Mobile Inverted Pendulum

Report

by

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2022

Intro:

Model:

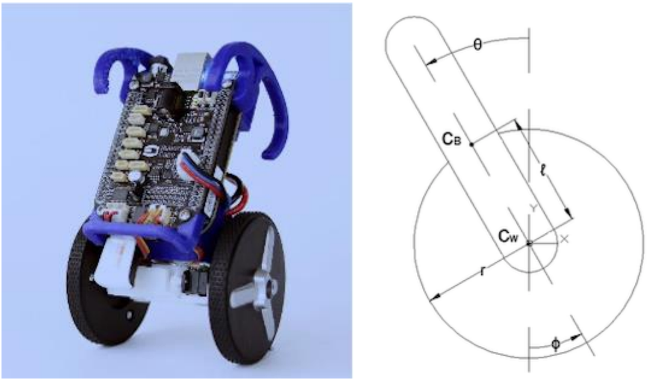


Figure 1 model of MIP

Equation:

$(I_B + m_B \cdot l^2)\cdot(d^2\theta/dt^2) + m_B \cdot l(-g\cdot\sin(\theta) + r\cdot\cos(\theta)\cdot(d^2\phi/dt^2)) = -\tau$

$m_B\cdot r\cdot l\cdot\cos(\theta)\cdot(d^2\theta/dt^2) - m_B\cdot r\cdot l\cdot\sin(\theta)\cdot(d\theta/dt)^2 + (2I_W + (m_B + 2m_W)\cdot r^2)\cdot(d^2\phi/dt^2) = \tau$

$\tau = 2G_r(\bar{s}\cdot u - k\cdot\omega_m)$

$\omega_W = d\phi/dt - d\theta/dt = \omega_m/G_r$

State Space:

linearize equations above at $\theta = 0$, $d\theta/dt = 0$, $\phi = 0$, $d\phi/dt = 0$, we can get

$\dot{x} = A\cdot x + B\cdot u$

$y = C\cdot x + D\cdot u$

$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \theta \end{bmatrix}$, u = input voltage

$A = \begin{bmatrix} -13.7015 & 13.7015 & 128.4352 \\ 21.0322 & -21.0322 & -83.5499 \\ 1 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -74.1271 \\ 113.7876 \\ 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Classical Control

To control the system, we get to choose which states we would like to feedback. The following are some different feedback states and their performances.

Feedback theta dot

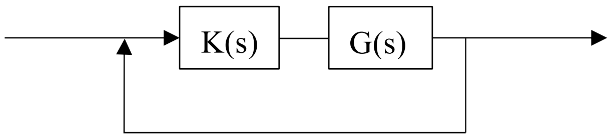
Transfer function

Use command `[num, den] = ss2tf(A,B,C,D)` to obtain transfer function.
By command `tf(num("inputnumber",den)` to obtain transfer function of input voltage to output theta dot.

$$\texttt{sys\_tf\_thetadot} = \frac{-74.13\ s\text{^}2}{s\text{^}3 + 34.73\ s\text{^}2 - 128.4\ s - 1557}$$

Root Locus

Take the following block diagram for example:



The example system's transfer function is K(s)*G(s) / (1+G(s)*K(s)). The main method of root locus is making 1 + G*K = 0, and plot lines by increasing K from 0 to infinity.

And then use command `rlocus("transfer function")` to get root locus plot

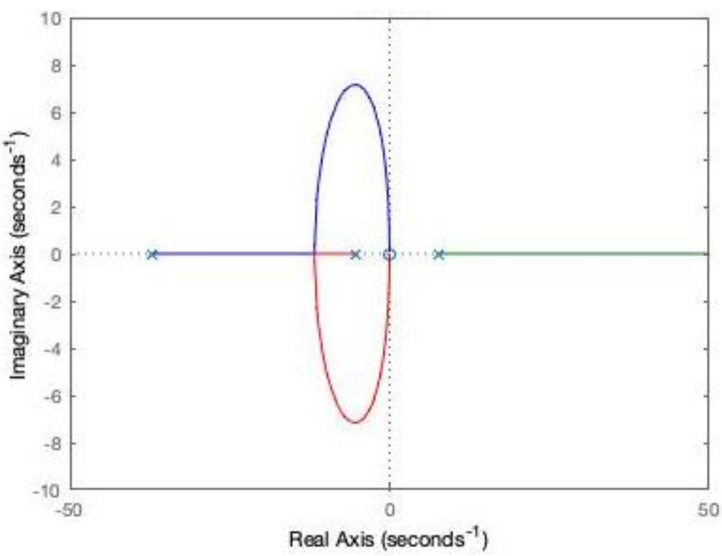


Figure 2 Root Locus plot of voltage to theta dot

By looking at root locus of theta dot, we notice there is a unstable pole in the RHP, to make this system stable, we can add a unstable pole in RHP.

Add unstable pole at 26.5:

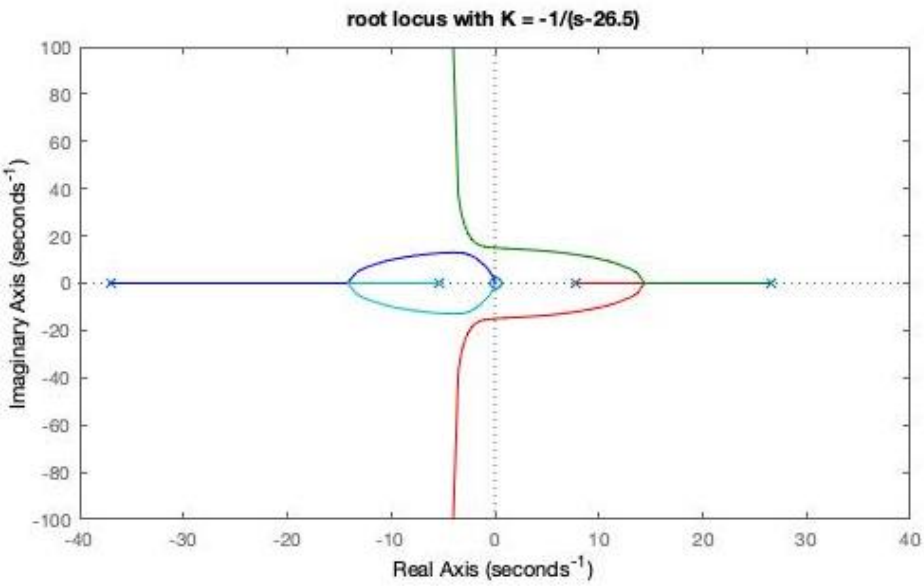


Figure 3 Root Locus after add unstable pole at 26.5

Choose proper gain

From Figure 3, we notice root locus goes to LHP, which means we can stabilize the system if gain is chosen properly. To get the right gain, I plot several gains in one figure to compare their performance.

command by `pzmap("transfer function")`

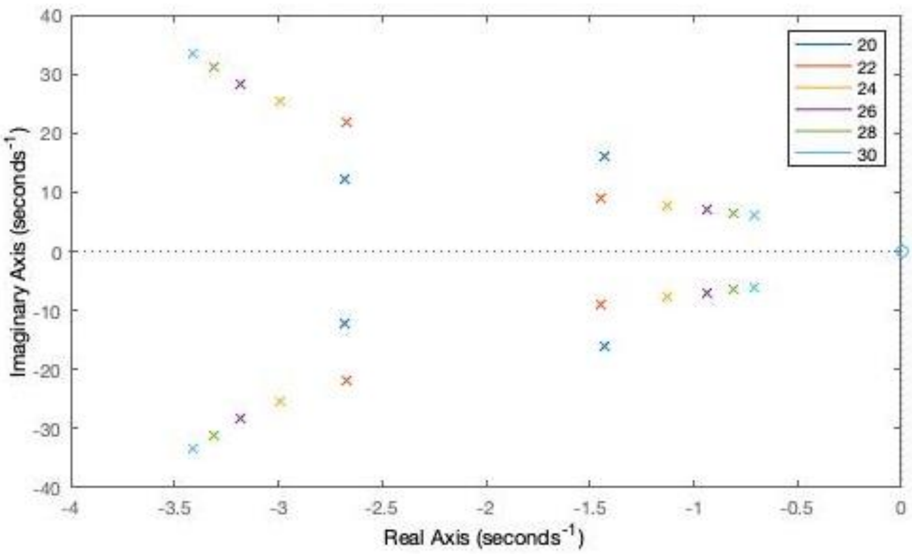


Figure 4 Poles and zeros with different value of gain

command by `step("system")`

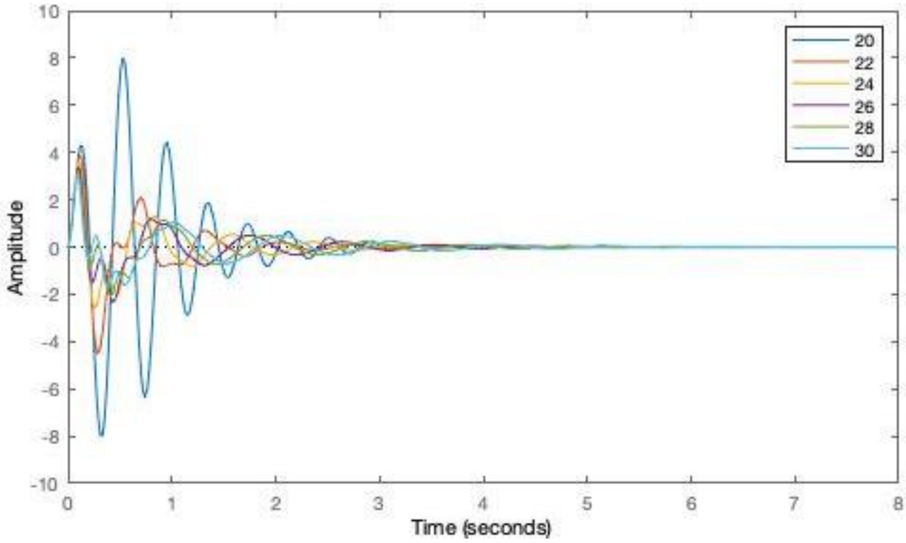
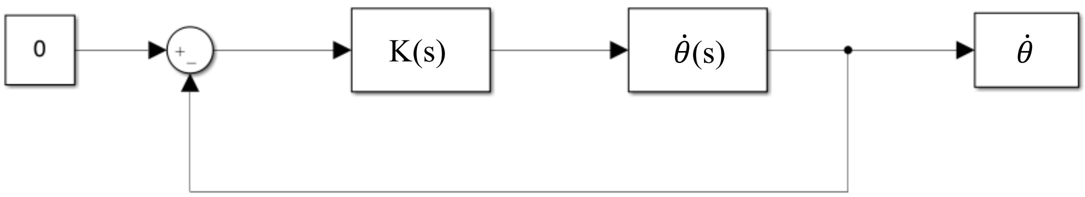


Figure 5 Step response with different gain

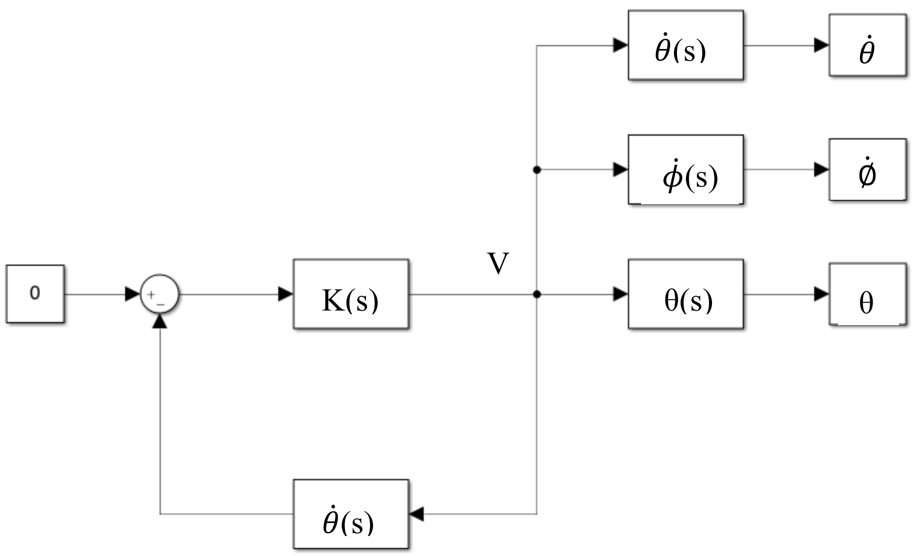
I chose gain to be 22, since it has largest damping ratio = 0.156, and smallest overshoot percentage = 60.9. Therefore, our K(s) = $\frac{-22}{s-26.5}$.

Stability of other states

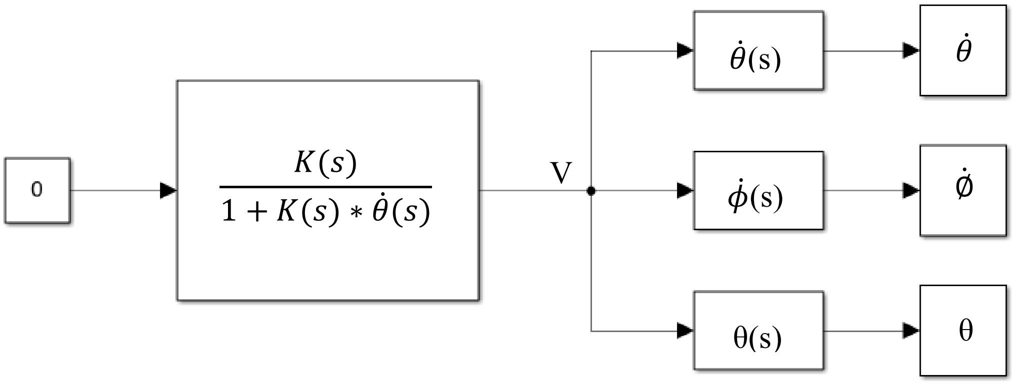
Now we are sure the state theta dot can be controlled, and its block diagram is:



But we also need to see other states' poles and zeros. The system's block diagram:



By the system block diagram above, we can reduce it to:

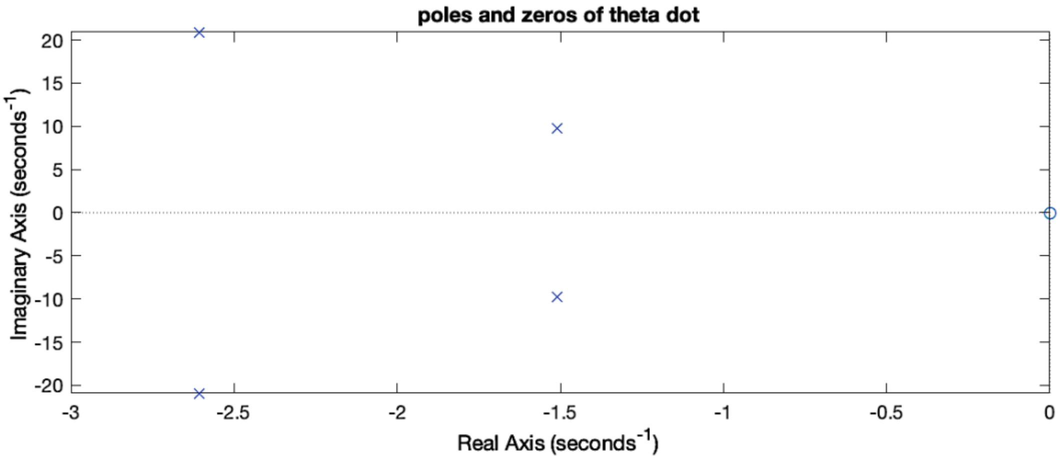


The first block is closed loop of theta dot. We can obtain by command `feedback(K,θ)`

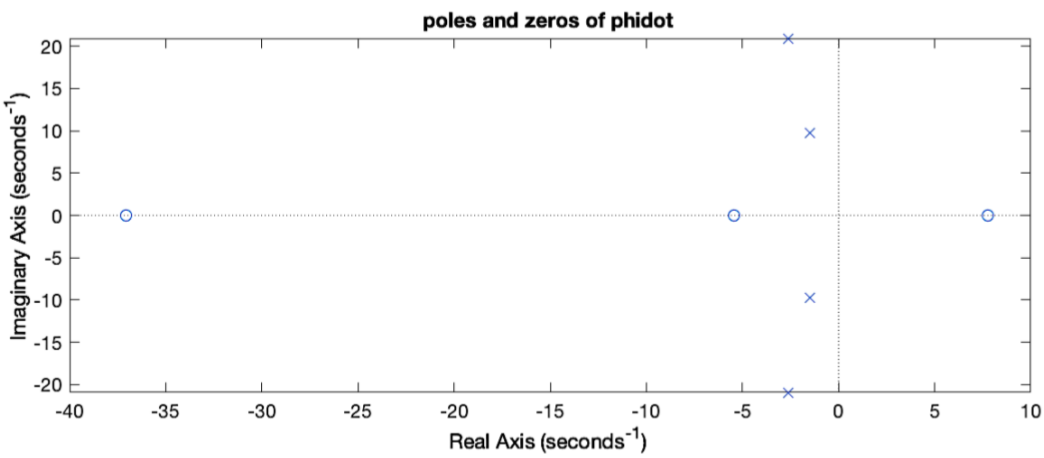
All states transfer functions' poles and zeros

With closed loop of theta dot, we can easily plot other transfer functions' poles and zeros

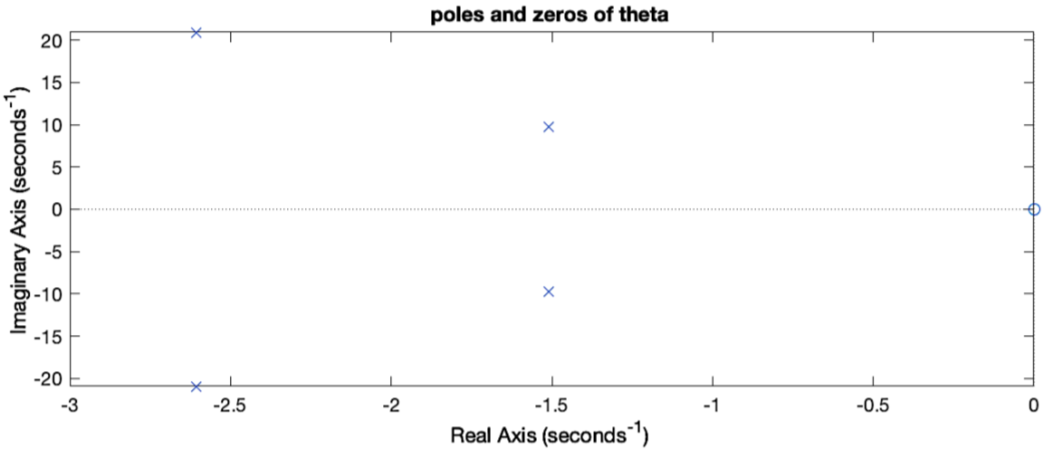
<u>theta dot:</u>



<u>phi dot:</u>



theta:



Performance of feedback controller

Now, applying this controller to our linearized system.

