The Catenary Curve of Constant Length

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1. Introduction

In many instances, we have parametric equations for one direct catenary length S chain, for example, suspension bridge or chain hanging between two loose ends at distance L and sag distance value h, see (Fig. 1). Our goal is to find a calculation method for a flexible, non-elastic (inextensible), catenary curve chain of constant length S=*const.*, shaped under influence of gravity force and derive a generalized equation independent of curve affiliation to one particular example or architecture design, by introducing initial angle Θ_0 values and x_0/b angle degree ratio analogy to catenary curve.

Keywords: *Hyperbolic cosine function, minimum tension curve, fractal curve*.

2. Theory and examples

Catenary (Latin l. – chain) is a curve between fixed points A and B, see (Fig. 1) that shapes into a hanging chain (string) <*The string will settle into a shape that minimizes the gravitational potential energy* [1] >. Catenary curve expressed as a trigonometric hyperbolic cosine (*hc*) function cosh:

$$y(x) = a\left(\frac{e^{x}+e^{-x}}{2}\right) = a\cosh(x), \qquad \begin{cases} x \in (-\infty,\ 0] \cap [0,\ +\infty) \\ y \in (-\infty,\ a] \cap [a,\ +\infty) \end{cases} \tag{1.0}$$

Function (1.0) resembles the parabola curve shape of such expression $y(x)=mx^2+a$, both are different functions, except both are even f-ions cosh(-x)=cosh(x), the same as a regular trigonometric cosine function.

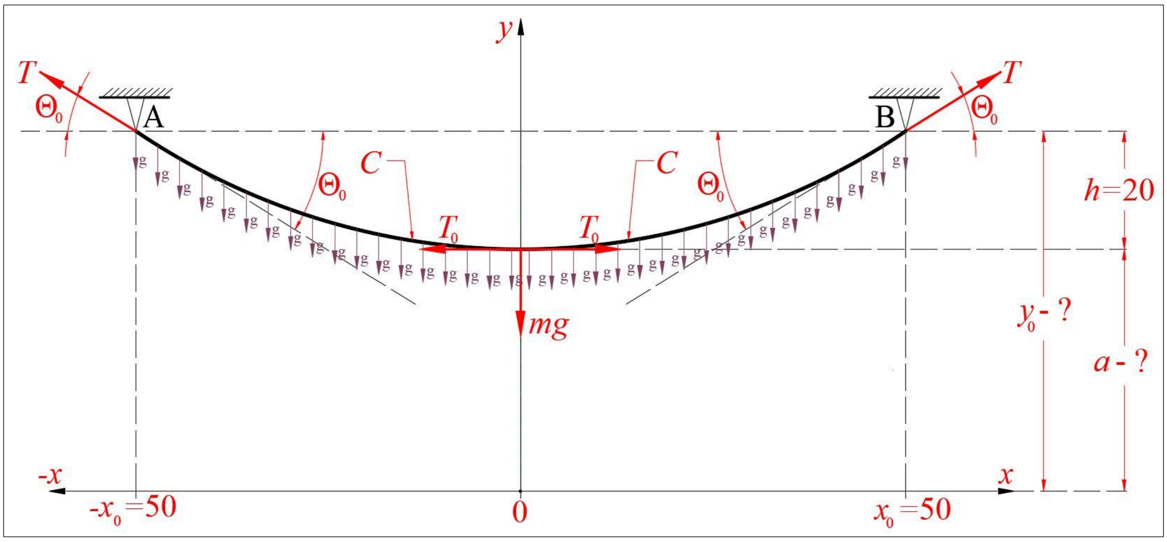

Fig. 1 Hanging S=2C length chain with mass m under the influence of Earth gravity force, where: AB=L=2x_0=100m, h=20m, y_0=a+h=a+20, [2].

