Mathematical model of vehicle fuel consumption solution based on car initial maximum speed

Author: Artūras Karbočius
Adress: phyarth@gmail.com

1. Introduction

In this paper, we thoroughly go to investigate energy terms of vehicles in motion, converted into fuel consumption results. Math model investigation will be symbolic and numerical. For example, does it take more fuel to start an engine than to keep it running? We investigate thoroughly concepts of inertia, static friction, kinetic friction, air resistance – car aerodynamics factor, and road slope angle gradient factors based on various car speed values. Compare the scale economy factor of freight trains versus freight trucks and find a reason why freight trains are more efficient.

Keywords: Static friction, kinetic friction, kinetic energy, inertia.

2. Most simplistic and symbolic calculation of fuel consumption

Here we compare the same car that goes let's say L=20 km to the city in the same direction on a completely flat road. This assumption is implausible but if you think of any road you see that road sometimes goes uphill and some segments of the road are downhill, thus we can average out and only consider flat roads or driving on Cartesian x-axis road type.

First-time car drives with constant speed v_1=*const.* entire road length L, second try the same car drives the same distance but with 2 times higher velocity $v_2=2v_1$ than the previous try. Fuel consumption of first-time drive is m_f, the second time is:

In this simplistic model, we get that the second time with 2 times higher velocity car kinetic energy is 4 times higher than the first try. The duration time of car driving is two times shorter than the second try $t_2=L/(2v_1)$, $t_2=1/2t_1$, engines operation time of travel distance L=20km requires two times shorter duration, thus at the end fuel consumption is not 4 times higher but two times higher $2m_f$. In our model, we skip the air resistance factor, thus $>2m_f$. Such simplification and logic behind are wrong. We need a reverse engineer to our physics problem and investigate every energy term that has an impact on car motion.

3. Static energy friction term

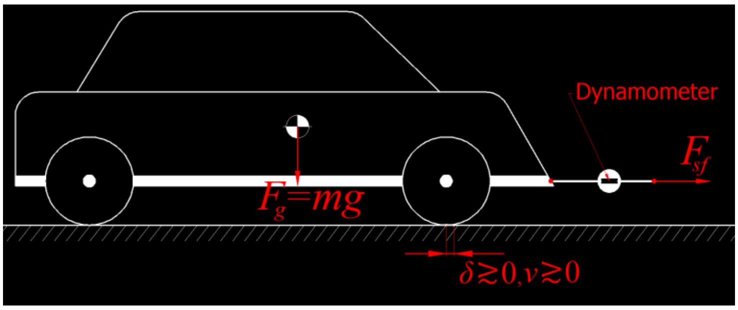

Fig. 1 Min. force F_{sf} required to move a particular car from a stillpoint.

Static force represents all friction resistance forces that exist for a particular car, (Fig. 1). And these factors are: tire friction with the ground, the "flatness" of tire factor [1], and pavement softness friction, for example, softer asphalt versus harder concrete, wheels bearings friction, piston movement friction inside the internal combustion engine, gears friction, and many more possible factors, that we don't care because we experimentally measure static friction force F_{sf} for a particular car measured in Newton's units. Static friction is always present and is the main factor of energy loss:

$$E_{sf} = F_{sf} \cdot L,\ [J] \tag{1.0}$$

In (1.0) eq. static friction force F_{sf} and driving distance L product gives static friction energy value. An analogy of static force friction is Earth's potential energy, which is always present; this fundamental law is not true for kinetic (speed) friction.

4. Earth gravity energy term

If a car goes uphill distance S with slope angle with the x-axis is positive Θ:

$$+E_g = F_g \sin(\Theta) S = mg \sin(\Theta) S \tag{2.0}$$

If a car goes downhill distance S with slope angle with the x-axis is negative $-\Theta$:

$$-E_g = F_g \sin(\Theta) S = mg \sin(\Theta) S \tag{2.1}$$

Downhill can be very helpful especially if a car is heavy you can transport many goods for free only using Earth's gravity force and the opposite is true for uphill motion, a vehicle must overcome Earth's gravity force.