Since the maximum value of our input voltage is 7.4 V, the largest initial body angle is 34 degrees (about 0.6 rad).

By scope triggers panel, there are two critical performances of theta we care

Settling time (the time body angle stays within ± 1 degree ≅ 0.02rad): 2.46 s

Maximum angle: 0.7281 rad (about 41.7 degrees)

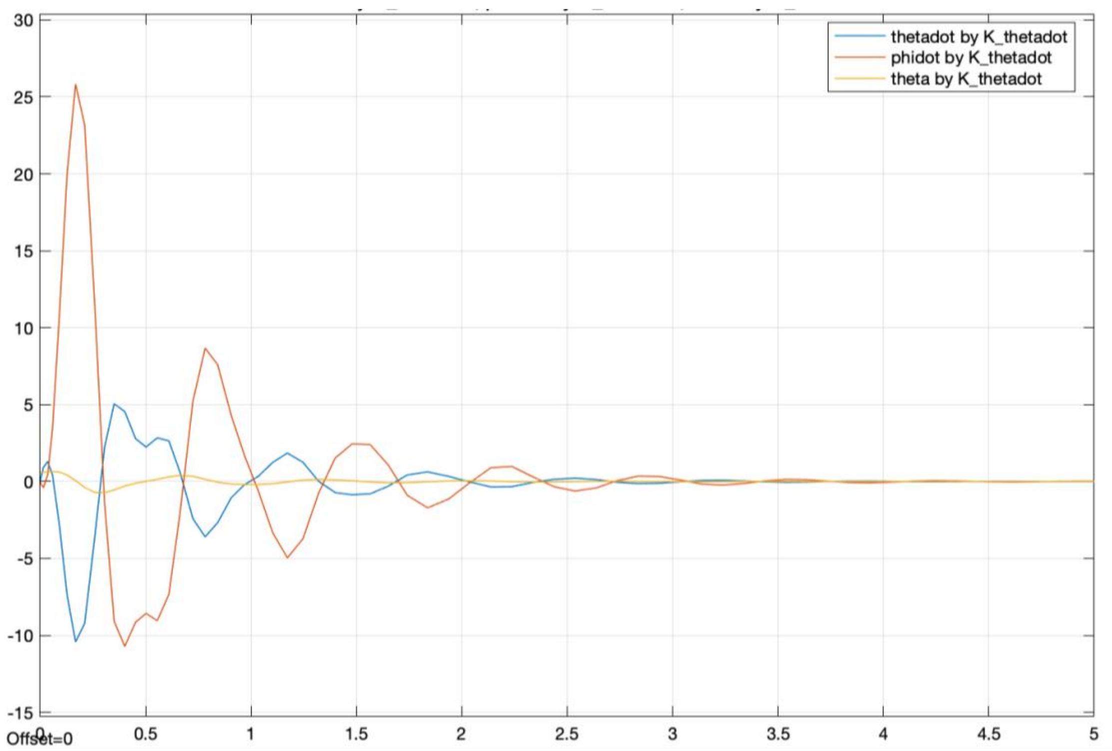


Figure 6 Performance of system by applying theta dot feedback

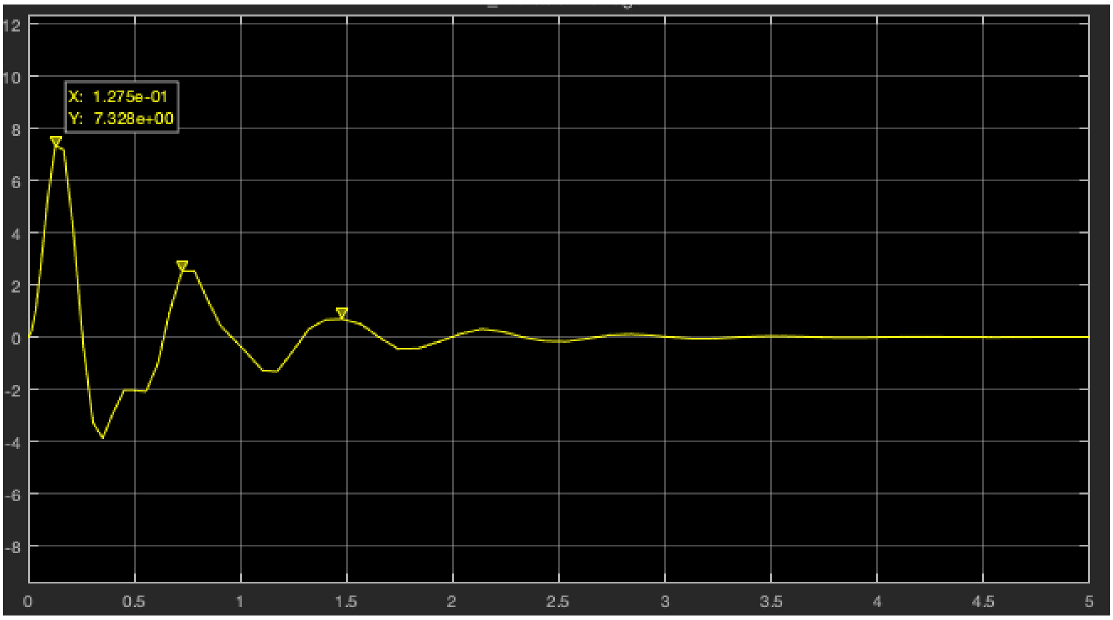


Figure 7 Voltage performance by applying theta dot feedback

Feedback phi dot

Transfer function

$$\text{sys\_tf\_phidot} = \frac{113.8\ s\text{^}2 - 8421}{s\text{^}3 + 34.73\ s\text{^}2 - 128.4\ s - 1557}$$

Root Locus

<u>negative feedback:</u>

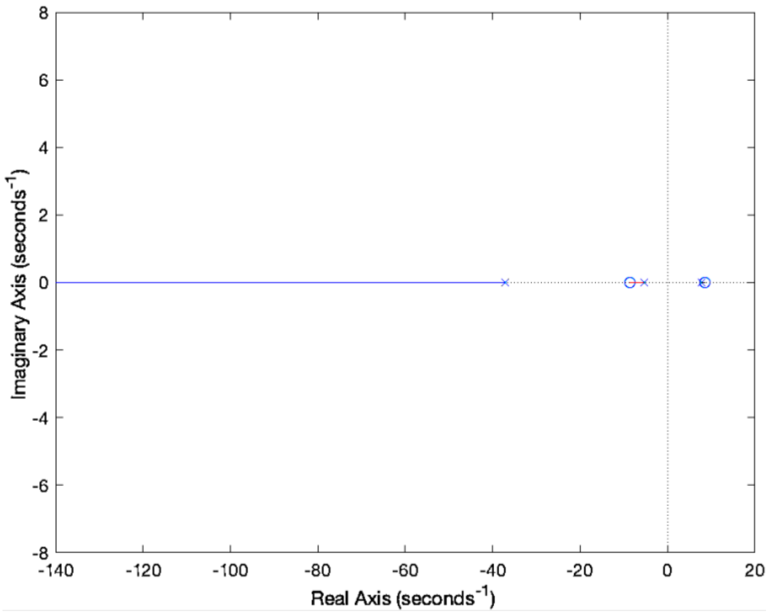


Figure 8 Root Locus of phi dot (negative feedback)

positive feedback:

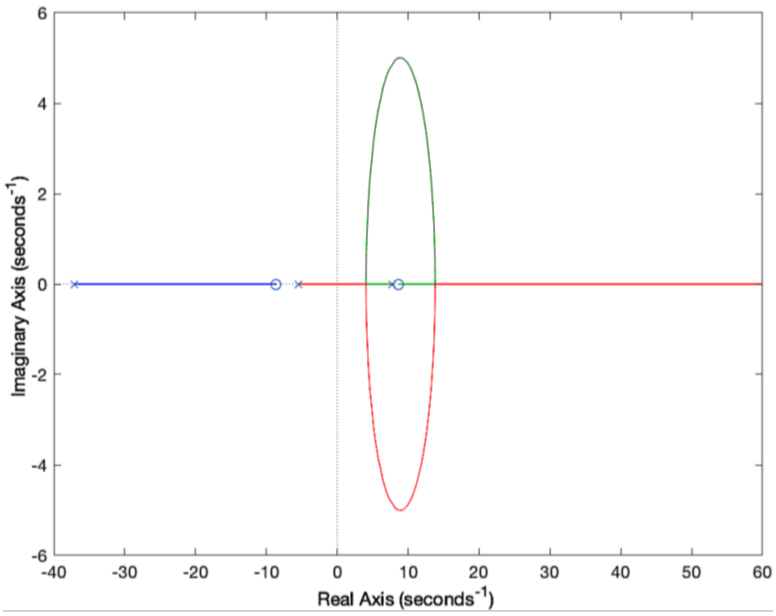


Figure 9 Root Locus of phi dot (positive feedback)

From Figure 8 and Figure 9, we notice there is a pole and a zero in RHP, and they are pretty close to each other. I tried to manipulate poles based on positive feedback, but I found it impossible to twerk root locus line to LHP.

Feedback theta

Transfer function

$$\text{sys\_tf\_theta} = \frac{-74.13\ s}{s\hat{}3 + 34.73\ s\hat{}2 - 128.4\ s - 1557}$$

Root Locus

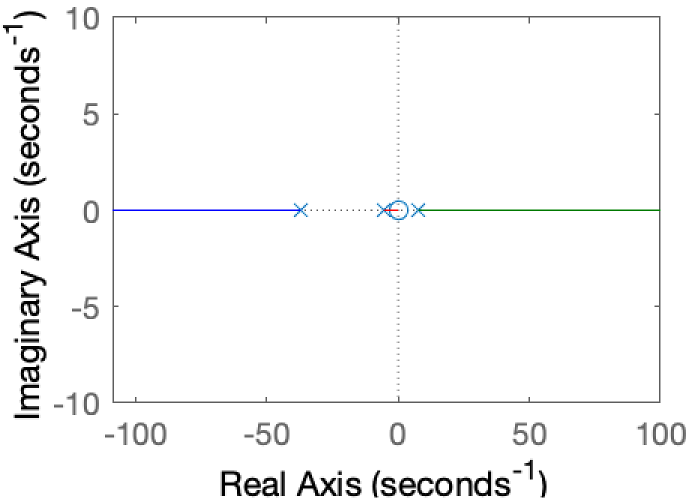


Figure 10 Root Locus of theta

By assigning an unstable pole at 0.5, and a zero at -6.5

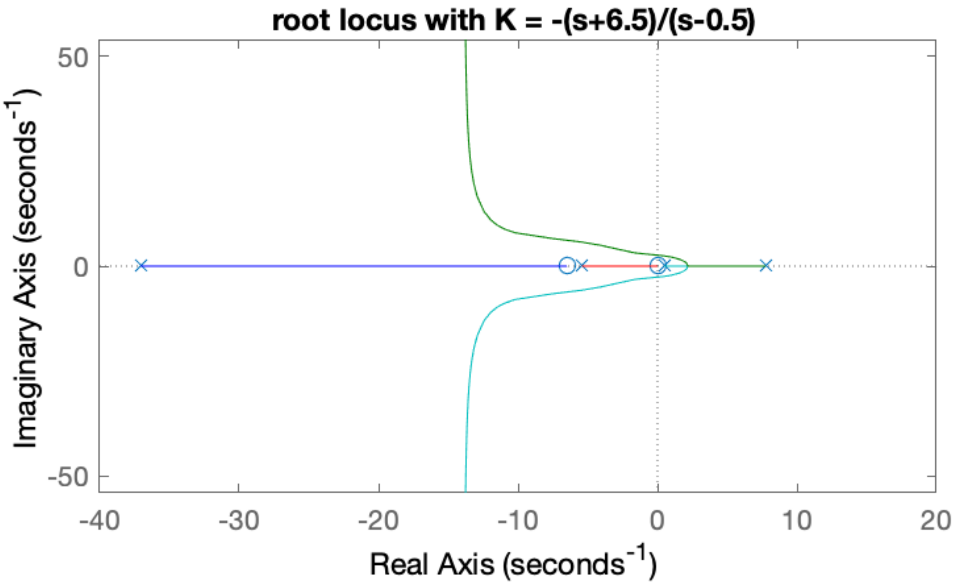


Figure 11 Root Locus of theta after pole and zero adding

Choose proper gain

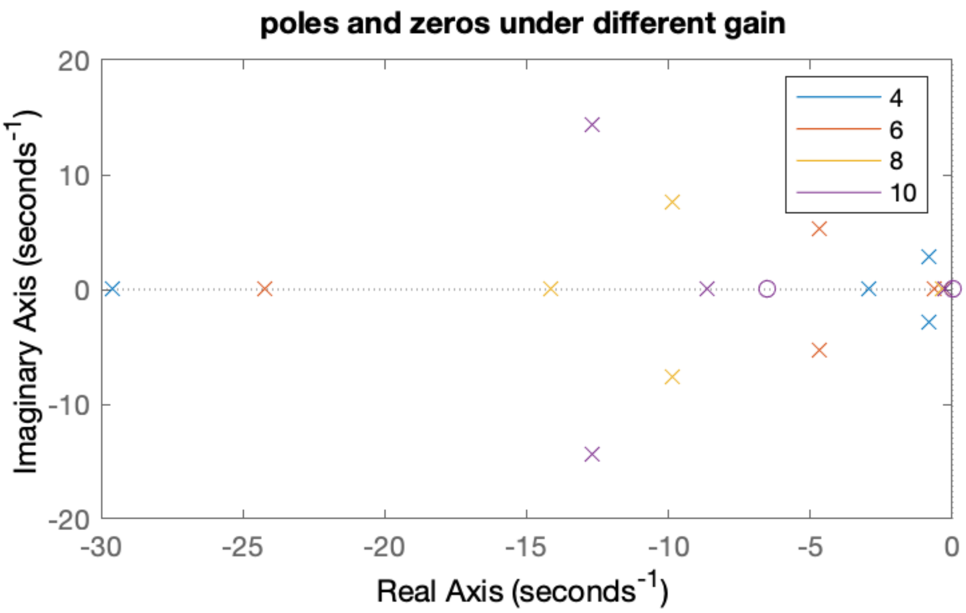


Figure 12 Poles and zeros under different gains

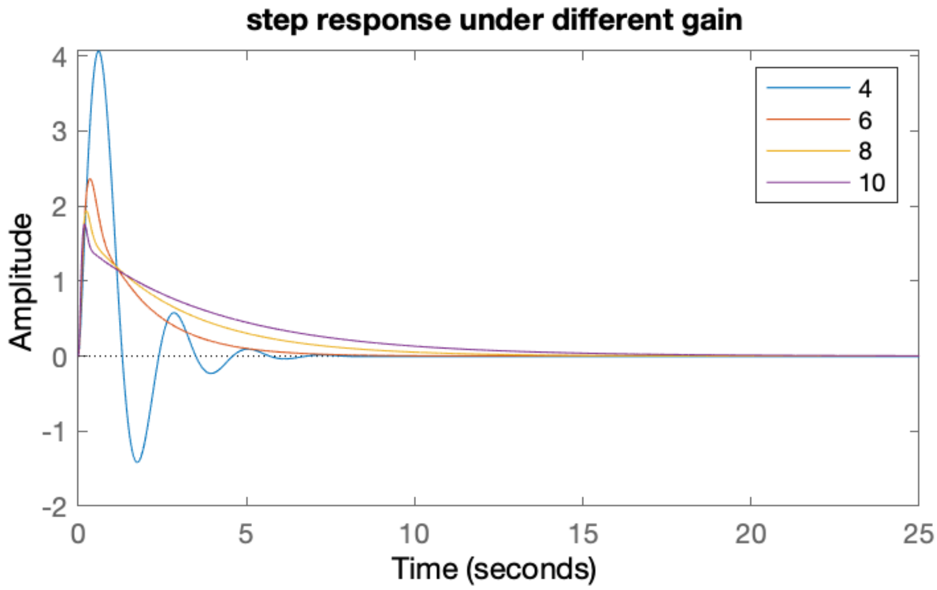
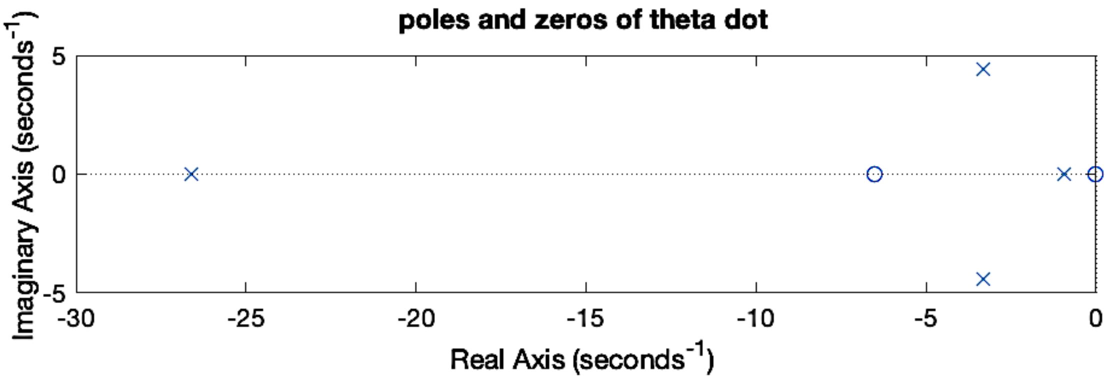


Figure 13 Step response under different gains

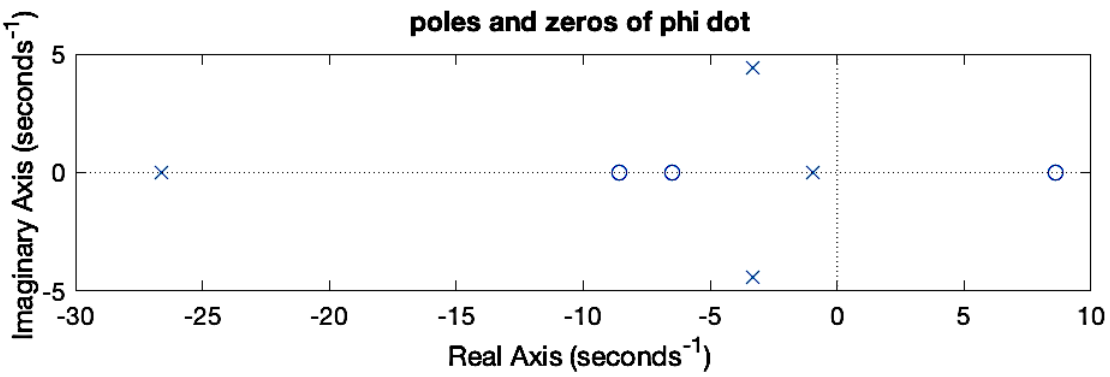
I chose G = 5.2. Therefore, K_theta = $\frac{-5.2(s+6.5)}{(s-0.5)}$

All states transfer functions' poles and zeros

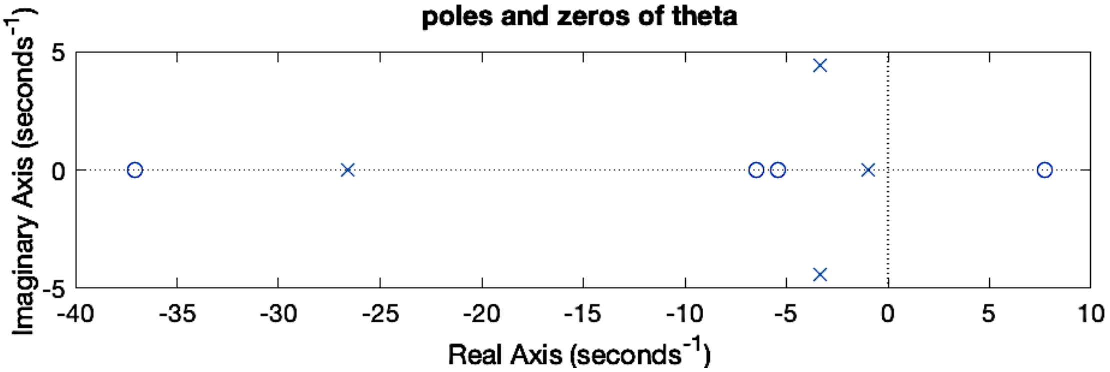
theta dot:



phi dot:



theta:



Performance of feedback controller

Under motor limitation (7.4V), we get to set our initial body angle to 74 degrees (about 1.3 rad). This makes sense since our initial tilt is body angle, and this feedback controller feedback θ (body angle) directly.

