

Magnetic Interactions of Wigner Crystal in Magnetic Field and Berry Curvature: Multi-Particle Tunneling through Complex Trajectories

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We study how a weak perpendicular magnetic field B and a Berry curvature Ω modify the magnetic interactions of a two-dimensional Wigner crystal (WC), using the semi-classical large- r_s expansion. When only a magnetic field is present, various ring-exchange interactions arise from electron tunneling along *complex* trajectories, which constitute *complex instanton* solutions of the coordinate-space path integral. To leading order in B , each ring-exchange constant acquires an Aharonov-Bohm (AB) phase equal to the magnetic flux enclosed by the real tunneling trajectory of the $B = 0$ problem. This effect is directly relevant to two-dimensional electron systems with a small g -factor ($g \ll 1$). In the presence of a Berry curvature, multi-particle tunneling must be considered in a (complexified) phase space (\mathbf{r}, \mathbf{k}) . To leading order in Ω , the exchange constants acquire the Berry phase enclosed by a *purely imaginary* trajectory in a momentum space. Finally, when both B and Ω are non-zero, in addition to having the AB and Berry phase factors, the magnitude of the exchange constant can also be renormalized by an effective-mass correction. These effects may be relevant for the WC and its proximate phases recently observed in tetra- and penta-layer rhombohedral graphene.

The two-dimensional electron gas (2DEG), a system of Coulomb-interacting electrons confined in two dimensions,

$$H_0 = \sum_i \frac{\mathbf{p}_i^2}{2m} + \sum_{i < j} \frac{e^2}{4\pi\epsilon} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (1)$$

is described by a dimensionless parameter characterizing the ratio of the interaction energy to the kinetic energy

$$r_s = \frac{e^2/8\pi\epsilon r_0}{\hbar^2/2mr_0^2} = r_0/a_B. \quad (2)$$

$r_0 = 1/\sqrt{\pi n}$ is the average interparticle distance (for an electron density n) and $a_B = 4\pi\epsilon\hbar^2/me^2$ is the effective Bohr radius. The 2DEG forms a Fermi liquid (FL) in the weak-coupling regime ($r_s \ll 1$) and a Wigner crystal (WC) in the strong-coupling regime $r_s \gg 1$ [1–4]. [5]

The magnetism within each such phase of the 2DEG is an interesting and subtle problem that has attracted long-standing attention [2–4, 6, 7]. In particular, in the WC phase, the magnetism is determined by various multi-spin ring-exchange processes [8–15] (See Ref. [16] for the most recent investigation). The effective spin Hamiltonian is written as a sum over various ring-exchange interactions:

$$\begin{aligned} H_{\text{eff}} &= - \sum_a (-1)^{P_a} J_a \hat{P}_a \\ &= \sum_{\text{---}} (J_2 \hat{P}_2 + J_2^* \hat{P}_2^{-1}) - \sum_{\triangle} (J_3 \hat{P}_3 + J_3^* \hat{P}_3^{-1}) \\ &+ \sum_{\text{---}} (J_4 \hat{P}_4 + J_4^* \hat{P}_4^{-1}) - \sum_{\text{---}} (J_5 \hat{P}_5 + J_5^* \hat{P}_5^{-1}) \\ &+ \sum_{\text{---}} (J_6 \hat{P}_6 + J_6^* \hat{P}_6^{-1}) + \sum_{\text{---}} (J'_6 \hat{P}_6 + J'^*_6 \hat{P}_6^{-1}) + \dots, \end{aligned} \quad (3)$$

where P_a refers to an n_a -particle exchange that permutes spins in each polygon in (3) in a counter-clockwise direction and $(-1)^{P_a} = (-1)^{n_a+1}$ is the sign of the permutation. \hat{P}_a is the permutation operator for P_a , which could in turn be expressed in terms of \mathbf{S}_i as in (25–28). J_a is the exchange constant for P_a that is in general complex when the magnetic field (B) or Berry curvature (Ω) is present (J_a^* is the complex conjugate). When $B = \Omega = 0$, the semi-classical large- r_s approximation gives [11–13, 16]

$$J_a(B = 0, \Omega = 0) \equiv J_a^{(0)} = \frac{e^2}{4\pi\epsilon a_B} \frac{A_a^{(0)}}{r_s^{5/4}} \sqrt{\frac{\tilde{S}_a^{(0)}}{2\pi}} e^{-\sqrt{r_s} \tilde{S}_a^{(0)}}, \quad (4)$$

where $\tilde{S}_a^{(0)}$ and $A_a^{(0)}$ are, respectively, the (normalized) action and the fluctuation determinant shown in Table I.

How does the presence of a uniform out-of-plane magnetic field B and a Berry curvature Ω affect these ring-exchange interactions? We address this question in the 2DEG system in the presence of a uniform B and/or Ω , where the Berry curvature is thought to be derived from some parent band [17]. Throughout this paper, we consider a system with a small effective g factor ($g \ll 1$) and ignore the Zeeman splitting. We derive the formula for the exchange constants J_a using the asymptotic large- r_s expansion (i.e., deep in the WC phase) for three cases: (i) $B \neq 0$ and $\Omega = 0$; (ii) $B = 0$ and $\Omega \neq 0$; and (iii) $B \neq 0$ and $\Omega \neq 0$. We summarize our main results as follows.

(i) When only the (uniform) magnetic field is present, we solve the multi-particle tunneling problem in the large- r_s limit at a fixed “Landau-level filling factor”

$$\nu \equiv n \cdot 2\pi l_B^2 = 2(l_B/r_0)^2 = O(1), \quad (5)$$

where $l_B \equiv \sqrt{\hbar/eB}$ is the magnetic length. We assume ν to be of order 1 and use $(\nu\sqrt{r_s})^{-1} \ll 1$ as a small expansion parameter [18]. To leading order, the exchange

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Process	$\tilde{S}_a^{(0)}$	$A_a^{(0)}$	$\tilde{\Sigma}_a^{(r)}$	$\tilde{\Sigma}_a^{(k)}$
J_2	1.63	1.28	1.43	-1.69
J_3	1.53	1.18	3.03	-0.73
J_4	1.65	1.32	4.62	-0.57
J_5	1.90	1.72	6.22	-0.53
J_6	1.79	1.62	10.87	-0.29
J'_6	2.11	2.48	7.79	-0.48

