Vol. 67, No. 3 May, 1960

THE PSYCHOLOGICAL REVIEW

A THEORY OF DATA

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Introduction

The behavioral sciences in general and psychology in particular are laden with methods for collecting and analyzing data. These methods usually have names associated with them which do not always clearly imply whether they are methods for collecting data, or for analyzing data, or for both. Thus the method of pair comparison is a method for collecting data, the law of comparative judgment (Thurstone, 1927) is a model for analyzing data, but psychophysical methods imply both. It is probably generally true that a method for analyzing data implies certain conditions that must be met by the method of collecting the data but there are many variations in the methods of collecting data that may satisfy the same conditions. When these conditions become clear the full generality of the methods for analyzing data becomes apparent. This generality is obscured by terminology particular to a context as in psychophysics or attitude scaling and it would seem desirable to abstract the properties of all methods and see thereby what is common among them and how they differ.

The theory of data proposes to do this. It proposes to provide a foundation for models of psychological measurement and classify, systematize, and interrelate them. It is by no means proposed that this is the only schema or the best one.

The domain of discourse of the theory of data includes the methodologies in what psychologists speak of as the areas of psychophysics, mental testing, attitude scaling, latent structure analysis (Green, 1954), scalogram analysis (Green, 1954), preferential choice behavior, rating scales, factor analysis, multidimensional psychophysics (Torgerson, 1958), etc.

Many of the psychologist's methodologies have been constructed with a particular content in mind such as mental testing or attitude scaling and hence are identified with content areas and use the vocabulary of such content. Courses in these various methodologies are frequently content oriented and the student may not be aware of the identities and differences among them. When such content-oriented models are cast in abstract form they are recognizable as miniature behavior theories, the scope of their applicability

[1] The preparation of this theory of data was supported in part by a grant from the National Science Foundation and in part by Project MICHIGAN, a Department of Army-Sponsored Project of the University of Michigan in the field of Combat Surveillance. The contract (DA-36-039 ac 78801) is administered by the United States Army Signal Corps.

is broadened, and alternative theories immediately spring to mind. There is perhaps less of a tendency to feel "this is the way to analyze that kind of data."

Initial steps toward a theory of data were first published in 1952 (Coombs, 1952) and a more explicit formulation of a theory of data a year later (Coombs, 1953). This current formulation, while intimately related to the previous, is a thorough revision and, I hope, a vast improvement.

The next three sections describe, in turn, the theory of data, the application of the theory to the classification of models and their relations within classes, and finally, a general discussion of the classes of psychological data and some of the relations between classes. At the end is a brief glossary and a mathematical appendix.

The Theory of Data

The fundamental ingredients of those psychological observations to which measurement models are applied are surprisingly simple. From the point of view of measurement models, basically all a person can do is compare stimuli with each other or against some absolute standard or personal reference point, the stimuli may come singly or in pairs, and the comparison is one of dominance or consonance. For example, an individual passing or failing an arithmetic problem may be regarded as a comparison between his ability (a personal reference point) and the problem's difficulty (a stimulus) and the comparison is one of dominance.

Before illustrating these basic ingredients of behavior it will be useful to abstract them explicitly. This is done by the fourth axiom (see Appendix) as follows:

Every measurement model may be regarded as satisfying each of three dichotomies:

(*a*) A relation exists on a pair of points or on a pair of pairs of points.
(*b*) The elements of a pair of points are drawn from two distinct sets (A) or from one set (B).
(*c*) The relation is either an order relation ($>$) or a proximity relation (O).

Consider the following two illustrations:

(*a*) An individual endorses an attitude statement. A model might regard the attitude of the individual as a point drawn from one set of points in a psychological space, and the attitude statement as a stimulus point drawn from another distinct set of points in the same space, and the observation ("endorses") as a proximity (O) relation on this pair of points. That is, that the stimulus point is "near" or "in the neighborhood of" the point corresponding to the attitude of the individual. It is, of course, by no means implied that this is the necessary model for that observation, but rather that it is a "theory" as to how this behavior is generated.
(*b*) An individual says that one pair of color patches are more alike than another pair. A model might regard each pair of stimuli as a pair of points and the four points contained in the two pairs of points as being drawn from the same set, and the observation (more alike) as signifying that the "distance" in the psychological space between one pair of points is greater than ($>$) the distance between the other pair of points.

