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Answer 1

(a) x is to be an *n-tuple* such that $x = (x_1, x_2, \dots x_n)$

$$\begin{aligned}
\prod_{k=1}^{n} A_k =& A_1 \times A_2 \times \cdots \times A_n \\
=& \{x \in E \mid x_1 \in A_1 \wedge x_2 \in A_2 \wedge \cdots \wedge x_n \in A_n\} \\
=& \{x \in E \mid f_1(x) \in A_1 \wedge f_2(x) \in A_2 \wedge \cdots \wedge f_n(x) \in A_n\} \\
=& \{x \in E \mid f_1(x) \in A_1\} \cap \{x \in E \mid f_2(x) \in A_2\} \cap \cdots \cap \{x \in E \mid f_n(x) \in A_n\} \\
=& f_1^{-1}(A_1) \cap f_2^{-1}(A_2) \cap \cdots \cap f_n^{-1}(A_n) \\
=& \cap_{k=1}^{n} f_k^{-1}(A_k)
\end{aligned}$$

(b) Take $n = 2$, $A_1 = \{a_1, a_2\}$ and $A_2 = \{b_1, b_2\}$
x_α and x_β are to be an *n-tuples*, take $x_\alpha = (a_1, b_2) \in E$ and $x_\beta = (a_2, b_2) \in E$.
Then $f_2(x_\alpha) = f_2(x_\beta) = b_2$, hence there exists some $x_\alpha \neq x_\beta$ such that $f(x_\alpha) = f(x_\beta)$.
Then by definiton of 1-1 function, f_2 is not 1-1.

(c) For example, every element in E_1 exists in at least one of the *n-tuples* of E because of the definition of *cartesian product*. Let $x \in E$ be an *n-tuple*. Then we can conclude;
For all x_k's where $k = 1, 2, \dots, cardinality(E_1)$; $\quad \exists x(x_k = f_1(x)$ where $x_k \in E_1 = codomain(f_1))$.
Then f_i is an onto function.

(d)

$$\begin{aligned}
\overline{f_k^{-1}(A_k)} =& \{x \in E \mid f_k(x) \notin A_k\} && \text{by the definition of complement} \\
=& \{x \in E \mid f_k(x) \in \overline{A_k}\} && \text{by the definition of complement} \\
=& f_k^{-1}(\overline{A_k})
\end{aligned}$$

(e) Depending on n, there are 2 different cases in question for *cartesian product*.

(i) when $n = 1$:
$\overline{A_1} \times \prod_{k=2}^{n} E_k = \overline{A_1} = \overline{A_1 \times \prod_{k=2}^{n} E_k}$

(ii) Say $P = \prod_{k=2}^{n} E_k$.

$$\begin{aligned}\overline{A_1} \times \prod_{k=2}^{n} E_k =& (E_1 \setminus A_1) \times (\prod_{k=2}^{n} E_k) \qquad \text{by the definition of complement}\\ =& (E_1 \setminus A_1) \times P\end{aligned}$$

Take an arbitrary $x \in E_1 \setminus A_1$ and an arbitrary y such that $y \in P = \prod_{k=2}^{n}$.

(Note: *When* $n = 2$, y *is an element of* E_2 *and in that case* (x, y) *is a pair. When* $n > 2$, y *is an* $n-1$ *tuple such that* $y = (y_2, y_3, \ldots, y_n)$ *and* $y_k \in E_k$ *for every* $k = 2, 3, \ldots, n$. *In that case* $(x, y) = (x_1, y_2, y_3, \ldots, y_n)$ *is an* $n-$ *tuple.*)

We know $y \in P$ and since $x \in E_1 \setminus A_1$ we also know $x \in E_1$ and $x \notin A_1$. Hence $(x, y) \in E_1 \times P$ and $(x, y) \notin A_1 \times P$, i.e $(x, y) \in (E_1 \times P) \setminus (A_1 \times P)$.
Then $(x, y) \in (E_1 \times E_2 \times E_3 \times \cdots \times E_n) \setminus (A_1 \times E_2 \times E_3 \times \cdots \times E_n) = E \setminus (A_1 \times E_2 \times E_3 \times \cdots \times E_n = \overline{A_1 \times \prod_{k=2}^{n} E_k})$
Then we conclude that $\overline{A_1} \times \prod_{k=2}^{n} E_k \subset \overline{A_1 \times \prod_{k=2}^{n} E_k}$

$$\begin{aligned}\overline{A_1 \times \prod_{k=2}^{n} E_k} =& E \setminus (A_1 \times \prod_{k=2}^{n} E_k) \qquad \text{by the definition of compl.}\\ =& (E_1 \times E_2 \times \cdots \times E_n) \setminus (A_1 \times E_2 \times E_3 \times \cdots \times E_n)\\ =& (E_1 \times P) \setminus (A_1 \times P)\end{aligned}$$

Take an arbitrary $(x, y) \in (E_1 \times P) \setminus (A_1 \times P)$.
Then $(x, y) \in (E_1 \times P)$ and $(x, y) \notin (A_1 \times P)$ hence $x \in E_1$ and $y \in P$.

