Singular Value Decomposition (SVD)
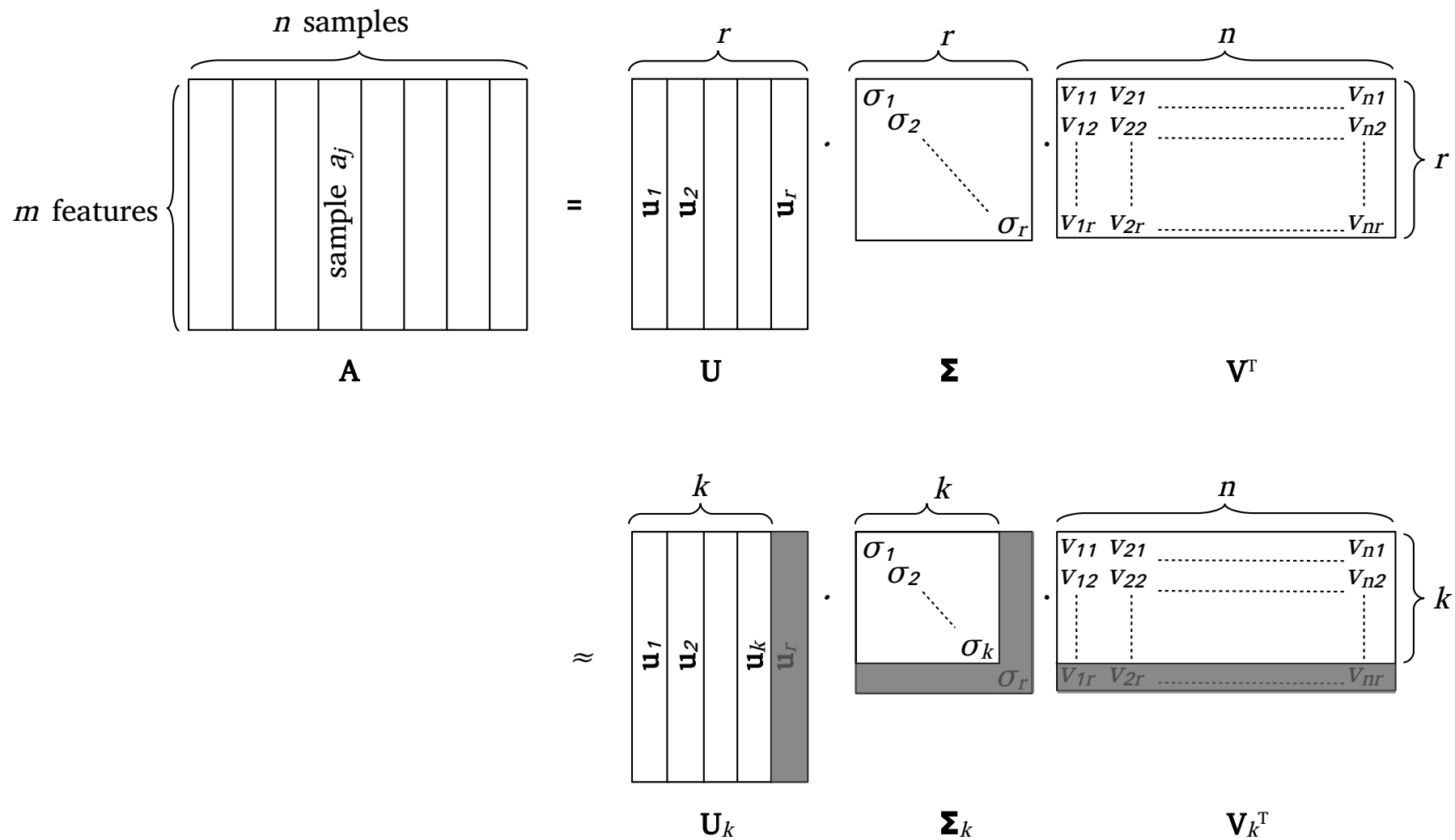
For any matrix $\mathbf{A}$, there exist $\mathbf{U}$ and $\mathbf{V}$ such that

$$\underset{\mathbf{A}}{} = \underset{\mathbf{U}}{} \cdot \underset{\mathbf{\Sigma}}{} \cdot \underset{\mathbf{V}^{\mathrm{T}}}{}$$

$\mathbf{A}$: $m \times n$, columns n, rows m

$\mathbf{U}$: columns r, with $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r$

$\mathbf{\Sigma}$: $r \times r$ diagonal, with $\sigma_1, \sigma_2, \dots, \sigma_r$

$\mathbf{V}^{\mathrm{T}}$: r rows, n columns

$$\begin{pmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & v_{22} & \cdots & v_{n2} \\ \vdots & \vdots & & \vdots \\ v_{1r} & v_{2r} & \cdots & v_{nr} \end{pmatrix}$$

$$\mathbf{A} \approx \mathbf{U}_k \cdot \mathbf{\Sigma}_k \cdot \mathbf{V}_k^{\mathrm{T}}$$

$\mathbf{U}_k$: columns k, with $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \mathbf{u}_r$

$\mathbf{\Sigma}_k$: $k \times k$, with $\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_r$

$\mathbf{V}_k^{\mathrm{T}}$: k rows, n columns

$$\begin{pmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & v_{22} & \cdots & v_{n2} \\ \vdots & \vdots & & \vdots \\ v_{1r} & v_{2r} & \cdots & v_{nr} \end{pmatrix}$$

In data science, we can apply singular value decomposition in two ways.