This 4th axiom is the critically significant axiom for the theory of data but clearly some prior mathematical machinery is necessary—so we begin at the beginning.

The essential objective of every psychological measurement model is to associate with each object of interest, individual or stimulus, a point in a psychological space, and the purpose of the model is to construct a calculus which will permit the recovery of the space, given the observations and the preconceptions of the space. A great variety of preconceptions of the properties of a psychological space are to be anticipated so the postulates of the theory of data are very general in order to accommodate this variety.

The four sets D, H, I, and J given in the appendix are merely label sets. The set D is for designation of dimensions and the set H for designation of trials—a quantized temporal variable. The sets I and J are for the designation of two distinct sets of objects. An illustration might be a set of individuals and a set of stimuli, in which case we shall adopt the convention of using the label set I for individuals and the label set J for stimuli. It is by no means necessary, however, that this always be the case. Sometimes, for example, individuals may be used as stimuli in which case the label set J would be used. Furthermore, some models label only the stimuli but deal with two distinct sets of stimuli, in which case the label sets I and J will be used.

As stated previously, the purpose of a model is to recover a psychological space given the data. Thus one model is concerned with scaling statements of opinion on a one dimensional continuum (that is, locating points on a line) and another model is concerned with the number of dimensions characterizing a set of statements of opinion and locating points in a multidimensional space which will account for the behavior observed. Hence the first axiom with its accompanying definition simply postulates the existence of such a space, with r dimensions and each point an r-tuple.

Note that the axiom says each dimension "is a segment of the real line," that is to say, inherently a ratio scale. This is quite a strong statement and could lead to endless interesting and futile philosophical argument. It is quite true that for some models it is sufficient to postulate merely an ordinal scale for the elements of a dimension K^d but other models would require an ordered metric, others an interval, and others a ratio scale. Rather than a series of successive versions of this axiom strengthening it as necessary, I chose to assume as much as is necessary to accommodate all the models, and then one may speak of weaker measurement models as "recovering" this space at lower levels of scales.

Measurement models variously identify points in this space K with stimuli and/or individuals so it is convenient to construct some sets of points in the space K. Two subsets of points are constructed which are called C and Q. C is the subset of points which are labelled by the I set and Q is the subset of points labelled by the J set. It might be well to point out that while the sets I and J are label sets for two distinct sets of objects (in the real world) it is not necessary that the subsets of points C and Q be distinct. For example, one might conceive of a statement of opinion that precisely reflects how an individual feels and while it is desirable to distinguish between the individual and the statement of opinion there is no necessary distinction between the points in the psychological space that correspond to them.

Having the two sets of points C and Q it is useful to construct sets of pairs of points. Consequently we conceive of a set of pairs of points where one is a member of the set C and the other a member of the set Q and call such a set the set A of ordered pairs (c_i, q_j). In accordance with the convention established, such a pair of points might correspond to an individual and stimulus, respectively, or a pair of stimuli from two sets as the two lines in a Mueller-Lyer illusion which terminate in a feather or arrow.

Sometimes the observations are made on pairs of such pairs where the same individual enters into both pairs, as, for example, when an individual is asked which of two statements he prefers to endorse. So it will be convenient to construct a subset of the set A, consisting of those pairs of points (c_i, q_j) where i is fixed, such a subset is labelled A_i. The subset A_i then consists of pairs of elements, one a fixed individual i, and the other a stimulus j.

For some models the members of a pair of points are drawn from the same set. It makes no difference whether such a pair is regarded as being drawn from the set C or the set Q but inasmuch as such pairs of points are usually identified with stimuli they will be regarded as drawn from the set Q and the set of such pairs of points drawn from the same set is called B. A typical example is in the scaling by pair comparison of lifted weights or brightness of lights.

It is interesting to note that when models deal with observations on pairs of pairs of points the pairs of points are always drawn from the same set, A or B, never is one drawn from A and the other from B. There is nothing logically necessary about this, of course, it is just that there are no psychological measurement models specifically constructed for such data.

The second axiom postulates the existence of a "distance" function in the space K—that is, between every pair of points in the space K there is a "distance" between them. It is to be noted that nothing is said about this distance concept other than that it satisfies the minimum conditions for a distance function. It is not required, for example, that the space be Euclidean or any other particular geometry. Some models do not require a metric space at all and may be spoken of as "recovering" the space at a lower level.