The catenary curve depicted in the picture (Fig. 1) is expressed as a trigonometric hyperbolic cosine (*hc*) function:

$$y = a\left(\frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}\right) = a\cosh\left(\frac{x}{a}\right), \qquad \begin{cases} x \in [-x_0,\ 0] \cap [0,\ x_0] \\ y \in [y_0,\ a] \cap [a,\ y_0] \end{cases} \tag{1.1}$$

In (1.1) eq. constant a is an unknown parameter and to find this value is done using computer or manual iteration [2] when such equality is satisfied:

$$\begin{aligned} & y_0 = a\cosh\left(\frac{x_0}{a}\right) \ \Rightarrow\ 1 + \frac{20}{a} = \cosh\left(\frac{50}{a}\right) \ \Rightarrow\ a \cong 65.5\ m \\ & y_0 = a + h \cong 85.5\ m \end{aligned} \tag{1.2}$$

A particular approach to calculate a and y_0 values can be applied to all catenary-shaped: strings, ropes, chains, and suspension bridges when distances L and h are well known. The total length S=2C of the catenary chain can be found from the Pythagoroes identity variant applied to hyperbolic cosine and sine functions $\cosh(x)^2 - \sinh(x)^2 = 1$:

$$\begin{cases} y_0 = a\cosh(x_0 / a) \\ C = a\sinh(x_0 / a) \end{cases} \ \Rightarrow\ y_0^2 - C^2 = a^2 \ \Rightarrow\ S = 2C = 2\cdot\sqrt{y_0^2 - a^2} \cong 110\ m$$

The shortfall of such a particular calculation method is that every hanging chain configuration is an exceptional example defined for specific distance parameters of L and h. Our goal is to calculate all possible curvature shape configurations for individual constant length chain S=*const.* from one distance variable L.

We have an arbitrary length chain S=*const*., by changing the distance $L=2x_0$ between two loose ends A and B, we observe how the catenary curve changes shape $y(x)$ from x variable:

$$y = a\left(\frac{e^{\frac{x}{b}} + e^{-\frac{x}{b}}}{2}\right) = a\cosh\left(\frac{x}{b}\right), \qquad \begin{cases} x \in [-x_0,\ 0] \cap [0,\ x_0] \\ y \in [y_0,\ a] \cap [a,\ y_0] \\ \Theta \in [\Theta_0,\ 0] \cap [0,\ \Theta_0] \end{cases} \tag{1.3}$$

3. Parameters *a*, *b*, *h*, and initial angle Θ_0 definitions

The parameters a and b and the initial angle Θ_0 are found during the experiment by measuring the hanging chain or calculating with the help of a computer under the conditions that the hanging chain is made from entirely flexible and non-elastic material, with a total and constant length of $S=2y_0=200$ distance units: millimeters, centimeters, meters, inches, etc. Two points are known in advance, see (Fig. 2), the vertical line **1)**, a=0, L=0, initial angle $\Theta_0=90^o$ and horizontal line **6)**, length L=S=200, initial angle Θ_0=0, this horizontal line represents a “curve” with radius $R\rightarrow\infty$. Lines **1** and **6** give a rectangular boundary box with height y_0=100 and width S=200, it is always a unique measurement constraint for a specific constant length S=*const*. chain.

The algorithm in Excel and Python is in the GitHub repository [13], there is a short video that shows constant length elastic chain shape change under Earth gravity force from variable $S=f(L)$=*const.* in real-time, also for more details see §9 chapter.