Settling time: 1.6 s

Maximum angle: initial angle, second large angle is 28.1 degrees

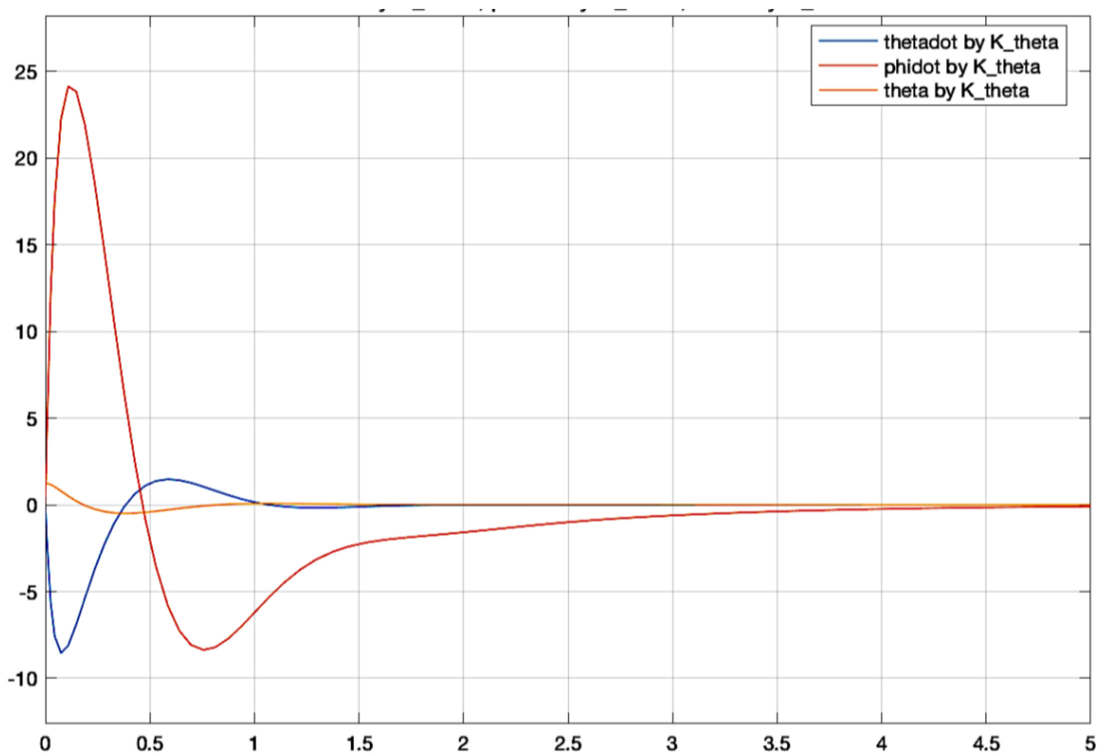


Figure 14 Performance of system by applying theta feedback controller

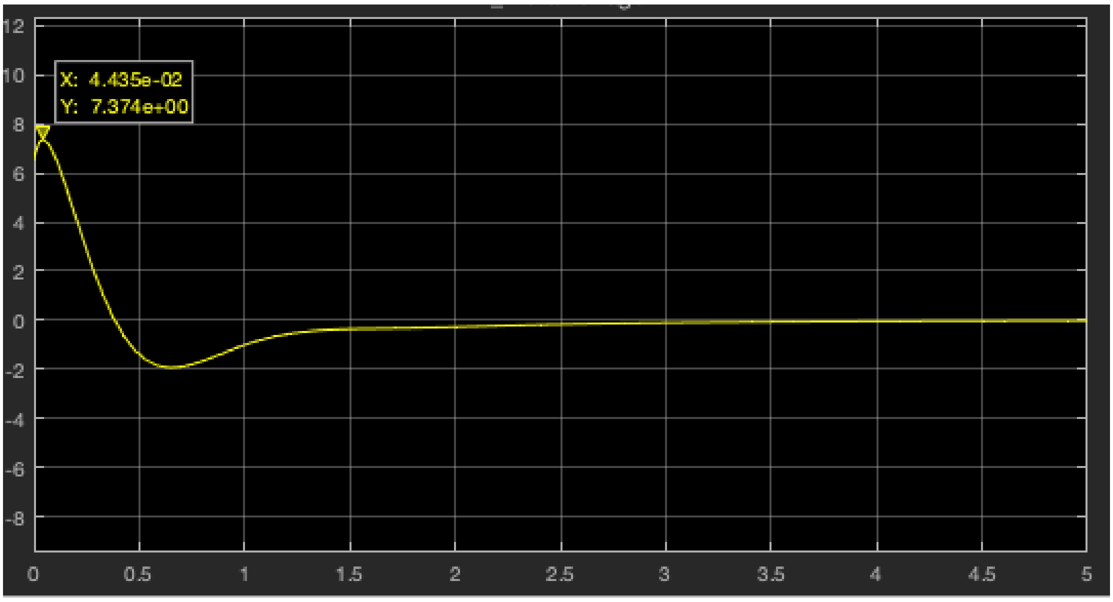
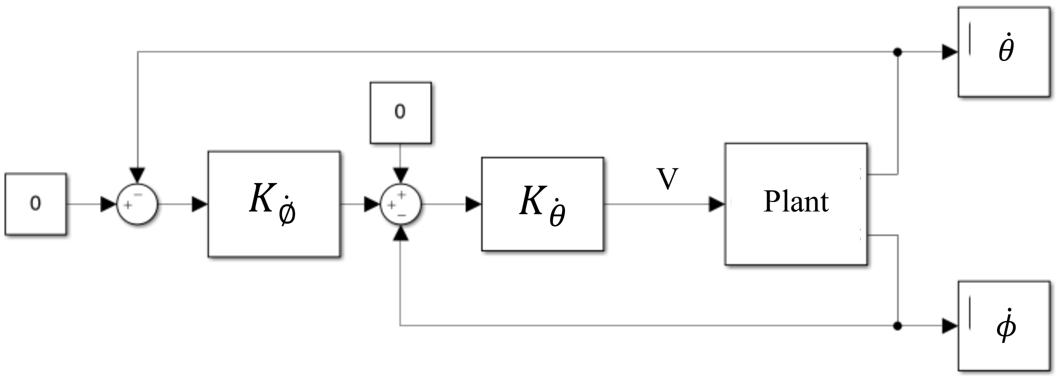


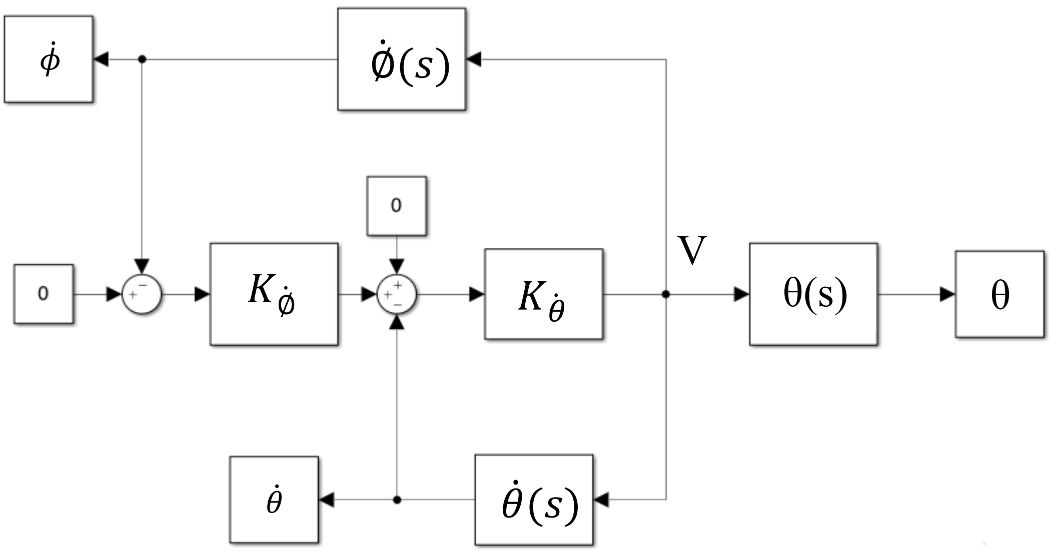
Figure 15 Voltage performance by applying theta feedback controller

Feedback theta dot and phi dot

Block diagram

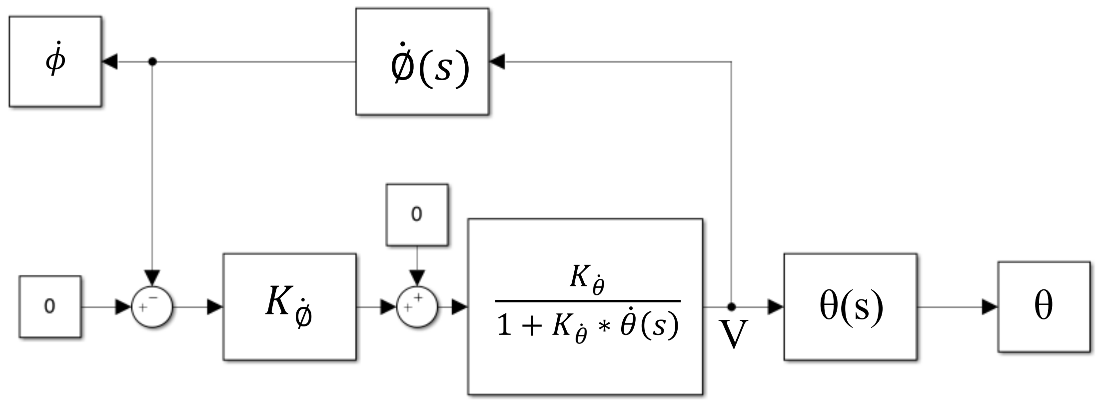


we can rearrange above block diagram into



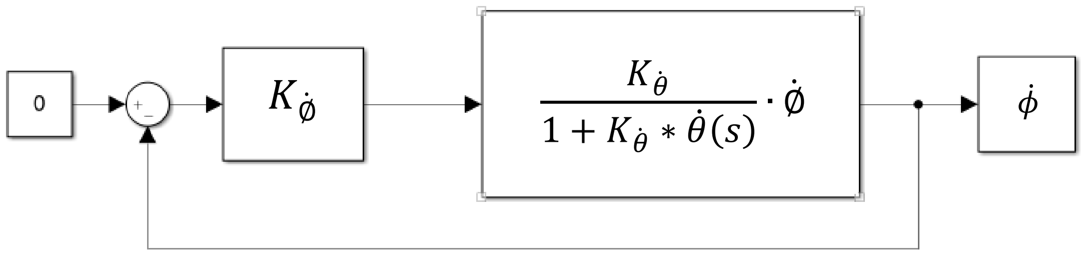
we can first close theta dot loop, which we did before (K_thetadot = $\frac{-5.2(s+6.5)}{(s-0.5)}$).

Therefore, we can simplify block diagram into



Now, we can design our K_phidot by root locus.

Let's rearrange block diagram again



Root Locus

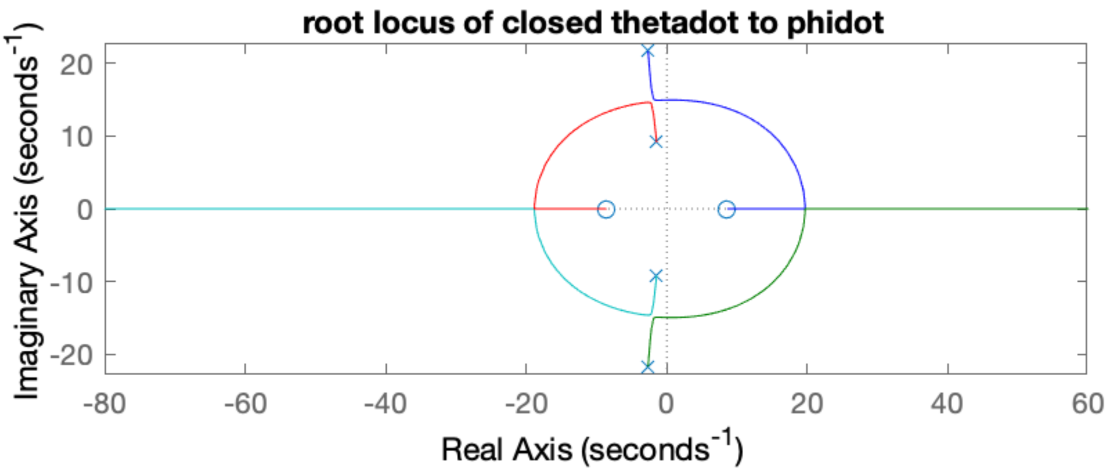


Figure 16 Root Locus of phi dot

Choose proper gain

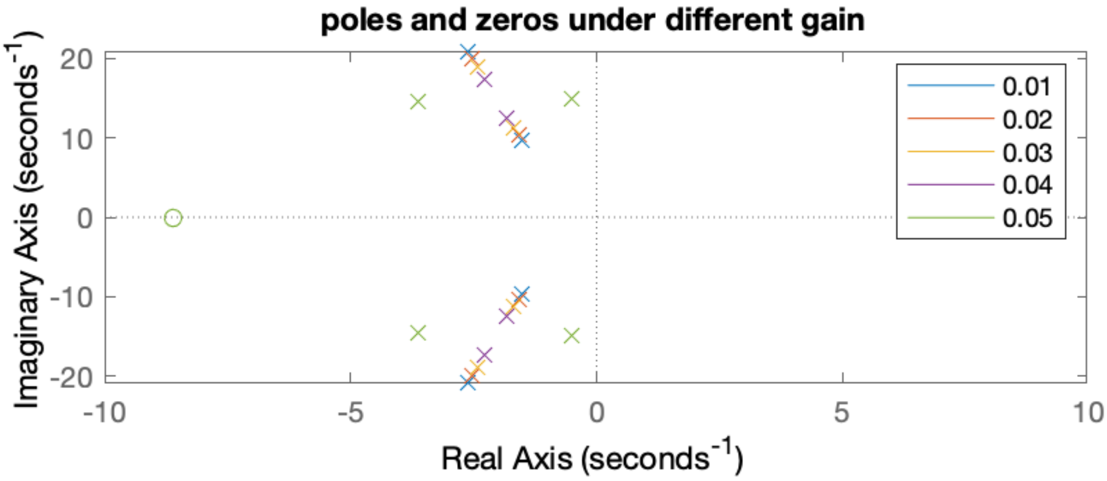


Figure 17 Poles and zeros of phi dot under different gains

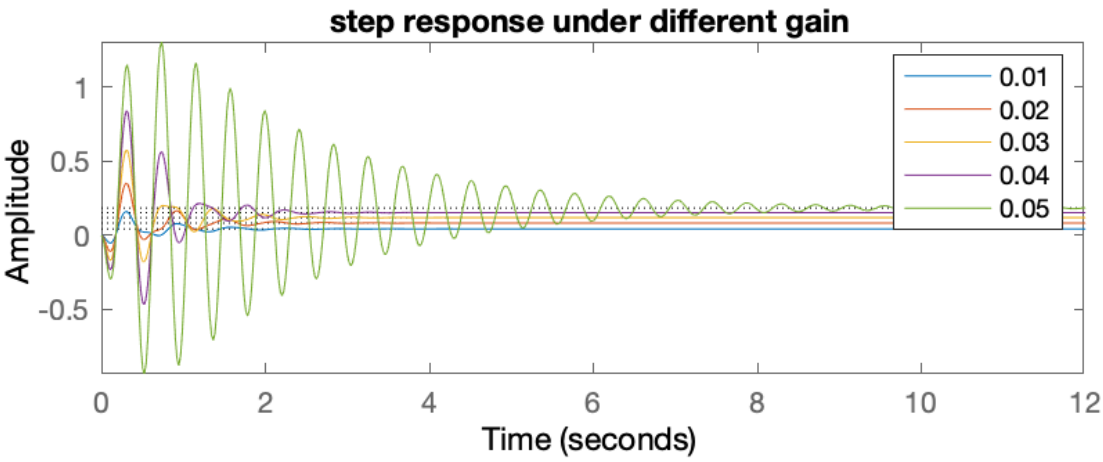
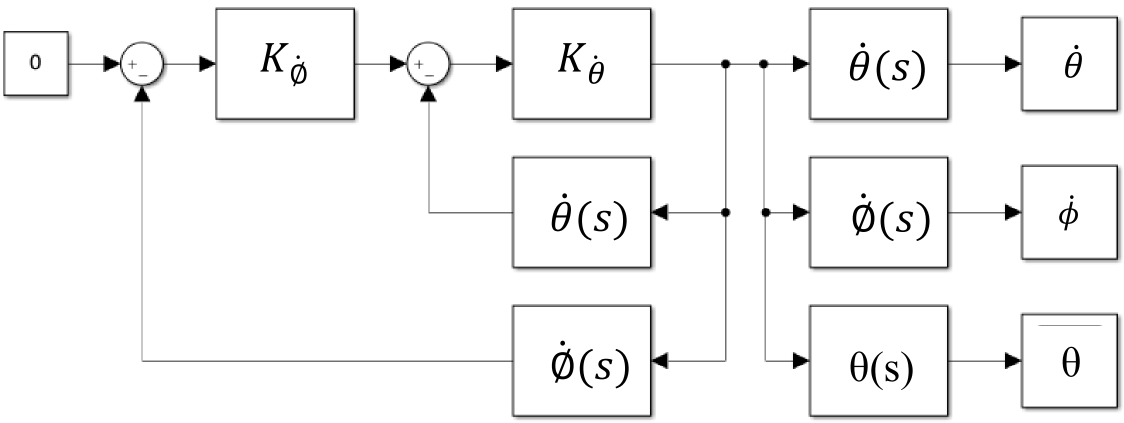


