Université de Nantes — UFR Sciences et Techniques
Master informatique parcours "optimisation en recherche opérationnelle (ORO)"
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Projet d'optimisation globale

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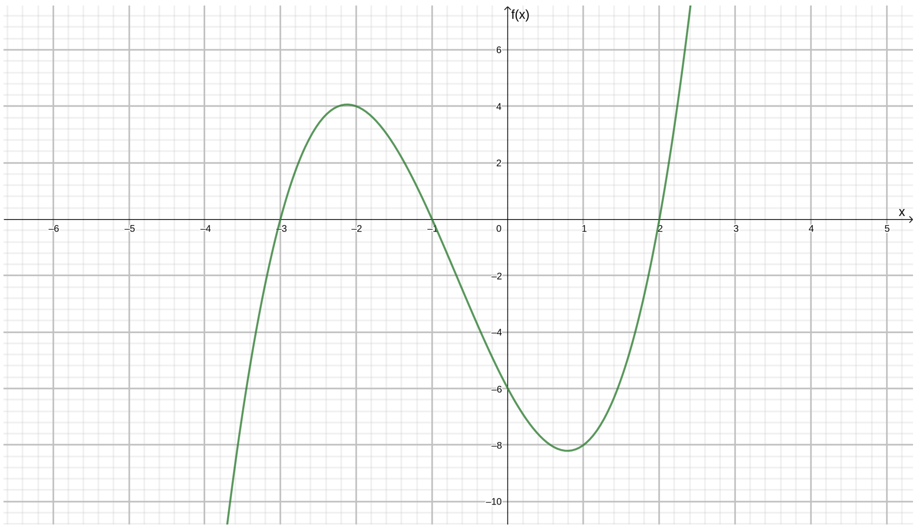
Let

$$f(x) = x^3 + 2x^2 - 5x - 6$$

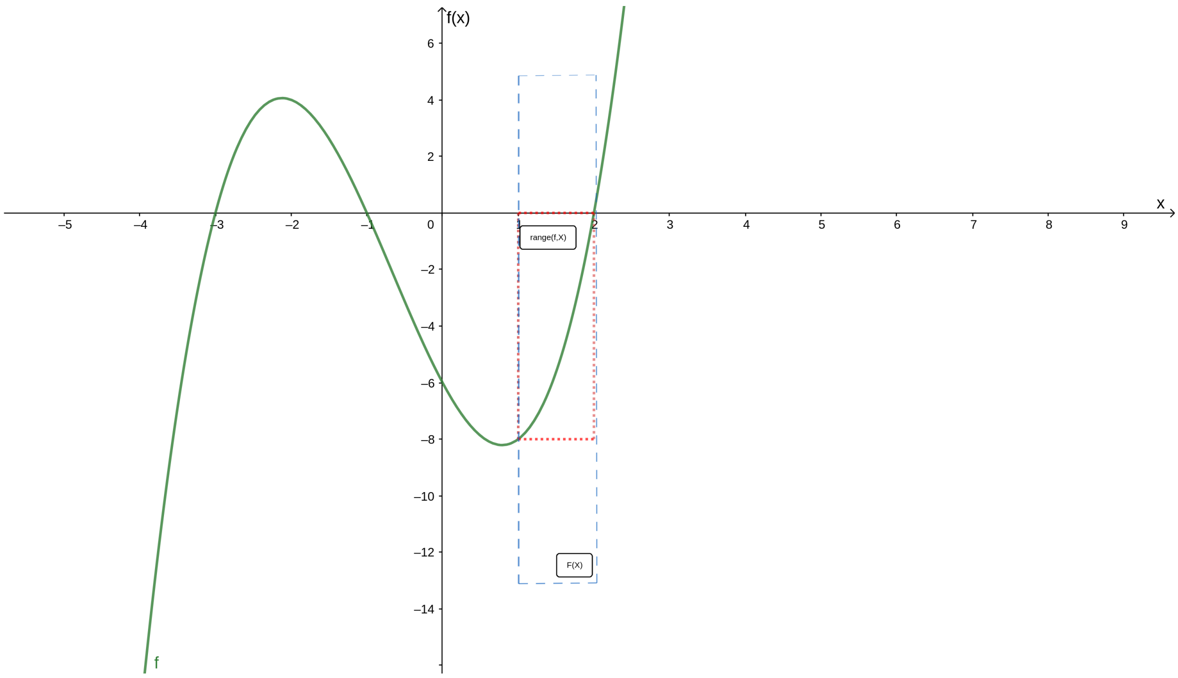
be a real function whose graph is plotted below :