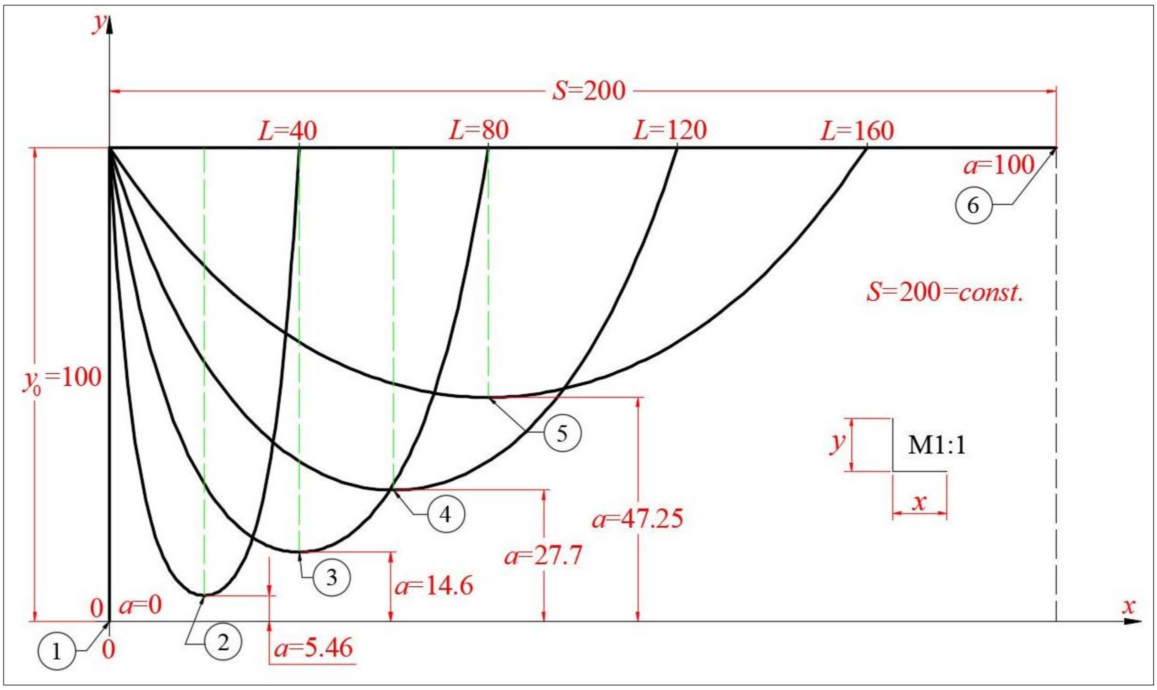

Fig. 2 Constant length chain S=200, with six different values of $L=2x_0$.

To allow observe a change of catenary chain shape from variable x, we transform (1.3) eq. in such a way that one loose end of the chain is permanently anchored to coordinates system point $(0, y_0)$, the second loose end of the chain is variable x, (1.3) equation has been shifted to the right side at a distance x_0:

$$y(x) = a\cosh\left(\frac{x_0 - x}{b}\right), \qquad \begin{cases} x \in [0,\ x_0] \cap [x_0,\ L] \\ y \in [y_0,\ a] \cap [a,\ y_0] \\ \Theta \in [\Theta_0,\ 0] \cap [0,\ \Theta_0] \end{cases} \tag{1.4}$$

Parameter $a \in (0, y_0)$ shows the curve's lowest tipping point height. Even $b \in (0, \infty)$ is non-present in the catenary curve definition, like distance parameters: y, x_0, a, h. Distance b shows hyperbolic cosine f-ion's curvature sharpness angle, parameter b always goes together as ratio x/b and the ratio is an analogy of angle measurement unit $\Theta \sim x/b$, see §4 chapter. The max. height of the chain $-$ $y_0=h+a=const.$ Chain sag distance $-$ $h = y_0 - a$, range of sag distance $-$ $h \in (y_0,\ 0)$.

4. Parameters a, b, and initial angle Θ_0 solutions

Parameters a and initial angles Θ_0 calculated using constant length S=200 chain, as initial reference constant length S chain, thus $y_0=S/2=100$, each one step $\Delta x=1$, we calculate values of x_0 hundred times using (1.3) eq., each time parameters a and initial angle Θ_0 values we plug into an array, matrix or spreadsheet table, see (Table 1). This initial length constant length $y_0=S/2=100$ chain will be the reference data set used to extrapolate distance a and initial angle Θ_0 from (Table 2). In (Table 2) is depicts an example when $S=2y_0=600=const.$ and distance is in range $L \in (0, 600)$, under conditions of these two enter values, we will get results of distance a and initial angle Θ_0, such operation can be done manually, but more easily can be done with help of math software. In (Table 1) the fourth column $\Sigma S \cong 200$, shows the calculation accuracy of our preferable ideal $S=200=const.$ at value x_0, an average error value of length ΣS=200=*const*. chain is $\varepsilon=\mp 0.036$. Table 1 and Table 2 will be attached as an MS Excel file, see Appendix §9.