The third axiom merely links the first two together and defines a positive direction for each dimension K^d.

The fourth axiom has already been discussed, so we conclude this section with a discussion of the fifth axiom and its accompanying definition. On a given trial or moment (h) when the behavior is interpreted as a relation between a pair of points from distinct sets (conveniently referred to as an individual, i, and a stimulus, j) we conceive of the behavior as being generated by some but not necessarily all of their attributes. Hence, the distance between the pair of points is a distance in a subspace of the total space called *the relevant dimensions, D'*.

In a similar manner, if the behavior of an individual (i), on a given trial or moment (h) is interpreted as a relation on a pair of points from the same set (the stimuli j and k), we conceive of the behavior as being generated by some but not necessarily all of the attributes in the space K. Hence, the distance between the pair of points j and k is a distance in the subspace called *the relevant dimensions, D''*.

The discussion of these axioms and their accompanying definitions complete the description of the basis of the theory of data. We proceed next to construct the eight types of data and the definition of the information in each. The objective here is to illustrate the mapping between this abstract model on the one hand and the types of observations made by psychologists.

The Classification of Psychological Measurement Models and Methodologies

Taking the conjunction or cross-partition of the three dichotomies given in Axiom 4 yields eight classes—a cube which is $2 \times 2 \times 2$. For simplicity of portrayal it may be drawn in a plane with one of the three dichotomies projected onto the other two. I have chosen to project the third because this dichotomy is collasped when an intermediate category of response of the form "I can't decide" is used. This point will be made clear in the discussion of the kinds of data below.

The distinctions of the theory of data may be portrayed, then, as in Fig. 1.

The four classes generated by the cross-partition of the first two dichotomies have been labelled as quadrants I to IV, and the additional distinction, an order (>) vs. a proximity (*O*) relation, divides each quadrant into an *a* or *b* octant, respectively. In each octant the set from which the points are drawn is indicated in the figure.

In general, for all the octants, a quantity *p* is assumed to exist which is a distance between a pair of points, and an observation is made concerning the magnitude of such *p*'s or pairs of such *p*'s. The precise definition of this information is different for each octant and they are all given in the appendix. The measurement problem may be simply put as the problem of how to construct the space *K* given the information about the *p*'s. Any psychological measurement model constitutes a calculus for that purpose, adding axioms to those given here, such as specifying that the dimensionality of the space *K* is *one,* for example, or for a dimensionality greater than one, specifying that the distance function is Euclidean or otherwise (e.g., cf. Householder & Landahl, 1945, Ch. 8 & Attneave, 1950). It is clear from this why psychological measurement has so often been spoken of as distance measurement (Bentley, 1950).

In the remainder of this section the quadrants will be discussed in turn, first the two octants separately and then the quadrant as a whole. Examples of the kinds of behavior that could be mapped into each octant are given with their corresponding interpretations as data in the form of relations on points. A brief summary of the models available for the data of each octant is also presented.

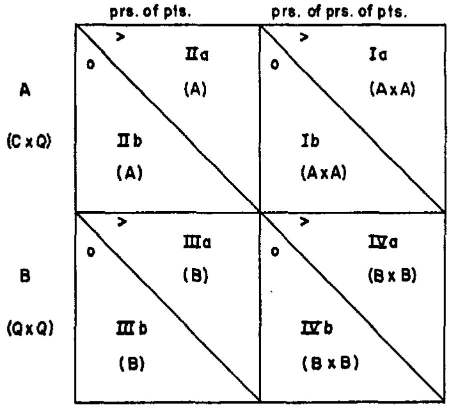

FIG. 1. Distinctions of the theory of data.

Quadrant Ia

An illustration of behavior which might be mapped into this octant is the behavior of an individual when asked which of two alternatives he prefers. When such behavior is so mapped it is assumed that there is a point corresponding to the individual representing an "ideal" point and each alternative also corresponds to a point and the individual will prefer that alternative "nearer" to his ideal point. Such data are order relations on pairs of distances.