TABLE I. $\tilde{S}_a^{(0)}$: Dimensionless action; $A_a^{(0)}$: fluctuation determinant; $\tilde{S}_a^{(0)}$ and $A_a^{(0)}$ enter (4); $\tilde{\Sigma}_a^{(r)}$: dimensionless area enclosed by the real-space trajectory $\tilde{\mathbf{r}}^{(0)}(\tau)$ (8); $\tilde{\Sigma}_a^{(k)}$: dimensionless area enclosed by (imaginary) momentum-space trajectory $\tilde{\mathbf{k}}^{(0)}(\tau)$ (13). These values are obtained from semi-classical trajectories—solutions to (50)—by discretizing the imaginary time into $N_{\text{time}} = 50$ slices, with $N_{\text{move}} = 30, 40, \dots, 90$ electrons allowed to move during the tunneling process, and then extrapolating to $N_{\text{move}} \rightarrow \infty$. The details on how the semi-classical trajectory is obtained can be found in Ref. [16].

constants in (3) are modified from their zero-field value by an Aharonov-Bohm phase factor $\phi_a(B)$ [19]:

$$J_a^{(0)} \rightarrow J_a(B) = J_a^{(0)} e^{i\phi_a(B)}, \quad (6)$$

$$\phi_a(B) = -\frac{eB}{\hbar} \Sigma_a^{(r)} = -\frac{2}{\nu} \tilde{\Sigma}_a^{(r)}, \quad (7)$$

$$\Sigma_a^{(r)} \equiv \sum_i \frac{1}{2} \int \mathbf{r}_{i,a}^{(0)} \times d\mathbf{r}_{i,a}^{(0)} \equiv r_0^2 \cdot \tilde{\Sigma}_a^{(r)}. \quad (8)$$

Here, $\Sigma_a^{(r)}$ is the area enclosed by the position-space tunneling trajectory $\mathbf{r}_a^{(0)}(\tau) \equiv (\mathbf{r}_{1,a}^{(0)}(\tau), \dots, \mathbf{r}_{N,a}^{(0)}(\tau))$ in the absence of a magnetic field and is positive for counter-clockwise motion. $\tilde{\Sigma}_a^{(r)}$ is the dimensionless area whose value (shown in Table I) is independent of r_s and ν .

(ii) In the case where only the (uniform) Berry curvature is present, we solve the problem at large r_s for a fixed

$$\alpha \equiv r_s \nu \Omega = \frac{2\Omega}{a_B^2 r_s} = O(1), \quad (9)$$

which is assumed to be of order 1. Here we defined the ‘‘Chern filling factor,’’

$$\nu_\Omega \equiv \Omega \cdot 2\pi n = \text{sgn}(\Omega) \cdot 2(l_\Omega/r_0)^2 = O(1/r_s), \quad (10)$$

with the Berry length scale $l_\Omega \equiv \sqrt{|\Omega|}$. To leading order in $1/r_s$, J_a acquires a Berry phase $\gamma_a(\Omega)$ enclosed by the momentum-space tunneling trajectory $\mathbf{k}^{(0)}(\tau)$ at $\Omega = 0$:

$$J_a^{(0)} \rightarrow J_a(\Omega) = J_a^{(0)} e^{i\gamma_a(\Omega)}, \quad (11)$$

$$\gamma_a(\Omega) = \Omega \cdot \Sigma_a^{(k)} = \frac{\alpha}{2} \tilde{\Sigma}_a^{(k)}, \quad (12)$$

$$\begin{aligned} \Sigma_a^{(k)} &\equiv \sum_i \frac{1}{2} \int \mathbf{k}_{i,a}^{(0)} \times d\mathbf{k}_{i,a}^{(0)} \\ &= -\sum_i \frac{1}{2} \int \mathcal{I}m \mathbf{k}_{i,a}^{(0)} \times d\mathcal{I}m \mathbf{k}_{i,a}^{(0)} \equiv \frac{1}{a_B r_0} \tilde{\Sigma}_a^{(k)}. \end{aligned} \quad (13)$$

Interestingly, since the momentum-space trajectory is purely imaginary at leading order, $\mathbf{k}^{(0)} = i\mathcal{I}m \mathbf{k}^{(0)}$, $\tilde{\Sigma}_a^{(k)} < 0$ is *negative* for counter-clockwise motion in a position space. $\tilde{\Sigma}_a^{(k)}$ is again independent of r_s and ν_Ω (Table I).

(iii) Finally, when both B and Ω are finite, the Aharonov-Bohm and Berry phases contribute additively

$$J_a^{(0)} \rightarrow J_a(B, \Omega) = |J_a(B, \Omega)| e^{i[\phi_a(B) + \gamma_a(\Omega)]} \quad (14)$$

with a possible modification of the magnitude of the exchange constant. This arises because the orbital magnetic moment that couples to the magnetic field

$$\Delta H = -M_z(\mathbf{k})B \quad (15)$$

can renormalize the effective band mass,

$$m \rightarrow m^* = \left(\frac{1}{m} - \frac{B}{\hbar^2} \frac{d^2 M_z}{dk^2} \bigg|_{\mathbf{k}=\mathbf{Q}_0} \right)^{-1}, \quad (16)$$

which in turn renormalizes $r_s \rightarrow r_s^*$ (2) and $a_B \rightarrow a_B^*$, and hence the tunneling rate

$$J_a^{(0)} \rightarrow |J_a(B, \Omega)| = \frac{e^2}{4\pi\epsilon a_B^*} \cdot \frac{A_a^{(0)}}{r_s^{*5/4}} \sqrt{\frac{\tilde{S}_a^{(0)}}{2\pi}} e^{-\sqrt{r_s^*} \tilde{S}_a^{(0)}}. \quad (17)$$

Here, \mathbf{Q}_0 is the crystal momentum at the band bottom (which is assumed to be unique) and the effective mass tensor is taken to be isotropic for simplicity.