Initial data: $y_0=100$			
x_0	Θ_0. deg.	a	ΣS
0	90	0	200
1	89.92	0.150	199.966
2	89.82	0.32	199.970
3	89.71	0.53	199.922
⋮	⋮	⋮	⋮
97	24.27	79	199.995
98	19.76	82.95	199.935
99	14.02	87.9	199.972
100	0	100	200

Table 1

Extrapolation: y_0 – ?		
x_0	Θ_0. deg.	a
0	90	0
3	89.92	0.45
6	89.82	0.96
9	89.71	1.59
⋮	⋮	⋮
291	24.27	237
294	19.76	248.85
297	14.02	263.7
300	0	300.00

Table 2

Length chain S=*const.*, when $x_0=2/L$ and a parameter extrapolated using an algorithm [13] or manually from (Table 2), thus parameter b is equal to:

$$\frac{x_0}{b}=\cosh^{-1}\left(\frac{y_0}{a}\right) \Rightarrow b=\frac{x_0}{\cosh^{-1}(y_0/a)} \tag{1.5}$$

Parameter b is nowhere present to define a catenary curve, thus (1.5) eq. is informative because ratio x_0/b is an arccosh(y_0/a) or inverse function of trigonometric hyperbolic cosine (*hc*). As we know from mathematics, regular inverse cosine function or any regular inverse trig. functions result gives angle magnitude, except for inverse trigonometric hyperbolic cosine function x_0/b shows the sharpness of curvature change from a vertical position to the horizontal straight line and is not the same as regular inverse cosine f-ion, and non-dimensional degree measurement units don't match. An alternative equation of inverse $\cosh^{-1}(y_0/a)$ f-ion can be directly derived from (1.3) eq. [3]:

$$\frac{x_0}{b}=\cosh^{-1}\left(\frac{y_0}{a}\right)=\ln\left(\frac{y_0+\sqrt{y_0^2-a^2}}{a}\right) \tag{1.6}$$

When in (1.6) eq. the, $y_0\to a$, thus $x_0/b=\ln(1)=0$, $b\to\infty$, a curvature of the arch with radius $R\to\infty$. Another boundary condition is curve **1)** when a=0, $x_0/b=\ln(\infty)\to$undefined, $b\to$undefined, this is a trivial boundary condition because we have a fixed condition x_0=0, variable x=0=*const*. becomes fixed value too. Boundary conditions for non-elastic chains can be solved from a physics (statics) point of view. Statics is a physics problem when all forces projections into the x-axis and y-axis are equal to zero, in (Fig. 1) force tension T at the ends of chain anchor points A and B and force T_0 at a middle point of the chain is equal to [4]:

$$\begin{cases}\sum F_x=0 \Rightarrow T\cos(\Theta_0)-T_0=0 \Rightarrow T_0=T\cos(\Theta_0) \\ \sum F_y=0 \Rightarrow -mg+2T\sin(\Theta_0)=0 \Rightarrow T=\dfrac{mg}{2}\dfrac{1}{\sin(\Theta_0)}\end{cases} \Rightarrow T_0=\frac{mg}{2}\frac{1}{\tan(\Theta_0)} \tag{1.7}$$

For non-elastic chains in (1.7) eq., when angle $\Theta_0\to 0$, force $T\to\infty$ and the curve radius go to infinity $R\to\infty$ too. A curve with an infinite radius is defined as a straight line. Exists and cases when a catenary chain made from elastic material and under a tension of force stretches in length < *The string is assumed to stretch by Hooke's Law.* [5] >.

5. <u>The catenary shape is curved with minimum tension condition</u>

We apply the physics laws for catenary shape <The *string will settle into a shape that minimizes the gravitational potential energy* [1].> We can rephrase this quote differently, catenary curve shape always has minimum tension force in the entire length S, no matter how this shape changes by varying distance L between two loose ends. Boundary conditions show that it goes to infinity, but we can calculate values close to infinity value. For this purpose, we will use an osculating circle – < *The circle that has the same tangent, and the same curvature at the point on the curve* [6]>, at point x=0 [7] osculating circle center points – (x_c=x_0, y_c=$b^2/a+a$), a radius of a circle – R=b^2/a, see (Fig. 3).