Figure 18 Step response of phi dot under different gains

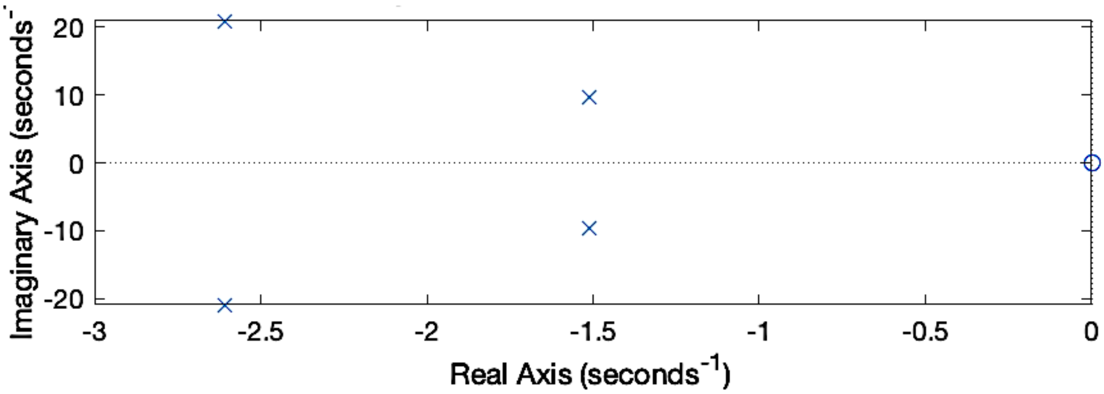
I chose gain to be 0.01. This makes our K_thetadot_phidot = 0.01

Now we have

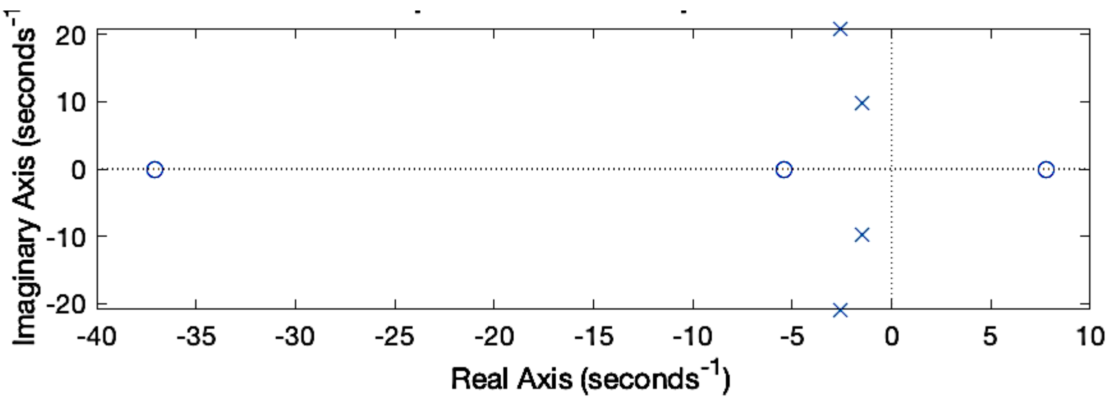


All states transfer function's poles and zeros

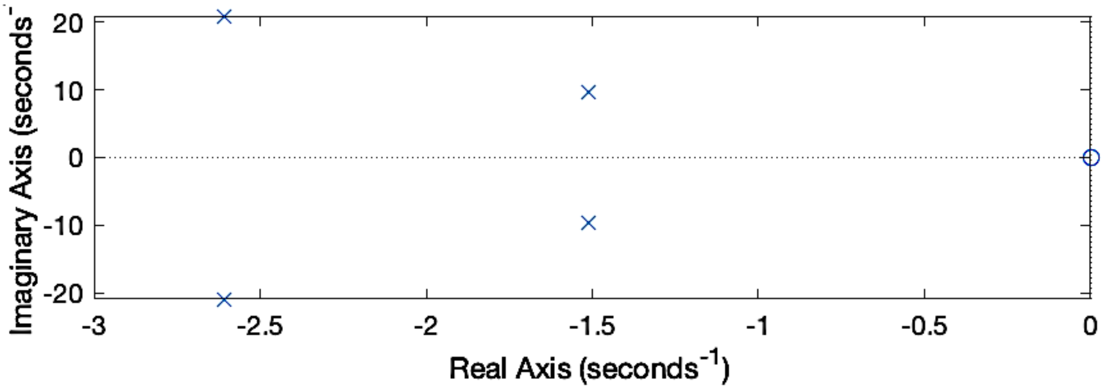
theta dot:



phi dot:



theta:



Performance of feedback controller

Under limitation of motor voltage (7.4V), maximum initial body angle is 31 degrees.

Settling time: 2.32 s

Maximum angle: 0.7727 rad (about 44.3 degrees)

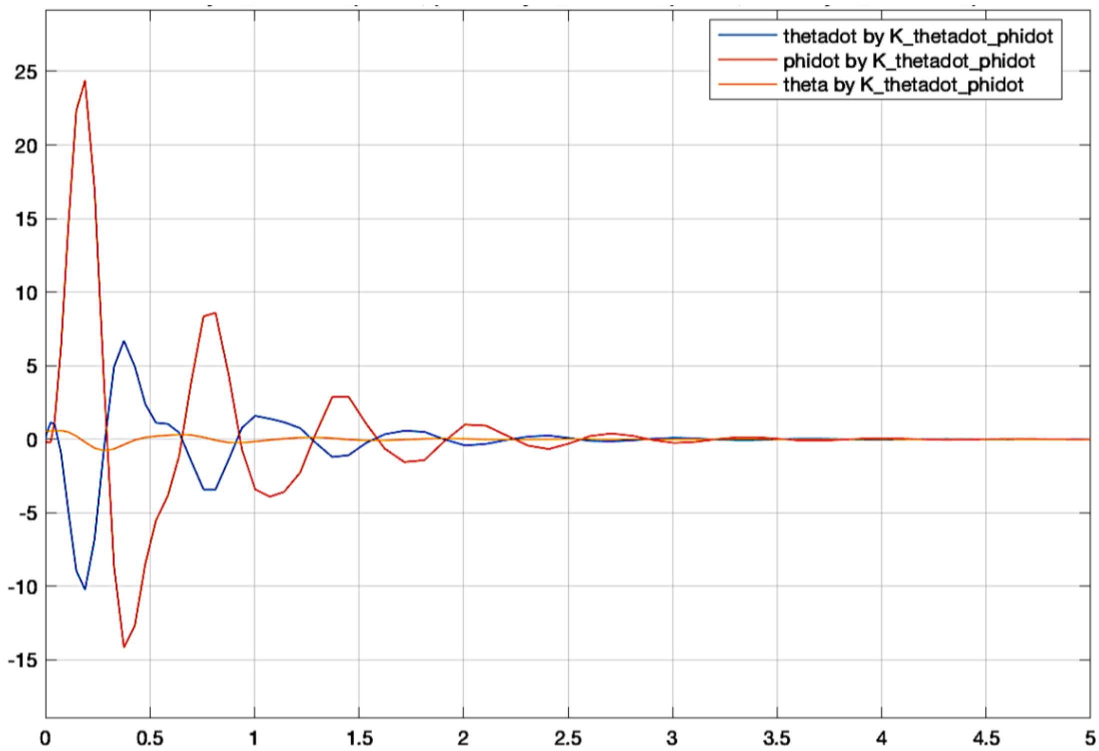


Figure 19 System performance under applying theta dot and phi dot feedback controller

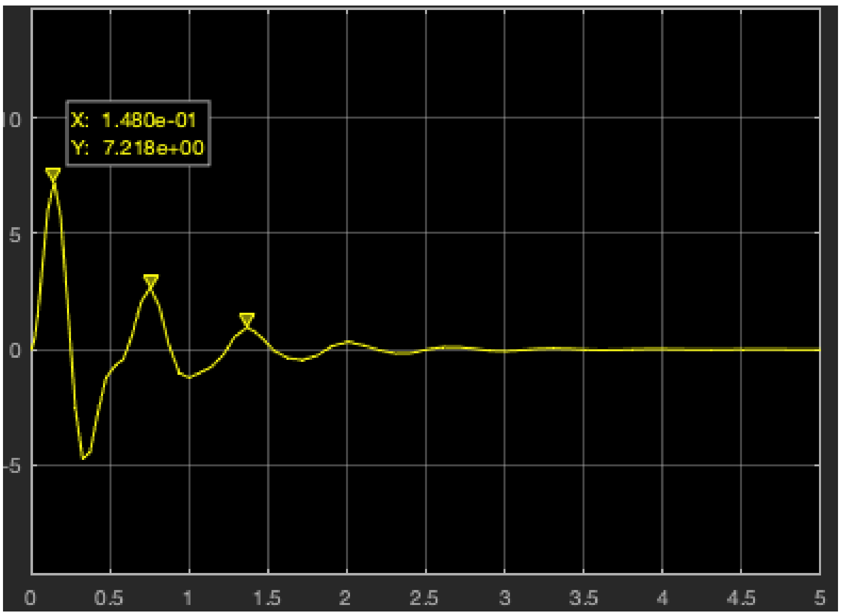
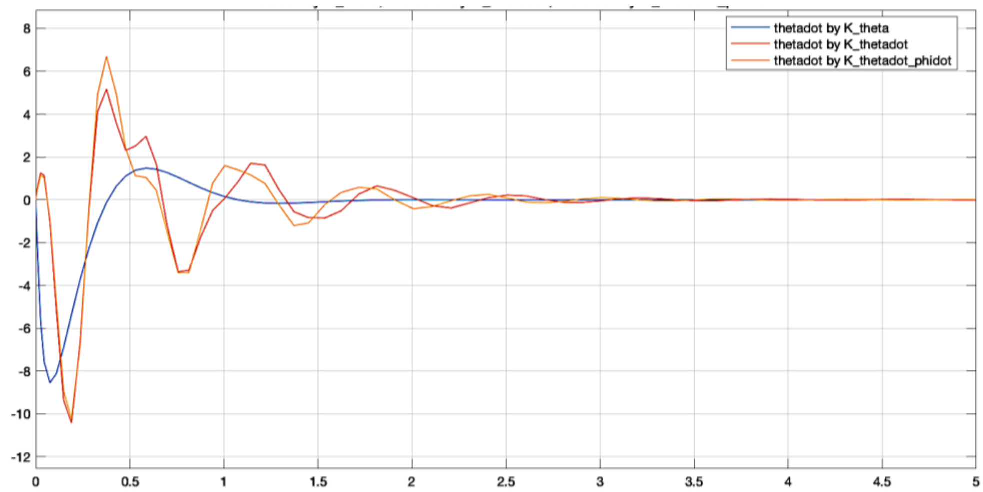


Figure 20 Voltage performance under applying theta dot and phi dot feedback controller

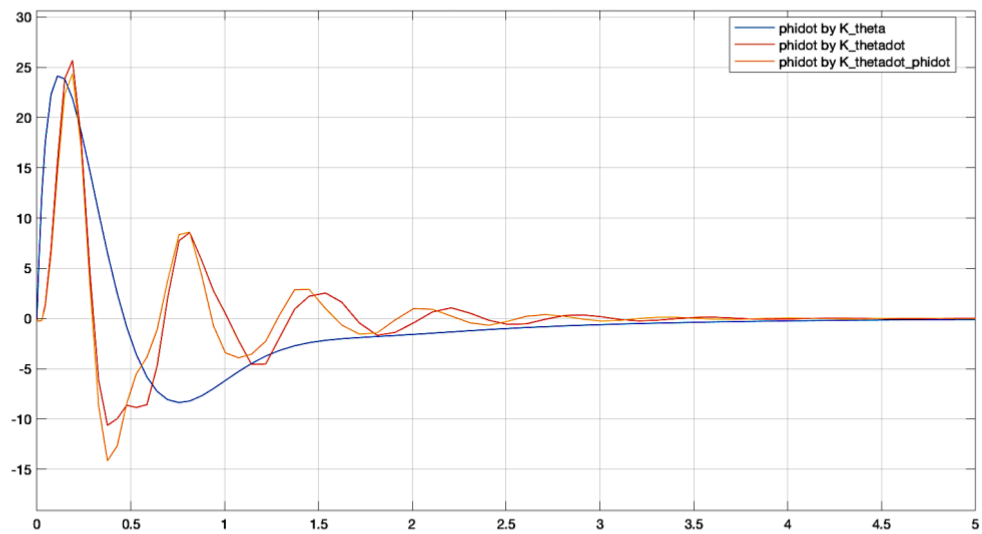
Controller comparison

	Maximum initial angle	Settling time	Max angle
Feedback theta dot	34 degrees	2.46 s	41.7 degrees
Feedback theta	74 degrees	1.6 s	28.1 degrees
Feedback theta dot and phi dot	31 degrees	2.32 s	44.3 degrees

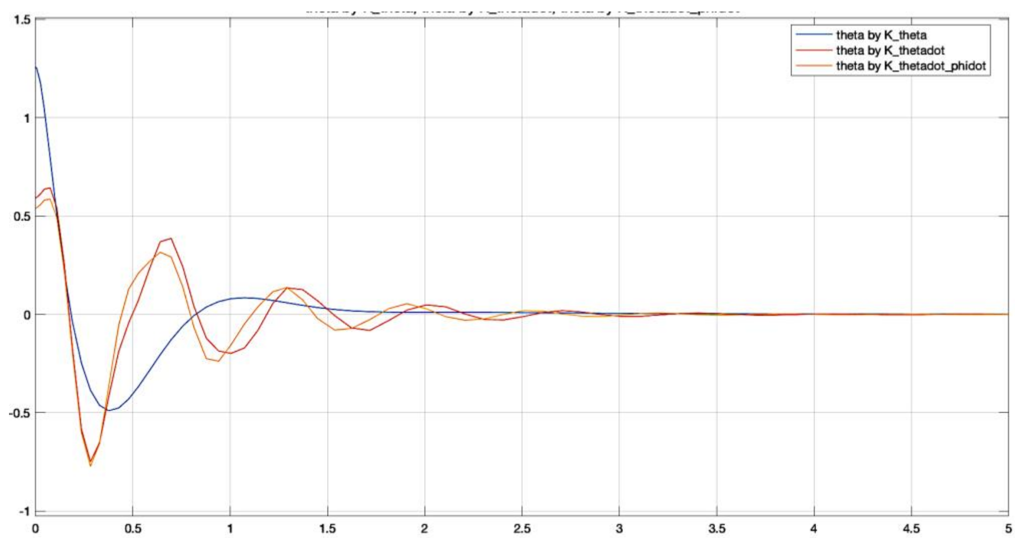
theta dot comparison:



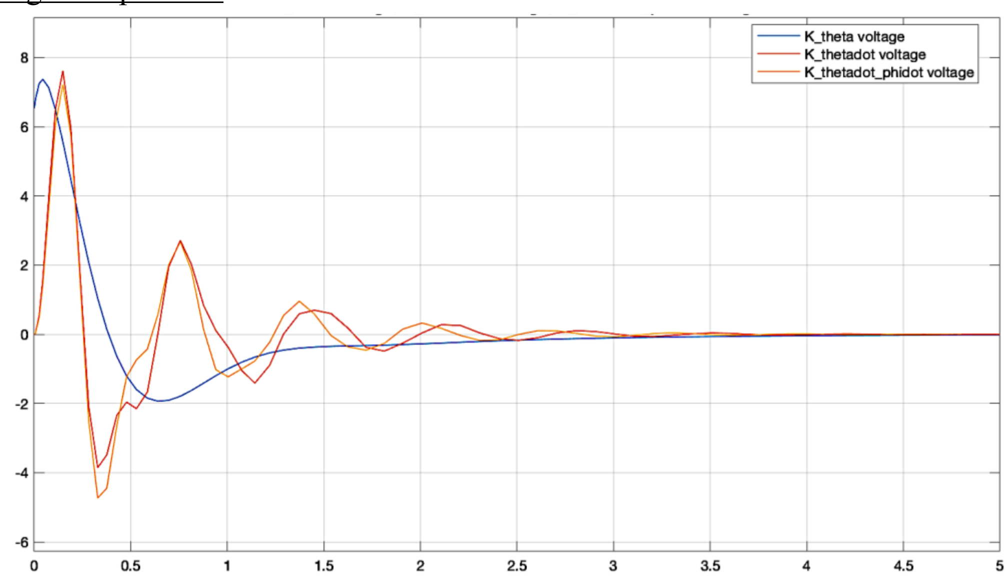
phi dot comparison:



theta comparison:



voltage comparison:



Modern control

LQG controller

LTI system:

$$dx(t)/dt = Ax(t) + B_u u(t) + B_w w(t)$$
$$y(t) = C_y x(t) + D_{yw} w(t)$$
$$z(t) = C_z x(t) + D_{zu} u(t)$$

Controller:

$$d\hat{x}(t)/dt = A\hat{x}(t) + B_u u(t) + L(\hat{y}(t) - y(t))$$
$$\hat{y}(t) = C_y \hat{x}(t)$$
$$u(t) = K\hat{x}(t)$$

find K:

$$R = D_{zu}' \cdot D_{zu} = 1$$

$$Q = C_z' \cdot C_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

command `[K,X,S] = lqr(A,B,Q,R)`

notice K obtained by `lqr()` is the K makes A-BK < 0

find L:

$$w(t) = \begin{bmatrix} \textit{input noise} \\ x1\ \textit{measure noise} \\ x2\ \textit{measure noise} \\ x3\ \textit{measure noise} \end{bmatrix}$$

$$W \text{ (covariance of noise)} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

$$R = D_{yw}\, W\, D_{yw}'$$
$$Q = B_w\, W\, B_w'$$

command `[L,X,S] = lqr(A',C',Q,R)`

remember to let `L = -L` to get L that makes A+LC < 0

Stabilizable and Detectable

To apply LQG controller, the system needs to meet some assumptions:

Assumptions:

(A, B_u) stabilizable

(A, C_y) detectable

$w(t)$ is a Gaussian zero mean

$C_z' D_{zu} = 0$ and $D_{zu}' D_{zu} > 0$

$B_w\, W\, D_{yw}' = 0$ and $D_{yw}\, W\, D_{yw}' > 0$

Stabilizable and Detectable:

The system is stabilizable if its unstable subsystem is controllable. The system is detectable if its unstable subsystem is observable.