Question 1 :

$$F(X) = X^3 + 2X^2 - 5X - 6$$

Let $F(X)$ the natural interval extension of $f(x)$
Is F optimal ?
We have seen in the course a theorem which tells us : "The natural extension of a function is optimal if each of its variables occurs only once" ; But the variable X appears three times in our $F(X)$. So we have no proof for optimality
We look for a counter example by comparing exact range and natural interval extension :
Let $X = [1, 2]$; $range(f, X) = [-8, 0]$; $F(X) = [-13, 5]$
As can be seen in the following graph : $range(f, X) \subset F(X)$ so $F(X)$ is not optimal.



Question 2 The algorithm implementing the optimal interval extension of $f(x)$. Given any interval X as input, the result must be equal to the following interval : $[\inf_{x\in X} f(x), \sup_{x\in X} f(x)]$
Let $a \in \mathbb{R}$; $b \in \mathbb{R}$ such that $f'(a) = 0, f'(b) = 0$ and $a < b$

study of monocity :

$$f'(x) = 3x^2 + 4x - 5$$

$f'(x) = 0 \Longrightarrow 3x^2 + 4x - 5 = 0 \Longrightarrow x = \frac{-4\pm\sqrt{76}}{6}$

x	$-\infty$		$\frac{-4-\sqrt{76}}{6}$		$\frac{-4+\sqrt{76}}{6}$		$+\infty$
$f'(x)$		+	0	-	0	+	
$f(x)$	$-\infty$	$\nearrow$	4.06	$\searrow$	-8.21	$\nearrow$	$+\infty$

Monotonicity is : $\nearrow a \searrow b \nearrow$ and the optimal interval extention is :

$$\left[\min\{f(\inf_{x\in X} x), f(b), f(\sup_{x\in X} x)\}, \max\{f(\inf_{x\in X} x), f(a), f(\sup_{x\in X} x)\}\right]$$

Algorithme 1 : Optimal interval extension F

Input : $X = [\inf_X, \sup_X]$ **,**$a \in \mathbb{R}$**,** $b \in \mathbb{R}$
if $\inf_X < a$ *and* $\sup_{x\in X} x < a$ **then**
    return $[f(\inf_X), f(\sup_X)]$ `/* Specific motonus cases */`
end
if $\inf_X > b$ *and* $\sup_X > b$ **then**
    return $[f(\inf_X), f(\sup_X)]$
end
if $\inf_X > a$ *and* $\sup_X < b$ **then**
    return $[f(\sup_{x\in X} x), f(\inf_X)]$
end
if $\inf_X > a$ **then**
    $a = \inf_X$ `/* Generic cases */`
end
if $\sup_X < b$ **then**
    $b = \sup_X$
end
return $\min[f(\inf_X), f(b), f(\sup_X)), max(f(\inf_X), f(a), f(\sup_X))]$

Question 3 Let

$$f'(x) = 3x^2 + 4x - 5 \Longrightarrow F'(X) = 3X^2 + 4X - 5; \;\; X = [\inf X, \sup X]$$