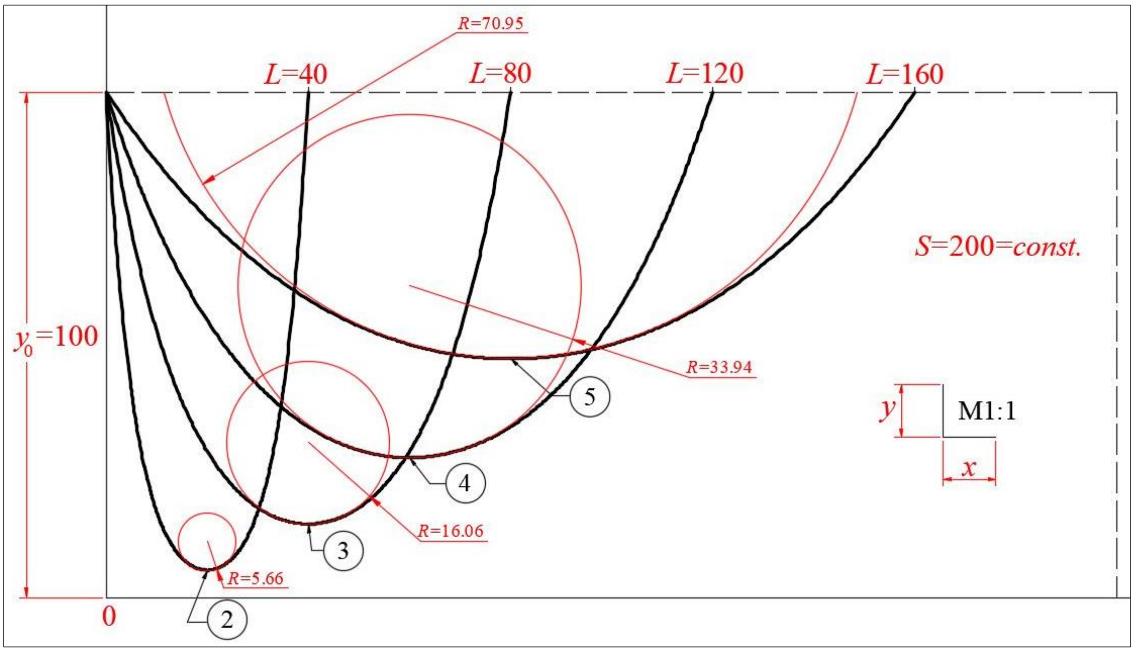

Fig. 3 Radius R of osculating circles.

We see in (Fig. 3) that by an increasing distance between two ends of catenary curve osculating circles approaching perfect circle chord with sag value h [8], the limit when catenary becomes exact circle chord is when initial angle Θ_0=22°, at this initial angle the catenary curve becomes the exact chord of the circle with sag distance h. The result of 22° was obtained via iteration calculation method using (Tables 1 and 2), with the condition that the circle which is osculating circle chords with radius R and sag distance h become equal to the catenary curve distance between loose ends:

$$x_0 = \sqrt{2Rh - h^2},\ \Theta_0 = 22^\circ \tag{1.8}$$

Equation (1.8) shows that below $\Theta_0 \le 22^\circ$, the catenary is always the chord of the circle with sag height h and length L=2x_0. Above this value $\Theta_0 > 22^\circ$, the catenary curve can be approximated by drawing arcs tangent to the osculating circle, and other points tangent to the line at the initial angle Θ_0 at point (0, y_0) and point (L, y_0). And when $L\to S$, approximation arcs near-perfectly resemble catenary curves, see (Fig. 4).