Effects of a magnetic field—We first derive the exchange interactions of the WC in the presence of a magnetic field,

$$H = \sum_i \frac{[\mathbf{p}_i + e\mathbf{A}(\mathbf{r}_i)]^2}{2m} + \sum_{i < j} \frac{e^2}{4\pi\epsilon} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (18)$$

where $\mathbf{A}(\mathbf{r}) = B(-y/2, x/2)$, closely following the analogous derivation for $B = 0$ (see e.g., Refs. [12, 16]). For a moment, let us consider a single tunneling process P_a that cyclically permutes n_a electrons at WC sites \mathbf{R}_{i_k} (with spins σ_k) to $\mathbf{R}_{i_{k+1}}$ ($i_{n_a+1} \equiv i_1$)

$$(\mathbf{R}_{i_k}, \sigma_k) \xrightarrow{P_a} (\mathbf{R}_{i_{k+1}}, \sigma_k), \quad k = 1, \dots, n_a. \quad (19)$$

The effective Hamiltonian matrix element describing this process is expressed as

$$\langle P_a \sigma | h_a | \sigma \rangle = -J_a(B), \quad (20)$$

where $\sigma \equiv (\sigma_1, \sigma_2, \dots, \sigma_N)$ is a spin configuration whose i th component σ_i is the spin at a WC site \mathbf{R}_i and

$$P_a \sigma \equiv (\sigma_{P_a^{-1}(1)}, \sigma_{P_a^{-1}(2)}, \dots, \sigma_{P_a^{-1}(N)}) \quad (21)$$

is the permuted configuration.

$$|\sigma\rangle \equiv |(\mathbf{R}_i, \sigma_i)\rangle \quad (22)$$

is a shorthand for a state where the site \mathbf{R}_i is occupied by an electron with a spin σ_i . In terms of operators [20],

$$h_a = -J_a \sum_{\{\sigma_i\}} f_{i_2, \sigma_1}^\dagger f_{i_3, \sigma_2}^\dagger \cdots f_{i_1, \sigma_{n_a}}^\dagger f_{i_{n_a}, \sigma_{n_a}} \cdots f_{i_2, \sigma_2} f_{i_1, \sigma_1} \\ = -(-1)^{P_a} J_a \hat{P}_a, \quad (23)$$

$$\hat{P}_a \equiv \sum_{\{\sigma_i\}} f_{i_{n_a}, \sigma_{n_a-1}}^\dagger f_{i_{n_a}, \sigma_{n_a}} \cdots f_{i_1, \sigma_{n_a}}^\dagger f_{i_1, \sigma_1} \\ = \hat{P}_{i_1 i_2} \hat{P}_{i_2 i_3} \cdots \hat{P}_{i_{n_a-1} i_{n_a}}, \quad (24)$$

where $f_{i, \sigma}^\dagger$ and $f_{i, \sigma}$ are, respectively, the electron creation and annihilation operator at a WC site \mathbf{R}_i with a spin σ . The sign factor $(-1)^{P_a}$ in (23) comes from reordering the fermionic operators. \hat{P}_a can in turn be expressed in terms of spin operators as

$$\hat{P}_{ij} = 2 \mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{2}, \quad (25)$$

$$\hat{P}_{ijk} = \mathbf{S}_i \cdot \mathbf{S}_j + \mathbf{S}_j \cdot \mathbf{S}_k + \mathbf{S}_k \cdot \mathbf{S}_i - 2i\chi_{ijk} + \frac{1}{4}, \quad (26)$$

$$\chi_{ijk} \equiv \mathbf{S}_i \cdot \mathbf{S}_j \times \mathbf{S}_k, \quad (27)$$

$$\hat{P}_{ijkl} = \frac{1}{2} (\mathbf{S}_i \cdot \mathbf{S}_j + \cdots) - i (\chi_{ijk} + \chi_{jkl} + \chi_{kli} + \chi_{lij}) \\ + 2 h_{ijkl} + \frac{1}{8}, \quad (28)$$

$$h_{ijkl} \equiv (\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{S}_k \cdot \mathbf{S}_l) + (j \leftrightarrow l) - (j \leftrightarrow k), \quad (29)$$

etc. Here, the first term in \hat{P}_{ijkl} involves the sum over all Heisenberg couplings between any two spins in i, j, k, l and χ_{ijk} is the spin-chirality operator. Summing over all h_a (23), we recover the effective spin Hamiltonian (3).

The objective now is to obtain the expression for $J_a(B)$ in the $r_s \rightarrow \infty$ limit analogously as in the $B = 0$ case [16]. This is done by first obtaining the asymptotic behavior of the off-diagonal and diagonal propagators at an intermediate time scale

$$1/E_{\text{Debye}} \ll \Delta\tau \ll 1/|J_a|, \quad (30)$$

where $E_{\text{Debye}} \equiv e^2/(4\pi\epsilon a_B r_s^{3/2})$ sets the zero-point energy scale (per particle) of the WC. By projecting to the Hilbert space spanned by classical ground-state manifold and expanding the exponential in Taylor series [21],

$$\langle \sigma | e^{-\Delta\tau H} | \sigma \rangle \sim |\psi(\mathbf{R})|^2 e^{-\beta E_0} \quad (31)$$

$$\langle P_a \sigma | e^{-\Delta\tau H} | \sigma \rangle \sim |\psi(\mathbf{R})|^2 e^{-\beta E_0} \Delta\tau J_a, \quad (32)$$

where $|\psi(\mathbf{R})|^2 = |\psi(P_a \mathbf{R})|^2$ is the probability density of electrons occupying the WC sites \mathbf{R} and $E_0 = cNE_{\text{Debye}}$ is the total zero-point energy of the system [c is some $O(1)$ number]. By taking the ratio of the two, we obtain

$$J_a \sim \Delta\tau^{-1} \frac{\langle P_a \sigma | e^{-\Delta\tau H} | \sigma \rangle}{\langle \sigma | e^{-\Delta\tau H} | \sigma \rangle}. \quad (33)$$

The propagators on the right-hand side can in turn be represented as a path integral,

$$\langle P \sigma | e^{-\beta H} | \sigma \rangle = \delta_{\sigma, P\sigma} \int_{\tilde{\mathbf{r}}(0)=\tilde{\mathbf{R}}}^{\tilde{\mathbf{r}}(\tilde{\beta})=P\tilde{\mathbf{R}}} D\tilde{\mathbf{r}}(\tilde{\tau}) e^{-\sqrt{r_s} \tilde{S}}, \quad (34)$$

$$\tilde{S}[\tilde{\mathbf{r}}(\tilde{\tau})] = \int_0^{\tilde{\beta}} d\tilde{\tau} \left[\sum_i \left\{ \frac{\dot{\tilde{\mathbf{r}}}_i^2}{2} + i \frac{2}{\nu \sqrt{r_s}} \tilde{x}_i \dot{\tilde{y}}_i \right\} + V(\tilde{\mathbf{r}}) - \tilde{V}_0 \right], \quad (35)$$

$$V(\tilde{\mathbf{r}}) \equiv \sum_{i < j} \frac{1}{|\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j|}. \quad (36)$$

