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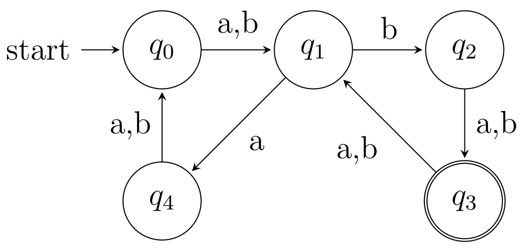
Answer 1

a.



$M = \{\{q_0, q_1, q_2, q_3, q_4\}, \{a, b\}, \delta, q_0, \{q_3\}\}$ where,

$\delta(q_0, a) = q_1$
$\delta(q_0, b) = q_1$
$\delta(q_1, a) = q_4$
$\delta(q_1, b) = q_2$
$\delta(q_2, a) = q_3$
$\delta(q_2, b) = q_3$
$\delta(q_3, a) = q_1$
$\delta(q_3, b) = q_1$
$\delta(q_4, a) = q_0$
$\delta(q_4, b) = q_0$

Applying partitioning procedure for states, initially we have two partitions, $A_1 = \{q_3\}$ and $A_2 = \{q_0, q_1, q_2, q_4\}$.
$(q_0, a, q_1 \in A_2), (q_0, b, q_1 \in A_2), (q_1, a, q_4 \in A_2), (q_1, b, q_2 \in A_2) \in M$. Hence q_0 and q_1 can be in the same partition.
$(q_1, a, q_4 \in A_2), (q_2, a, q_3 \in A_1) \in M$. Hence q_1 and q_2 cannot be in the same partition.
$(q_2, a, q_3 \in A_1), (q_4, a, q_0 \in A_2) \in M$. q_2 and q_4 they cannot be in the same partition.
$(q_0, a, q_1 \in A_2), (q_0, b, q_1 \in A_2), (q_4, a, q_0 \in A_2), (q_4, b, q_0 \in A_2) \in M$ in the automaton. Hence q_0 and q_4 can be in the same partition.

Now we have three possible partitions $A_1 = \{q_0, q_1, q_4\}$, $A_2 = \{q_2\}$ and $A_3 = \{q_3\}$.
$(q_0, b, q_1 \in A_1), (q_1, b, q_2 \in A_2) \in M$. Hence q_0 and q_2 cannot be in the same partition.
$(q_1, b, q_2 \in A_2), (q_4, b, q_0 \in A_1) \in M$. Hence q_1 and q_4 cannot be in the same partition.

$(q_0, a, q_1 \in A_1), (q_0, b, q_1 \in A_1), (q_4, a, q_0 \in A_1), (q_4, b, q_0 \in A_1) \in M$. Hence q_0 and q_1 can be in the same partition.

Now we have four possible partitions $A_1 = \{q_0, q_4\}$, $A_2 = \{q_2\}$, $A_3 = \{q_3\}$ and $A_4 = \{q_1\}$. $(q_0, b, q_1 \in A_4), (q_4, b, q_0 \in A_1) \in M$. Hence q_0 and q_2 cannot be in the same partition.

So it can be said that we already have the minimal number of states in our automaton.

b.

$$E_{q_0} = \Big(\Sigma a \Sigma\Big)^* \cup L\Sigma a \Sigma$$
$$E_{q_1} = \Sigma\Big(a\Sigma\Sigma\Big)^* \cup L\Sigma$$
$$E_{q_2} = L\Sigma b \cup \Sigma\Big(a\Sigma\Sigma\Big)^* b$$
$$E_{q_3} = L$$
$$E_{q_4} = L\Sigma a \cup \Sigma a\Big(\Sigma\Sigma a\Big)^*$$

Answer 2

a.

Say $w_n, v_n \in \{0,1\}^*$ are the string of length $n \geq 2$ and w_n includes the substring 01 only as its suffix. So $\forall w_n v_n \in L$.

Claim: For any $\forall p_1, p_2 : 2 \leq p_1 < p_2$, $\forall z \in v_{p_1}$, $\forall x \in w_{p_1}$ and $\forall y \in w_{p_2}$, while $xz \in L$, $yz \notin L$.

