Mathematical model of time solutions of round object rolling on various curvature profile ramps

Author: Artūras Karbočius
Adress: phyarth@gmail.com

1. Introduction

In this paper, we solve symbolically and numerically round objects rolling time on various inclination curvature profile ramps. Later, gathered information we apply it to calculate the unwinding time of the cylinder rod from the thread and to calculate the drainage time of liquid from a storage tank. Moreover, calculate numerically a function or various inclination profiles with the shortest time of ball rolling property, an extension of the Johann Bernoulli brachistochrone curve, which has the shortest time rolling property of all possible mathematics functions, which is the *cycloid* function.

Keywords: Cycloid function, the brachistochrone curve.

2. Theory

In all calculations and physics thought experiments, no external forces are present that can intact ball rolling time results; thus no friction ball with a platform, no air resistance, no slippage. The secondary ball, cylinder, or wheel is homogeneous which means that the center mass of gravity of the rolling object exactly matches concentrically the rotation axis point. In such case shape, diameter, density, and moment of inertia of the rolling ball, cylinder or wheel have no impact on the rolling time result.

The minimum time that a rolling ball can take is equal to a ball rolling (falling) perpendicular to the ground x-axis from height (altitude) y is:

$$t_0 = \sqrt{\frac{2y}{g}},\ [s] \tag{1.0}$$

According to energy conservation law $E_p=mgy=const.$, thus velocity is always the same and depends only on the y variable, and not on the shape of the curve profile, thus:

$$E_p = E_k \ \Rightarrow\ mgy = \frac{mv^2}{2} \ \Rightarrow\ v = \sqrt{2gy} = const.,\ [m/s] \tag{2.0}$$

Velocity change (2.0) equation suggests that final velocity is independent of f-ion $y=f(x)$ curvature profile shape, final velocity only depends on maximum altitude height y value, which in the end sustains energy conservation law condition $E_p = E_k = const.$

Velocity is the ball distance changing over time changing:

$$v = \frac{dS}{dt} \tag{3.0}$$

Curves length according to Pythagoras' theorem is:

$$dS = \sqrt{dx^2 + dy^2} \ \Rightarrow\ dS = \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \sqrt{1+(y')^2}dx \tag{4.0}$$

3. <u>Time solution of ball rolling on a straight inclined ramp</u>

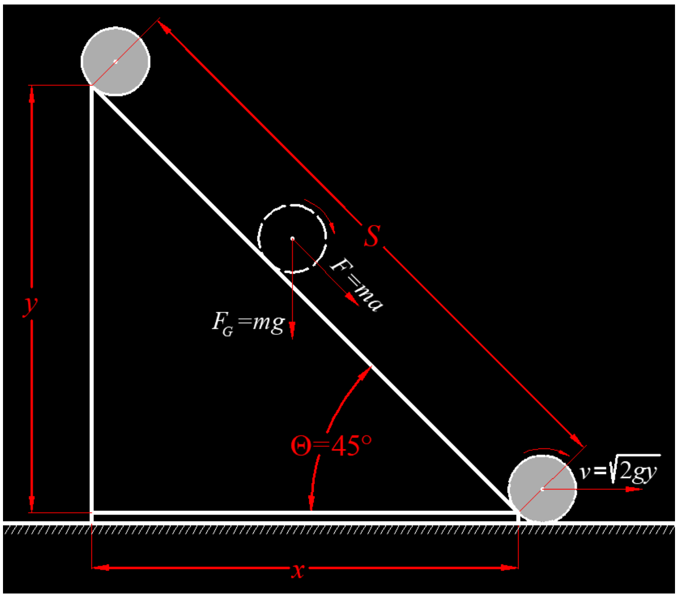

Fig.1 Ball rolls on an inclined ramp under a constant angle Θ=45°.

Rolling time on a straight ramp (Fig. 1) is defined by a linear equation, where c is the ramp inclination slope constant:

$$y = cx \tag{5.0}$$

Derivative of a linear function (5.0) eq. is:

$$y' = c = const. \tag{6.0}$$

From velocity expression (3.0) eq., we solve the ball rolling time t:

$$\int dt = \frac{1}{v}\int dS = \frac{1}{\sqrt{2gy}}\int\sqrt{1+(y')^2}\,dx = \frac{1}{\sqrt{2g}}\int\frac{\sqrt{1+c^2}}{\sqrt{y}}dx \;\Rightarrow \tag{7.0}$$

$$\frac{1}{\sqrt{2g}}\int\frac{\sqrt{1+c^2}}{\sqrt{cx}}dx = \frac{\sqrt{1+c^2}}{\sqrt{2gc}}\int\frac{dx}{\sqrt{x}} = \sqrt{\frac{1+c^2}{2gc}}\times 2\sqrt{x} \;\Rightarrow \tag{7.1}$$