Here, the action is rescaled to make the r_s dependence manifest by introducing the dimensionless coordinates, $\tilde{\mathbf{r}} \equiv \mathbf{r}/r_0$, and the dimensionless imaginary time $\tilde{\tau} = \tau E_{\text{Debye}}$. The corresponding dimensionless inverse temperature is $\tilde{\beta} \equiv \beta E_{\text{Debye}}$. The $\delta_{\sigma, P\sigma}$ factor in the first line appears because each electron's spin is conserved under the dynamics of (18). $(P\mathbf{R})_i \equiv \mathbf{R}_{P(i)}$. The dot indicates a derivative with respect to $\tilde{\tau}$, e.g., $\dot{\tilde{x}}_i \equiv d\tilde{x}_i/d\tilde{\tau}$. ν is the Landau-level filling factor (5). The minimum (classical) Coulomb energy $\tilde{V}_0 = \min_{\tilde{\mathbf{r}}} V(\tilde{\mathbf{r}})$ corresponding to the classical WC configuration is subtracted for convenience. The long-range Coulomb interaction V is calculated using the standard Ewald method [22].

At large r_s , the dominant paths in the path integral (34) are given by small oscillations around the classical WC configurations together with instanton events that connect one WC configuration to another via multi-particle exchange processes. Since $1/E_{\text{Debye}}$ is the typical size of instantons in imaginary time and $1/|J_a|$ is the average distance between them, at most one instanton event can occur for the intermediate time scale $\Delta\tau$ (30). Therefore,

$$J_a \sim \Delta\tau^{-1} \frac{\langle P_a \sigma | e^{-\Delta\tau H} | \sigma \rangle |_{1\text{-inst}}}{\langle \sigma | e^{-\Delta\tau H} | \sigma \rangle |_{0\text{-inst}}}. \quad (37)$$

The denominator is evaluated by considering small oscillation contributions around a WC configuration, $\tilde{\mathbf{r}}(\tilde{\tau}) = \tilde{\mathbf{R}} + \delta\tilde{\mathbf{r}}(\tilde{\tau})$. Noting that $S[\tilde{\mathbf{R}}] = 0$,

$$\langle \mathbf{R} | e^{-\Delta\tau H} | \mathbf{R} \rangle |_{0\text{-inst}} \sim \left[\det(\sqrt{r_s} \mathbf{M}[\tilde{\mathbf{R}}]) \right]^{-1/2} \quad (38)$$

$$\equiv \int_{\delta\tilde{\mathbf{r}}(0)=0}^{\delta\tilde{\mathbf{r}}(\Delta\tilde{\tau})=0} D\delta\tilde{\mathbf{r}}(\tilde{\tau}) \exp \left(-\sqrt{r_s} \int d\tilde{\tau} \frac{1}{2} \delta\tilde{\mathbf{r}}(\tilde{\tau})^T \mathbf{M}[\tilde{\mathbf{R}}] \delta\tilde{\mathbf{r}}(\tilde{\tau}) \right),$$

$$M_{ij}[\tilde{\mathbf{R}}] \equiv \frac{\delta^2 \tilde{S}}{\delta\tilde{r}_i(\tilde{\tau}) \delta\tilde{r}_j(\tilde{\tau})} \bigg|_{\tilde{\mathbf{r}}(\tilde{\tau})=\tilde{\mathbf{R}}} \quad (39)$$

$$= -\delta_{ij} \frac{\partial^2}{\partial\tilde{\tau}^2} + \frac{2i}{\nu\sqrt{r_s}} \mathcal{E}_{ij} \frac{\partial}{\partial\tilde{\tau}} + \frac{\partial^2 V}{\partial\tilde{r}_i \partial\tilde{r}_j} [\tilde{\mathbf{R}}],$$

$$\mathcal{E} \equiv \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \ddots & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}. \quad (40)$$

The numerator of (37) is obtained from the semi-classical path $\tilde{\mathbf{r}}_a(\tilde{\tau})$ connecting $\tilde{\mathbf{R}}$ to $P\tilde{\mathbf{R}}$, together with harmonic fluctuations about it. Here $\tilde{\mathbf{r}}_a(\tilde{\tau})$ is the solution to the saddle-point equation of motion of the action (35),

$$\frac{d^2 \tilde{x}_i}{d\tilde{\tau}^2} - i \frac{2}{\nu \sqrt{r_s}} \frac{d\tilde{y}_i}{d\tilde{\tau}} - \partial_{\tilde{x}_i} V[\tilde{\mathbf{r}}(\tilde{\tau})] = 0, \quad (41)$$

$$\frac{d^2 \tilde{y}_i}{d\tilde{\tau}^2} + i \frac{2}{\nu \sqrt{r_s}} \frac{d\tilde{x}_i}{d\tilde{\tau}} - \partial_{\tilde{y}_i} V[\tilde{\mathbf{r}}(\tilde{\tau})] = 0, \quad (42)$$

satisfying the boundary condition

$$\tilde{\mathbf{r}}_a(0) = \tilde{\mathbf{R}} \equiv \mathbf{R}/r_0 \text{ and } \tilde{\mathbf{r}}_a(\Delta\tilde{\tau}) = P_a \tilde{\mathbf{R}} \equiv P_a \mathbf{R}/r_0. \quad (43)$$

Note that, when $B \neq 0$ (i.e., $1/\nu \neq 0$), no real solution of the above instanton equations exists, forcing us to look for a complex solution. This amounts to a *complex instanton* saddle of the path integral. Defining $\tilde{S}_a \equiv \tilde{S}[\tilde{\mathbf{r}}_a(\tilde{\tau})]$,

$$\begin{aligned} \langle P_a \boldsymbol{\sigma} | e^{-\Delta\tau H} | \boldsymbol{\sigma} \rangle_{|a,1\text{-inst}} &\sim e^{-\sqrt{r_s} \tilde{S}_a} \times \\ &\int_{\delta\tilde{\mathbf{r}}(0)=\tilde{\mathbf{R}}}^{\delta\tilde{\mathbf{r}}(\Delta\tilde{\tau})=P\tilde{\mathbf{R}}} D\delta\tilde{\mathbf{r}}(\tilde{\tau}) \exp \left(-\frac{\sqrt{r_s}}{2} \int_0^{\tilde{\tau}} d\tilde{\tau} \delta\tilde{\mathbf{r}}(\tilde{\tau})^T \mathbf{M}[\tilde{\mathbf{r}}_a(\tilde{\tau})] \delta\tilde{\mathbf{r}}(\tilde{\tau}) \right) \\ &= e^{-\sqrt{r_s} \tilde{S}_a} [\det(\sqrt{r_s} \mathbf{M}[\tilde{\mathbf{r}}_a(\tilde{\tau})])]^{-1/2}. \end{aligned} \quad (44)$$