Solve min. a static force from the perspective of min. slope angle instead of min. displacement $\delta \gtrsim 0$ and min. velocity $v \gtrsim 0$:

$$F_{sf} = F_g \sin(\Theta_{\min}) \Rightarrow F_{sf} = mg \sin(\Theta_{\min}) \tag{2.2}$$

Put a car on a platform and increase the angle very slowly, the angle at which a car starts slightly move will be the min. angle Θ_{min} that requires overcoming static force. The road can be built at that angle that requires minimum engine assistants $\Theta > \Theta_{min}$. Building such roads that have forward and backward directions always downhill is a very challenging task.

Experimental measure the road gradient of a particular interval of the road, for example, L=20 *km*, using some kind of "sonar" measurement device, which also measures the sloped distance δS, thus average "mountainish" angle gradient over total distance S is:

$$\pm\overline{\Theta} = \frac{\pm\Theta_1 \pm \Theta_2 \pm \Theta_3 \pm \ldots \pm \Theta_N}{N} \tag{2.3}$$

If a car travels forth and back on the same road, the average angle gradient cancels each out. Road $S > L$, where L – is the total distance on the flat 2D map.

5. Kinetic energy friction term

The difference between kinetic and static and kinetic energy friction kinetic can be avoided in motion E_{kf}, where static force in any travel distance L is a constant value.

< *... the friction is nearly independent of the velocity. Many people believe that the friction to be overcome to get something started (static friction) exceeds the force required to keep it sliding (sliding friction), but with dry metals it is very hard to show any difference*. [2]>

From Prof. Richard Feynman's lecture quote, we see that is very difficult to separate to concepts of friction static and kinetic. He is right, that is very confusing at least. My thesis is that in rigid body mechanics, static friction is the analogy of non-Newtonian liquids, to allow fluid to start to flow from a stillpoint requires some kind of threshold force or pressure

F_{sf} to overcome internal liquid elasticity. In fluid mechanics exist and Newtonian fluids, all types of gases, water, kerosene, light diesel, and many more, which have zero static force value. In rigid body mechanics, such perfect objects with zero static force are achievable only in laboratory conditions, motion in a vacuum chamber of a perfectly round smooth surface ball or cylinder on a perfectly flat hardened metal with a smooth surface. Such a condition implies the 1st – Newton law of motion, the ball pushed once with initial velocity $v>0$ will roll inside the chamber never slowing down. From such implication, we can conclude that starting the automobile from zero speed point does not require more energy, but every object has internal static friction $F_{sf} > 0$.

Proof of static and kinetic friction will be based on boundary condition $v \to 0$:

$$\sum E = E_k \pm E_g + E_D + E_{sf} \quad \Rightarrow \tag{3.0}$$

Here: $E_k = mv^2/2$ – max. the kinetic energy of the vehicle, is also the car inertia term, $E_D = C_D A L \rho v^2/2$ – air drag energy losses, car aerodynamics term.

$$\sum E = \frac{mv^2}{2} + C_D \frac{A\rho v^2 L}{2} + F_{sf} L \;\Rightarrow \tag{3.1}$$

Numerical example. Car velocity has a “snail pace” $v=0.01\ m/s$ and the static force to move from the stillpoint is $F_{sf}=70\ N$, total driving distance is $L=20\ km=2\times10^4\ m$. Thus results of total energy in (3.1) eq. is:

$$\sum E = F_{sf} L\Big|_{v\to 0} = 1.4\times10^6\ J \tag{3.2}$$

Fuel consumption is:

$$m_f = \frac{\sum E}{\eta q} = 0.11\ kg \tag{4.0}$$

Here: $m=2t=2000\ kg$ – car’s total mass, $A=1.5\ m^2$ – frontal cross-section area of the car, $C_D=0.65$ – average frontal drag coefficient of the car, for more details see chapter §8, $\rho=1.2\ kg/m^3$ – air density, $\eta=0.3$ – internal combustion engine efficiency coefficient, $q=43\ MJ/kg$ – gasoline specific heat constant, m_f – fuel consumption in kilograms.