Proof. Basis step, $l = 3$. Take $x = 01 \in w_2$, $y \in w_3$, $z \in v_2$. For any x, y and z, $xz \in L$ but $yz \notin L$. Inductive step. Assume claim given above holds for all p values $l \geq p > 3$. Prove it also holds for $l + 1$. For any $2 \leq k < l + 1$, for any $x \in w_k$, $y \in w_{l+1}$, $z \in v_k$, while $xz \in L$ $yz \notin L$. Then we can say that $\forall p_1, p_2 : 2 \leq p_1 < p_2 \leq l + 1$, for any $x \in w_{p_1}$, $y \in w_{p_2}$, $z \in v_{p_1}$, while $xz \in L$ $yz \notin L$. Then claim holds for $l + 1$ under the assumption that it holds for l. $\square$

Since the claim is true, then we can say that *no two strings $x \in w_i, y \in w_j$ -where $i \neq j$- are equivalent under $\approx_L$*. Hence $\approx_L$ has infinitely many equivalence classes and thus by the corollary of the *Myhill-Nerode Theorem* given in the book, L is not regular.

b.

$M = \{\{s, f\}, \{0, 1\}, \{0, 1, c\}, \Delta, s, \{f\}\}$ where

$$
\begin{aligned}
\Delta = \{ & ((s, 1, c), (s, c1)), \\
& ((s, 1, 0), (s, c10)), \\
& ((s, 1, e), (s, 1)), \\
& ((s, 0, c), (s, c0)), \\
& ((s, 0, e), (s, 0)), \\
& ((s, e, c), (f, e)), \\
& ((f, 0, 1), (f, e)), \\
& ((f, 1, 0), (f, e)), \\
& ((f, 1, 1), (f, e)), \\
& ((f, 0, 0), (f, e))\}
\end{aligned}
$$

c.

$$
\begin{aligned}
(s, 10100001, e) & \vdash_M (s, 0100001, 1) \\
& \vdash_M (s, 100001, 01) \\
& \vdash_M (s, 00001, c101) \\
& \vdash_M (s, 0001, c0101) \\
& \vdash_M (f, 0001, 0101) \\
& \vdash_M (f, 001, 101) \\
& \vdash_M (f, 01, 01) \\
& \vdash_M (f, 1, 1) \\
& \vdash_M (f, e, e)
\end{aligned}
$$

w_1 is in the language.

$$
\begin{aligned}
(s, 1101, e) & \vdash_M (s, 101, 1) \\
& \vdash_M (s, 01, 11) \\
& \vdash_M (s, 1, 011) \qquad \text{Now we have two possible transitions rules} \\
& \vdash_M (s, e, c1011) \textbf{ or } \vdash_M (s, e, 1011)
\end{aligned}
$$

In either case, stack is not empty. Then w_2 is not in the language.

Answer 3

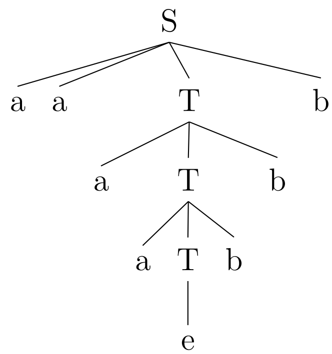
a.

The property P is "to be a string w such that $w = a^i b^j$ and $2j \geq i > j > 0$".

b.

We have only one derivation and so only one parse tree:

$$S \Longrightarrow aaTb \Longrightarrow aaaTbb \Longrightarrow aaaaTbbb \Longrightarrow aaaabbb$$



c.

i:

G is ambiguous. For example take the string $w = aaaaabbb$. We have 2 different derivations:
$S \Longrightarrow aaTb \Longrightarrow aaaTbb \Longrightarrow aaaSbb \Longrightarrow aaaaaTbbb \Longrightarrow aaaaabbb$ and
$S \Longrightarrow aaTb \Longrightarrow aaSb \Longrightarrow aaaaTbb \Longrightarrow aaaaabbb$.

Now define an unambiguous grammar $G_u = \{\{S, T, P, a, b\}, \{a, b\}, R, S\}$ for L, where

$R = \{S \to T,$
$T \to aaTb,$
$T \to aab,$
$T \to aaKb,$
$K \to P,$
$P \to aPb,$
$P \to ab\}$

Now try to prove G_u is unambiguous. Remember L $= \{w \mid w = a^i b^j$ and $2j \geq i \geq j > 0\}$.