The only currently available model for analyzing Quadrant Ia data is the unfolding technique (Coombs, 1950, 1952; Bennett, 1951; Hays, 1954). Analysis of such preferential choice data by the unfolding technique yields a joint space (in one dimension it is called a *J scale*) in which are located points associated with individuals and points associated with the stimuli. This is a genotypic structure. The development of the method for Euclidean multidimensional spaces is due to Bennett (1951) and Hays (1954). Papers presenting this development are now in press. More recently Coombs and Kao (in press) have shown how the technique of multiple factor analysis can be used for the multidimensional analysis of this same data.

Quadrant Ib

An illustration of such data as this is the behavior of an individual when asked whether or not he can choose between a pair of alternatives. For every pair of alternatives he answers merely that he would or would not prefer one alternative more than the other. Such data are proximity relations on pairs of distances. There are no models specifically designed to analyze such data, perhaps because the amount to be learned from such data is small compared with Quadrant Ia data. The primary purpose in mentioning it here, aside from formal completeness, is its relevance to the interpretation of preferential choice data when an intermediate category of judgment is used. Such data constitute what might be called Quadrant I in that there are data in each of the two octants.

Quadrant I

If an individual is asked which of two alternatives he prefers and is permitted to answer "I don't know" or "I can't say" we may speak of this as using an intermediate category of judgment.

The result would be that when an individual made a choice the data would be Quadrant Ia and when he failed to make a choice it would be Quadrant Ib. No models exist for the analysis of such data to yield a joint space; one would be compelled to neglect responses using the intermediate category of judgment or to make assumptions which would map it into Quadrant Ia data for which models are available.

Quadrant IIa

This is the most prevalent kind of psychological data so a number of examples of it will be given. One illustration of this type of data is mental test behavior—for example, the response of an individual to an arithmetic item. An individual passing an item implies that he has more of the ability involved than is required by the difficulty of the item. The phenotypic behavior, pass or fail, is interpreted as an order relation on a pair of points—one the individual's ability and the other the difficulty of the item.

In another context, the study by Janis (1949) on psychophysiological correlates of fear is interpreted as this type of data. When the individual said "yes" or "no" to having experienced a particular fear symptom in combat, this was interpreted as an order relation on a pair of points, one corresponding to the amount of fear he had experienced and the other corresponding to the amount implied by the symptom.

Responses of individuals to neurotic inventory type items are typically interpreted as this kind of data also. An individual answers "yes" or "no" to the question "Do you wet your bed frequently?" The answer is interpreted as an order relation between a pair of points, one corresponding to the individual's perception of how frequently he wets his bed and the other corresponding to his decision as to how often one must wet his bed to do it "frequently."

Psychophysical threshold studies deal with identically equivalent data in a formal sense. When an individual is asked whether or not he perceives a stimulus, the behavior is interpreted as an order relation on a pair of points one corresponding to his threshold and the other corresponding to the magnitude of the stimulus. In the case of determining a difference limen, the same thing holds if the stimulus is defined to be a *difference* between two stimuli.

Clearly the data of psychophysical

studies, mental tests, and neurotic inventories all have the same formal character of consisting of order relations on pairs of points from distinct sets. But certain further differences remain. The great advantage that psychophysics has over mental testing is the apparent possibility of experimentally independent replication of the individual-stimulus pair. The great advantage that mental testing has over that of the neurotic inventory is that the individual is not able to decide for himself how difficult an arithmetic problem is. The difficulty of an arithmetic problem is presumably more compelling than how shy you have to be to say yes. In abstract terms, the point corresponding to the difficulty of an arithmetic item is, presumably, relatively stable over different individuals compared with the point corresponding to the item, "Are you shy?" But in a formal sense the data from these separate areas are all equivalent, the same basic measurement models apply, and the concepts and problems of one suggest equivalences in the other.

There are a variety of methods and models for the analysis of this kind of data. These include Guttman's scalogram analysis (Green, 1954), Lazarsfeld's latent distance model (Green, 1954), test theory (Gulliksen, 1950), the law of categorical judgment (Torgerson, 1958), Coombs-Kao nonmetric factor analysis models (1955), and multiple factor analysis (Thurstone, 1947).

These models all yield a joint space in which the elements of two distinct sets are located, typically individuals and stimuli, except in the case of the law of categorical judgment, as explained below. They vary in their assumptions and hence correspond to different theories of how the behavior is generated. Some models, like test theory yield a one-dimensional *J* scale no matter how the individuals behave, and then measures of reliability and homogeneity are constructed which are related to how good a fit is obtained by the one dimension. Another model, like scalogram analysis, yields one dimension but is highly sensitive to imprecision or lack of unidimensional homogeneity. This method is essentially a method for testing whether certain conditions for unidimensionality in the genotypic structure generating the behavior are met. One has one's choice here between models which are deterministic or probabilistic and which are unidimensional or multidimensional.