However, this expression diverges due to a zero mode of $\mathbf{M}[\tilde{\mathbf{r}}_a(\tilde{\tau})]$ corresponding to the translation of instanton in an imaginary time: $\mathbf{M}[\tilde{\mathbf{r}}_a(\tilde{\tau})] \partial_{\tilde{\tau}} \tilde{\mathbf{r}}_a(\tilde{\tau}) = 0$. After properly regularizing this divergence [16, 23, 24],

$$\begin{aligned} \langle P_a \mathbf{r}_0 | e^{-\Delta\tau H} | \mathbf{r}_0 \rangle_{|a,1\text{-inst}} & \quad (45) \\ &= \Delta\tau \frac{e^2}{4\pi\epsilon a_B r_s^{3/2}} \sqrt{\frac{\tilde{S}_a}{2\pi}} e^{-\sqrt{r_s} \tilde{S}_a} [\det'(\sqrt{r_s} \mathbf{M}[\tilde{\mathbf{r}}_a(\tilde{\tau})])]^{-\frac{1}{2}}, \end{aligned}$$

where prime in the determinant indicates that the zero mode contribution is omitted. Using the identity,

$$\left(\frac{\det'(\sqrt{r_s} \mathbf{M}[\tilde{\mathbf{r}}_a(\tilde{\tau})])}{\det(\sqrt{r_s} \mathbf{M}[\tilde{\mathbf{R}}])} \right)^{-\frac{1}{2}} = r_s^{1/4} \left(\frac{\det'(\mathbf{M}[\tilde{\mathbf{r}}_a(\tilde{\tau})])}{\det(\mathbf{M}[\tilde{\mathbf{R}}])} \right)^{-\frac{1}{2}}, \quad (46)$$

we obtain

$$J_a = \frac{e^2}{4\pi\epsilon a_B} \cdot \frac{A_a}{r_s^{5/4}} \sqrt{\frac{\tilde{S}_a}{2\pi}} e^{-\sqrt{r_s} \tilde{S}_a}, \quad (47)$$

$$A_a = \left(\frac{\det' \mathbf{M}[\tilde{\mathbf{r}}_a(\tilde{\tau})]}{\det \mathbf{M}[\tilde{\mathbf{R}}]} \right)^{-\frac{1}{2}}. \quad (48)$$

The effective Hamiltonian (3) is then obtained by summing over all the possible tunneling processes P_a [25]

This expression for J_a applies for any value of B as long as the complex solutions of (41) and (42) are found (which is a complicated task). But since the real instanton solutions $[\equiv \tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})]$ for the $B = 0$ problem were already obtained [12, 13, 16], we would like to obtain a simplified

expression for J_a by expanding $\tilde{\mathbf{r}}_a(\tilde{\tau})$ around the zero-field solution $\tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})$ in powers of $1/\sqrt{r_s}$:

$$\tilde{\mathbf{r}}_a(\tilde{\tau}) = \tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau}) + \frac{1}{\sqrt{r_s}} \delta\tilde{\mathbf{r}}_a^{(1)}(\tilde{\tau}) + O(r_s^{-1}). \quad (49)$$

By substituting this in (41, 42), we obtain the real instanton equation at zeroth order

$$\frac{d^2 \tilde{\mathbf{r}}_a^{(0)}}{d\tilde{\tau}^2} - \tilde{\nabla} V[\tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})] = 0, \quad (50)$$

subject to $\tilde{\mathbf{r}}_a^{(0)}(0) = \tilde{\mathbf{R}} \equiv \mathbf{R}/r_0$ and $\tilde{\mathbf{r}}_a^{(0)}(\Delta\tilde{\tau}) = P_a \tilde{\mathbf{R}} \equiv P_a \mathbf{R}/r_0$. The first-order equation is

$$\frac{d^2 \delta\tilde{\mathbf{r}}_{i,a}^{(1)}}{d\tilde{\tau}^2} - \left(\partial_{\tilde{r}_i} \partial_{\tilde{r}_j} V[\tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})] \right) \delta\tilde{\mathbf{r}}_{a,j}^{(1)} = \frac{2i}{\nu} \mathcal{E}_{ij} \frac{d\tilde{\mathbf{r}}_{a,j}^{(0)}}{d\tilde{\tau}}, \quad (51)$$

with $\delta\tilde{\mathbf{r}}_a^{(1)}(0) = 0$ and $\delta\tilde{\mathbf{r}}_a^{(1)}(\Delta\tilde{\tau}) = 0$, where the repeated indices are implicitly summed over. Since (51) is a second-order differential equation for $\tilde{\mathbf{r}}_a^{(1)}(\tilde{\tau})/i$ with real coefficients, $\tilde{\mathbf{r}}_a^{(1)}(\tilde{\tau})$ is purely imaginary.

We also expand the instanton action in powers of $r_s^{-1/2}$:

$$\begin{aligned} \sqrt{r_s} \tilde{S}_a &= \sqrt{r_s} \int_0^{\Delta\tilde{\tau}} d\tilde{\tau} \left(\dot{\tilde{\mathbf{r}}}_a^2 + \frac{i}{\nu \sqrt{r_s}} \tilde{\mathbf{r}}_a^T \mathcal{E} \dot{\tilde{\mathbf{r}}}_a \right) \\ &= \sqrt{r_s} \tilde{S}_a^{(0)} + i \frac{2}{\nu} \tilde{\Sigma}_a^{(r)} + O(1/\sqrt{r_s}), \end{aligned} \quad (52)$$

$$\tilde{S}_a^{(0)} = \int_0^{\Delta\tilde{\tau}} d\tilde{\tau} \left(\dot{\tilde{\mathbf{r}}}_a^{(0)} \right)^2 = \int d\tilde{\mathbf{r}}_a^{(0)} \sqrt{2V[\tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})]}, \quad (53)$$

$$\tilde{\Sigma}_a^{(r)} \equiv \sum_i \frac{1}{2} \int \tilde{\mathbf{r}}_{i,a}^{(0)} \times d\tilde{\mathbf{r}}_{i,a}^{(0)}, \quad (54)$$

where $\tilde{S}_a^{(0)} \equiv \tilde{S}[\mathbf{r}_a^{(0)}(\tilde{\tau})] > 0$ is the instanton action of the $B = 0$ problem and $\tilde{\Sigma}_a$ is the signed area (positive for the counter-clockwise motion) enclosed by $\tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})$. The first line in (52) follows from the constant of motion of the zeroth-order instanton equation (50)

$$\frac{1}{2} \left(\dot{\tilde{\mathbf{r}}}_a^{(0)} \right)^2 - V[\tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})] + \tilde{V}_0 = 0. \quad (55)$$

