Algorithm for Calculation of Earth Atmosphere Mass.

According to the I – law of thermodynamics heat change dq depends on the system's internal energy change du and object work change dw, thus total heat energy is $dq = du + dw$. For ideal gasses, I – law of thermodynamics heat change dq equation is:

$$\overbrace{pdv}^{dw} + \overbrace{vdp}^{du} = \overbrace{RdT}^{dq}, \quad [J/kg] \tag{1.0}$$

Here: $R = 287 \; J/(kg \times K)$ – universal gas constant, p – gas pressure, $[Pa]$, $v = 1/\rho$ – gas specific volume, $[m^3/kg]$, ρ – gas density, $[kg/m^3]$.

Isothermal – $dq = 0$, $dT = const.$ – slow gas expansion – compression then afterward gas temperature stays unchanged. Earth pressure density change from height is based on **isothermal** gas expansion from height change. Both are similar units: $\rho(h) \sim \rho(h)$ based on the ideal gas equation, for more details see (1.05) equation.
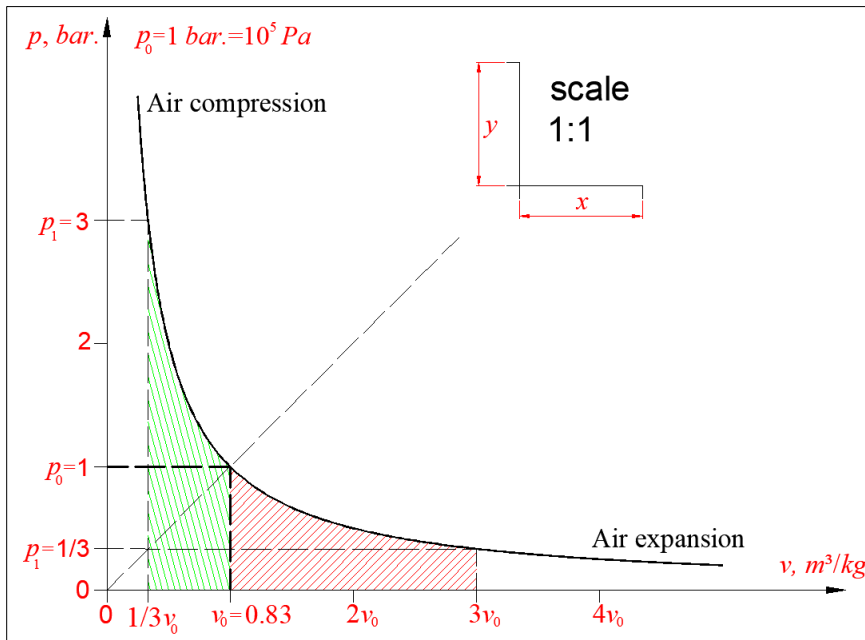

Fig. 1 Isothermal process $p = const./v$ of air compression – expansion.

Equation (1.04) is a square (isosceles) hyperbola math function and it defines a thermodynamic isothermal process of air compression – expansion.
The enthalpy equation of an ideal gas is:

$$pv = RT, \; [J/kg] \; \Rightarrow \; p/\rho = RT. \tag{1.01}$$

First, from ideal gas (1.01) eq. express gas (air) density term and plug this term into a differential equation for compressible gas:

$$\rho = \frac{p}{RT} \Rightarrow dp = \rho g dh \Rightarrow dp = \frac{p}{RT} g dh. \tag{1.02}$$

Multiply the (1.02) equation by $1/p$ and later integrate the left side from dp in a range from p_0 to p_1 and the right side from dh in a range from $h_0 = 0$ to h:

$$\int_{p_0}^{p} \frac{dp}{p} = \frac{g}{RT} \int_{0}^{h} dh \; \Rightarrow \; -\ln\left(\frac{p}{p_0}\right) = \frac{gh}{RT} \; \Rightarrow \; p = p_0 e^{\frac{-gh}{RT}}. \tag{1.03}$$

Equation (1.03) shows isothermal air pressure decreases from height altitude change $p(h)$. The density of air change $\rho(h)$ from height (altitude) change is:

$$\rho = \rho_0 e^{\frac{-gh}{RT}} \tag{1.04}$$

We can prove that pressure and density are similar units $p \sim \rho$ or $p \sim 1/v$:

$$p = p_0 e^{\frac{-gh}{RT}}; \quad v = v_0 e^{\frac{gh}{RT}}; \quad p_0 e^{\frac{-gh}{RT}} \cdot v_0 e^{\frac{gh}{RT}} = RT \quad \Rightarrow \quad pv = RT = const. \tag{1.05}$$

Calculation of average air density based on intensive property of density. Average air temp. is $T = 273 + 20°C = 293\,K$, at $h_0 = 0$, thus $\rho(h_0) = 1.205$, to $h = 10^6$ meters, thus $h \to \infty$, $\rho(h) \to 0$, a sum of air density performed by an increment of 1 meter $\delta = h/N = 1$ till $h = N\delta$:

$$\overline{\rho} = \frac{\rho_0}{N} \sum_{n=0}^{N=10^6} e^{\frac{-g\delta}{RT}n} \tag{1.06}$$