with $F'(X)$ the natural interval extension of $f'(x)$

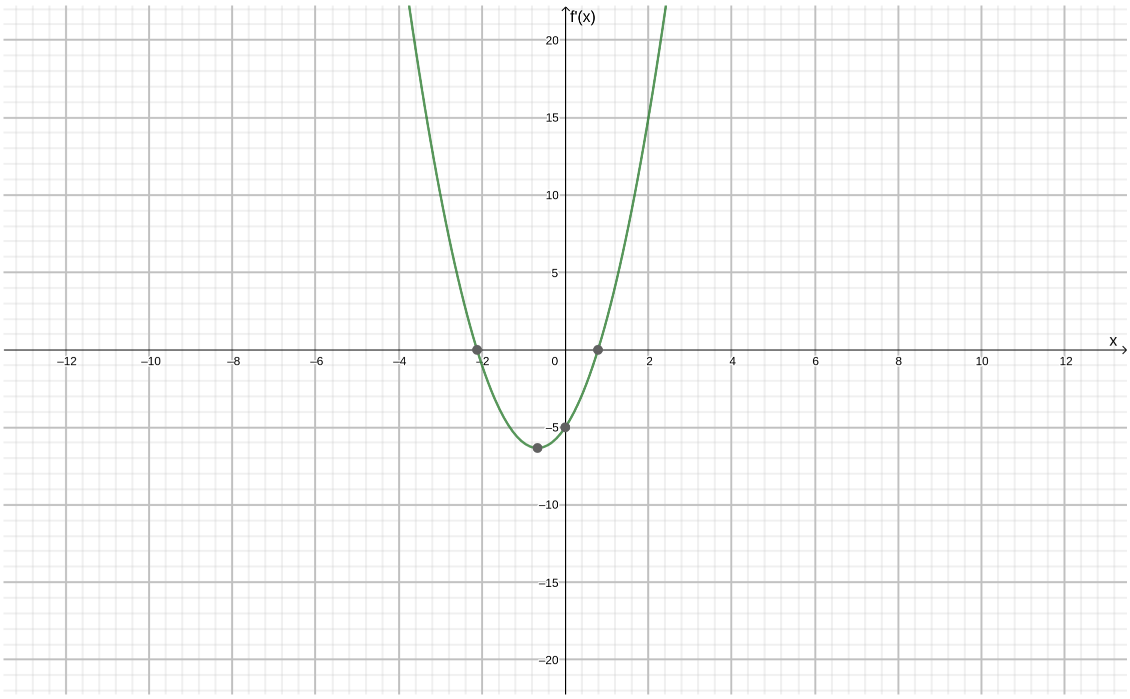
x	$-\infty$		$\frac{-2}{3}$		$+\infty$
$f''(x)$		-	0	+	
$f'(x)$	$+\infty$	$\searrow$	-6.33	$\nearrow$	$+\infty$

Then $f'(x)$ is decreasing in the interval $]-\infty, -2/3]$ and increasing in $[-2/3, +\infty[$
The optimal interval extension algorithm of the derivative is defined as :

Algorithme 2 : Optimal interval extension F'

Input : $X = [\inf_X, \sup_X]$ **,** $a = \frac{-2}{3}$
if $\sup_{x \in X} x < \frac{-2}{3}$ **then**
  return $[f'(\sup_{x \in X} x), f'(\inf_X)]$
end
if $\inf_X > \frac{-2}{3}$ **then**
  return $[f'(\inf_X), f'(\sup_X)]$
end
if $a \in [\inf_X, \sup_X]$ **then**
  return $[f'(a), max\{f'(\inf_X), f'(\sup_X)\}]$
end

The graph of $f'(x)$ is ploted below :



Question 4 : We want to compute hull $\{x \in X \mid (\exists y \in Y) \ y = f(x)\}$
which is in fact the projection of the equation $y = f(x)$ over x within the given box $X \times Y$.

$$Proj(y = f(x), X \times Y, 1) = Y \cap X^3 + 2X^2 - 5X - 6 \ \ i.e \ \ Y \cap F(X)$$

and

$$Proj(y = f(x), X \times Y, 2) = X \cap F^{-1}(Y)$$

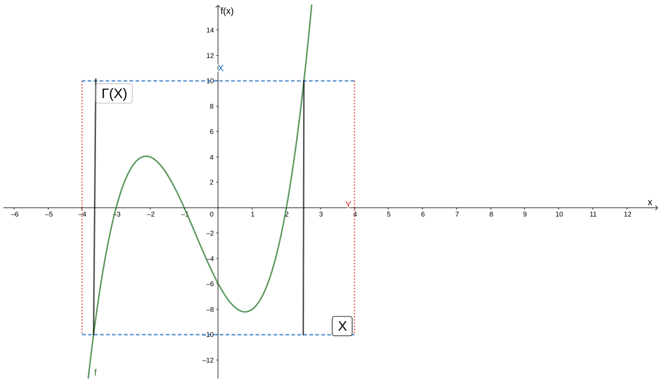
We illustrated below the possible cases of $proj(y = f(x), X \times Y, 2)$

First we compute the $F^{-1}(Y)$, to do so, we search for the intersection points x such that $f(x) = \sup_Y$ or $f(x) = \inf_Y$. We have the 3 following cases :