(Note: *When* $n = 2$, y *is an element of* $P = E_2$ *and in that case* (x, y) *is a pair. When* $n > 2$, y *is an* $n-1$ *tuple such that* $y = (y_2, y_3, \ldots, y_n)$ *and* $y_k \in E_k$ *for every* $k = 2, 3, \ldots, n$. *In that case* $(x, y) = (x_1, y_2, y_3, \ldots, y_n)$ *is an* $n-$ *tuple.*)

Since $(x, y) \notin (A_1 \times P)$, either $x \notin A_1$ or $y \notin P$. But we know $y \in P$, which implies $x \notin A_1$. Hence $x \in (E_1 \setminus A_1)$ and $y \in P$ then $(x, y) \in (E_1 \setminus A_1) \times P$. Then $(x, y) \in \overline{A_1} \times P = \overline{A_1} \times \prod_{k=2}^{n} E_k$.
Then we conclude that $\overline{A_1 \times \prod_{k=2}^{n} E_k} \subset \overline{A_1} \times \prod_{k=2}^{n} E_k$.

Since both $\overline{A_1 \times \prod_{k=2}^{n} E_k}$ and $\overline{A_1} \times \prod_{k=2}^{n} E_k$ are subsets of each other, then by definition;

$\overline{A_1 \times \prod_{k=2}^{n} E_k} = \overline{A_1} \times \prod_{k=2}^{n} E_k$

Answer 2

(a)

$$\begin{aligned} &\forall x < 0, |x| = -x, \textit{ then} \quad f(x) = -2x. \\ &\forall x \geq 0, \qquad\qquad\qquad\quad f(x) = 2x + 1. \end{aligned}$$

(i) *Proving $f(x)$ is 1-1*

Taking an arbitrary pair (x_1, x_2) such that $x_1 \in dom(f(x))$ and $x_2 \in dom(f(x))$, I will consider three cases to prove $f(x)$ is 1-1.

(i) Choose an (x_1, x_2) pair such that $f(x_1) = f(x_2)$ and $x_1 < 0, x_2 < 0$.
Assuming that $x_1 \neq x_2$,
if $f(x_1) = f(x_2)$ then $-2x_1 = -2x_2$ then $x_1 = x_2$, which contradicts with assumption.

(ii) Choose an (x_1, x_2) pair such that $f(x_1) = f(x_2)$ and $x_1 \geq 0, x_2 \geq 0$.
Assuming that $x_1 \neq x_2$,
if $f(x_1) = f(x_2)$ then $2x_1 + 1 = 2x_2 + 1$ then $x_1 = x_2$, which contradicts with assumption.

(iii) Choose an (x_1, x_2) pair such that $x_1 \geq 0$ and $x_2 < 0$. We know $x_1 \neq x_2$ as $x_1 \geq 0$ and $x_2 < 0$.
Assuming that $f(x_1) = f(x_2)$ then $2x_1 + 1 = -2x_2$ where $x_1 \in \mathbb{Z}$ and $x_2 \in \mathbb{Z}$. When we divide both sides of equation, we get $x_1 = -x_2 - \frac{1}{2}$ then $x_1 \notin \mathbb{Z}$, which contradicts with the definition of $f(x)$.

So we can say that $\forall (x_1, x_2) \in dom(f(x)), f(x_1) = f(x_2) \rightarrow x_1 = x_2$. Then by definition, $f(x)$ is a 1-1 function.

(ii) *Proving $f(x)$ is onto*

Pick up an arbitrary $x_1 \in dom(f(x))$ and an arbitrary $y_1 \in codomain(f(x))$. Since $y_1 \in \mathbb{N}^+$, we have two possibilities;

(i) y_1 is an odd number. Then $y_1 = 2k + 1$ where $k \in \mathbb{Z}^+ \cup \{0\}$ hence $k \in dom(f(x))$. Then we can easily take $x_1 = k$ and it turns out that $y_1 = 2x_1 + 1 = f(x_1)$ when $x_1 \geq 0$.

(ii) y_1 is an even number. Then $y_1 = -2k$ where $k \in \mathbb{Z}^-$ hence $k \in dom(f(x))$.
Then we can easily take $x_1 = k$ and it turns out that $y_1 = -2x_1 = f(x_1)$ when $x_1 < 0$.

So we can say that $\forall y_1 \in codomain(f(x)), \exists x_1$ such that $y_1 = f(x_1)$. Then by definition, $f(x)$ is an onto function.

Since $f(x)$ is 1-1 and onto, then it has an inverse.

(b) Since $f(x)$ has an inverse, then $\exists! x_1$ such that $f(x_1) = 26$, so $f^{-1}(26) = x_1$.
Assuming that $x_1 < 0$, $f(x_1) = -2x_1 = 26$. Then $x_1 = -13 < 0$(satisfies our assumption).
Since x_1 is unique, answer is found. Then $f^{-1}(26) = x_1 = -13$.