The sum of (1.06) eq. express via infinite geometric series, then $N = 10^6 \to \infty$, and $x < 1$:

$$S = \sum_{n=0}^{N \to \infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}, \quad e^{-x} = e^{\frac{-g\delta}{RT}} = 0.999883 < 1 \tag{1.07}$$

$$\overline{\rho} = \frac{\rho_0}{N} \sum_{n=0}^{N} e^{-xn} = \frac{\rho_0}{N} \sum_{n=0}^{N} e^{\frac{-g\delta}{RT}n} = \frac{\rho_0}{N} \cdot \frac{1}{1 - e^{-g\delta/RT}} = 0.0103298 \; kg/m^3 \tag{1.08}$$

$$e^{-x} = \sum_{n=0}^{N \to \infty} (-1)^n \frac{x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots \tag{1.09}$$

For $x = g\delta/RT = 1.183 \cdot 10^{-4} \approx 0$ we approximate (1.09) eq. to such expression:

$$e^{-x} \approx 1 - x \quad \Rightarrow \quad e^{\frac{-g\delta}{RT}} \approx 1 - g\delta/RT. \tag{1.10}$$

Plug (1.10) eq. into (1.08) eq., we get:

$$\overline{\rho} = \frac{\rho_0}{N} \cdot \frac{1}{1 - e^{-g\delta/RT}} = \frac{\rho_0}{N} \cdot \frac{1}{1 - (1 - g\delta/RT)} = \frac{\rho_0 RT}{gN\delta} = \frac{p_0}{gh} \tag{1.11}$$

The result of (1.11) eq. is known, see (1.08) eq., thus atmospheric pressure is:

$$p_0 = \overline{\rho} gh = 1.01335 \cdot 10^5 \cong 10^5 \; Pa \tag{1.12}$$

Atmospheric pressure result in (1.12) eq. is accurate to the actual measured value – $p_0 = 1.01325 \times 10^5 \, Pa$ [1]. Mass of air is calculated by summing air density from range $h_0 = 0$ till $h = 10^6\,m$, $N = 10^6$, over $\delta = h/N = 1$ meter increments, and multiplying by sphere volumes in the shape of "onion rings" over $\delta = h/N = 1$ meter radius increments starting from a radius of Earth – $R_{eth} = 6371000\,m$, for more details see (Fig .2):

$$M_{air} = \rho_0 \sum_{n=0}^{N} e^{\frac{-g\delta}{RT}n} \cdot \frac{4}{3}\pi \sum_{n=0}^{N} \left[(R_{eth} + \delta(n+1))^3 - (R_{eth} + \delta n)^3 \right] = 5.283 \times 10^{18} \; kg \tag{1.13}$$

Convergence condition to (1.13) equation is:

$$M_{air} = \lim_{N \to \infty} \left(\sum_{n=0}^{N} \rho_n \cdot \sum_{n=0}^{N} V_n \right) \to \text{finite} \tag{1.14}$$
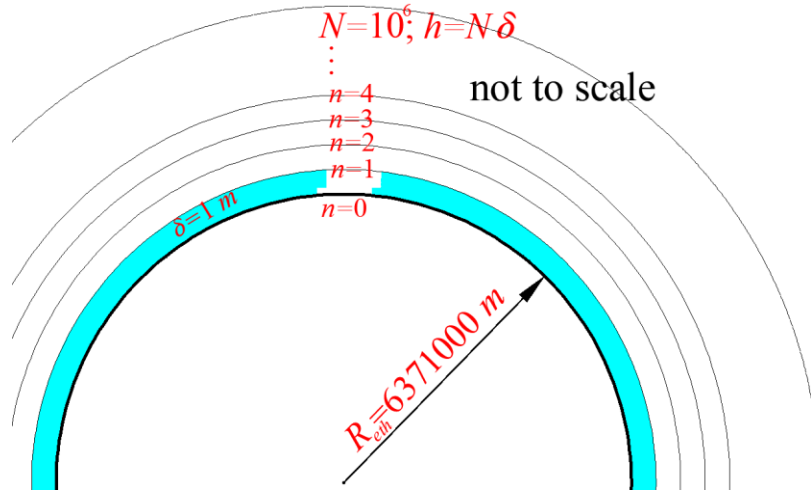
Fig. 2   Sketch for Earth air mass calculation over 1-meter increment.

Error calculation two facts increase the total mass of atmosphere and one factor decrease. Temperature increases starting form $h=0$ About $T(h)=0.0065$ K/m till $100km$ height – Karman line also has name Thermosphere from that point temperature starts again increasing. And free fall acceleration constant dependable from height:

$$g(h) = \frac{GM_{earth}}{(R_{erth}+h)^2}, \quad \left[\frac{m}{s^2}\right] \tag{1.15}$$

Most of Earth's atmosphere is close to Earth's ground thus 2 km where atmosphere is dense. A negative factor is a landmass, which is any object that is above sea level, mountains peaks displace Earth's atmosphere mostly. By all consideration, Earth's mass is 10^{15} kg bigger than calculated by the (1.13) equation:

$$M_{air} = 5.283 \times 10^{18} kg + 10^{15} \ kg = 5.284 \times 10^{18} kg \tag{1.16}$$

Errors are 1/1000th or 0.1% bigger than calculated.

According to Google search, Earth's mass is $M_{air}=5.15\times10^{18}$ kg, ratio very close to calculated here. Chat-gpt inquiry is very close to the same result of $M_{air}=5.14\times10^{18}$ kg.