We see that result of (4.0) eq. that of “snail pace” velocity kinetic energy and air drag force F_D has a close zero impact on fuel consumption because both terms are proportional to velocity squared $E_k \propto E_D \propto v^2 \to 0$. Travel time is $t=L/(v\times3600\times24)=23.1$ days ☺. Kinetic energy (speed) $v=0.01 m/s$ and air resistance drag force have close to zero impact on total fuel consumption of $m_f = 0.11\ kg$ to travel $20 km$ was consumed by internal static friction energy of the car. However, to me is very perplexing, velocity is a term of motion that has close to zero impact $E_{sf} >> E_k$, $E_{sf} >> E_D$ on total energy consumption, even though we know for a fact that after 23.2 days the car will reach a destination of 20 kilometers!

6. Kinetic energy is an inertia force term. Inertia is energy for free

From physics we know, that the mass of an object is the magnitude (scalar) of inertia. In addition, if an object in our case car with mass m moves with velocity v, inertia wants to move with that velocity, to change or more precisely, to slow down we need internal force, for example, to use car breaks, which is additional force to existing internal static friction or slow down car with air drag resistance which is an external force. The previous example was “snail pace” speed, this chapter will be the extreme opposite, the car with a “crazy speed” of

v=50 m/s=180 km/h. First, we calculate fuel consumption by applying (3.1) eq. later, we reverse engineer the result by estimating the inertia factor of the car in motion:

$$\sum E = \frac{mv^2}{2} + C_D \frac{A\rho v^2 L}{2} + F_{sf} L = 3.3 \times 10^7 \;\; J \tag{5.0}$$

$$m_f = \frac{\sum E}{\eta q} = 2.6 \;\; kg \tag{5.1}$$

The algorithm [4] to estimate driving distance L_i by the inertia force of a car's speed. At moment t_0=0, v=v_0=50, time interval $-\ \delta t = 0.05$ seconds. If $\delta t \to 0$, thus L_i result converges to a precise value, for fuel consumption calculation, an error can be in the range of ±100 m.

Iteration step n=1:

$$L_1 = v_0 \delta t \tag{6.0}$$

$$E_{k1} = \frac{mv_0^2}{2};\;\; E_{D1} = C_D \frac{\rho v_0^2 A L_1}{2};\;\; E_{sf1} = F_{sf} L_1 \tag{6.1}$$

$$E_1 = E_{k1} - E_{D1} - E_{sf1} \tag{6.2}$$

$$v_1 = \sqrt{\frac{2E_1}{m}},\;\; v_1 < v_0 \tag{6.3}$$

Iteration step n=2:

$$L_2 = v_1 \delta t \tag{6.4}$$

$$E_{k2} = \frac{mv_1^2}{2};\;\; E_{D1} = C_D \frac{\rho v_1^2 A L_2}{2};\;\;\; E_{sf2} = F_{sf} L_2 \tag{6.5}$$

$$E_2 = E_{k2} - E_{D2} - E_{sf2} \tag{6.6}$$

$$v_2 = \sqrt{\frac{2E_2}{m}},\;\; v_2 < v_1 \tag{6.7}$$

$$\vdots$$

Iteration is done N times while a such condition is met:

$$v_N = 0 \tag{6.8}$$

The result of driving distance by inertia force is equal to:

$$L_i = \delta t \sum_{n=0}^{N} v_n = \sum_{n=0}^{N} L_n = 5273 \;\;\; m \tag{6.9}$$

We can trace back the algorithm and claim that the max. kinetic energy term $E_k \propto v^2$ is an energy conservation law condition term, also an object‘s in motion inertia storage term. Energy is stored in velocity, after a moment the engine is a turn-off, and the car is in neutral gear moving only by inertia force, result was $L_i \approx 5$ km driven for free. The reverse engineer our previous calculations and correct total energy and fuel consumption results in (5.0), (5.1) eqs.:

$$l = L - L_i = 14727 \;\;\; m \tag{7.0}$$

$$\sum E_i = \frac{mv^2}{2} + C_D \frac{A\rho v^2 l}{2} + F_{sf} l = 2.5 \times 10^7 \;\; J \tag{7.1}$$