Torgerson's law of categorical judgment is designed to analyze the data obtained by having individuals sort stimuli into ordered classes and the occurrence of a stimulus in a particular class is interpreted as an order relation between a point corresponding to the stimulus and a point corresponding to one of the boundaries of the class. Here the two distinct sets of real world objects are the stimuli being sorted and the boundaries of the classes.

Quadrant IIb

Typical of the kind of behavior mapped into this octant is that of an individual agreeing or disagreeing to endorse a statement of opinion. Such behavior is interpreted as a proximity relation on a pair of points, one corresponding to the attitude of the individual, the other corresponding to the attitude expressed by the statement of opinion. More broadly, the interpretation is that there are two sets of elements, and the elements of one set are being matched with the elements of another set. Clearly, in the most general case, a labelling process. Membership in an organization, marriage, and clinical diagnosis are all further examples of matching between elements of distinct sets.

A classic experiment that may be used to illustrate the data of Quadrant IIb is the study of Watson and Watson (1921) of the generalization of a conditioned fear response to a white rat. The child, Albert, having been conditioned to fear a white rat, may be represented by a point in a psychological space and the occurrence of a conditioned response to other stimuli, such as a white rabbit, (represented by another point in the space) reflects a proximity relation on the pair of points.

In all of these cases there are two sets of points and the elements of one set are matched with the elements of another set. In the case of endorsing attitude statements there is a set of points corresponding to individuals and another set of points corresponding to statements of opinion and the individuals match themselves with the statements. But also it should be eminently clear that as far as the formal aspects of the data are concerned it doesn't matter who does the matching. Thus, rating scale behavior, magnitude estimation, and magnitude production are also illustrations of this kind of data in that one has a set of objects to be judged and another set of elements which constitute the response categories and the objects of judgment are matched with response categories.

When next to nothing is known a priori about the relation between elements within the two classes, Lazarsfeld's latent class model is the most appropriate. A good illustration of this is in the study by Gibson of the preferences of radio listeners for 13 types of evening programs (Lazarsfeld, 1959). However, in certain cases, as in rating scales, the elements of one of the sets, the set of response categories, has an a priori order relation on the elements of the set. Thus, a foreman who is rated "superior" is presumed better than one rated "mediocre" and by virtue of this order relation on the elements of one of the sets there is generated an order relation between pairs of elements from two sets. The consequence of this is that one has the option of analyzing such data by the models of Quadrant IIa. Thus rating scale data may be analyzed by the law of categorical judgment in which one does not need to assume the classes are equally spaced on the continuum, and a scale may be recovered in which both the stimuli and the boundaries of the classes are scaled.

If the elements of one of the sets are not only simply ordered but are numbers, as in some rating scales and in magnitude estimation, then numbers may be associated with the objects being judged and these numbers have the properties assumed to be true for the numbers constituting the response set.

Probably the oldest model for collecting and analyzing a special kind of data in Quadrant IIb is the method of average error as applied, for example, to measure the extent of the Mueller-Lyer illusion. The two distinct sets are the two sets of line segments, one with terminal arrowheads and one with terminal feathers. The elements of one set are matched vs. the elements of the other set. The method requires that the elements of both sets be measurable on the same physical scale and then the mean difference between matched elements is the extent of the illusion for that individual or collection of individuals.

Quadrant II

Just as in Quadrant I where an intermediate category of response generated data in both Quadrant Ia and Quadrant Ib, so also is this true of Quadrant II. The behavior would have to be of such a nature that if the point corresponding to the individual exceeded the point corresponding to the stimulus by more than an amount ϵ_{hij} he reacted positively, if by less than an amount $-\epsilon_{hij}$ he reacted negatively, and otherwise intermediately. Thus, if an individual, in evaluating candidates for office, had the following three categories of response available:

(*a*) No, he's too liberal
(*b*) Yes
(*c*) No, he's too conservative

such behavior could be interpreted as Quadrant II data including both IIa and IIb.