Subsystem:

To obtain unstable subsystem, we need ordered schur decomposition

command `[U,T] = schur(A)` to get U and T such that T = U'AU

command `[US,TS] = ordschur(U,T,'lhp')` to get US and TS such that TS = US'AUS, and ordered eigenvalues appear on TS main diagonal.

$$\begin{bmatrix} TS11 & TS12 & Bs \\ 0 & TS22 & Bu \\ Cs & Cu & D \end{bmatrix} = \begin{bmatrix} US'AUS & US'B \\ CUS & D \end{bmatrix}$$

Unstable subsystem:

Find unstable subsystem T_{22} by eigenvalues.

$$\begin{bmatrix} T22 & Bu \\ Cu & 0 \end{bmatrix}$$

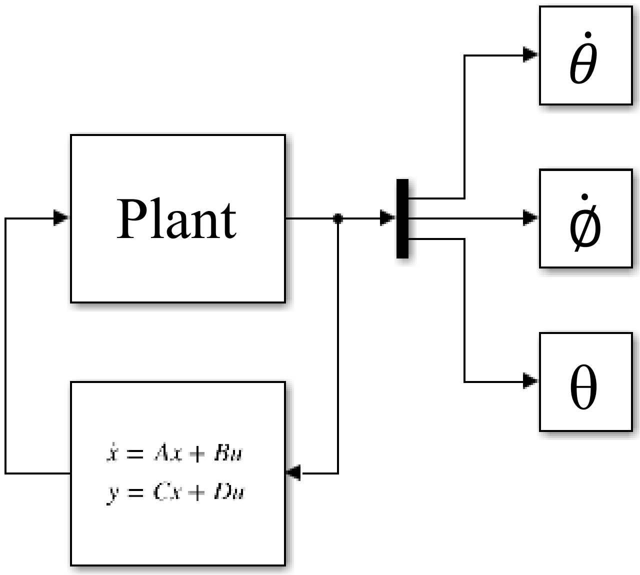
Controllable:

command `rank(ctrb(A,B))` to obtain rank of controllability matrix

Observable:

command `rank(obsv(A,C))` to obtain rank of observability matrix

Block Diagram



Controller

where A_controller = A+B_uK+LC_y

B_controller = -L

C_controller = K

D_controller = 0

Closed loop

$$\begin{bmatrix} \dot{e} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A + LC_y & 0 \\ -FC_y & A + B_u K \end{bmatrix} \cdot \begin{bmatrix} \dot{e} \\ \dot{\hat{x}} \end{bmatrix} + \begin{bmatrix} B_w + FD_{yw} \\ -FD_{yw} \end{bmatrix} \cdot w(t)$$

$$z = \begin{bmatrix} C_z & C_z + D_{zu} K \end{bmatrix} \cdot \begin{bmatrix} \dot{e} \\ \dot{\hat{x}} \end{bmatrix}$$

Controller dθ/dt

LQG controller

K_thetadot:

[3.4769   1.2018   31.0211]

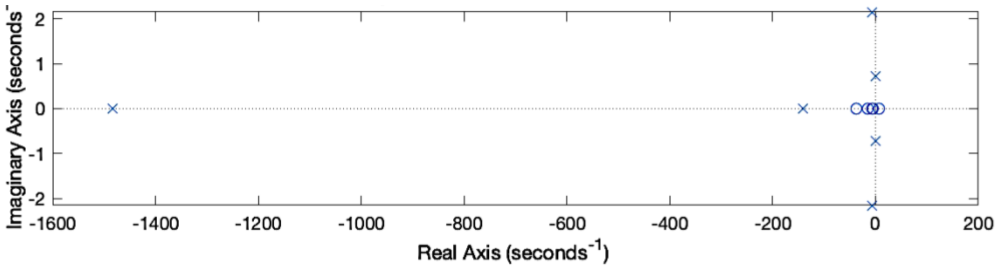
L_thetadot:

$$1.0e+03 * \begin{bmatrix} -1.4498 \\ 2.0758 \\ -0.0020 \end{bmatrix}$$

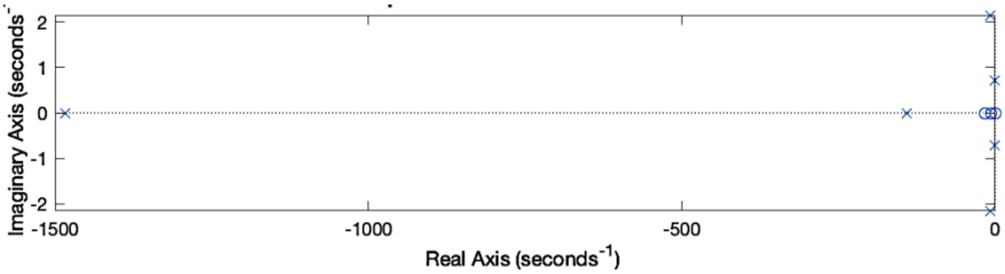
with K and L, we can compute transfer function of controller by
command `[num, den] = ss2tf()`
and then
command `tf(num, den)`
by transfer function of dθ/dt to V, we can plot poles and zeros for all states.

All states transfer functions' poles and zeros

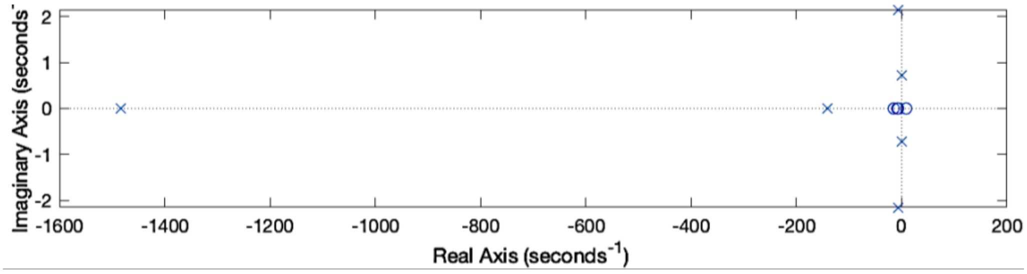
theta dot:



phi dot:



theta:



Performance

According to limitation of input voltage, maximum initial body angle is 30 degrees.

Settling time: 2.11 s

Maximum angle: 0.6884 rad (about 39.5 degrees)

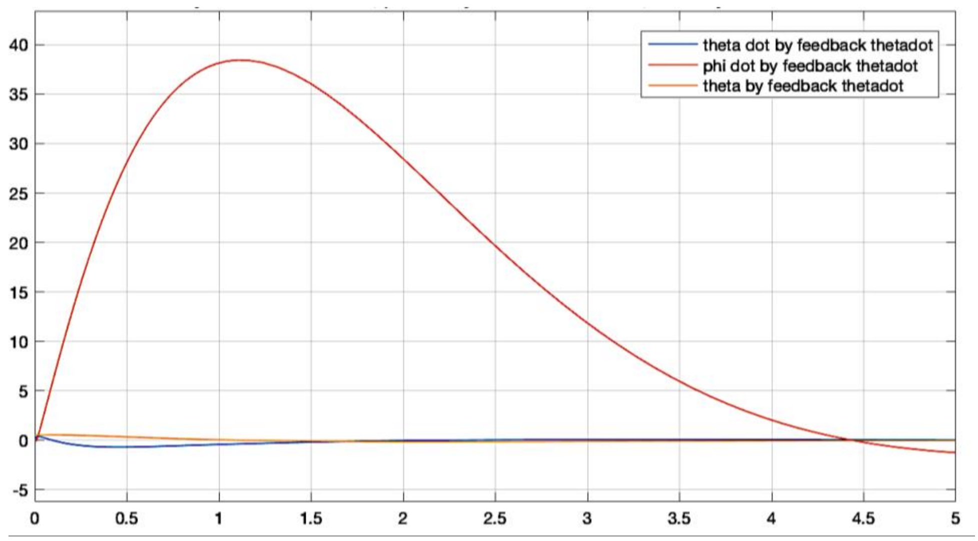


Figure 21 System performance by applying LQG controller (theta dot)

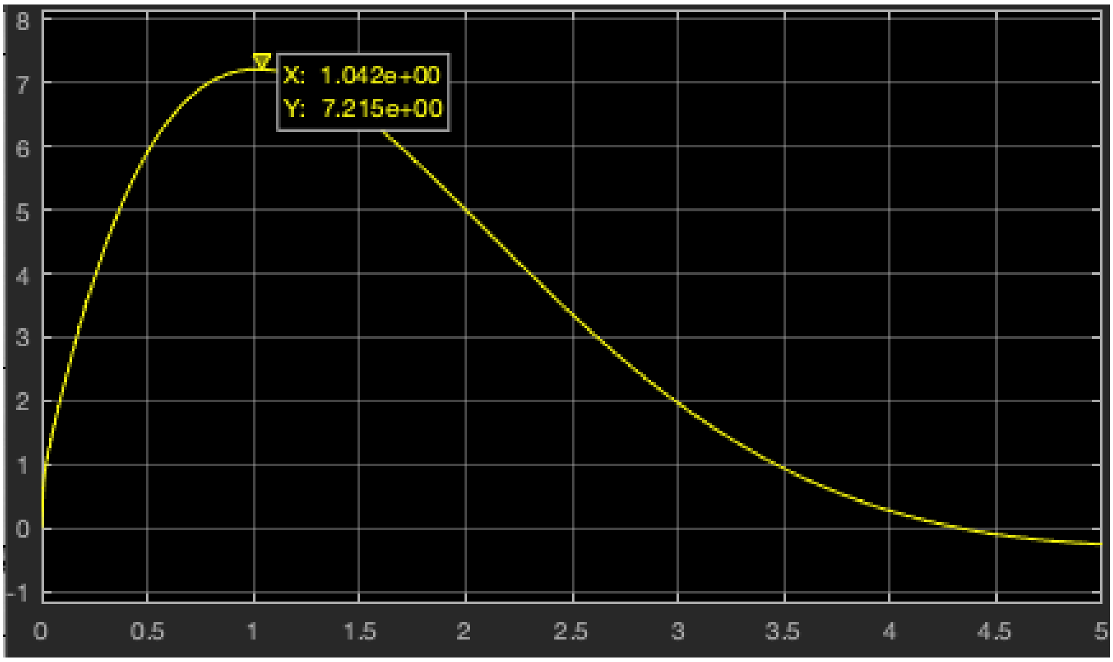


Figure 22 Voltage performance by applying LQG controller (theta)

Controller dϕ/dt

LQG controller

K_phidot:

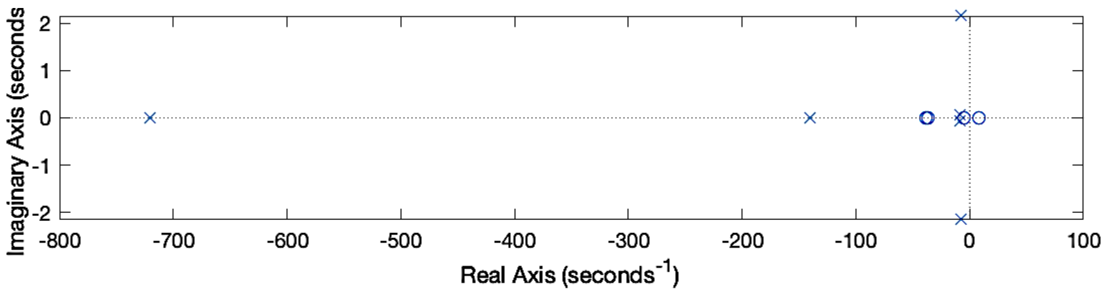
$$[3.4769 \quad 1.2018 \quad 31.0211]$$

L_phidot:

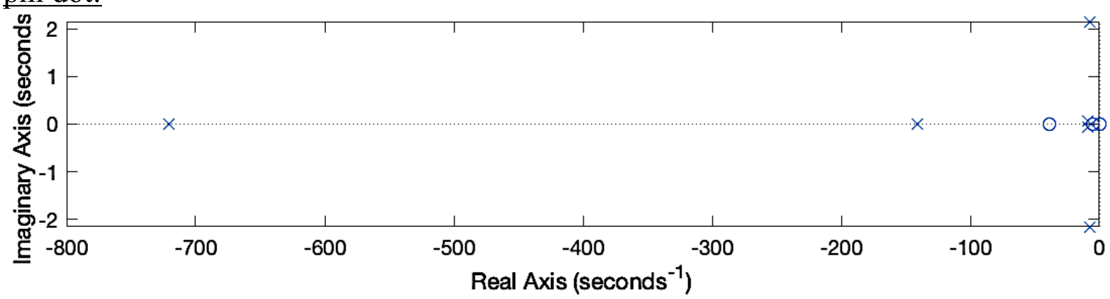
$$\begin{bmatrix} -649.7766 \\ -703.0331 \\ -128.1200 \end{bmatrix}$$

All states transfer functions’ poles and zeros

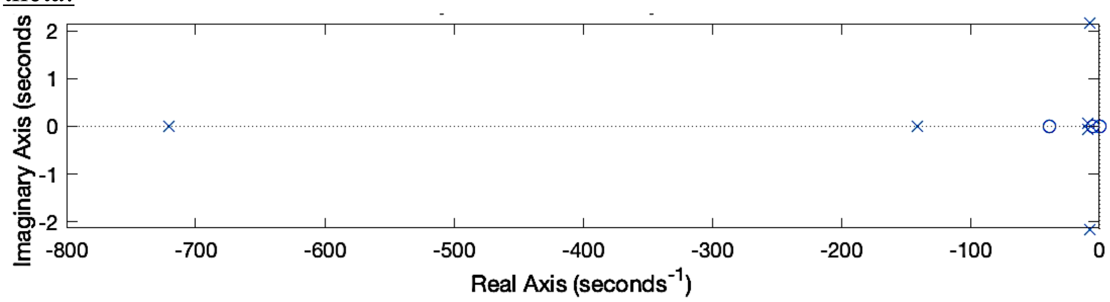
theta dot:



phi dot:



theta:



Performance

Under limitation of input voltage, maximum initial body angle is 29 degrees.

Settling time: 1.29 s

Maximum angle: 0.9123 rad (about 52.3 degrees)

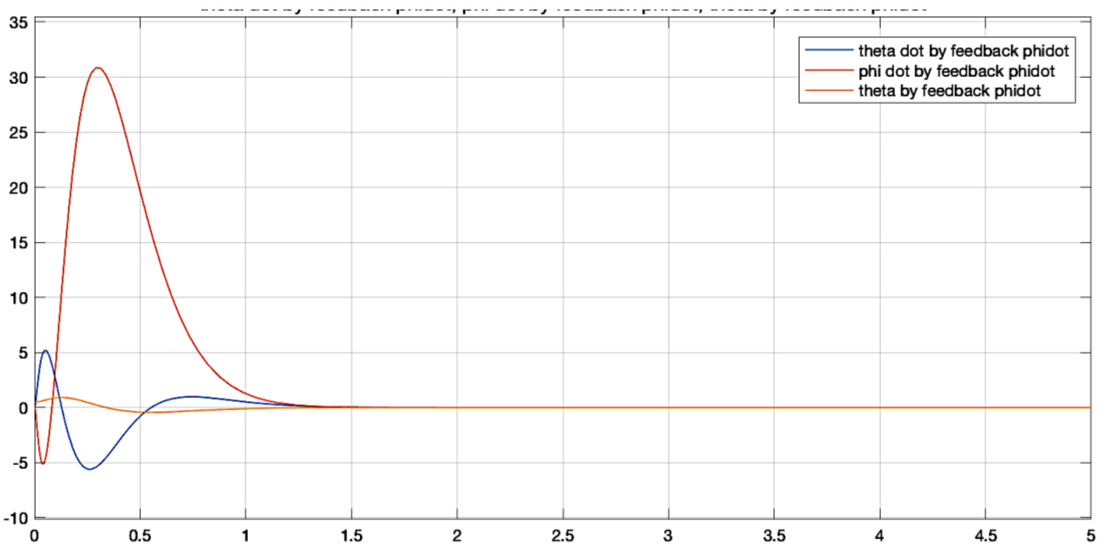


Figure 23 System performance by applying LQG controller (phi dot)

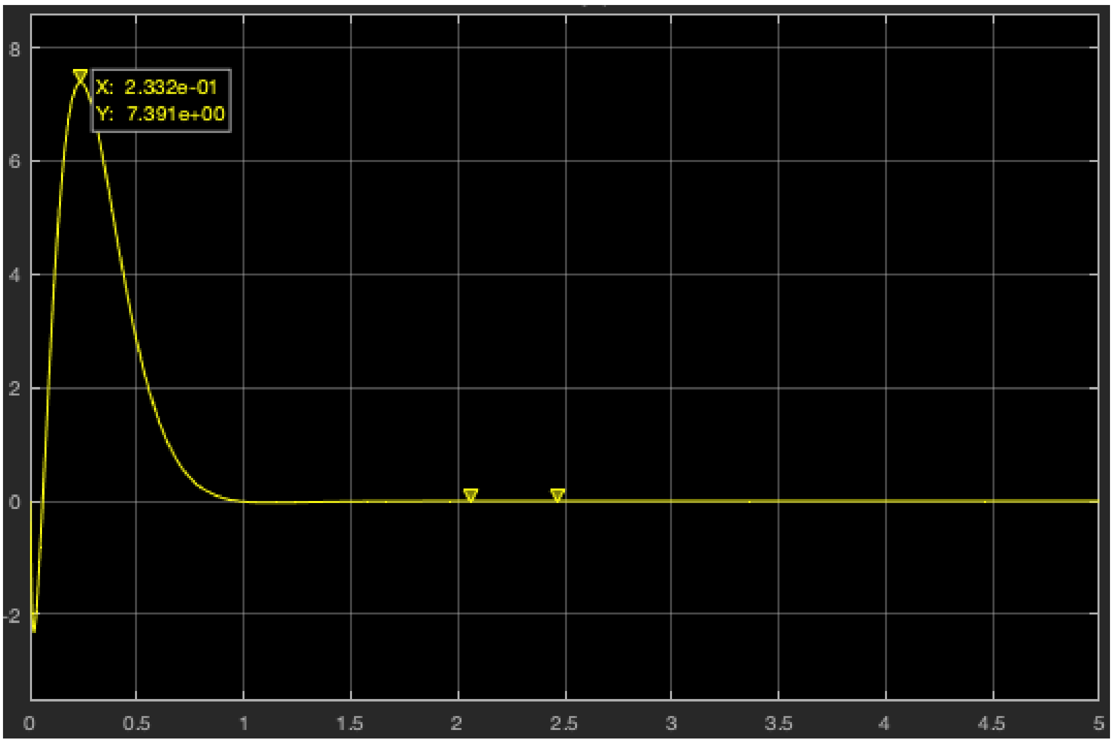


Figure 24 Voltage performance by applying LQG controller (phi dot)

