P(n)$ be the proposition that $H_n \leq 7^n$:

1. Basis Step

   P(0) is true: n=0, $H_0 = 1 \leq 7^0$.
   P(1) is true: n=1, $H_1 = 3 \leq 7^1$.
   P(2) is true: n=2, $H_2 = 5 \leq 7^2$.
   Basis step is done.

2. Inductive Step

   Assume that for any j such that $0 \leq j \leq k$, $P(j)$ is true, which means $H^j \leq 7^j$.
   Now prove that $H^{k+1} \leq 7^{k+1}$.
   $P(k)$, $P(k-1)$ and $P(k-2)$ are true under our assumption since k, $k-1$ and $k-2$ are in the interval $[0, n]$.

$$\begin{aligned}
H_{k+1} &= 7H_k + 5H_{k-1} + 63H_{k-2} \\
&\leq 7^k + 5.7^{k-1} + 63.7^{k-2} \\
&\leq 7^k + 5.7^{k-1} + 9.7^{k-1} \\
&\leq 7^k + 14.7^{k-1} \\
&\leq 7^k + 2.7^k \\
&= 3.7^k \\
&\leq 7^{k+1}
\end{aligned}$$

   So we have proven $P(k+1)$ is also true under our assumption by showing that $H_{k+1} \leq 7^{k+1}$. This completes inductive step.

Because we have completed basis and inductive steps, we know P(n) is true for any n such that $n \geq 0$. Proof is done.

Answer 3

a)

We have sets

E :{Options to choose 4 books from alll 12 books}.
A :{Options to choose 4 books from 7 Signals And Systems books}.

We need to find the cardinality of the set $E \setminus A$. Answer is $|E| - |A| = C(12, 4) - C(7, 4) = 460$.
b)

$$E : \{\text{Options to choose 4 books from alll 12 books}\}.$$
$$A : \{\text{Options to choose 4 books from 7 Signals And Systems books}\}.$$
$$B : \{\text{Options to choose 4 books from 5 Discrete Mathematics books}\}.$$

Note: A and B are disjoint sets.
We need to find the cardinality of the set $E \setminus (A \cup B)$. Answer is $|E| - (|A| + |B|) = C(12, 4) - C(7, 4) - C(5, 4) = 455$.

Answer 4

a_n is to be the number of strings having even number of 3's with the length n.
We can generate *n-strings* by appending either a 3 or 2 to the *valid n-1-strings*.

1. We can generate an *n-string* having even number of 3's by appending 2 to an *n-1-string* having even number of 3's.
   Then we can say $a_n = a_{n-1}$.

2. We can generate an *n-string* having even number of 3's by appending 3 to an *n-1-string* **not** having even number of 3's.
   We can find *n-1-strings* **not** having even number of 3's by excluding valid strings form all possible strings with the length *n-1*. Then we can say $a_n = 2^{n-1} - a_{n-1}$.

Because all valid strings can be generated in one of these two ways, it follows that our recurrence relation is:
$a_n = a_{n-1} + 2^{n-1} - a_{n-1} = 2^{n-1}$.

Answer 5

In the equation given in the question, take the terms with a_{n-1}, a_{n-2} and a_{n-3} to the left side and get $a_n - 4a_{n-1} - a_{n-2} + 4a_{n-3} = 0$. Then we have the characteristic equation $r^3 - 4r^2 - r + 4 = 0$ if and only if $(r - 4)(r + 1)(r - 1) = 0$. Then we get $r_1 = 4$, $r_2 = -1$ and $r_3 = 1$.
Now we have $a_n = \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3$.
Substituting r_1, r_2 and r_3 to this equation we get $a_n = \alpha_1 4^n + \alpha_2(-1)^n + \alpha_3(1)^n$.

Substituting $n = 0$, $n = 1$ and $n = 2$ in that equation, we have 3 equations with 3 unknowns such that

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 &= 4 \\
4\alpha_1 - \alpha_2 + \alpha_3 &= 8 \\
16\alpha_1 + \alpha_2 + \alpha_3 &= 34
\end{aligned}$$

Subtracting first equation from third equation we get $15\alpha_1 = 30$ and hence $\alpha_1 = 2$. Adding first two equations we get $5\alpha_1 + 2\alpha_3 = 12$ and hence $\alpha_3 = 1$. Putting α_1 and α_3 in the first equation we get $\alpha_2 = 1$. Then $a_n = 2.4^n + (-1)^n + 1$.

Answer 6

By the extended binomial theorem,

$$\begin{aligned}\sum_{n=0}^{\infty} C(10,n)x^n &= C(10,0)x^0 + C(10,1)x^1 + C(10,2)x^2 + \dots C(10,10)x^{10} + \dots \\ &= (1+x)^{10}\end{aligned}$$

Then,

$$< C(10,0), C(10,1), C(10,2) \cdots > \longleftrightarrow (1+x)^{10}$$

Shift the sequence above left,

$$\begin{aligned}\frac{(1+x)^{10} - C(10,0)x^0}{x} &= \frac{C(10,0)x^0 + C(10,1)x^1 + C(10,2)x^2 + C(10,3)x^3 \cdots - C(10,0)x^0}{x} \\ &= \frac{C(10,1)x^1 + C(10,2)x^2 + C(10,3)x^3 + C(10,4)x^4 \dots}{x} \\ &= C(10,1)x^0 + C(10,2)x^1 + C(10,3)x^2 + C(10,4)x^4 \dots \\ &= \sum_{n=0}^{\infty} C(10,n+1)x^n \\ &= \sum_{n=0}^{\infty} a_n x^n\end{aligned}$$

Then,

$$\begin{aligned}\{a_n\} &= \frac{(1+x)^{10} - C(10,0)x^0}{x} \\ &= \frac{(1+x)^{10} - 1}{x}\end{aligned}$$