From (5.0) eq. $c=y/x$, plug this expression into (7.1) eq., we get:

$$t = \sqrt{1+\left(\frac{y}{x}\right)^2}\times\sqrt{\frac{2x^2}{gy}} \;\Rightarrow \tag{7.2}$$

From (5.0) eq. $x=y/c$, plug this expression into (7.2) eq., we get:

$$t = \sqrt{1+\left(\frac{y}{x}\right)^2}\times\sqrt{\frac{2x^2}{gy}} = \sqrt{1+c^2}\times\sqrt{\frac{2y^2}{gc^2y}} = \sqrt{\left(1+\frac{1}{c^2}\right)}\times\sqrt{\frac{2y}{g}} \;\Rightarrow \tag{7.3}$$

$$t = \sqrt{1+\left(\frac{x}{y}\right)^2}\times t_0 \tag{7.4}$$

The time of the ball rolling on an inclined straight ramp (7.4) eq. is equal to the minimum time t_0 (1.0) eq. multiply by a factor of inclined ramp geometry.

From geometry we know that the cotangent of linear functions $y=cx$ can be written as a constant value from (x, y) coordinates:

$$\frac{1}{y'} = \frac{dx}{dy} = \cot(\Theta) \;\Rightarrow\; \frac{x}{y} = \cot(\Theta) = \frac{1}{c} = const. \tag{8.0}$$

From geometry we know that $1+\cot(\Theta)^2 = 1/\sin(\Theta)^2$, thus:

$$t = \sqrt{1+\left(\frac{x}{y}\right)^2} \times t_0 = \sqrt{1+\cot(\Theta)^2} \times t_0 = \frac{t_0}{\sin(\Theta)} \tag{9.0}$$

In (Fig. 1) we have ramp $x=y$, $\Theta=45^o$, $\sin(45^\circ)=1/\sqrt{2}$, time of ball rolling is:

$$t = \sqrt{2}t_0 \tag{9.1}$$

To solve the acceleration of the rolling ball we apply an alternative method to calculate rolling time t. Momentum same as energy sustains quantity conservation law condition, thus:

$$p = Ft = mv = const., \ [N \cdot s] \tag{10.0}$$

$$B_0 = mvy = F_0 t_0 y, \ [J \times s] \tag{11.0}$$

$$B = mvS = FtS, \ [J \times s] \tag{11.1}$$

Two equations above are energy impulse terms:

$$E_0 = \frac{B_0}{t_0} = F_0 y; \ E = \frac{B}{t} = FS \ \Rightarrow \ F_0 y = FS = const., \ [J] \ \Rightarrow \tag{12.0}$$

$$\frac{mvy}{t_0} = \frac{mvS}{t} \ \Rightarrow \ \frac{y}{t_0} = \frac{S}{t} \ \Rightarrow \ t = \frac{S}{y}t_0 \ \Rightarrow \ t = \frac{\sqrt{x^2+y^2}}{y}t_0 \tag{12.1}$$

From the (12.0) equation we solve the ball acceleration value:

$$F_0 y = FS \ \Rightarrow \ mgy = maS \ \Rightarrow \ a = \frac{gy}{S} = g\sin(\Theta) = const. \tag{13.0}$$

Ball acceleration a is a constant value over the entire S distance because ball motion is driven by Earth's gravity force $F_G = F_0 = mg = const.$ expressed through free fall acceleration $g = 9.81\,m/s^2$ physics constant.

Rolling ball velocity change $v = f(S)$ over the entire path $S = \sqrt{x^2+y^2}$ is equal to:

$$v = \sqrt{2gS\sin(\Theta)} \qquad \begin{cases} S \in [0, \ S] \\ v \in [0, \ v] \\ v(S) = \sqrt{2gy} \end{cases} \tag{14.0}$$