$$m_{fi} = \frac{\sum E_i}{\eta q} = 1.95 \;\; kg \tag{7.2}$$

Fuel savings:

$$\Delta m_f = m_f - m_{fi} = 0.627 \;\; kg \tag{7.3}$$

The total distance is 20*km*; additionally, ≈5*km* was driven by inertia alone, and fuel savings are based on the boundary condition $v = 0$. The car converts kinetic energy into energy losses (heat) and the rest of the energy into useful distance L_i, which is a plausible but doubtful scenario, for safety concerns car velocity can drop to min. 10*m/s* from a max. of 50*m/s* to save 0.5 *kg* of fuel. Such a scenario of v=0 is possible for trains. An Inertia is a magnitude (scalar) of mass, even though our calculation focuses mainly from the perspective of max. velocity v_0. Freight trains of course can pull much heavier massive cargo than freight trucks and can take into account the inertia force of their enormous mass. Another reason for energy savings is that railway tracks are dedicated to one train at a time. A freight train has an economic benefit, economy-of-scale, or in our case economy-of-mass. Additionally, freight trains also have low wheel rolling resistance coefficient: railroad steel wheel on steel rail is C_{Dr} = 0.0010 to 0.0024 and large truck (semi) tires are C_{Dr} = 0.0045 to 0.0080 [3].

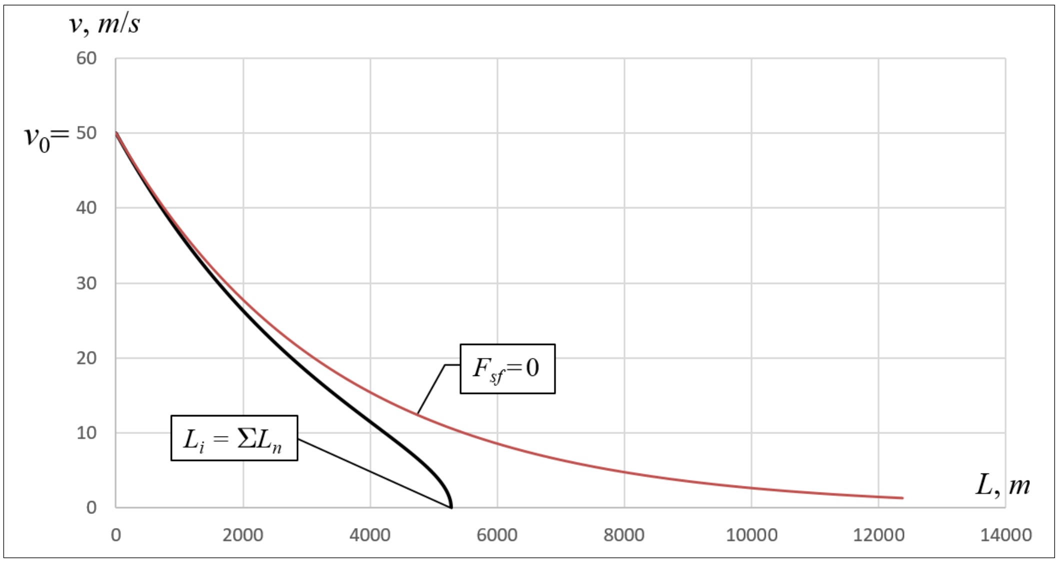

Fig. 2 Two charts of car max velocity v_0=50 drop from driving distance L.
The first chart is (6.9) eq. result, the second chart is an F_{sf} = 0.

When we have no static friction except air resistance force $F_D \propto v^2$, car velocity drops following the exponential decaying law from distance L variable:

$$v = v_0 e^{-\mu L}, \quad \text{while } F_{sf} = 0 \tag{8.0}$$

Here: μ – exponent coefficient, [m^{-1}]

If the term is F_{sf} =0, in a chart (Fig. 2) velocity is asymptotically approaching x-axis but never reaches zero value, a consequence of that is $L \to \infty$. From observation, we can prove that static friction of a car is always present $F_{sf} > 0$ because every vehicle or bicycle with initial velocity $v_0 > 0$ on a flat surface stops after the engine or internal force is absent.