Controller θ

LQG controller

K_theta:

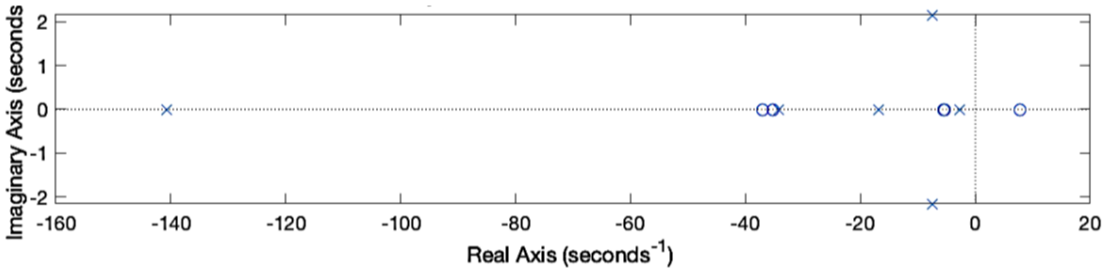
$$[3.4769 \quad 1.2018 \quad 31.0211]$$

F_theta:

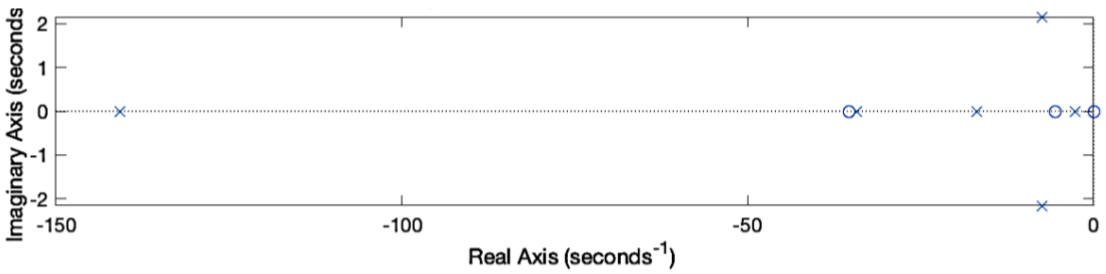
$$\begin{bmatrix} -181.7252 \\ 51.7494 \\ -19.0644 \end{bmatrix}$$

All states transfer functions' poles and zeros

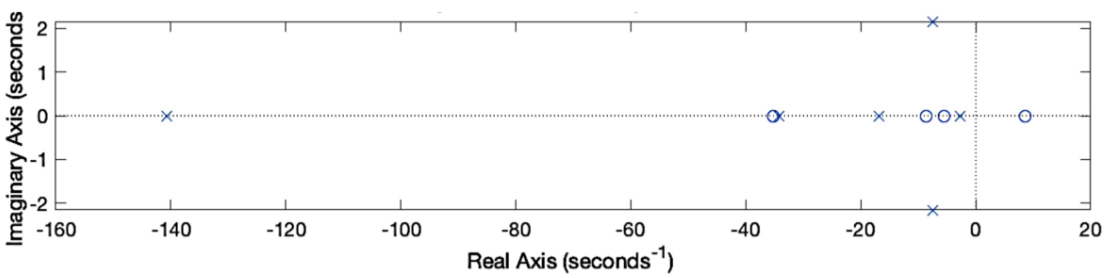
theta dot:



phi dot:



theta:



Performance

Under limitation of input voltage, maximum initial body angle is 49 degrees.

<u>Settling time:</u> 1.43 s

<u>Maximum angle:</u> initial body angle. Second large angle is 0.4011 rad (about 23 degrees)

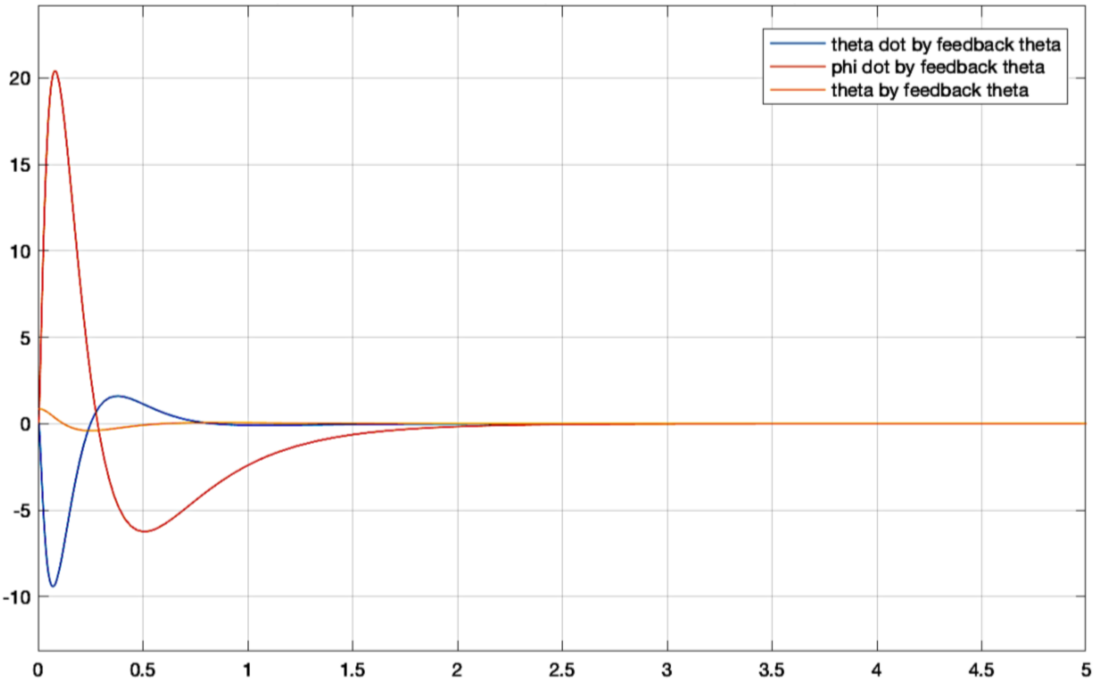


Figure 25 System performance under applying LQG controller (theta)

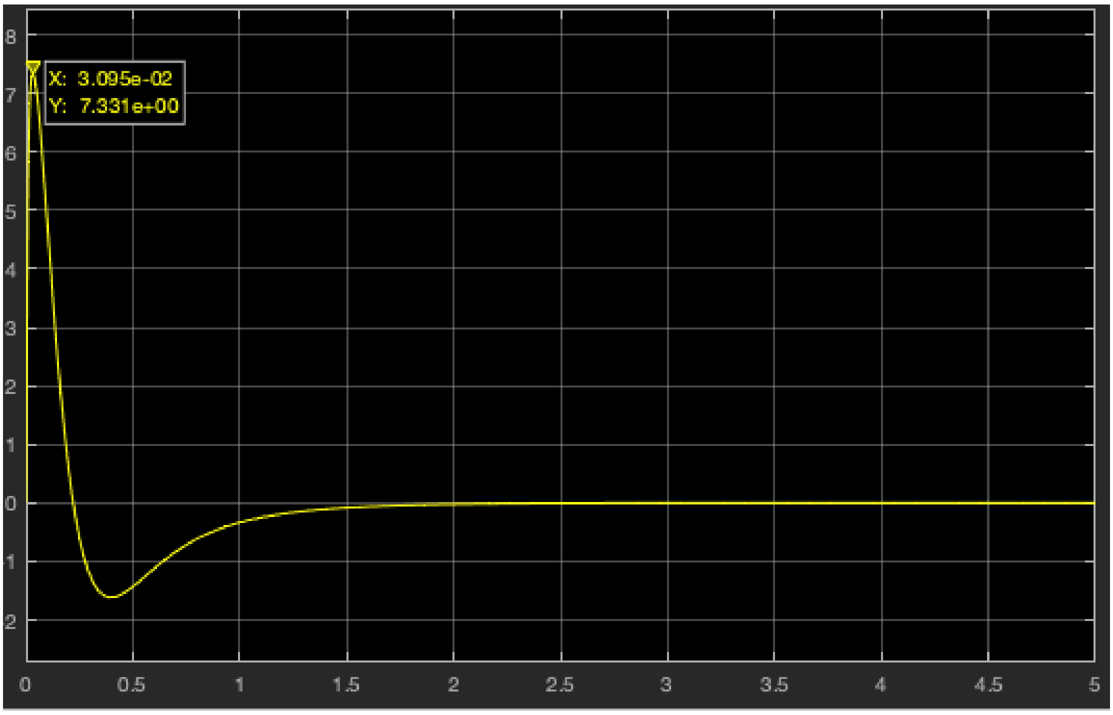
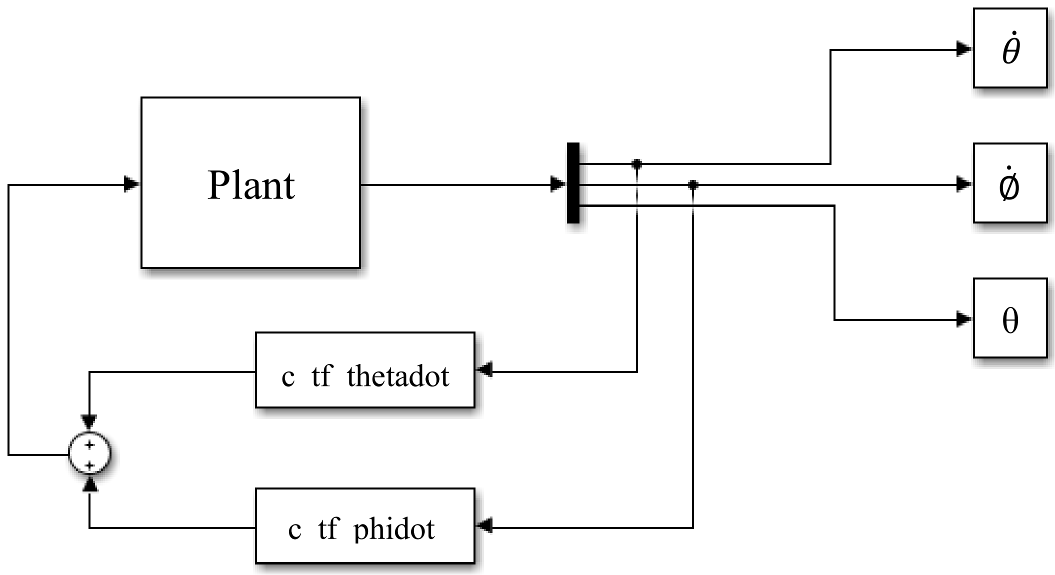


Figure 26 Voltage performance under applying LQG controller (theta)

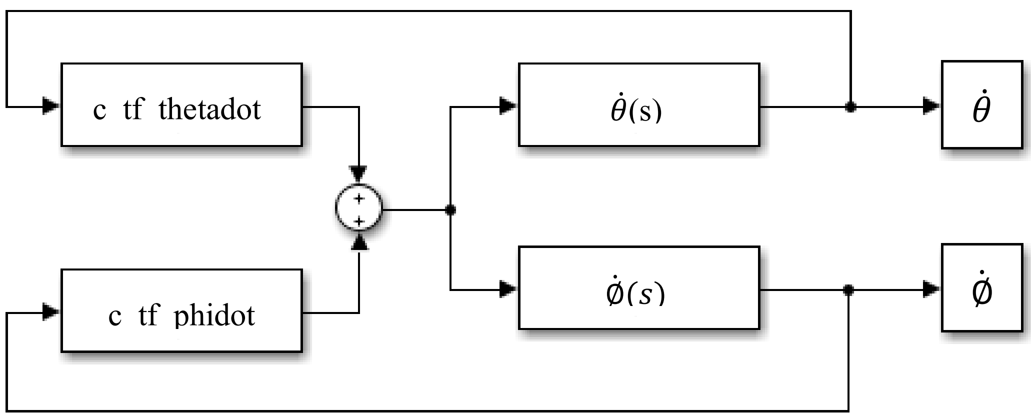
Controller dθ/dt and dϕ/dt

Multiple input controller block diagram

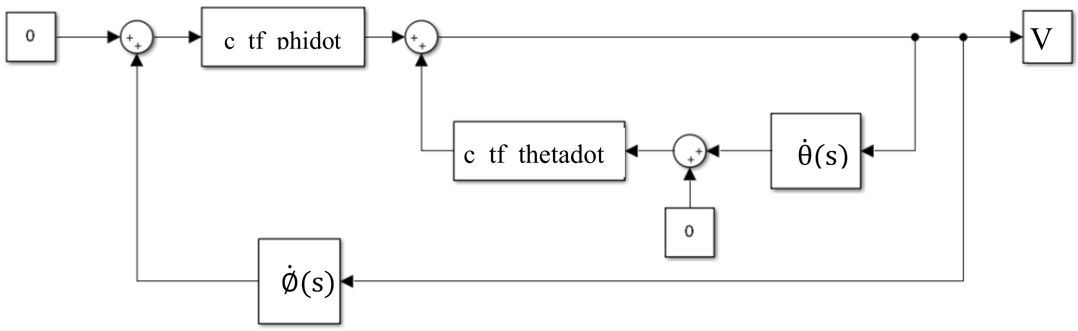
To get better understanding in classical control and modern control, I would like to represent modern control in block diagram. Let's take dθ/st and dϕ/dt for example. We first can draw the block diagram intuitively.



c_tf_thetadot can be obtained by `ss2tf("controller system", 1)`, and c_tf_phidot is obtained by `ss2tf("controller system", 2)`.
Now, try to draw plant block explicitly.



It's a block diagram that is easy to understand, but it's hard to determine closed loop system. Therefore, although we can sure that both dθ and dϕ will be stable after the loop close, we cannot know the poles and zeros of theta.
To illustrate closed loop better, I tried several methods, and the best one is



By the block diagram above, we can easily compute transfer function, and then multiply with sys_tf_theta (transfer function of theta).

LQG controller

K_thetadot_phidot:

[3.4769   1.2018   31.0211]

F_thetadot_phidot:

$$1.0e{+}03 * \begin{bmatrix} -1.3081 & 0.1992 \\ 1.9921 & -0.3153 \\ -0.0015 & -0.0003 \end{bmatrix}$$

Performance

Under limitation of input voltage, maximum initial body angle is 56 degrees.

Settling time: 1.36 s

Maximum angle: 1.011 rad (about 58 degrees)

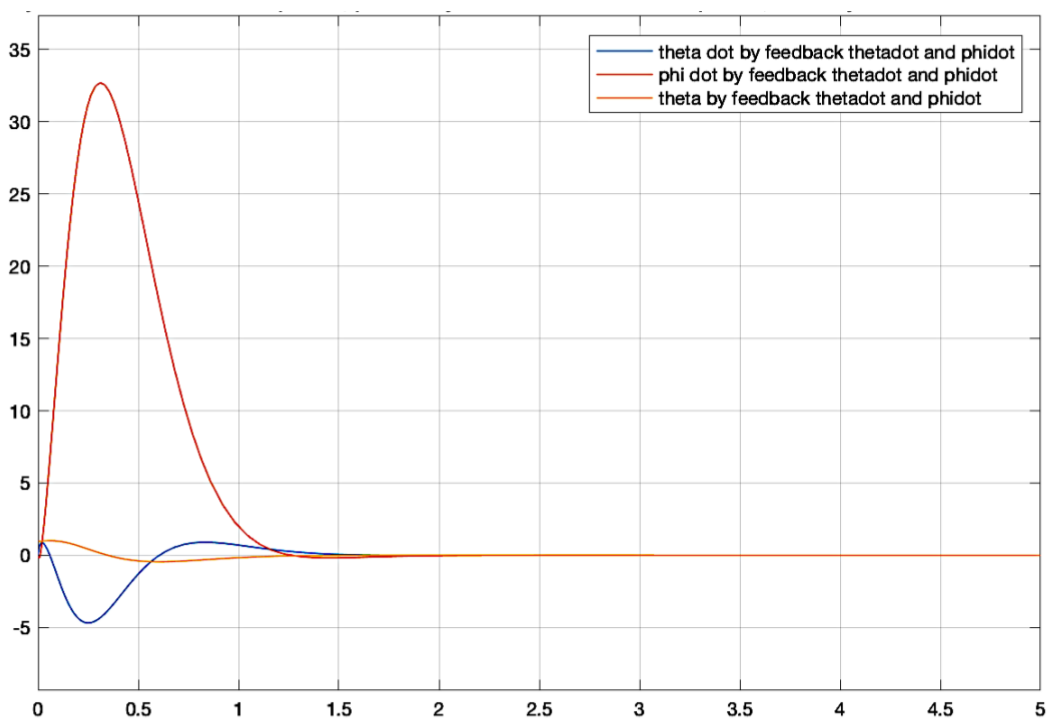


Figure 27 System performance under applying LQG controller (dθ/dt and dϕ/dt)

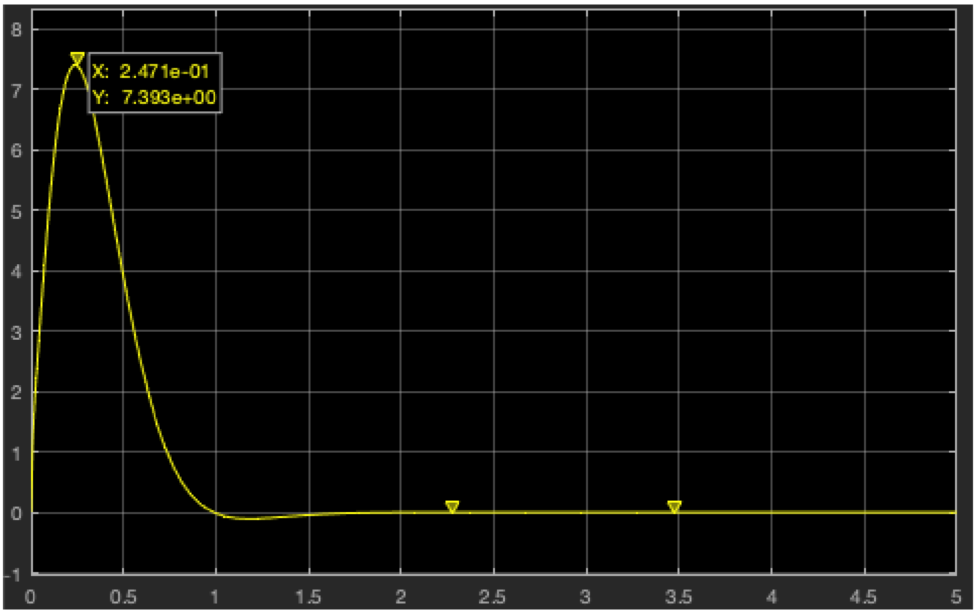


Figure 28 Voltage performance under applying LQG controller (dθ/dt and dϕ/dt)

Controller dθ/dt and θ

LQG controller

K_thetadot_theta:

$$[3.4769 \quad 1.2018 \quad 31.0211]$$

F_thetadot_theta:

$$\begin{bmatrix} -436.7090 & -12.4944 \\ 635.5742 & -66.0860 \\ -1.2494 & -3.0623 \end{bmatrix}$$

Performance

Under limitation of input voltage, maximum body angle is 74 degrees.