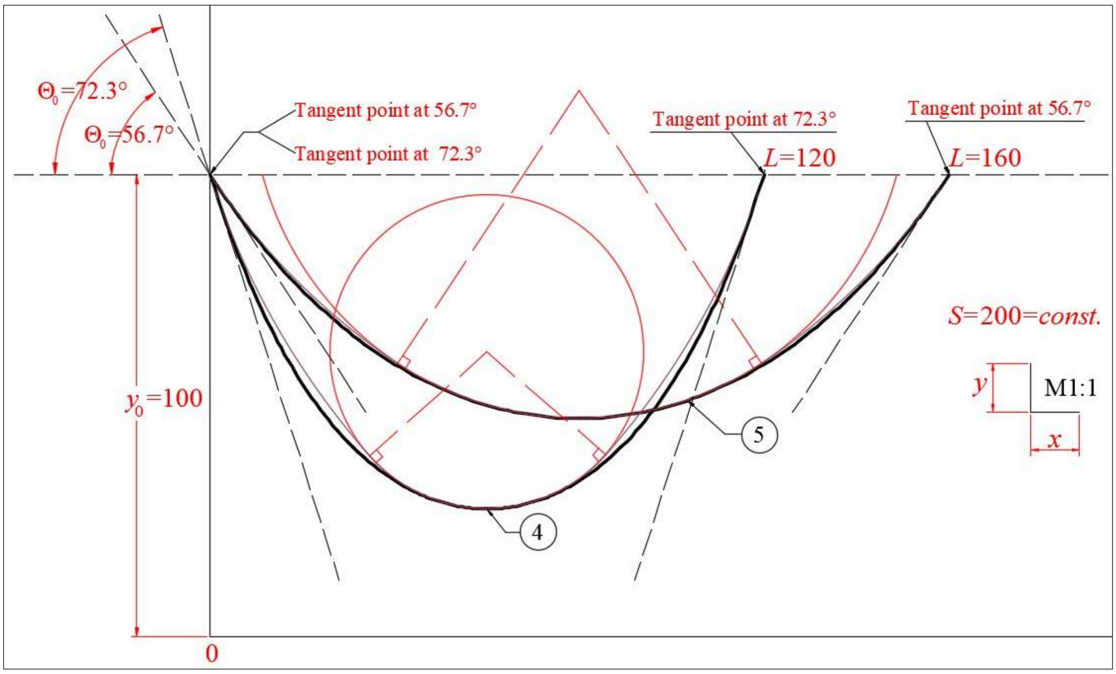

Fig. 4 Catenary curves approximations by drawing arcs.

In (Fig. 4) in the case of catenary curve L=160, Θ_0=72.3° the approximating arc is so close to the real catenary, two curves are nearly indistinguishable from each other, in case L=120, Θ_0=56.7° exist some small gaps that distinguish both curves, still it's a very close snug fit approximation of both curves. Such a "roundish" circle shape of catenary curves comes from tension optimization property when the distance between two loose ends increases, the curve maintains minimum tension forces condition, and the best form for that is circle chord curvature. < *The spherical shape minimizes then necessary "wall tension" of the surface layer according to Laplace law.* [9]>. A sphere to minimize surface layer tension for liquids is the 3D geometrical figure $x^2+y^2+z^2=R^2$, the catenary curve rotated 2π around the x-axis is called a catenoid, 3D geometrical figure and it has minimum surface area tension property same as sphere [10]. These two facts [9], [10], prove that in the case of a catenary curve, which is a two-dimensional object always maintains minimum tension force law, based on the shape of a circle chord, which is sphere related figure – $x^2+y^2=R^2$.

Example S=200, Θ_0=30°, R=183.7, L=190.6, the section of a circle defined as chord angle [8] is $\alpha=2\sin^{-1}(x_0/R)$=61.87°, approximated S length is equal to:

$$S=\frac{2\pi R\alpha}{360^\circ}=198.4\approx 200 \tag{1.9}$$

In (1.9) eq. by stretching constant length S=200=*const*. string wide apart from two loose ends $L\to S$ the catenary curve forms the chord.

6. The catenary is a fractal curve

The fractal definition is: <*a curve or geometrical figure, each part of which has the same statistical character as the whole*. [11]>, in practice, we mostly related to fractal geometrical curves of regular geometrical shapes, for example, the "Sierpinski triangle" constructed out of equilateral triangles, and the "Koch snowflake" fractal constructed out of equilateral triangles. "Sierpinski carpet" constructed out of squares, also exists fractals based on complex numbers, “Mandelbrot set”, “Newton fractal” etc. A curves-based fractal example is a logarithmic function (spiral). There are other examples of regular function curves with fractal properties, but the most famous is the circle.