In the second line of (52), we used the following identity

$$\begin{aligned} \int d\tilde{\tau} \dot{\tilde{\mathbf{r}}}_a^{(0)}(\tilde{\tau}) \cdot \delta\dot{\tilde{\mathbf{r}}}_a^{(1)}(\tilde{\tau}) &= \int d\tilde{\tau} \nabla V[\tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})] \cdot \delta\tilde{\mathbf{r}}_a^{(1)}(\tilde{\tau}) \\ &= \int d\tilde{\tau} \ddot{\tilde{\mathbf{r}}}_a^{(0)} \cdot \delta\tilde{\mathbf{r}}_a^{(1)} = - \int d\tilde{\tau} \dot{\tilde{\mathbf{r}}}_a^{(0)} \cdot \delta\dot{\tilde{\mathbf{r}}}_a^{(1)} = 0, \end{aligned} \quad (56)$$

where the first equality follows from the constant of motion of the first-order instanton equation (51)

$$\dot{\tilde{\mathbf{r}}}_a^{(0)} \cdot \delta\dot{\tilde{\mathbf{r}}}_a^{(1)} - \tilde{\nabla} V[\tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})] \cdot \delta\tilde{\mathbf{r}}_a^{(1)} = 0, \quad (57)$$

the second equality follows from (50) and the third equality is obtained by integration by parts. This expression is zero since the first and the last expression is identical up to a minus sign.

Due to the imaginary deformation of the instanton path, the fluctuation determinant (48) also acquires a $O(r_s^{-1/2})$ correction relative to its zero-field value:

$$A_a = A_a^{(0)} + O(1/\sqrt{r_s}), \quad (58)$$

$$A_a^{(0)} \equiv \left[\frac{\det'(-\delta_{ij} \frac{\partial^2}{\partial \tilde{\tau}^2} + \frac{\partial^2 V}{\partial \tilde{\mathbf{r}}_i \partial \tilde{\mathbf{r}}_j} [\tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})])}{\det(-\delta_{ij} \frac{\partial^2}{\partial \tilde{\tau}^2} + \frac{\partial^2 V}{\partial \tilde{\mathbf{r}}_i \partial \tilde{\mathbf{r}}_j} [\tilde{\mathbf{R}}])} \right]^{-\frac{1}{2}}. \quad (59)$$

Collecting the leading contributions, we recover (6-8):

$$J_a(B) = J_a^{(0)} \exp \left[-\frac{2i}{\nu} \tilde{\Sigma}_a^{(r)} \right], \quad (60)$$

where $J_a^{(0)} > 0$ is the exchange constant at $B = 0$

$$J_a^{(0)} = \frac{e^2}{4\pi\epsilon a_B} \cdot \frac{A_a^{(0)}}{r_s^{5/4}} \sqrt{\frac{\tilde{S}_a^{(0)}}{2\pi}} e^{-\sqrt{r_s} \tilde{S}_a^{(0)}}. \quad (61)$$

The values of $\tilde{S}_a^{(0)}$, $A_a^{(0)}$ and $\Sigma_a^{(0)}$ are numerically calculated and shown in Table I.

Effects of a Berry curvature—The previous result in a magnetic field (60-61) can be straightforwardly generalized to include the (uniform) Berry curvature effect. The Berry curvature effect on the real-time dynamics can be captured by considering a (phase-space) action $S_M = \int dt L_M$, where [26, 27] [28]

$$L_M = \sum_i \left[\hbar \mathbf{k}_i \cdot \frac{d\mathbf{r}_i}{dt} - \varepsilon_M(\mathbf{k}_i) - e \frac{d\mathbf{r}_i}{dt} \cdot \mathbf{A}(\mathbf{r}_i) + \frac{d\mathbf{k}_i}{dt} \cdot \mathcal{A}(\mathbf{k}_i) \right] - \sum_{i < j} \frac{e^2}{4\pi\epsilon} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} + V_0. \quad (62)$$

Here, $\mathbf{A}(\mathbf{r}) = B(-y/2, x/2)$ and $\mathcal{A}(\mathbf{k}) = \Omega(-k_y/2, k_x/2)$, and as in (35), the minimum classical Coulomb energy $V_0 = \frac{e^2}{4\pi\epsilon r_0} \tilde{V}_0$ is subtracted from the Coulomb energy for convenience. $\varepsilon_M(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} - BM_z(\mathbf{k} + \mathbf{Q}_0)$ is the modified single-electron energy due to the orbital magnetic moment M_z , where \mathbf{Q}_0 is the momentum at the band bottom. Within a continuum approximation near $\mathbf{k} \sim \mathbf{Q}_0$, this amounts to renormalizing the effective mass $m \rightarrow m^*$ through (16) with a constant shift in energy [29]:

$$\varepsilon_M(\mathbf{k}) \approx \frac{\hbar^2 k^2}{2m^*} - M_z(\mathbf{Q}_0)B. \quad (63)$$

We will drop the constant term $-M_z(\mathbf{Q}_0)B$ as it does not affect the spin physics when the band minimum occurs at a single valley [30].