Settling time: 2.17s

Maximum angle: initial angle.

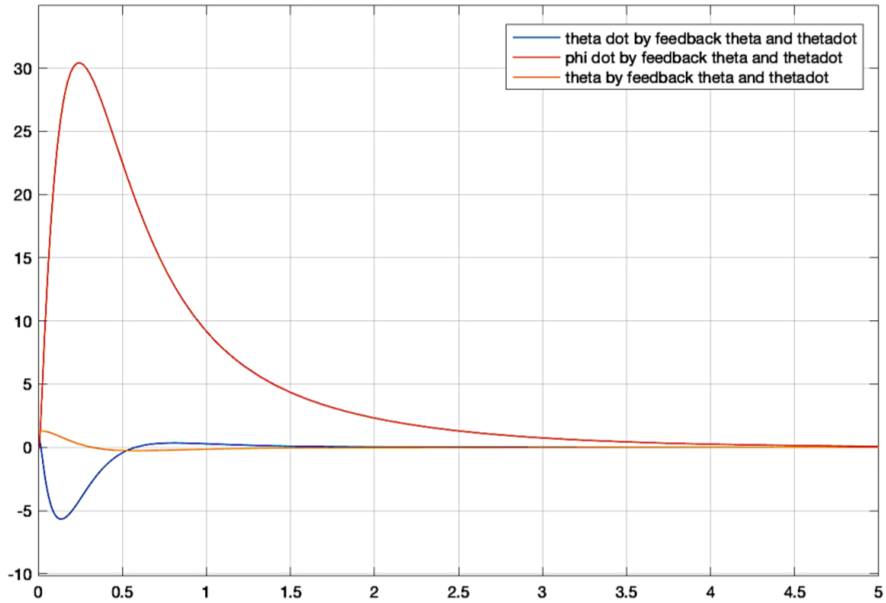


Figure 29 System performance under applying LQG controller (dθ/dt and θ)

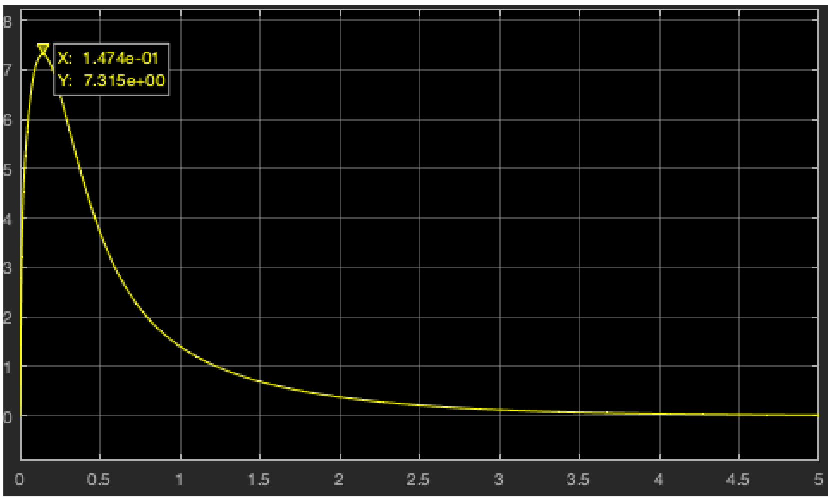


Figure 30 Voltage performance under applying LQG controller (dθ/dt and θ)

Controller $d\phi/dt$ and θ

LQG controller

<u>K_phidot_theta:</u>

$$[3.4769 \quad 1.2018 \quad 31.0211]$$

<u>F_phidot_theta:</u>

$$\begin{bmatrix} 444.7440 & -150.1839 \\ -685.8629 & 3.7743 \\ 0.3774 & -17.2900 \end{bmatrix}$$

Performance

Under limitation of input voltage, maximum body angle is 48 degrees.

<u>Settling time:</u> 1.15s

<u>Maximum angle:</u> initial angle.

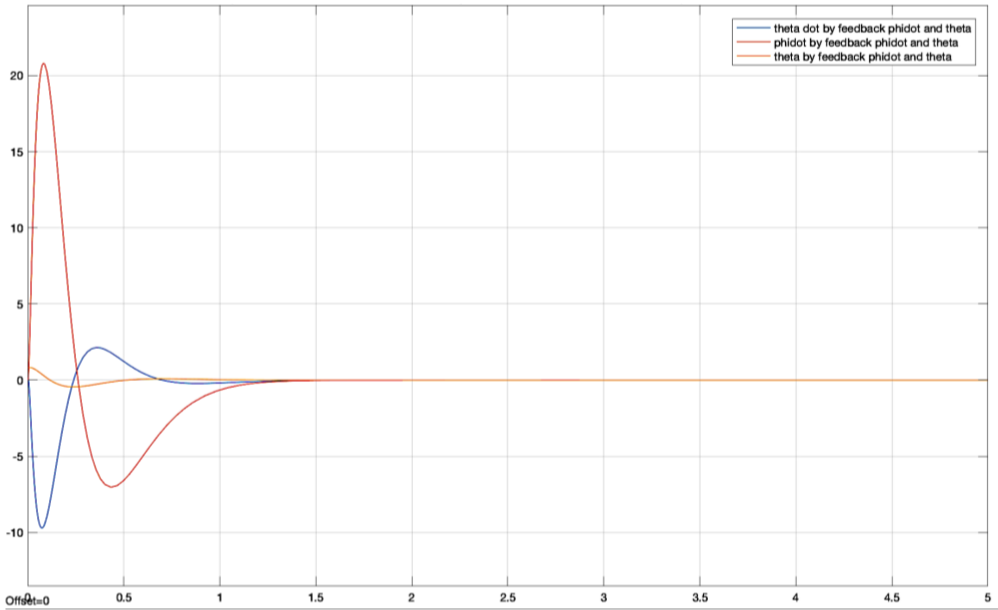


Figure 31 System performance under applying LQG controller ($d\phi/dt$ and θ)

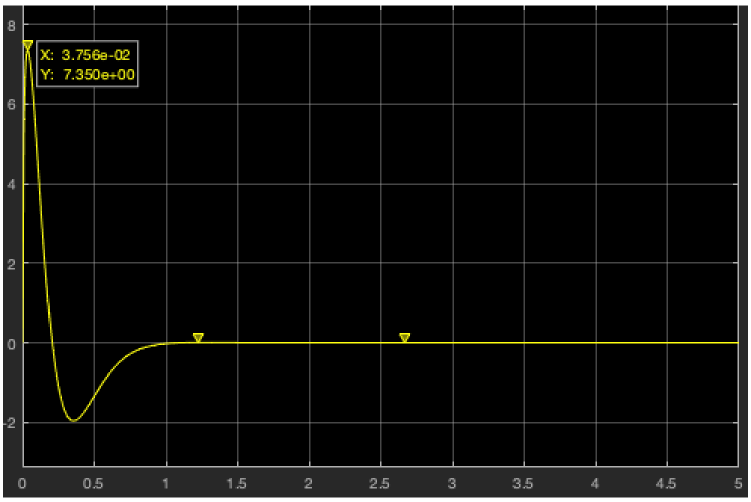


Figure 32 Voltage performance under applying LQG controller ($d\phi/dt$ and θ)

Controller dθ/dt, dϕ/dt and θ

LQG controller

K_all:

$$[3.4769 \quad 1.2018 \quad 31.0211]$$

F_all:

$$1.0e+03 * \begin{bmatrix} 0.1387 & 1.3871 & -0.1477 \\ -0.2139 & -2.1385 & -0.0001 \\ 0 & -0.0001 & -0.0172 \end{bmatrix}$$

Performance

Under limitation of input voltage, maximum body angle is 43 degrees.

Settling time: 1.03s

Maximum angle: initial angle.

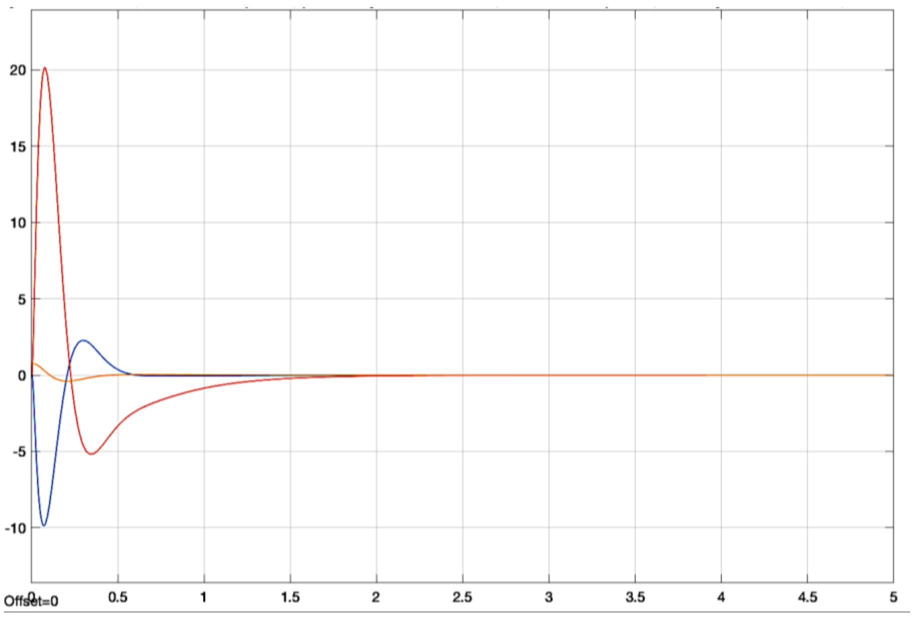


Figure 33 System performance under applying LQG controller (dθ/dt, dϕ/dt and θ)

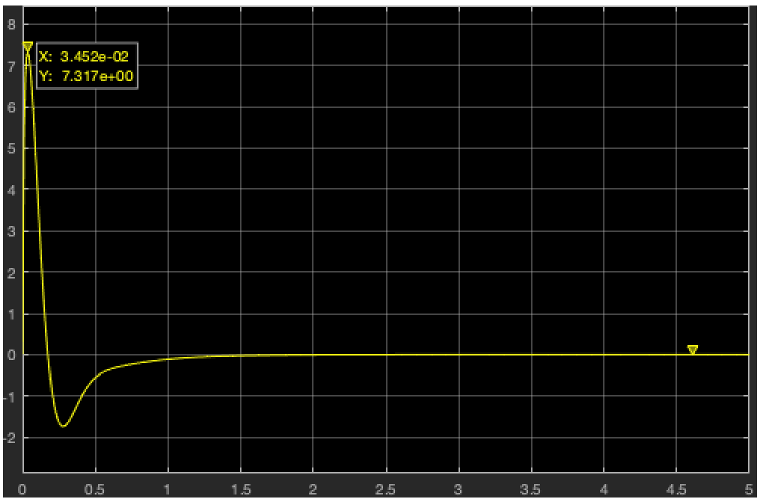
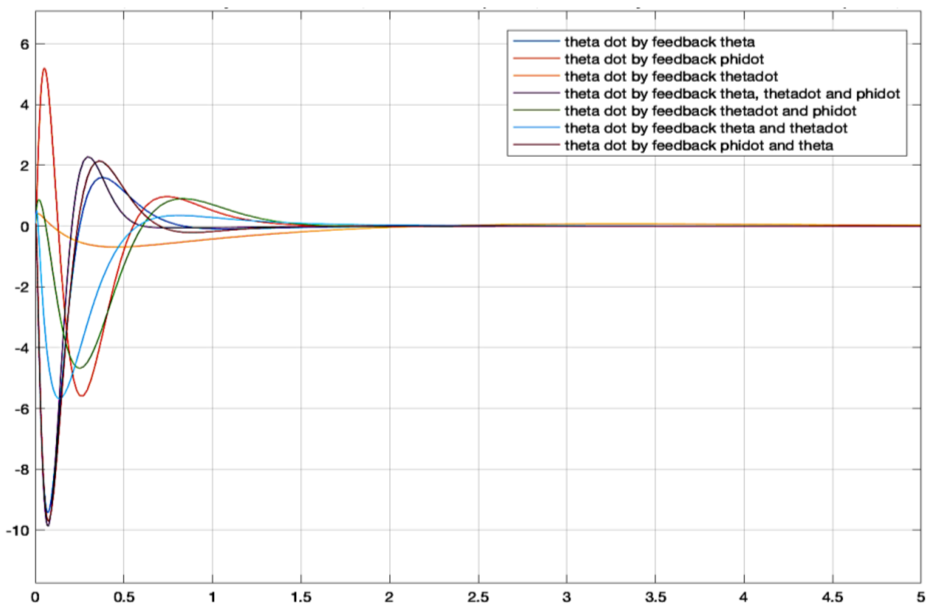


Figure 34 Voltage performance under applying LQG controller (dθ/dt, dϕ/dt and θ)

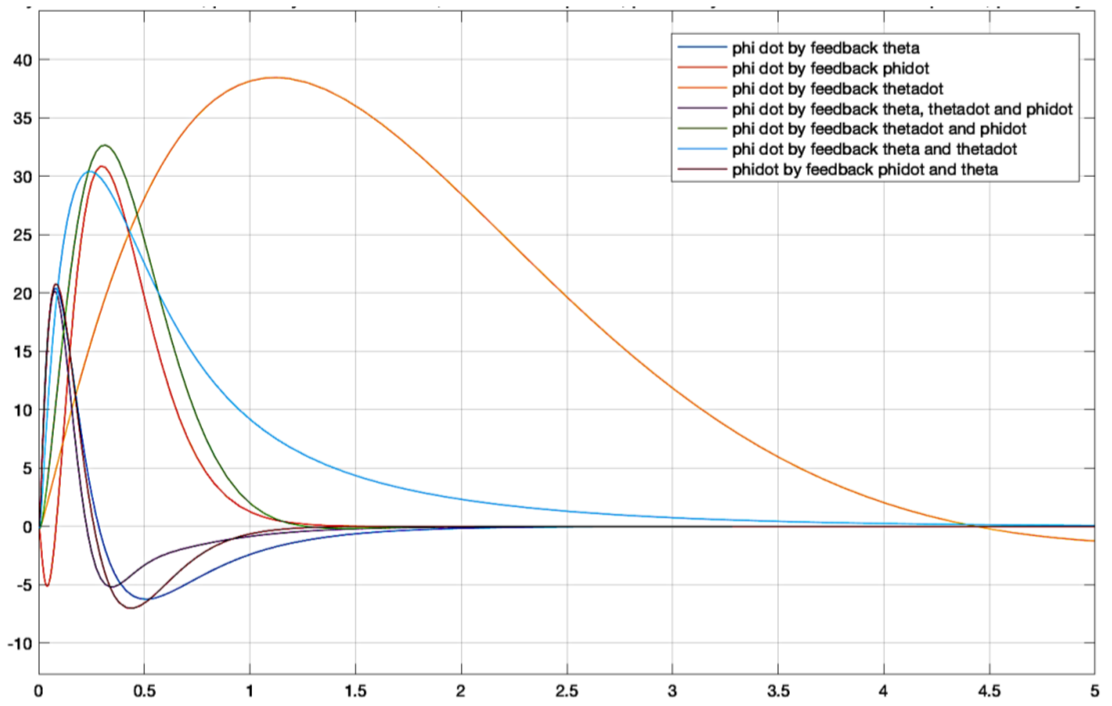
Comparison

	Initial angle	Settling	Maximum angle
Controller dθ/dt	30 degrees	2.11 s	39.5 degrees
Controller dϕ/dt	29 degrees	1.29 s	52.3 degrees
Controller θ	49 degrees	1.43 s	49 degrees
Controller dθ/dt and dϕ/dt	56 degrees	1.36 s	58 degrees
Controller dθ/dt and θ	74 degrees	2.17 s	74 degrees
Controller dϕ/dt and θ	48 degrees	1.15 s	48 degrees
Controller dθ/dt, dϕ/dt and θ	43 degrees	1.03 s	43 degrees