4. <u>Cylinder unwinding from a thread of length *y* time solution</u>

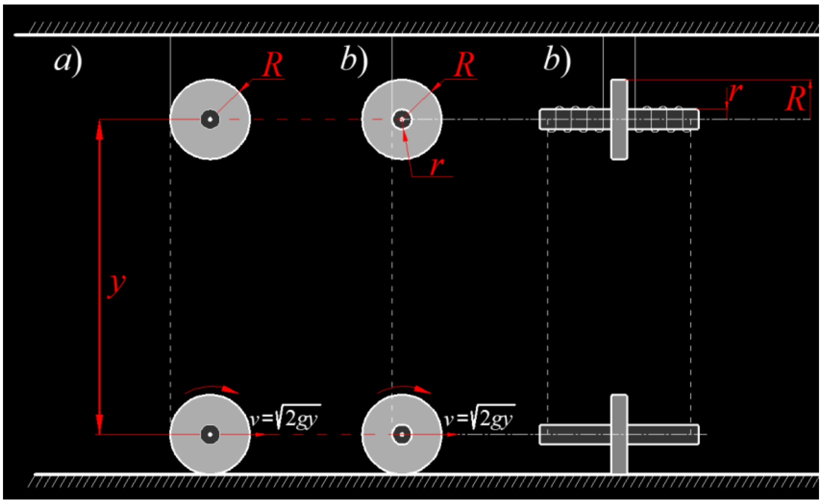

Fig. 2 a) Various radius cylinder unwinds from thread perpendicular to ground from R.
b) Various radius cylinder unwinds from thread perpendicular to ground from r.

In (Fig. 2 a)) the unwinding length of the thread is equal to the cylinder traveled path $S=y$, thus time solution is the same as (1.0) eq.:

$$t_0 = \sqrt{\frac{2S}{g}} = \sqrt{\frac{2y}{g}} \tag{15.0}$$

In (Fig. 2 a)) the unwinding length of the thread is $S=y$, but a bigger radius R cylinder travels a longer distance path $S_R > S_r$, thus to compensate for it, travel time will be longer:

$$t = \frac{R}{r} t_0 \tag{15.1}$$

Both cylinders at the touchdown point on the ground have a similar linear velocity $v = \sqrt{2gy}$, which only depends on maximum altitude value y, and is solved according to (2.0) eq. not to violate energy conservation law condition $E_p = E_k$.

5. Different radius cylinder rolling on inclined straight ramp

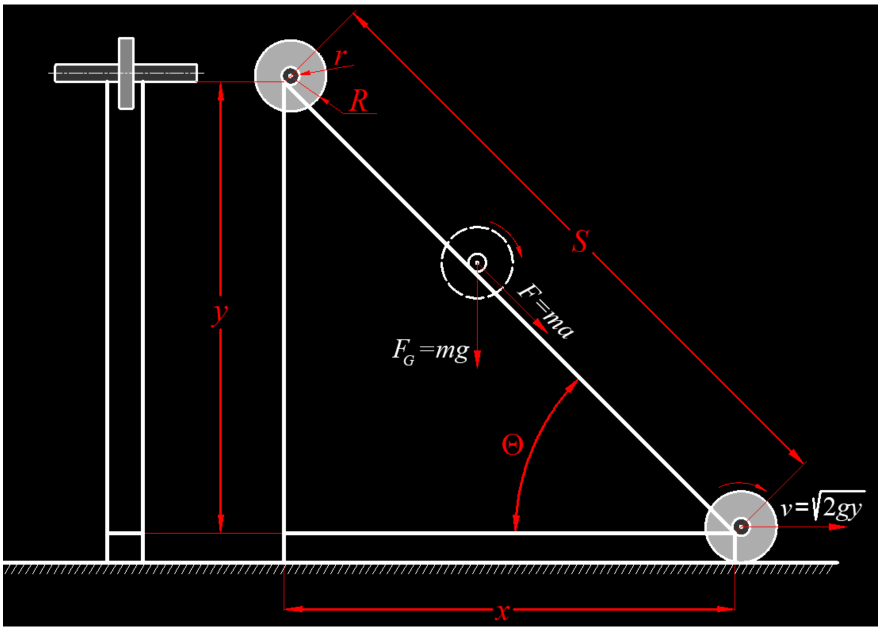

Fig. 3 Various radius cylinder rolls on the inclined ramp with angle Θ.

In (Fig. 3) the combined radius cylinder $R > r$ rolls on the inclined ramp with angle Θ, rolling occur on a smaller radius r cylinder side, if rolling occurs on the bigger side of radius R, then see (7.4) equation.

The total time of ball roll on the entire length S path is equal to:

$$t = \sqrt{\left(\frac{R}{r}\right)^2 + \left(\frac{x}{y}\right)^2} \times t_0 \tag{16.0}$$

The (16.0) equation is the general time calculation formula for a straight ramp, thus from (16.0) we can derive all possible combinations of previous examples:

- For $R = r$ and $x = 0$, we get (1.0) and (15.0) equations.
- For $R > r$ and $x = 0$, we get a (15.1) equation.
- For $R = r$ and $x > 0$, we get (7.4) and (9.0) equations
- For $R > r$ and $x > 0$, we have a (16.0) equation.