Answer 3

$f(n) = 12n\log_2 n + 36n\log_2^2 n + 12n^2 + 36n^2\log_2 n$

Say $p(n) = 12n\log_2 n$, $r(n) = 36n\log_2^2 n$, $t(n) = 12n^2$, $q(n) = 36n^2\log_2 n$.

(i) $\left|p(n)\right| = |12n\log_2 n| \leq \left|12n^2\log_2 n\right| = 12\left|n^2\log_n\right|$,
then $p(n)$ is $O(n^2\log_2 n)$, choosing $k = 2$ and $c_1 = 12$.

(ii) $\left|r(n)\right| = \left|36n\log_2^2 n\right| = |36n\log_2 n| \,.|\log_2 n| \leq |36n\log_2 n| \,.|n| = \left|36n^2\log_2 n\right| = 36\left|n^2\log_2 n\right|$,
then $r(n)$ is $O(n^2\log_2 n)$, choosing $k = 2$ and $c_2 = 36$.

(iii) $\left|t(n)\right| = \left|12n^2\right| \leq \left|12n^2\log_2 n\right| = 12\left|n^2\log_2 n\right|$,
then $t(n)$ is $O(n^2\log_2 n)$, choosing $k = 2$ and $c_3 = 12$.

(iv) $\left|q(n)\right| = \left|36n^2\log_2 n\right| = 36\left|n^2\log_2 n\right|$,
then $q(n)$ is $O(n^2\log_2 n)$, choosing $k = 2$ and $c_4 = 36$.

Hence $\left|f(n)\right| = \left|p(n) + r(n) + t(n) + q(n)\right| \leq max(c1, c2, c3, c4)\left|g(n)\right| = C\left|g(n)\right|$, then $f(n)$ is $O(g(n))$ by definiton of *Big O*.

Answer 4

Assume that $E \setminus S$ is countable.
Also we know S is countable.
Then $S \cup E \setminus S = E$ is also countable by *the theorem proven below*, which contradicts with the premise *"E is uncountable"*.
Then our assumption is false hence $E \setminus S$ is uncountable.

Theorem 1. *If A and B are countable sets then $A \cup B$ is also countable.*

Proof. We have three cases for the sets A and B.

(a) Assume that A and B are finite, then $A \cup B$ is also finite hence $A \cup B$ is countable.

(b) Assume that A is countably infinite and B is finite.
Since A is countably infinite then its elements can be listed in an infinite sequence such that $a_1, a_2, a_3, \ldots, a_m, \ldots$.

Since B is finite then its elements can be listed in a finite sequence such that $b_1, b_2, b_3, \ldots, b_n$. Then we can show the elements of $A \cup B$ as $b_1, b_2, b_3, \ldots, b_n, a_1, a_2, a_3, \ldots, a_m, \ldots$, which means $A \cup B$ is countable.

(c) Assume that A and B are countably infinite.
Since A is countably infinite then its elements can be listed in an infinite sequence such that $a_1, a_2, a_3, \ldots, a_m, \ldots$.
Since B is countably infinite then its elements can be listed in an infinite sequence such that $b_1, b_2, b_3, \ldots, b_n, \ldots$.
Then we can show the elements of the $A \cup B$, again, in an infinite sequence such that $a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_n, b_n, \ldots$, which means $A \cup B$ is countable.

As we see, if the sets A and B are countable then $A \cup B$ is also countable, regardless of whether one or two of the sets is infinite or not.

$\square$

Answer 5

(a) If $n \equiv 1(mod 3)$, then
$n^2 \equiv n.n \equiv 1.1 \equiv 1(mod3)$ then
$n(n+1) \equiv n^2 + n \equiv 1 + 1 \equiv 2(mod3)$

Otherwise(i.e $n \equiv 2(mod3)$ or $n \equiv 3(mod3)$):

If $n \equiv 2(mod3)$, then
$n^2 \equiv n.n \equiv 2.2 \equiv 4 \equiv 1(mod3)$ then
$n(n+1) \equiv n^2 + n \equiv 1 + 2 \equiv 0(mod3)$

If $n \equiv 0(mod3)$, then
$n^2 \equiv n.n \equiv 0.0 \equiv 0(mod3)$ then
$n(n+1) \equiv n^2 + n \equiv 0 + 0 \equiv 0(mod3)$

(b) $gcd(123, 277) = gcd(277, 123) = gcd(123, 31) = gcd(31, 30) = gcd(30, 1) = gcd(1, 0) = 0$

(c) Implication can be converted to the compound logic statement $r \rightarrow q$,
where *r: p is an even prime greater than 2, q: p is greater than* $2^{100} + 1$.

Since the only even prime number is 2 and $p \neq 2$ then r is false.
Then the implication $r \rightarrow q$ is true, regardless of q is true or not.