The most recent model constructed for analyzing this kind of data is Lazarsfeld's latent structure analysis. This is actually a very general model which can admit a mixture of monotone and nonmonotone trace lines and thus would be a model for Quadrant II as a whole.

Quadrant IIIa

A judgment as to which of two stimuli has more of some attribute is an example of the kind of behavior mapped into this octant. The behavior involves the comparison of stimuli, as in much of psychophysical scaling, and is interpreted as an order relation on a pair of points, both identified with stimuli. The distinction to be noted between Quadrant IIIa data and Quadrant Ia data is that in the latter the individual is also conceived of as being represented by a point in the space with the stimuli whereas in Quadrant IIIa data the individual is not represented by a point in the space. Athletic meets, such as tennis tournaments and professional baseball, are examples in which "nature" is making such a pair comparison (Mosteller, 1951).

The intent of all Quadrant IIIa data is to construct a subjective scale of stimulus magnitude and all of the models for analyzing Quadrant IIIa data yield spaces in which only the elements of a single set are located. Hence, these models may be spoken of as yielding a stimulus space (in one-dimension it is called a stimulus *scale*) in contrast with the joint spaces which generate data in Quadrants I or II.

If the order relation on pairs of points is transitive an ordinal scale of the stimuli follows immediately. If there are replications on each pair of stimuli which yield a probability other than zero or one that one member of the pair is greater than the other, then an ordinal scale follows immediately if weak stochastic transitivity is satisfied (Davidson & Marschak, 1958). Models exist which made transformations of these probabilities into psychological distances on a subjective stimulus scale. Thurstone's law of comparative judgment (Thurstone, 1927) yields an interval scale and Luce's recent model for choice behavior (Luce, 1959) yields a ratio scale. Each of these require that the probabilities satisfy strong stochastic transitivity among other things. In S. S. Stevens' (1957) method of ratio estimation the individual judges not only which stimulus is greater but how many times greater and the result is a ratio scale also. On a certain primitive level the data to which these models apply is the same—order relations on pairs of points from the same set. Data collected by Stevens' method could be analyzed by either Luce's or Thurstone's model but not vice versa. So Stevens' method requires more information in the response than do the others.

Quadrant IIIb

The kind of behavior that represents data in Quadrant IIIb is the response of an individual as to whether two stimuli are the same or not—that is, whether they *match*. This type of data has only recently become of interest through the book of Goodman's (1951).

Galanter (1956) has since begun the construction of a model for analyzing such data. Even more recent is the work of Hefner (1958) on the construction of a model for this octant for data obtained by degrading stimuli through brief time exposures.

The potential significance of this type of data (Quadrant IIIb) resides in the fact that the symmetric predicate (do the stimuli match or not) yields information about the distance between a pair of stimuli in the stimulus space and hence may lead to the exploration of multidimensional stimulus spaces.

The data of Quadrant IIIa, in contrast, is derived from an asymmetric predicate (which stimulus is more of something) and there is serious question as to whether such data could lead to anything more than a one-dimensional stimulus scale. These issues are discussed fully by Goodman.

Clearly, if an experimenter wanted to construct a one-dimensional stimulus scale, an asymmetric predicate should be used. Whereas, if he wants to explore the cognitive space in which the perceptions of the stimuli are imbedded, the symmetric predicate of Quadrant IIIb is superior.

However, the construction of multidimensional spaces is also possible with the asymmetric predicate of Quadrant IVa and will be discussed below.

Quadrant III

Finally, as in the case of the preceding quadrants, if an individual in judging which of two stimuli is greater were permitted to respond "I can't decide" the behavior constitutes Quadrant III data and a finite ϵ corresponding to a threshold for decision would be involved.

It is of interest to note that sociometric matrices contain data belonging in this quadrant. If the matrix is asymmetric, as in who bosses whom, the data are Quadrant IIIa; if the matrix is symmetric, as in whom do you go to the movies with, the data are Quadrant IIIb; and if the matrix is mixed, as in whom do you like, the data are Quadrant III. Thus once again we see relations between models apparently constructed for quite different purposes but now become suggestive for other real world content.