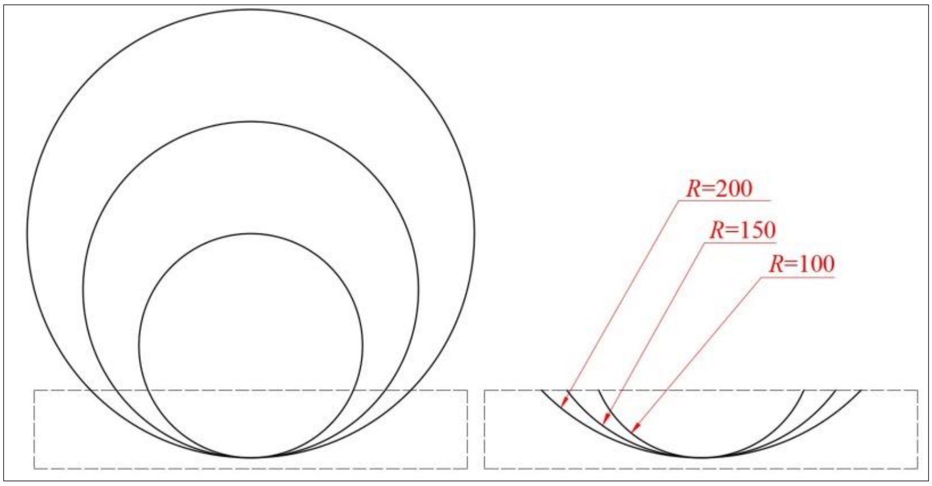

Fig. 5 Three circles with radius: R=200, R=150, R=100.

We seen (Fig. 5) what if the left side of the picture we see sections - chords of full circles on the right side, of course, a circle is an ultimate or unique figure in all mathematics described by math constant π=3.141…, this constant in physics is a factor of periodicity, in mathematics, it’s a factor of roundness. The catenary curve is described by a trigonometric hyperbolic function which contains another math constant e=2.718…, from here we follow the conclusion that the catenary curve must be fundamental and unique same as a circle.

< *It is worthwhile to note that catenary has only one shape (it is not a family of curves). If you hold the two ends of a string, and vary the distance of their endings, you will see shapes of different sharpness and wonder how can they all be catenary. In fact, you are merely seeing different scales of catenary. The phenomenon is the same as looking at part of a circle. The closer you look, the straighter it is but the whole circle never changes shape. At each instance you are holding the string, the law of physics dictates its shape to be part of the catenary (since the string is finite in length). The wider apart the endings, the smaller part of catenary you see. It is also a common misunderstanding to assume parabolas as a curve of many shapes.* [12]>

Even the catenary curve function is derived according to (1.3) eq., I wanted to prove that (1.4) eq. is better to describe the catenary of constant length S=*const.* when only one variable is present; L distance between loose ends, thus (1.4) eq. of constant length catenary S=*const.* is just a variable of the initial angle function from distance $\Theta_0 = f(L)\big|_{S=const.}$. Close to boundary condition $L \to S$, $\Theta_0 \to 0$ catenary curve becomes the exact shape of a chord, based on optimization of tension in the entire chain length. The fractal definition is <*a curve or geometrical figure, each part of which has the same statistical character as the whole.* [11]>, from definition follows that we can extract infinite many curves from one catenary curve, for example, **1)** S=200*cm*, L=80*cm*, Θ_0=81.2°, see (Fig. 6).

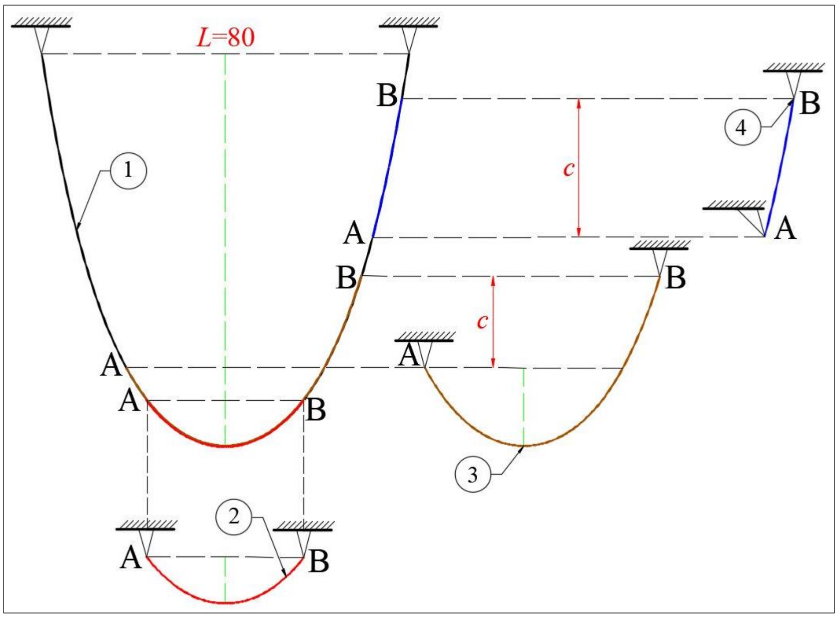

Fig.6 Catenary curves: **1)** S=200 cm, L=80 cm **2)** S=41.5, L=32.2 **3)** S=79.1, c=20, L=51.1 **4)** S=30.7, c=30.1, L=6.3.