1. *Generating unequal number of a's and b's and the case of* $i = 2j$.
   We first use the rule $S \Longrightarrow T$. After, if $i = 2$ and $j = 1$ then we directly use the rule

$T \implies aab$. If $j > 1$, then we can use the rule $T \implies aab$ only for termination of our string generating process. Before termination, we only use the rule $T \implies aaTb$ as we need. Note that in any step of string generation, we cannot use the rule $T \implies aaKb$ because if we use, we then have to use $K \implies P$ then have to use either $P \implies aPb$ or $P \implies ab$, which fails $i = 2j$. There are no other possibilities. We have only one derivation.

2. *Generating equal number of a's and b's. Not a real case, just will be used below.*
Suppose we somehow passed to P. If $i = 1$ and $j = 1$ then we directly use the rule $P \implies ab$. If $i > 1$, then we use the rule $P \implies ab$ only for termination of our string generation process. Before termination, we only use the rule $P \implies aPb$ as we need. There are no other possibilities. We have only one derivation.

3. *Case of $j < i < 2j$.* We first use the rule $S \implies T$. For such a string, we should do k_1 many $2\ a's - 1\ b$ type generations, $k_1 - 1$ of them will be done before passing to K, and k_2 many $1\ a - 1\ b$ type generations. According to G_u, in order to do $1\ a - 1\ b$ type generatings, we should first use the rule $T \implies aaKb$, then the rule $K \implies P$ to be able to use the rules $P \implies aPb$ and $P \implies ab$. Note that this provides us to put (at least one time) unequal number of $a's$ and $b's$ to make $j > i$. Notice that if we pass to P, we cannot go back to T. Hence we should first do these $k_1 - 1$ generations, they can be done in only one way as stated in (1), then use the rules $T \implies aaKb, K \implies P$ and after do those k_2 generations, again they can be done in only one way as stated in (2). So, again, we have only one derivation for this case.

Hence G_u is unambiguous.

Answer 4

$G = \{\{S, u, d, r, l\}, \{u, d, r, l\}, R, S\}$

$$
\begin{aligned}
R = \{&S \to uSd, && (1)\\
&S \to rSl, && (2)\\
&S \to SS, && (3)\\
&S \to e\} && (4)
\end{aligned}
$$

S generates strings of length of an even number, which can be deriven easily from the set R. From the rule (1) and (2), we can derive that we can do an u or r movement at first, done at an odd numbered pass. After even number of moves, it is even due to S, we immediately do a d (if an u movement was done at the beginnig) or l (if a r movement was done at the beginning) movement, done at an even numbered pass. It also provides us that whehever we go up or right of a point, after doing a set of movements included in S, we go back by a d or l movement(accordingly), so we pass even number of times from a line segment.

From the rule (4), we can derive that $e \in L$ and that whenever we satisfy the conditions i and ii given in the question, we can stop.

From the rule (3), we can derive that whenever a set of movements included in S is done, you can extend your path adding another S. This is because, in the first S, you already balanced your moves as desired in the question so you can do another set of S movements, reminding us of the **concept of generating balanced parantheses**.

Answer 5

a.

$L = \{\ w \mid w = a^n b^n,\ n \geq 0\}$

b.

$L_1 = \{\ w \mid w = a^i b^j,\ j \leq i \leq 2j,\ j > 0\}$
$L_2 = \{\ w \mid w = a^j b^i,\ j \leq i \leq 2j,\ j > 0\}$
$L = L_1 \cap L_2 = \{\ w \mid w = a^i b^i,\ i > 0\}$

c.

$L_1 = \{\ w \mid w = a^i b^i c^j,\ i, j \geq 0\}$
$L_2 = \{\ w \mid w = a^j b^i c^i,\ i, j \geq 0\}$
$L = L_1 \cap L_2 = \{\ w \mid w = a^i b^i c^i,\ i \geq 0\}$

Answer 6

a.

Not a context free language. To prove that, use *Pumping Lemma for CFL*.
Choose $w = 0^{3^K} 1^{K^2} = uvxyz \in L_1$ where $0 < |vy| \leq K$. Now assume L_1 is a context free language. Then we should able to pump v and y as many times we want such that $w' = uv^i xy^i z \in L_1$. We have different cases for v and y.