Since we are interested in the thermodynamic properties (governed by the imaginary-time dynamics), we consider the Euclidean action S_E upon Wick rotation

$$S_E = \int d\tau \left(\sum_i \left[-i \mathbf{k}_i \cdot \frac{d\mathbf{r}_i}{d\tau} + \frac{\hbar^2 k_i^2}{2m^*} + ie \frac{d\mathbf{r}_i}{d\tau} \cdot \mathbf{A}(\mathbf{r}_i) - i \frac{d\mathbf{k}_i}{d\tau} \cdot \mathcal{A}(\mathbf{k}_i) \right] + \sum_{i < j} \frac{e^2}{4\pi\epsilon} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} - V_0 \right) \equiv \sqrt{r_s} \tilde{S}, \quad (64)$$

$$\tilde{S} = \int d\tilde{\tau} \left(\sum_i \left[-i \tilde{\mathbf{k}}_i \cdot \dot{\tilde{\mathbf{r}}}_i + \frac{1}{2} \tilde{k}_i^2 + \frac{i}{\nu \sqrt{r_s}} \tilde{\mathbf{r}}_i \times \dot{\tilde{\mathbf{r}}}_i - \frac{i\alpha}{4\sqrt{r_s}} \tilde{\mathbf{k}}_i \times \dot{\tilde{\mathbf{k}}}_i \right] + V(\tilde{\mathbf{r}}) - \tilde{V}_0 \right). \quad (65)$$

Again, in order to see the r_s dependence manifest, we introduced the normalized action \tilde{S} in terms of dimensionless coordinates [as defined below (34-36)] and momenta $\tilde{\mathbf{k}} \equiv \sqrt{a_B r_0} \mathbf{k} = r_0 \mathbf{k} / \sqrt{r_s}$. $\alpha \equiv \nu_\Omega r_s = 2\Omega/(a_B^2 r_s)$ is an $O(1)$ parameter (9) introduced to capture the Berry phase effect, where $\nu_\Omega \equiv \Omega \cdot 2\pi n$ (10) is the Chern filling factor (analogous to the Landau-level filling factor ν).

With this setup, the exchange constants $J_a(B, \Omega)$ can be derived analogously, but with key modifications, from the large- r_s expansion. The saddle-point equations of the action (65) yield $4N$ first-order Hamilton's equations for the (complex) phase-space variables $(\tilde{\mathbf{r}}, \tilde{\mathbf{k}})$:

$$\dot{\tilde{\mathbf{r}}} = -i\tilde{\mathbf{k}} - \frac{\alpha}{2\sqrt{r_s}} \mathcal{E} \dot{\tilde{\mathbf{k}}} \quad (66)$$

$$\dot{\tilde{\mathbf{k}}} = i\nabla_{\tilde{\mathbf{r}}} V - \frac{2}{\nu \sqrt{r_s}} \mathcal{E} \dot{\tilde{\mathbf{r}}}, \quad (67)$$

which result in

$$\left(1 + \frac{\alpha}{\nu r_s} \right) \ddot{\tilde{\mathbf{r}}} - \frac{2i}{\nu \sqrt{r_s}} \mathcal{E} \dot{\tilde{\mathbf{r}}} + \frac{i\alpha}{2\sqrt{r_s}} \mathcal{E} \nabla \nabla V[\tilde{\mathbf{r}}] \dot{\tilde{\mathbf{r}}} - \nabla V[\tilde{\mathbf{r}}] = 0, \quad (68)$$

where $\nabla \nabla V[\tilde{\mathbf{r}}]$ is the Hessian matrix of V at $\tilde{\mathbf{r}}$. These equations can be solved perturbatively in powers of $1/\sqrt{r_s}$:

$$\tilde{\mathbf{r}}_a(\tilde{\tau}) = \tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau}) + \frac{1}{\sqrt{r_s}} \delta \tilde{\mathbf{r}}_a^{(1)}(\tilde{\tau}) + O(r_s^{-1}). \quad (69)$$

$$\tilde{\mathbf{k}}_a(\tilde{\tau}) = \tilde{\mathbf{k}}_a^{(0)}(\tilde{\tau}) + \frac{1}{\sqrt{r_s}} \delta \tilde{\mathbf{k}}_a^{(1)}(\tilde{\tau}) + O(r_s^{-1}). \quad (70)$$

The zeroth-order term for $\tilde{\mathbf{r}}_a(\tilde{\tau})$ again satisfies the zero-field and zero-Berry-curvature instanton equation (50) whereas the first-order equation is modified to

$$\delta \ddot{\tilde{\mathbf{r}}}_a^{(1)} - \nabla \nabla V[\tilde{\mathbf{r}}_a^{(0)}(\tilde{\tau})] \delta \tilde{\mathbf{r}}_a^{(1)} = \frac{2i}{\nu} \mathcal{E} \dot{\tilde{\mathbf{r}}}_a^{(0)} - \frac{i\alpha}{2} \mathcal{E} \ddot{\tilde{\mathbf{r}}}_a^{(0)}. \quad (71)$$

Again, whereas $\tilde{\mathbf{r}}_a^{(0)}$ is real, $\delta \tilde{\mathbf{r}}_a^{(1)}$ is purely imaginary since the right-hand side of (71) is purely imaginary. On the other hand, the zeroth-order term for $\tilde{\mathbf{k}}_a$ is purely imaginary and the first-order correction is purely real

$$\tilde{\mathbf{k}}_a^{(0)}(\tilde{\tau}) = i \dot{\tilde{\mathbf{r}}}_a^{(0)}(\tilde{\tau}), \quad (72)$$

$$\delta \tilde{\mathbf{k}}^{(1)}(\tilde{\tau}) = i \delta \dot{\tilde{\mathbf{r}}}_a^{(1)}(\tilde{\tau}) - \frac{\alpha}{2} \mathcal{E} \nabla V[\tilde{\mathbf{r}}^{(0)}(\tilde{\tau})]. \quad (73)$$

With the solutions to these equations, the expression for the tunneling action follows straightforwardly from the procedure analogous the one outlined below (52-54):

$$\sqrt{r_s} \tilde{S}_a = \sqrt{r_s} \tilde{S}_a^{(0)} + i \left(\frac{2}{\nu} \tilde{\Sigma}_a^{(r)} - \frac{\alpha}{2} \tilde{\Sigma}_a^{(k)} \right) + O(r_s^{-\frac{1}{2}}), \quad (74)$$

$$\tilde{\Sigma}_a^{(k)} \equiv \sum_i \frac{1}{2} \int \tilde{\mathbf{k}}_{i,a}^{(0)} \times d\tilde{\mathbf{k}}_{i,a}^{(0)}, \quad (75)$$