6. Liquid tank drainage time solution, Torricelli law [1]

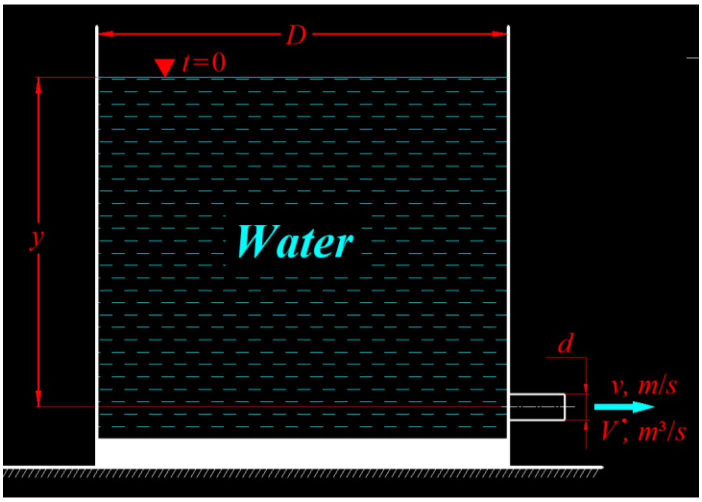

Fig. 4 Cylindrical water storage tank of maximum height y and diameter D.

Drainage time t solution of Newtonian-liquid from starting time t=0, at that time maximum height is y after time $t>0$ height of water column will be y=0.

Drainage time solution is based on a differential equation system of liquid volume:

$$dV = \begin{cases} A_T dy \\ A_p v dt \end{cases} \Rightarrow dV = \begin{cases} \pi D^2/4 dy \\ \pi d^2/4\sqrt{2gy}dt \end{cases}, \ [m^3] \Rightarrow \tag{16.0}$$

Here: A_T – cross-section area of the tank, A_p – cross-section area of drainage nozzle, v – liquid velocity through drainage nozzle, $V^{\bullet} = A_p v = A_p\sqrt{2gy}, \ [m^3/s]$ – volumetric flow rate.

$$\pi d^2/4\sqrt{2gy}\int dt = \pi D^2/4\int dy \Rightarrow \int dt = \frac{D^2}{d^2\sqrt{2g}}\int\frac{dy}{\sqrt{y}} \Rightarrow \tag{16.1}$$

$$t = \left(\frac{D}{d}\right)^2 \frac{1}{\sqrt{2g}} \times 2\sqrt{y} \Rightarrow t = \left(\frac{D}{d}\right)^2 \sqrt{\frac{2y}{g}} \Rightarrow \tag{16.2}$$

$$t = \left(\frac{D}{d}\right)^2 t_0 = \left(\frac{R}{r}\right)^2 t_0 \tag{16.3}$$

Additionally, fluid mechanics and rigid body mechanics have one thing in common, (16.3) eq. is proportional to radius or diameter squared $t \propto r^2$, whereas in rigid body mechanics cylinder thread unwinding time example (Fig. 2 b)), (15.1) eq., time t is directly proportional to smaller cylinder radius $t \propto r$.

The heuristic approach of solution can be explained based on the fact that at time t=0, initial liquid mass in the tank is $m_0 = \rho A_T y, \ [kg]$, the initial mass flow rate is $M_0^{\bullet} = \rho A_p\sqrt{2gy}, \ [kg/s]$, here ρ – liquid density. After drainage is completed y=0, the flow rate will be zero $M^{\bullet} = 0$, total duration of drainage $t > 0$, thus mean mass flow rate value is between these two values:

$$\overline{M^{\bullet}} = \frac{M_0^{\bullet} + M^{\bullet}}{2} = \frac{M_0^{\bullet}}{2}, \ [kg/s] \tag{17.0}$$

Free fall g=9.81 m/s^2 physics constant pushing liquid down at every height dy point at a constant value g=9.81 m/s^2, thus (17.0) has a linear trend of mean value between two mass flow rate values of maximum and minimum.

The heuristic solution of drainage time t (Fig. 3) is equal to:

$$t = \frac{m_0}{\overline{M}^{\bullet}} = \frac{2m_0}{M_0^{\bullet}} \quad \Rightarrow \tag{17.1}$$

$$t = \frac{2m_0}{M_0^{\bullet}} = \frac{2\rho\pi D^2/4y}{\rho\pi d^2/4\sqrt{2gy}} = \left(\frac{D}{d}\right)^2\sqrt{\frac{2y}{g}} = \left(\frac{D}{d}\right)^2 t_0 \tag{17.2}$$

Equation (17.1) can be applied to solve drainage time for all abstract shape volume storage tanks, under the condition that we know the initial mass of liquid m_0, initial liquid column height y, and any shape cross-section area A_p of drainage nozzle.