1. v and y are homogenous strings. Say we pumped v and y only once such that $w' = uv^2 xy^2 z$. In that case, $v = 0^{K-m}$ and $y = 1^m$ where $K \geq m$ and m depends on how we split w into $uvxyz$ such that v and y are homogenous.
Then $3^K + K^2 < |uv^2 xy^2 z| \leq 3^K + K^2 + K < 3^{K+1} + (K+1)^2$ so $w' \notin L_1$.

2. v and y are not homogenous strings. That means either v or y is a string of $0s$ and $1s$ and the other one is empty string, a string of $0s$ or a string of $1s$ depending on the other string's position in w. Here the point, actually, is when one of v or y is a string of $0s$ and $1s$, then pumping this string fails the order of $0s$ and $1s$ so $w' \notin L_1$.

b.

$L_2 = L_{21} \cup L_{22}$ such that: $L_{21} = \{w \in \{a,b,c\}^* : n_a(w) \leq n_b(w)\}$ and $L_{22} = \{w \in \{a,b,c\}^* : n_a(w) > n_c(w)\}$. Since L_{21} and L_{22} are context free languages, then by the closure property of the context free languages L_2 is a context free language as well. Now first provide CFGs for L_{21} and L_{22}.

$G_{21} = \{V_{21}, \{a,b,c\}, R_{21}, S_{21}\}$ where

$V_{21} = \{S_{21}, T_{21}, a, b, c\}$

$$
\begin{aligned}
R_{21} =\{ & S_{21} \rightarrow T_{21}T_{21}, \\
& T_{21} \rightarrow e, \\
& T_{21} \rightarrow cT_{21}, \\
& T_{21} \rightarrow T_{21}c, \\
& T_{21} \rightarrow T_{21}b, \\
& T_{21} \rightarrow bT_{21}, \\
& T_{21} \rightarrow bT_{21}a, \} \\
& T_{21} \rightarrow aT_{21}b, \}
\end{aligned}
$$

$G_{22} = \{V_{22}, \{a,b,c\}, R_{22}, S_{22}\}$ where

$V_{22} = \{S_{22}, T_{22}, a, b, c\}$

$$
\begin{aligned}
R_{22} =\{ & S_{22} \rightarrow T_{22}aT_{22}, \\
& T_{22} \rightarrow e, \\
& T_{22} \rightarrow bT_{22}, \\
& T_{22} \rightarrow T_{22}b, \\
& T_{22} \rightarrow T_{22}a, \\
& T_{22} \rightarrow aT_{22}, \\
& T_{22} \rightarrow cT_{22}a, \} \\
& T_{22} \rightarrow aT_{22}c, \}
\end{aligned}
$$

Hence $G_2 = \{V_{21} \cup V_{22} \cup \{S_2\}, \{a,b,c\}, R_{21} \cup R_{22} \cup \{S_2 \rightarrow S_{21}, S_2 \rightarrow S_{22}\}, S_2\}$.

c.

Not a context free language. Say $L' = \{w | w = a^{10n}b^{6n}c^{15n}, n \geq 0\}$. Observe that $L' = L_3 \cap a^*b^*c^*$. If L_3 was a CFL then L' would also be a CFL. However, since L' is not a CFL then L_3 is not a CFL. Now prove L' is not a CFL:

Proof. Proof by pumping lemma. Choose $w = a^{10k}b^{6k}c^{15k} = uvxyz \in L_3$ where $0 < |vy| \leq k$. Now assume L_3 is a context free language. Then we should able to pump v and y as many times we want such that $w' = uv^ixy^iz \in L'$. We have different cases for v and y.

First case. v and y includes all the letters a, b, c from alphabet such that either v or y includes two of the letters in an arbitraty length. In that case, as we pump the one including two letters of the alphabet fails the order of as, bs and cs so this case creates strings which do not belong to L'.

Second case. v and y do not include at least one of the letters from the alphabet such that either one or two of the letters cannot be pumped as we pump v and y. Hence the string obtained by the pumping indicated in that case results in a string, not obeying the rule $10k - 6k - 15k$, in other words $w' \notin L'$.
As a result, L' is not a CFL. $\square$