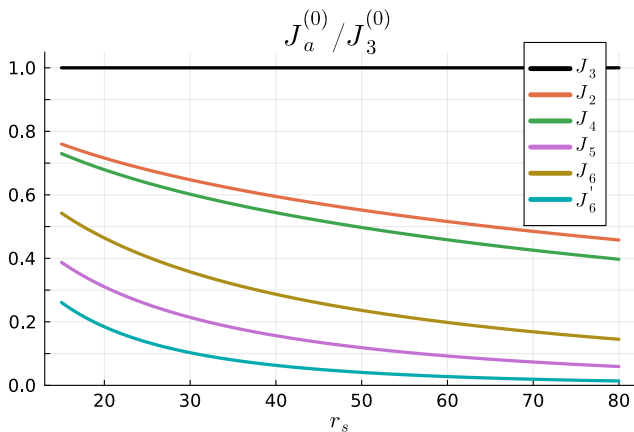


FIG. 1. Magnitude of exchange constants $J_a^{(0)}$ normalized by the largest one $J_3^{(0)}$ as a function of $15 \leq r_s \leq 80$. $J_a^{(0)}$ is obtained from (4) with $\tilde{S}_a^{(0)}$ and $\tilde{A}_a^{(0)}$ taken from Table I.

with $\tilde{S}_a^{(0)}$ and $\tilde{\Sigma}_a^{(r)}$ given by (53) and (54). The fluctuation determinant similarly acquires only a subdominant $O(1/\sqrt{r_s})$ correction which we ignore. Therefore,

$$J_a(B, \Omega) = |J_a(B, \Omega)| \exp \left[i \left(-\frac{2}{\nu} \tilde{\Sigma}_a^{(r)} + \frac{\alpha}{2} \tilde{\Sigma}_a^{(k)} \right) \right], \quad (76)$$

where the expression for the magnitude $|J_a(B, \Omega)|$ is given in (17) and is obtained from (61) by substituting $m \rightarrow m^*$ [see the relevant discussions below (62)].

Discussion—We analyzed how a magnetic field (B) and a Berry curvature (Ω) modify the magnetism of a Wigner crystal (WC) through an asymptotic large- r_s expansion. When either B or Ω is present individually, the effect is simply to multiply the zero-field and zero-Berry-curvature exchange constant $J_a^{(0)}$ by a phase factor: the Aharonov-Bohm phase $\phi_a(B)$ for $B \neq 0$ (6-8) and the Berry phase $\gamma_a(\Omega)$ for $\Omega \neq 0$ (11-13). When both B and Ω are present, in addition to having both phases, the magnitude of the exchange constant is also modified due to an effective mass renormalization originating from the coupling of the orbital magnetic moment to the magnetic field (15-17).

In the absence of a magnetic field or Berry curvature, the WC is thought to occur for $r_s \gtrsim 31$ [4], aside from the subtleties of possible intermediate phases [16, 31–33]. This critical value of r_s will in general be shifted when B or Ω is finite (as a function of dimensionless parameters ν and α). Fig. 1 shows the relative magnitudes of exchange constants $J_a^{(0)}$ compared to the largest one $J_3^{(0)}$ over a physically relevant regime, e.g., $15 \leq r_s \leq 80$. In this range, at least four processes (J_3 , J_2 , J_4 , J_6) are appreciable, giving rise to a highly frustrated generalized

Heisenberg model. To the best of our knowledge, the phase diagram of this J_2 - J_3 - J_4 - J_6 Hamiltonian in the presence of the Aharonov-Bohm (7) and/or Berry phase (12) has not been systematically investigated (even without the J_6 term). In contrast, the simpler J_2 - J_3 - J_4 model at $B = \Omega = 0$ (i.e., $\phi_a = \gamma_a = 0$) has been studied in [34–38]. A further truncation to only J_2 and J_3 interactions yields a nearest-neighbor Heisenberg model supplemented by a spin-chirality term, which was studied in [39, 40],

$$H_{\text{eff}} \approx 4(|J_2| \cos \Phi_2 - |J_3| \cos \Phi_3) \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - 4|J_3| \sin \Phi_3 \sum_{\langle i,j,k \rangle \in \Delta} \mathbf{S}_i \cdot \mathbf{S}_j \times \mathbf{S}_k + \dots, \quad (77)$$

$$\Phi_a(\nu, \alpha) \equiv -\frac{2}{\nu} \tilde{\Sigma}_a^{(r)} + \frac{\alpha}{2} \tilde{\Sigma}_a^{(k)}. \quad (78)$$

Here, $|J_a|$ is given by (4) [or by (17) when both $B \neq 0$ and $\Omega \neq 0$]. Interestingly, both the spin-chirality term χ_{ijk} and the four-spin term h_{ijkl} (29) are shown to stabilize the Kalmeyer-Laughlin chiral spin liquid (CSL) phase [37, 39, 40]. Since both interactions appear in our effective Hamiltonian, the CSL phase could be realized over a broader region of the parameter space.

Recently, an extended region of the WC phase has been observed in tetra- and penta-layer rhombohedral graphene systems [41], suggesting that our considerations may be directly relevant to the WC and proximate correlated phases in this system. More broadly, the present results highlight that geometric phases manifest in a fundamentally different manner in correlated electron crystals compared to weakly interacting systems or Hubbard-type lattice models.

Note Added: Upon completion of this work, a preprint [42] appeared that partially overlaps with this work. The results are largely in agreement where they overlap.

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 - [19] The existence of the Aharonov-Bohm phase in a magnetic field was suggested previously in [45, 46]. However, the theoretical analysis in [46] did not consider a complexified path integral needed to obtain the correct asymptotic expression of the exchange constants. In our approach, the tunneling trajectory is determined by solving the complex instanton equations of motion (41) and (42). As a result, while the leading-order expression (i.e., the phase factor) in [46] coincides with ours, their subleading corrections are not valid.
 - [20] The f operators in the first line are ‘normal-ordered’ such that electrons with spins $\sigma_1, \sigma_2, \dots, \sigma_{n_a}$ are annihilated in order and then created in a reverse order. For a more detailed discussion, see relevant discussions, e.g., in [47, 48].
 - [21] Here, the tilde denotes that the left-hand side is asymptotically equivalent to the right-hand side for $1/E_{\text{Debye}} \ll \Delta\tau \ll 1/|J_a|$.
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 - [29] The details of how the effective mass renormalizes depend on the parent band structure and are not explored here. We treat m^* (or M_z) as a given parameter in this work.
 - [30] When there are multiple valleys with different orbital magnetic moments, $-M_z(\mathbf{k})B$ acts as a Zeeman-like term for valley degrees of freedom and must be included. Relatedly, a trigonal warping correction to the otherwise isotropic dispersion $\mathbf{p}^2/2m$ is shown to lead to a valley polarization of the WC even without this Zeeman-like coupling of the orbital magnetic moment to the magnetic field [51].
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