7. The Cycloid function is applicable for the shortest rolling time curve design

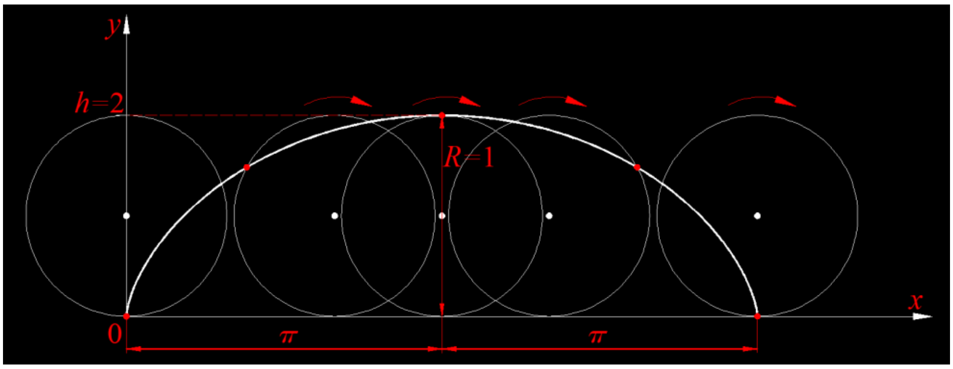

Fig. 5 The cycloid f-ion is drawn as a circle of radius R rolling along the x-axis.

The cycloid of the R=1 (Fig. 5) in polar coordinates, a system of equations are:

$$\begin{cases} x(\Theta) = R[\Theta - \sin(\Theta)] \\ y(\Theta) = R[1 - \cos(\Theta)] \end{cases} \qquad \Theta \in [0,\ 2\pi] \tag{18.0}$$

<The brachistochrone problem is considered to be one of the foundational problems of the calculus of variations. The standard method in that field now uses the Euler–Lagrange equation. Bernoulli's 1697 solution, which did not use the Euler–Lagrange equation, is of great interest to both mathematicians and physicists . [2]>

8. The brachistochrone shortest ball rolling time curve

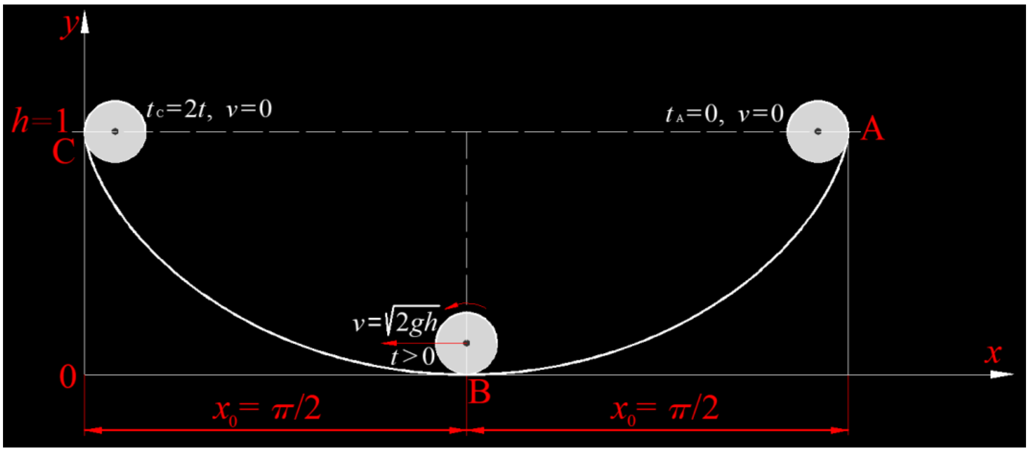

Fig. 6 Ball rolling on brachistochrone curve which is inverted cycloid f-ion of R=1/2 m.

<In Bernoulli's brachistochrone problem one has two points at different elevations and one seeks the minimum-time curve for a particle to slide frictionlessly from the higher point to the lower point. [2] >

Our choice function that resembles an inverted cycloid function is:

$$x^k + y^k = h^k \tag{19.0}$$

Our new function (19.0) eq. hold similarity with the cycloid function in one significant way, in (Fig. 6) at point A, the initial angle with the x-axis is Θ_0=90°, and at point B angle with the x-axis angle is Θ=0 degrees.

The physics problem of time roll calculation can be reversed, the total time t to roll downhill from height h=1m to zero h=0, is the same as the ball rolling uphill with initial maximum velocity $v=\sqrt{2gh}$, from moment $t=0$ till point C time will be the same $t_C=t$.