Another property related to fractal geometrical figures and curves is self-similarity; from one curve we can extract as many as possible curves that preserve similar features of the original curve. By all accounts in (Fig. 6), the shown curves are not generalized catenary curves; even we can extract an infinite many catenary curves from it. Circle circumference length is proportional to radius $S \propto R$, but it is just a special case of ellipse function:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \text{ for } a=b=R \;\Rightarrow\; x^2 + y^2 = R^2$$

The same rule can be applied to the (1.4) equation of hyperbolic function S=*const*. and distance parameters in (Fig. 6) Catenary curve: **1)** S=200 cm, L=80 cm, a=14.6 cm, and b=15.31 cm can give infinity more different variations of catenary curves, (Fig. 6) have infinity many catenary curves but it also has infinite more. Same as ellipse equation parameters a and b variations can give infinitely more different ellipse curve variations and just special case of a circle defined by one parameter radius R magnitude is unique to all other circles. The same principle applies to the catenary curve by varying distance $\Theta_0 = f(L)\big|_{S=const.}$ we get different sharpness angles, but it stays the same unique catenary curve, fractal scale factor can have every curve, but self-similarity extrapolation property have only constant length S=*const.* catenary curve or radius R circle.

The fractal property of the catenary curve is not obvious, but as we know due to gravity force, flexible material gains a catenary curve shape. And gravity force expressed as free fall magnitude g=9.81m/s^2 permeates every spatial coordinate, and at various configurations between two loose ends points of A and B gravity forces creates infinite many variations of curves.

7. Conclusions

Constant length S=*const.* catenary chain defined by hyperbolic function (1.4) eq., where distance parameters a and b extrapolated using (Table1 and Table 2), non-dimensional magnitude x_0/b=cosh^{-1}(y_0/a) represents curvature degree of S=*const.* catenary curve and is an analogy of angle $\Theta_0 \sim x_0/b$ in the sense of regular trigonometric functions. When the constant length is S=*const.* and $L \rightarrow S$, thus $\Theta_0 \rightarrow 0$ to maintain minimum tension in the entire length of chain S catenary curve gains the shape of a circle chord.

8. References

http://www1.phys.vt.edu/~takeuchi/Tools/CSAAPT-Spring2019-Catenary&SuspensionBridge.pdf [1]

https://www.youtube.com/watch?v=OBoPpTExyBI&ab_channel=MichelvanBiezen [2]

http://www.sosmath.com/trig/hyper/hyper03/hyper03.html [3]

https://www.youtube.com/watch?v=N1LRgHYxMKo&ab_channel=PhysicsNinja [4]

https://en.wikipedia.org/wiki/Catenary#Elastic_catenary [5]

https://en.wiktionary.org/wiki/osculating_circle [6]

In WolframAlpha page enter text: Osculating circle y=a*cosh(x/b) at x=0 [7]

https://www.liutaiomottola.com/formulae/sag.htm [8]

http://hyperphysics.phy-astr.gsu.edu/hbase/surten2.html [9]

https://mathworld.wolfram.com/MinimalSurfaceofRevolution.html [10]

https://www.lexico.com/definition/fractal [11]

http://xahlee.info/SpecialPlaneCurves_dir/Catenary_dir/catenary.html [12]

https://github.com/ArthurKarbocius/catenary_curve [13]

9. Appendix

MS Excel file "Constant length S catenary curve calculator.xlsx" – 72 KB.
https://github.com/ArthurKarbocius/catenary_curve/blob/master/Constant%20length%20S%20catenary%20curve%20calculator.xlsx

Python file "catenary_curve.py" – 8 KB.
https://github.com/ArthurKarbocius/catenary_curve/blob/master/catenary_curve.py

Video file "Catenary curve S=200.mp4 " – 1.1 MB, video length: 46 *s*.
https://github.com/ArthurKarbocius/catenary_curve/blob/master/Catenary%20curve%20S%3D200.mp4