Application 1 (as in the slides):

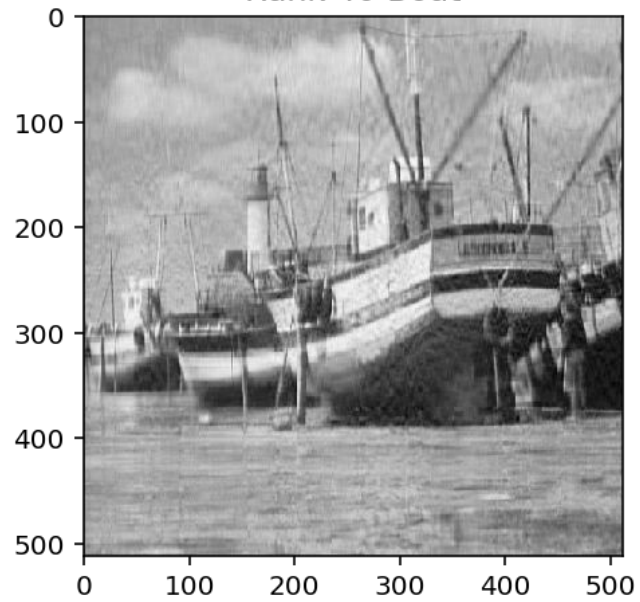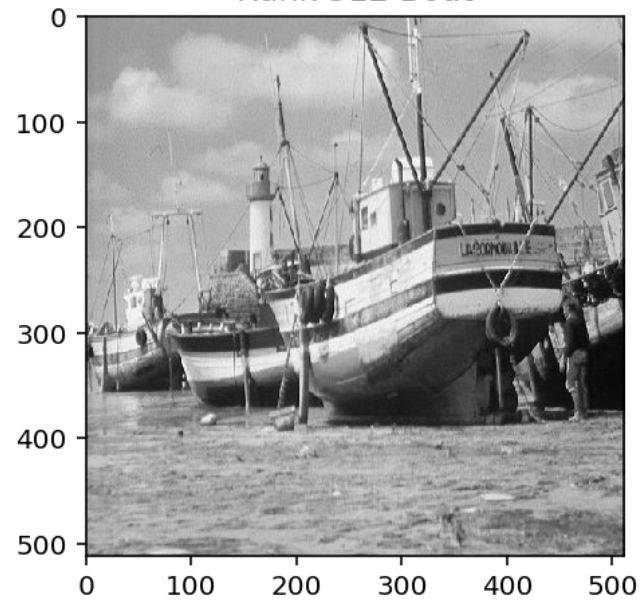Application 2 (as in the code or as in this Wikipedia page):