Equation (19.0) differs from cycloid f-ion in that (19.0) eq. curve is always isosceles function $x_0=h$, whereas a cycloid function is always $x_0>h$. The Isosceles f-ion property will allow us additionally to compare the inverted quarter of circle curve ball rolling time t, which solution has a well-established formula [3], which is same as the pendulum period formula.

9. Function's $x^k + y^k = h^k$ numerical solution of shortest time of ball rolling

The function of various $k\in(0,\ 1]$, which resembles inverted cycloid curve is such:

$$x^k + y^k = h^k \ \Rightarrow\ y=(h^k - x^k)^{\frac{1}{k}} \qquad \begin{cases} x\in[x_0,\ 0] \\ y\in[0,\ h] \\ h=x_0 \end{cases} \tag{19.1}$$

Ball rolling time t solution for (19.1) eq. is the same method as for a straight ramp described in §2 and §3 chapters:

$$t=\frac{1}{v}\int dS=\frac{1}{\sqrt{2gy}}\int\sqrt{1+(y')^2}\,dx=\frac{1}{\sqrt{2g}}\int\frac{\sqrt{1+\left(-x^{k-1}(h^k-x^k)^{(1-k)/k}\right)^2}}{\sqrt{(h^k-x^k)^{1/k}}}dx \tag{20.0}$$

Above, the result is an undefined integral, that does not have the solution simply by applying integration rules and methods which are available today. Professional mathematical software of WolframAlpha, MathCad, and Desmos also have no definite result.

Our goal is by using discrete number-based computer code [6], instead of calculus to find power coefficient k, which satisfies the minimum rolling time t_{min} condition k=[t_{min}].

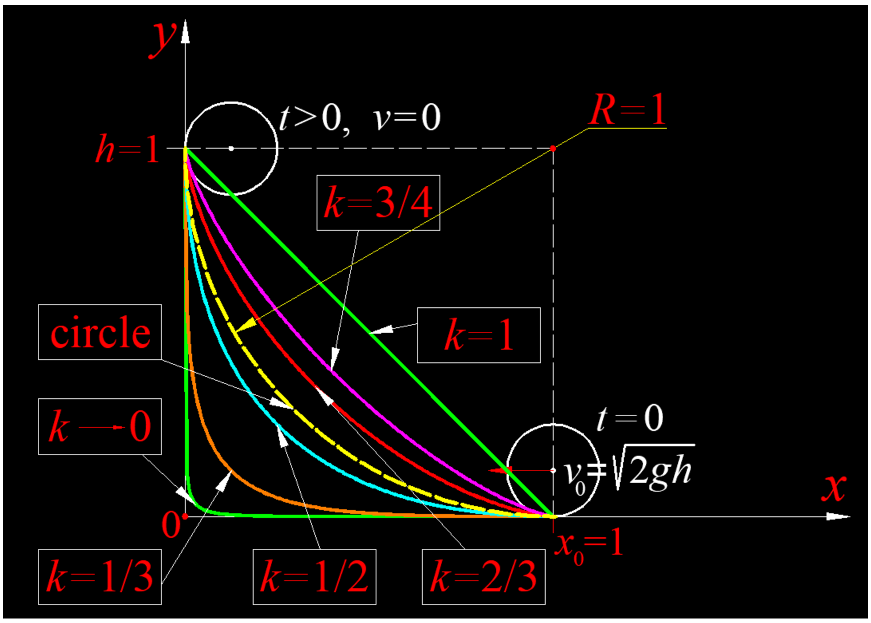

Fig. 7 Various f-ion $y(x)=(h^k-x^k)^{1/k}$ curves from variable $k\in(0,\ 1]$.

A coefficient $k \to 0$ is the maximum time rolling f-ion with maximum rolling length $S \to 2\ m$ and $k=1$ is the shortest length curve S_{min}:

$$k \to 0: \ t_{max} = t_0 + \frac{h}{v} = \sqrt{\frac{2h}{g}} + \frac{h}{\sqrt{2gh}} = 0.45152 + \frac{1}{4.4294} = 0.67729 \ \ s \tag{20.1}$$

$$k = 1: \ t = \sqrt{2} t_0 = \sqrt{2}\sqrt{\frac{2h}{g}} = \sqrt{2} \cdot 0.45152 = 0.63855\ s, \ S_{min} = \sqrt{2} = 1.414\ m \tag{20.2}$$

Search for the shortest time curve that will be bounded between these two curves. Also (20.2) eq. is different from all variable curves $k \in (0,\ 1]$ in that it is a straight ramp with a constant curve slope angle $\Theta = 45^{o}$ in the entire path S.