The first case is when $\inf_Y < b$ and $\sup_Y > a$
By applying

$f(x_1) = \inf_Y$ and $f(x_2) = \sup_Y$

We can directly deduce the interval $[x_1, x_2]$

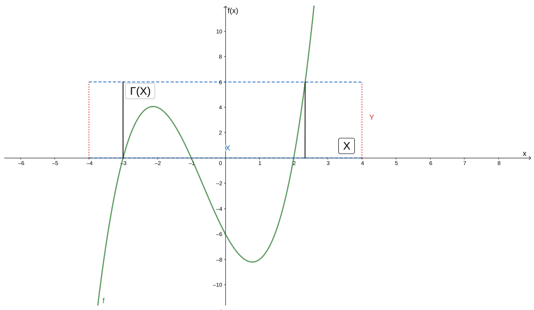


2 intersect

The second case is when $\inf_Y > b$ or $\sup_Y < a$ but not both at the same time
For example, let $\inf_Y > b$ and $\sup_Y > a$.
By applying

$f(S) = \inf_Y$ and $f(x_4) = \sup_Y$

We will find three solution for $S : \{x_1, x_2, x_3\}$ and we can deduce from them the projected intervals $[x_1, x_2]$ and $[x_3, x_4]$

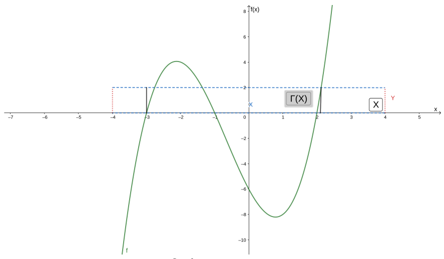


4 intersect

The last case is when $\inf_Y > b$ and $\sup_Y < a$
By applying

$f(S_1) = \inf_Y$ and $f(S_2) = \sup_Y$, $S = S_1 \cup S_2$

We have $S_1 : \{x_1, x_2, x_3\}$ and $S_2 : \{x_4, x_5, x_6\}$.
We deduce the projected intervals $[x_1, x_4]$, $[x_2, x_5]$, and $[x_3, x_6]$



6 intersect

Remark :

- We use the predefined function on julia poly_solve_cubic() in the *GSL* library to compute the cubic root of the equation $f(S) = y$
- We note that the projected interval bounds are linked two by two in sorted order to make the final intervals

The optimal interval projection algorithm of F^{-1} is defined as :

Algorithme 3 : Interval Projection F^{-1}

Input : $X = [\inf_X, \sup_X]$, $Y = [\inf_Y, \sup_Y]$, $S = \{\}$, $a \in \mathbb{R} | f'(a) = 0$, $b \in \mathbb{R} | f'(b) = 0, a < b$
Push(S, poly_solve_cubic($f(x) = \sup_Y$))
Push(S, poly_solve_cubic($f(x) = \inf_Y$))
$S \leftarrow S \setminus \{a, b\}$ `/*` We remove from S the critical points a and b if they are solutions `*/`
Sort(S)
return $X \cap ([s_1, s_2] \cup [s_3, s_4] \cup [s_5, s_6]) \quad \forall s \in S$

Question 5 We define the contractor enforcing hull consistency on the equation $y = f(x)$ given a box $X * Y$ as a mapping on boxes :

$$\Gamma : X * Y \Longrightarrow (X \cap F^{-1}(Y)) * (Y \cap F(X))$$

Proof : Hull consistency states that each domain $X\&Y$ is equal to the hull of its projection.

$$\begin{cases} Proj(y = f(x), X * Y, 1) & = & Y \cap F(X) \\ Proj(y = f(x), X * Y, 2) & = & X \cap F^{-1}(Y) \end{cases}$$

We have the hull of the projection on X from question 4. And we have the optimal extention on Y from question 2. Wich is, because of the continuity of f, the hull of the projection on Y.

Question 6 To execute the code, the following julia libraries are needed :

```
using IntervalArithmetic,GSL
```

Thanks to the GSL library, all our results comes with a 12 digits numerical precision.

The precision of the Intervals can be choosed

Algorithme 4 : Contractor

1 **Input :**$X = [\inf_X, \sup X]$, $Y = [\inf_Y, \sup Y]$
2 $X' \leftarrow IntervalProjection(X, Y)$
3 $Y' \leftarrow OptimalExtension(X')$
4 **return** $X \cap X'$,$Y \cap Y'$

Question 7

We could study the same way any other polynomial function.

By studying the montonicity and the critical points, we could build an extension and a projection for any polynomial function as long as we have an algorithm to compute $x = F^{-1}(y)$. But the roots of a polynomial function are harder to compute as the polynomial function gain degree.

Considering this, we could deduce the corresponding Contractor.