m features

n samples

sample a_j

A

r

$\mathbf{u}_1$ $\mathbf{u}_2$ $\mathbf{u}_r$

U

r

σ_1
σ_2
σ_r

Σ

m

v_{11}	v_{21}		v_{m1}
v_{12}	v_{22}		v_{m2}
v_{1r}	v_{2r}		v_{mr}

r

V$^\mathrm{T}$
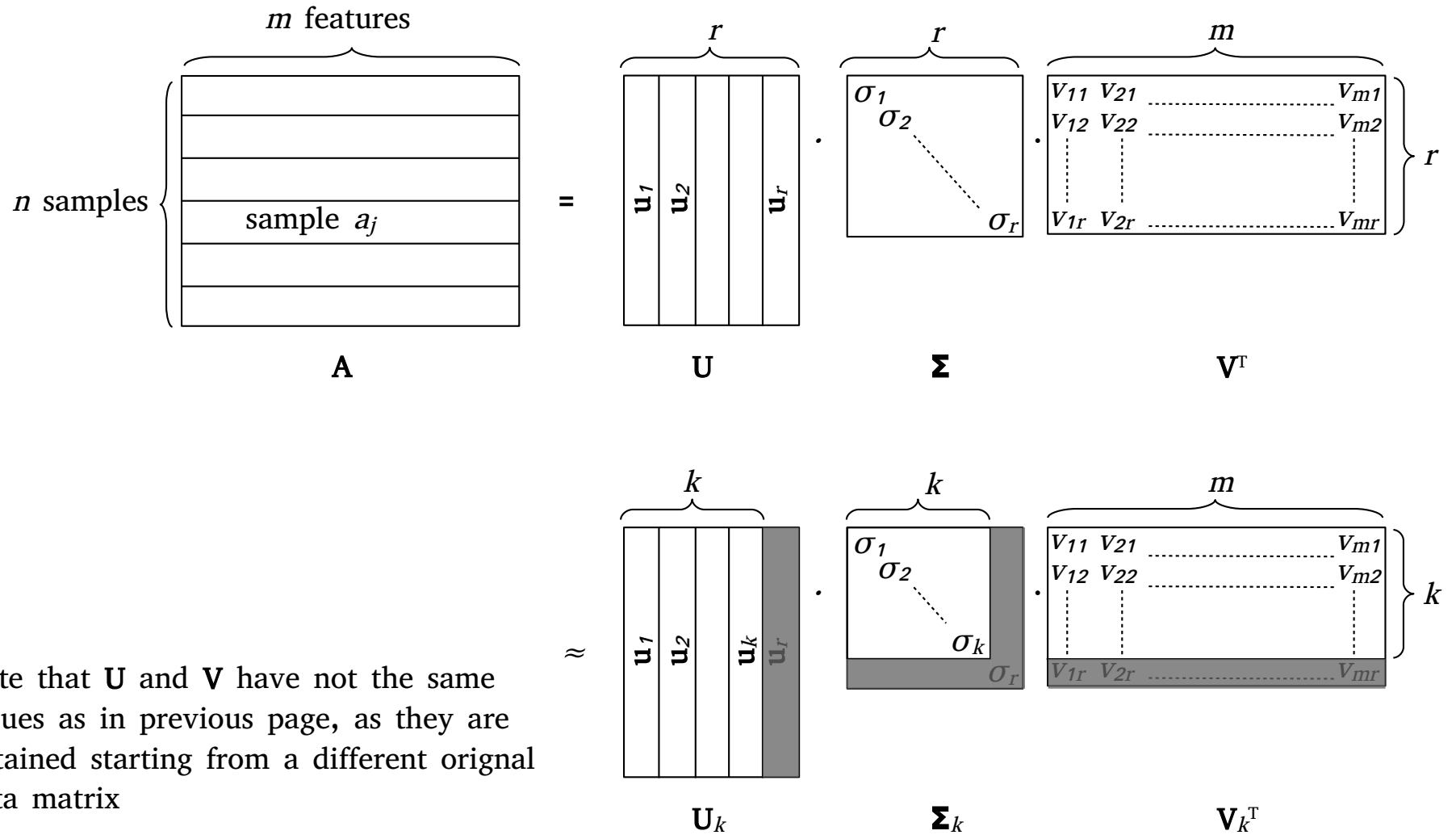
Note that **U** and **V** have not the same values as in previous page, as they are obtained starting from a different orignal data matrix

k

$\mathbf{u}_1$ $\mathbf{u}_2$ $\mathbf{u}_k$ $\mathbf{u}_r$

U$_k$

k

σ_1
σ_2
σ_k
σ_r

Σ$_k$

m

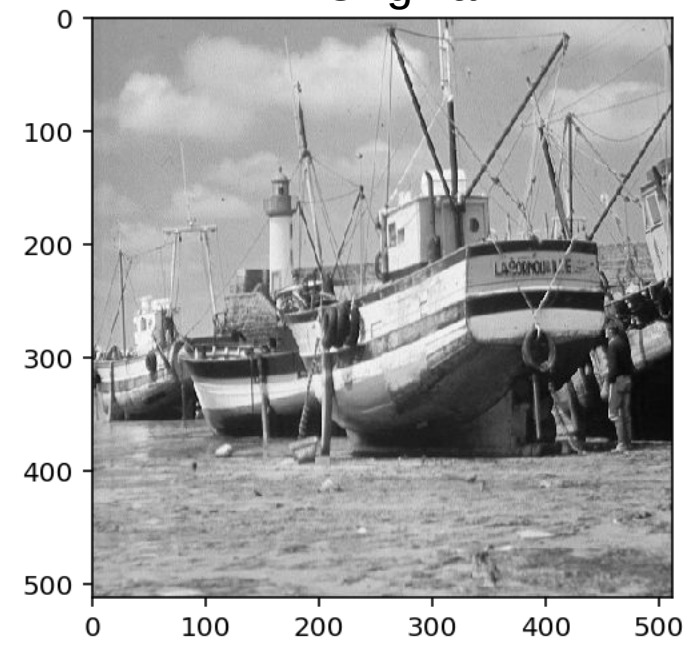| v_{11} | v_{21} | | v_{m1} |
v_{12}	v_{22}		v_{m2}
v_{1r}	v_{2r}		v_{mr}

k

V$_k^\mathrm{T}$

Reduced

Rank 40 Boat

Rank 512 Boat

Original

Source: https://www.alimentipedia.it/come-si-fa-il-formaggio.html