Outside our new f-ions, we also compare the inverted quarter of the circle curve which time calculation is a well-established fact and equal to pendulum swing time t [3] from the maximum possible angle of $\Theta = 90^{o} = \pi/2$:

$$t = \frac{\pi}{2}\sqrt{\frac{h}{g}} \cdot \sum_{n=0}^{N=20}\left[\frac{(2n)!}{\left(2^n n!\right)^2}\sin\left(\frac{\Theta}{2}\right)^n\right]^2 = 0.59196 s; \ S = \frac{\pi h}{2} = 1.57080 m \tag{20.3}$$

Interesting fact $k=1/2$, function and an inverted quarter of circle are two different functions, common feature is that both f-ions at height h slope angle is $\Theta = 90^{o}$, and at x_0 angle is $\Theta = 0^{o}$, nor or less both are two different f-ions with different curve properties, see (Fig. 7). The reason why the circle has a calculus-based solution (20.3) eq. is that that radius of the circle $h=R=const.$ is a constant value in the entire S path and is put in front of the summation sign as a scalar, such simplification can not be applied to our new functions. Therefore time results will be obtained by the process of the brutal force of computation power.

The algorithm [6] for every possible coefficient $k \in (0,\ 1]$ is such:

Initial values and conditions:

$$n = 0,\ 1\ \dots\ N,\ \ N \to \infty,\ k \in \mathbb{R}, \text{ for } n = 0: \ x_0 = 1, \ h = x_0, \ v_0 = \sqrt{2gh}, \ S_0 = 0, \ \Theta_0 = 0.$$

Coordinates of x_n and y_n:

$$\begin{aligned} x_n &= x_0(1 - n/N) \\ y_n &= [h^k - (x_n)^k]^{1/k} \end{aligned} \tag{21.0}$$

Curves segments lengths and total curve length are:

$$S_{n+1} = \sqrt{(y_{n+1} - y_n)^2 + (x_n - x_{n+1})^2} \tag{21.1}$$

$$S = \sum_{n=0}^{N} S_n \tag{21.2}$$

Curve segments slope angles:

$$\Theta_{n+1} = \tan^{-1}\left(\frac{y_{n+1} - y_n}{x_n - x_{n+1}}\right), \ N \to \infty, \ \Theta_N \to \frac{\pi}{2} \tag{21.3}$$

Ball velocity at every curve segment length S_n and every angle Θ_n:

$$v_{n+1} = v_n - \frac{g S_{n+1}}{v_n}\sin(\Theta_{n+1}), \ N \to \infty, \ v_N \to 0 \tag{21.4}$$

Ball roll times at every curve segment length S_n and total time are:

$$t_n = \frac{S_n}{v_n} \tag{21.5}$$

$$t = \sum_{n=0}^{N} t_n \tag{21.6}$$

N	k	t, s	S, m	comments
none	$k \to 0$	**0.67729**	$S_{max}=2$	**Max. time *t* curve**
10^6 10^7 10^8	1/3	0.63161 0.63252 **0.63318**	1.80214	**None**
10^6 10^7 10^8	1/2	0.59794 0.59897 **0.59956**	1.62323	**None**
10^6 10^7 10^8	2/3	0.58253 0.58334 **0.58373**	1.5	**Astroid curve**
10^6 10^7 10^8	3/4	0.58329 0.58399 **0.58430**	1.46009	**None**
none	1	**0.63855**	$S_{min}=1.41421$	**45° slope straight ramp**
none	none	**0.59196**	1.57080	**Circle curve**
none	none	**0.45152**	1	**Min. *t* perpendicualar to ground path**

Table 1 Function's $x^k + y^k = h^k$ total time t and total length S results from a total iteration number N.

We have a very interesting race happening between f-ion curve angle Θ_n versus total distance S. From geometry we know the fact that the shortest path is a straight line, except we see that at k=1 curve time is over $t > 0.6$ seconds mark. Another bounding curve $k \to 0$ has a very steep 90° angle and the total time is even worse than a straight's ramp.