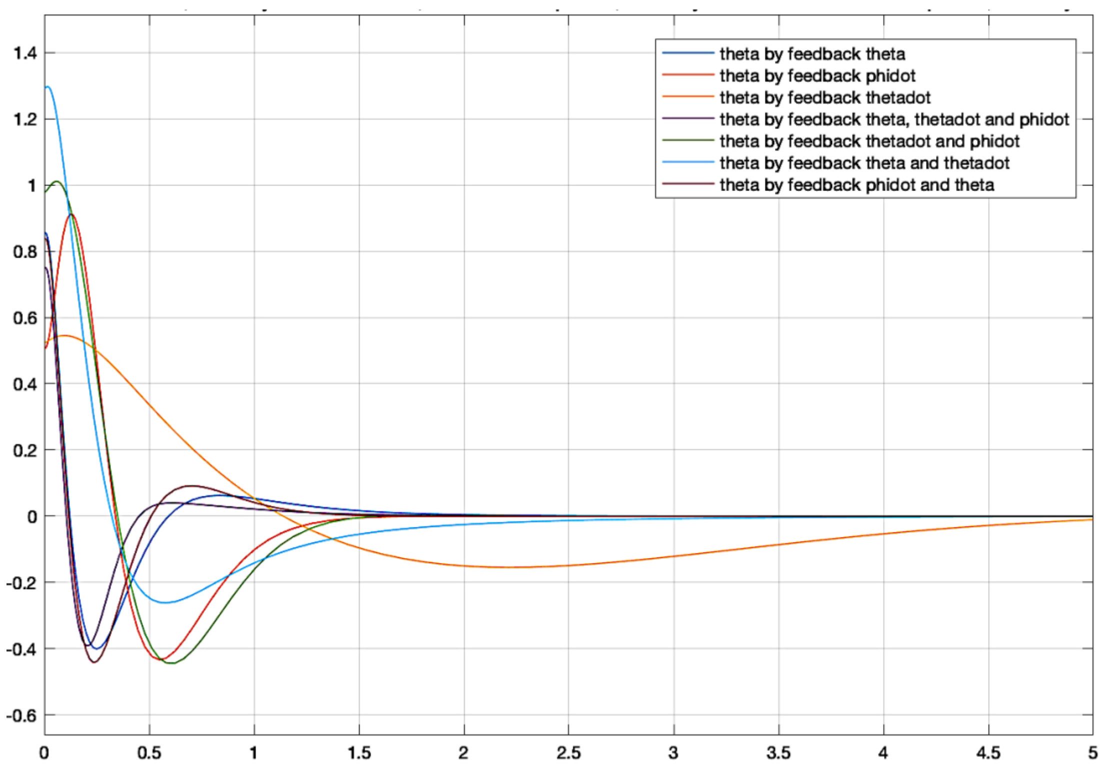
<u>theta dot:</u>



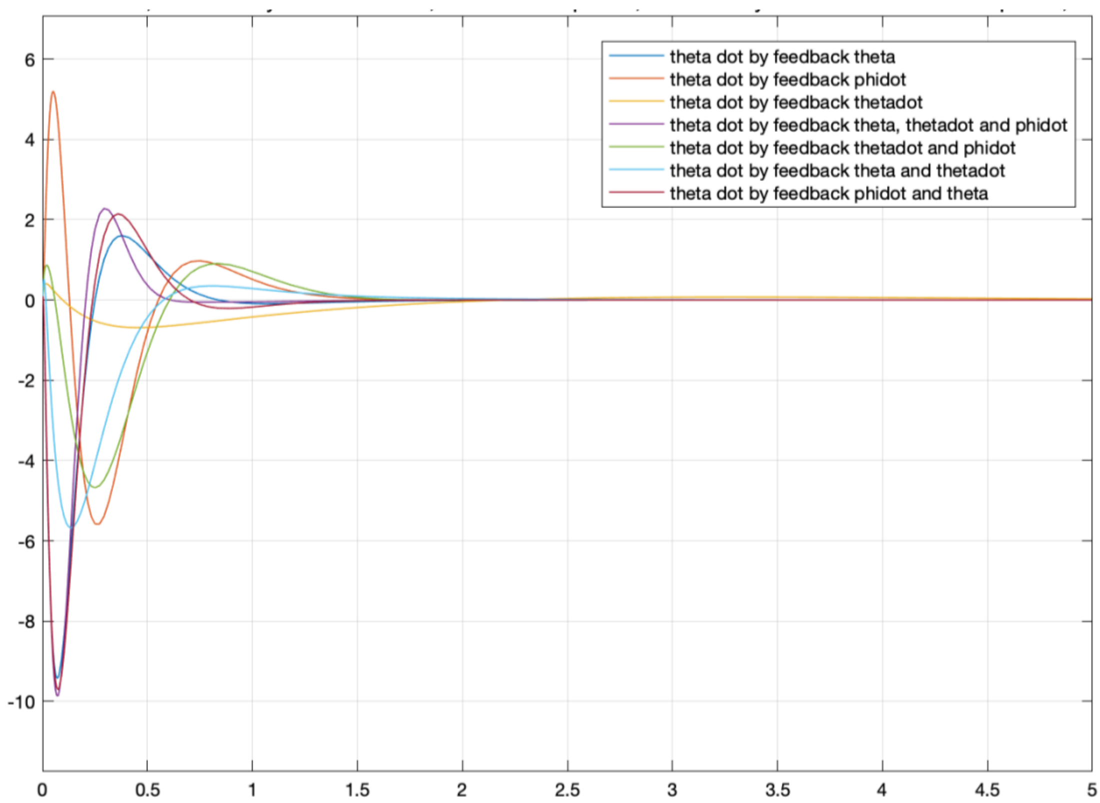
phi dot:



theta:



voltage:



theta dot by feedback theta
theta dot by feedback phidot
theta dot by feedback thetadot
theta dot by feedback theta, thetadot and phidot
theta dot by feedback thetadot and phidot
theta dot by feedback theta and thetadot
theta dot by feedback phidot and theta
6
4
2
0
-2
-4
-6
-8
-10
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Apply controller to non-linear system

We have several controllers right now, but all of them are designed through linearized system. Therefore, I would like to control original non-linear system by our controller.

Classical control controller

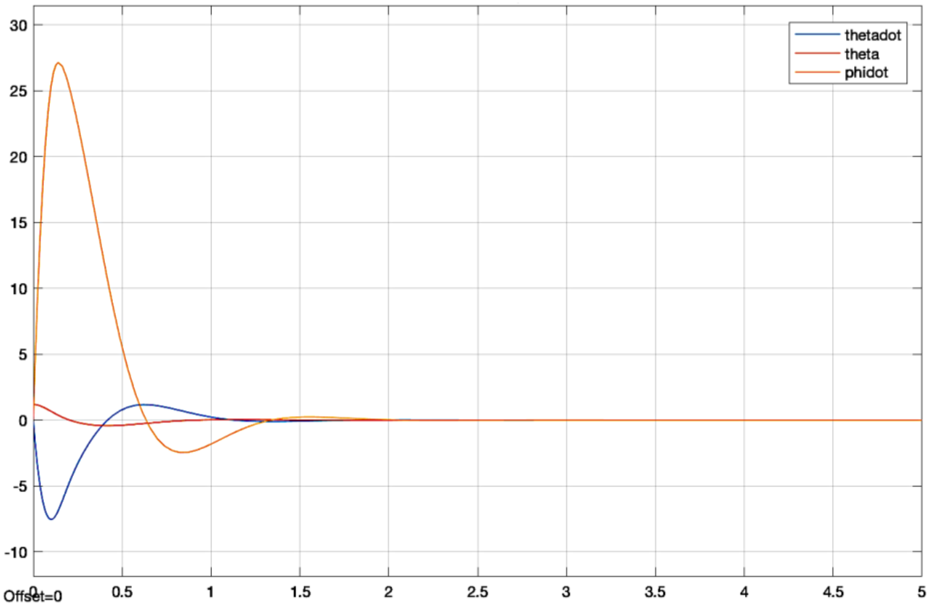
Feedback theta controller:

<u>System performance:</u>

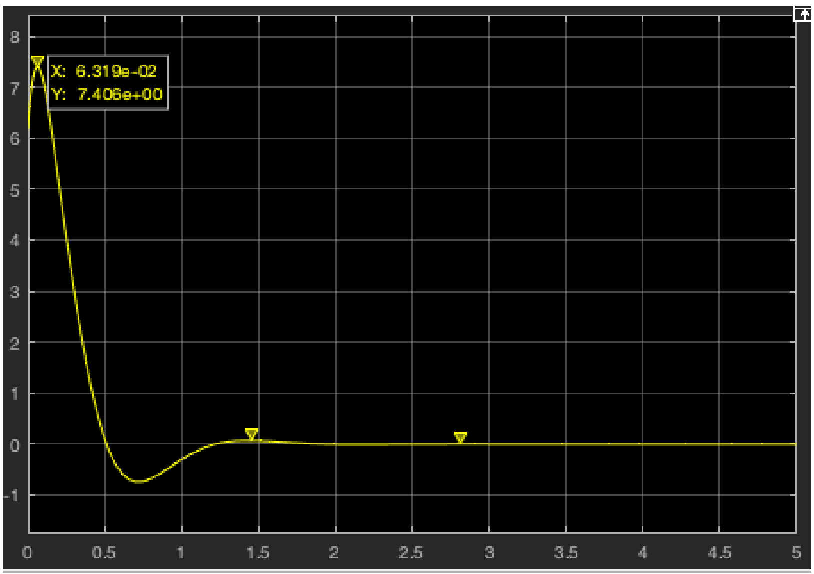
Initial angle = 68 degrees

Settling time: 1.37 s

Maximum angle: initial angle



<u>Voltage performance:</u>



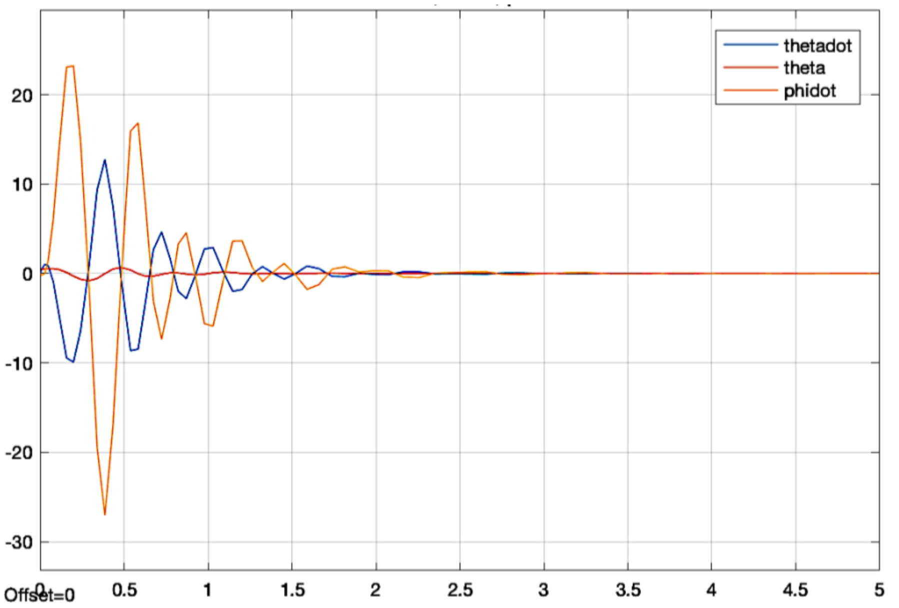
Feedback theta dot and phi dot controller

<u>System performance:</u>

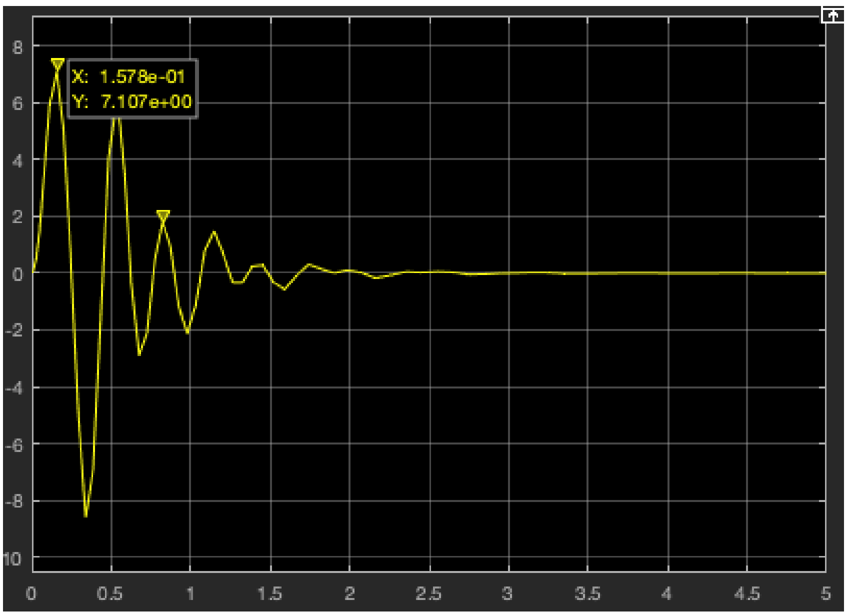
Initial angle = 28 degrees

Settling time: 1.79 s

Maximum angle: 0.6638 rad (about 38 degrees)



<u>Voltage performance:</u>



LQG controller

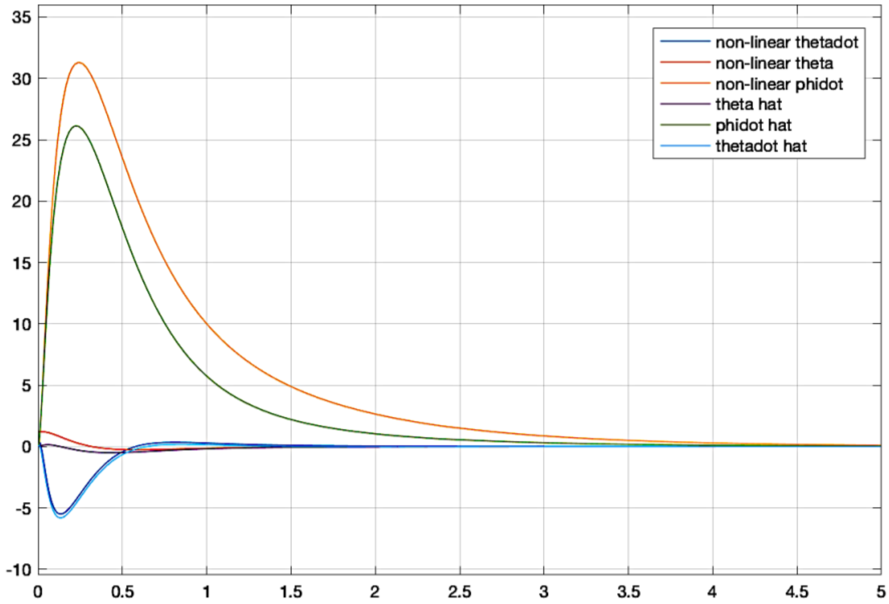
Controller dθ/dt and θ

<u>System performance:</u>

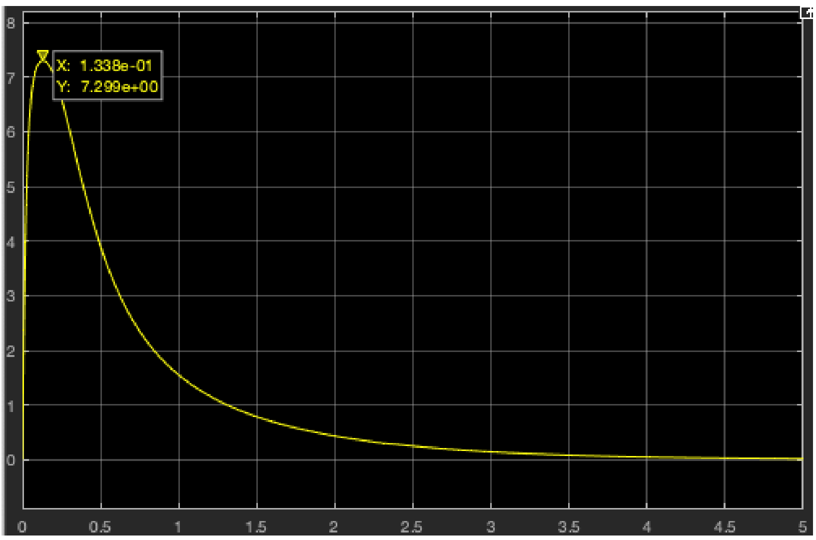
Initial angle = 69 degrees

Settling time: 2.27 s

Maximum angle: initial angle



<u>Voltage performance:</u>



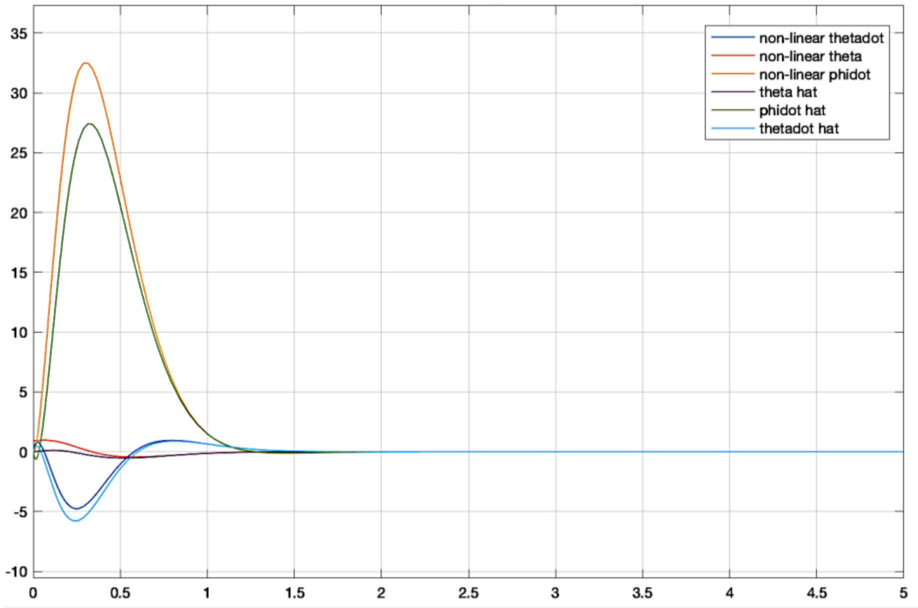
Controller dθ/dt and dϕ/dt

<u>System performance:</u>

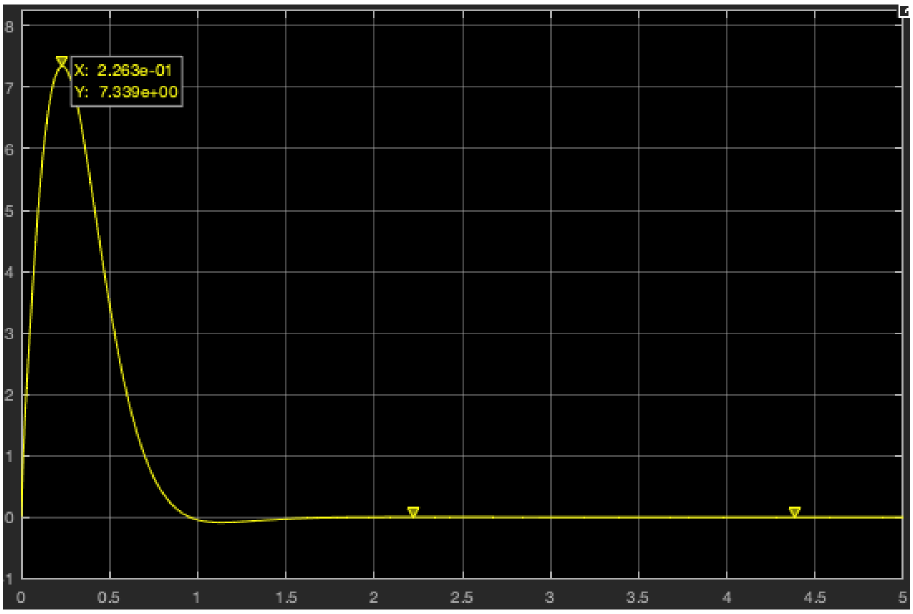
Initial angle = 53 degrees

Settling time: 1.32 s

Maximum angle: 0.9584 rad (about 54.94 degrees)



<u>Voltage performance:</u>



Linear/ non-linear performance comparison

	Linear			Non-linear		
	Initial angle	Settling time	Maximum angle	Initial angle	Settling time	Maximum angle
Feedback theta	74 degrees	1.6 s	74 degrees	68 degrees	1.37 s	68 degrees
Feedback theta dot and phi dot	31 degrees	2.32 s	44 degrees	28 degrees	1.79 s	38 degrees
Controller dθ/dt and θ	74 degrees	2.17 s	74 degrees	69 degrees	2.27 s	69 degrees
Controller dθ/dt and dϕ/dt	56 degrees	1.36 s	58 degrees	53 degrees	1.32 s	55 degrees