The last example is if we have a city speed of v_0=15*m/s*=54 *km/h*. By applying provided algorithm [4] above, we get such results:

$$L_i = \delta t \sum_{n=0}^{N} v_n = \sum_{n=0}^{N} L_n = 1808 \ \ m \tag{9.0}$$

Fuel consumption without inertia factor:

$$m_f = \frac{\sum E}{\eta q} = 0.330 \ \ kg \tag{9.1}$$

The fuel consumption after estimating inertia distance L_i factor:

$$m_{fi} = \frac{\sum E_i}{\eta q} = 0.302 \;\; kg \tag{9.2}$$

Fuel savings:

$$\Delta m_f = m_f - m_{fi} = 0.028 \; kg \tag{9.3}$$

Fuel savings due to inertia force v_0=15 *m/s* is negligible, whereas, in the case of 50*m/s*, it was 0.627 *kg*. Of course, in that case, initial fuel consumption for a 20*km* distance was high at 2.6 *kg*. Fuel consumption has no optimization sweet spot from a speed perspective, a freight train with enormous mass is a specialized example for cargo transportation. For human's transportation to save money and fuel, speed must be slow as possible, or use bikes and legs for short-distance trips ☺.

7. Conclusions

The total energy for all types of vehicles can be calculated by the (3.0) equation:

$$\sum E = E_k \pm E_g + E_D + E_{sf}, \;\; [J]$$

E_k – max. the kinetic energy of vehicles is the magnitude of inertia term. The other two terms E_D and E_{sf} are energy losses and must be avoided if possible, $\pm E_g$ – Earth gravity term, can be negative energy loss, neutral, or positive which benefits the kinetic energy term.

8. Appendix

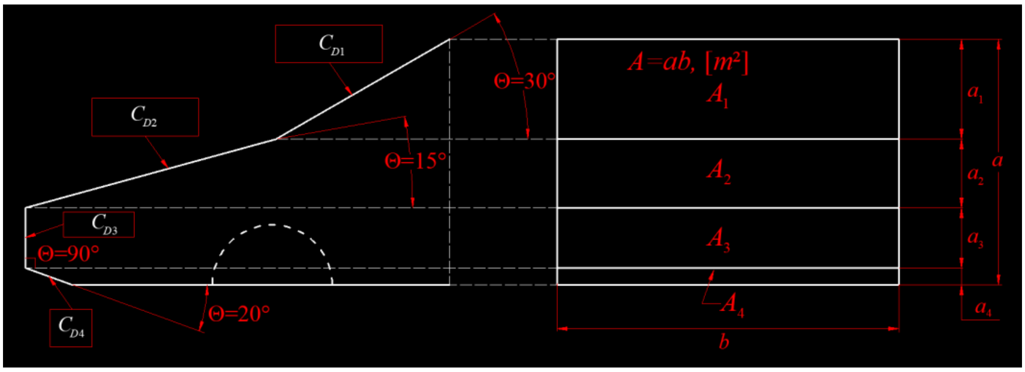

Fig. 3 Particular car's frontal cross-section area *A*=*ab* drag coefficient values.

Average frontal cross-section area's drag coefficient equation for all possible shapes of aerodynamic bodies:

$$\overline{C_D} = \frac{1}{A}\left(\frac{A_1C_{D1} + A_2C_{D3} + A_3C_{D3} + \ldots + A_NC_{DN}}{N}\right) \leq 1 \tag{10.0}$$

9. References

Apiwat Suyabodha "A Relationship between Tyre Pressure and Rolling Resistance Force under Different Vehicle Speed", *Department of Automotive Engineering, Rangsit University, Lak-hok, Pathumthani, Thailand* [1]

https://www.feynmanlectures.caltech.edu/I_12.html [2]

https://en.wikipedia.org/wiki/Rolling_resistance#Rolling_resistance_coefficient_examples [3]

GitHub - algorithm in Excel file [4]