Thus, the winner must be a compromise between two of k bounding values, a considerably steeper angle curve and a considerably shorter length curve:

$$k = 0.7005 \;\Rightarrow\; t_{\min} = \sum_{n=0}^{N=10^8} t_n = 0.58319237s,\;\; S = 1.48213m. \qquad (21.7)$$

10. Astroid and brachistochrone are quite similar curves.

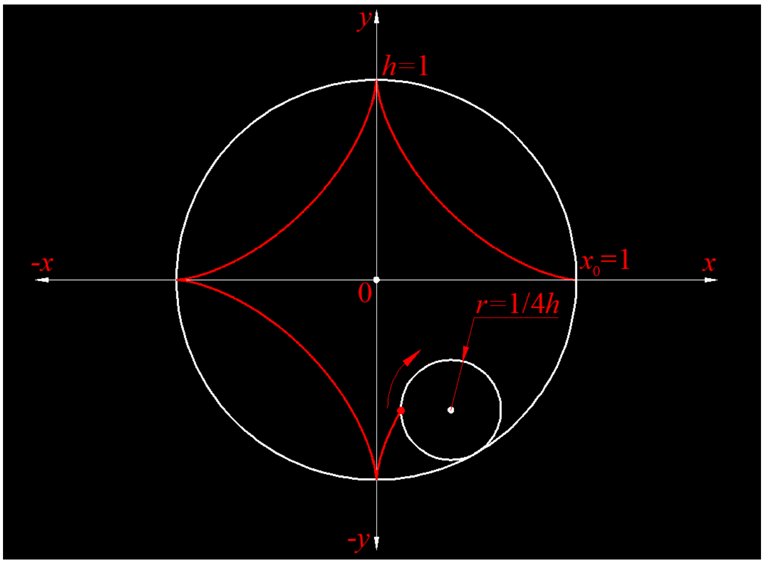

Fig. 8 Astroid curve is a function of $x^{2/3}+y^{2/3}=h^{2/3}$.

The Astroid curve [4] is similar to the inverted cycloid function see (Fig. 5) and (Fig. 6) pictures. The Astroid curve produced see (Fig. 8) by a fixed red dot on the circumference of a small circle of radius r=1/4h, rolling around the inside of a large circle of radius h. The astroid curve is one of many a special family of curves called *hypocycloids* [5].

In (Table 1) shows rolling time accuracy from N iteration number value, for $k=2/3$ and for $k=0.7005$, (21.5) eq. which is the holder of the shortest time curve, both have very similar times $t=0.583\,s$. Accuracy will be constant at three digits value and independent of iteration number $N\rightarrow\infty$, therefore both curves have similar time results to 1/1000th of the second accuracy. I am aware that our math problem is a thought experiment based on ideal conditions where no friction, air resistance, or slippage occurs, and we care about horse race photo-finish or even closer to quantum physics precision results. But as I mentioned before the Astroid has exactly an S=1.5 m length and thus a longer curve length than k=0.7005 but the angles of the astroid curve are also steeper than k=0.7005, which is a reason why both time results are approximately equal. My hypothesis same is for the cycloid function on which the shortest time brachistochrone curve is based, it also has many close resembling possible brachistochrone curves in the range of 1/1000th of the second accuracy.

11. Scalarity

We choose to calculate time t at distances x_0=1 meter, h=x_0=1 meter. This was our arbitrary choice. Equations (20.2) and (20.3) show that time is $t\propto\sqrt{h}$. This proportionality relation is fundamental to all curve shapes, 1-meter height is already calculated value if we want any other different height $0\geq H\geq 1$ than h=1 m, we can use such equation:

$$t_H=\sqrt{\frac{H}{h}}\times t=\sqrt{\frac{H}{1}}\times t=\sqrt{H}\times t\,,\ \text{for } H=h=1\,m\ \Rightarrow\ t_H=t \qquad (22.0)$$

The equation above can be used to calculate all times results when t time is already calculated and the result is recorded, for example, time t results depicted in (Table 1).

12. Conclusions

The general case of a round object rolling on a straight inclined ramp can be solved using the (16.0) equation. From this equation can be derived other cases.

The shortest time t of ball rolling on the function $x^k + y^k = h^k$ family curves is when at maximum iteration number N=10^8 power coefficient k value is equal to:

$$k=0.7005\ \Rightarrow\ t_{\min}=0.58319237\quad s.$$

13. References

https://en.wikipedia.org/wiki/Torricelli%27s_law [1]

Herman Erlichson "*Johann Bernoulli's brachistochrone solution using Fermat's principle of least time*" Department of Engineering Science and Physics, The College of Staten Island, The City University of New York, Staten Island, NY l0314, USA [2]

https://en.wikipedia.org/wiki/Pendulum_(mechanics)#Legendre_polynomial_solution_for_the_elliptic_integral [3]

https://mathworld.wolfram.com/Astroid.html [4]

https://mathworld.wolfram.com/Hypocycloid.html [5]

https://github.com/ArthurKarbocius/Ball-rolling-shortes-time-solution/blob/master/shortest_time.py [6]