Quadrant IVa

The behavior of individuals when presented with two pairs of stimuli and asked which pair is more alike is representative of behavior typically mapped into Quadrant IVa. The basic observations are the comparative similarities of pairs of stimuli. The individual is presumed to be responding to the distance between the members of a pair of stimuli. Such data may lead via the unfolding technique to a one-dimensional ordered metric stimulus scale (Coombs, 1954a) if certain conditions are satisfied by the data. If one dimension will not satisfy the data then a multidimensional solution may be sought.

Hays, in a paper being prepared, has adapted his multidimensional unfolding solution of Quadrant Ia data to the data of Quadrant IVa and the result is a nonmetric model for multidimensional psychophysics which requires only an order relation on the distances between pairs of points in the space (assumed to be Euclidean). This model yields the stimulus space recovered only at the level of a product of simple orders. An example is contained in Coombs (1958).

Torgerson's model (1958) assumes more and yields more: (*a*) the order relations on distances are transformed into measures of the distances on a ratio scale by a Quadrant IIIa model (*b*) these distances between pairs of points are transformed into scalar products (c) which are then factor analyzed. The result is the recovery of a real Euclidean space.

Quadrant IVb

The type of behavior which would be mapped into Quadrant IVb would be the response of an individual to two pairs of stimuli which was interpreted to mean that one pair was no more alike (or different) than the other pair. Again, as in Quadrant IVa, the individual is presumed to be responding to the distances between the members of a pair and the response is interpreted as a proximity relation on these distances. Observations collected by the methods of equisection in psychophysics are representative of behavior mapped into this octant.

The data of this quadrant seem to be of only slightly more interest than that of Quadrant Ib.

Quadrant IV

If an individual were permitted to judge that one pair is more alike than the other or that he can't decide, the behavior would fall into both Quadrant IVa and IVb and would constitute Quadrant IV data again introducing a finite ϵ as a parameter.

General Discussion

The Four Basic Kinds of Psychological Data

From the preceding discussion of the information in data, it is apparent that the dichotomy of whether an order relation or a proximity relation is observed is a subordinate dichotomy to the others in the sense that it is not satisfied when an intermediate category of response is used by the experimenter. Hence the four quadrants rather than the eight octants may be seen as representing four primary kinds of psychological distance observations. As a mnemonic convenience, these four quadrants may be given names to signify descriptively the type of behavior that is mapped into each.

In Quadrant I, the relation observed is on a pair of distances where each distance is between a pair of points from distinct sets, usually an individual and a stimulus. The real world context in which this kind of data is most commonly obtained is in observing the preferential choices of an individual over a set of stimuli. The data may be referred to as individual-stimulus differences comparison or, more meaningfully, *Preferential Choice* data.

In Quadrant II, the relation observed is on a pair of points which are from distinct sets, typically an individual and a stimulus. Such data may be called individual-stimulus comparison data, more commonly known in psychology as *Single Stimulus* data. It is important to note that this includes not only mental test data, endorsing statements of opinion, and psychophysical threshold data, but also rating scales, the method of successive intervals, magnitude estimation, and, in general what is known as absolute judgment data.

In Quadrant III, the relation observed is on a pair of points which are from the same set, called stimuli. So such data may be called *Stimuli Comparison* data. One is tempted to call methods for collecting such data psychophysical methods, but this would lead to confusion with psychophysical studies of thresholds which belong in Quadrant II. Also, such a name is content-bound and there are many examples outside of conventional psychophysics which deal with identically the same kind of data—as for example, sociometric matrices.

In Quadrant IV, the relation observed is on a pair of distances where each distance is between a pair of stim-

Single Stimulus Data or Individual-Stimulus Comparison	Preferential Choice Data or Individual-Stimulus Differences Comparison
Stimuli Comparison Data	Similarities Data or Stimuli-Differences Comparison

FIG. 2. The four quadrants.

uli. This kind of data has led to the construction of models under the rubric of multidimensional psychophysics. The data could be called stimuli-differences comparison. Inasmuch as the behavior is typically a response to the relative similarity of stimuli, the name *Similarities* data is proposed.

Figure 2 contains the four quadrants with their suggested labels indicated.

Some Interrelations of Quadrants

It is important to note that there is no unique mapping of behavior into these quadrants. An experimenter, when he analyzes his data, has made a choice of a behavior theory. He has decided, for example, that the behavior was generated in a space in which both the individuals and stimuli were points or just the stimuli are points, he may decide that the behavior is generated by one-dimension or he may ask if it is in a space of more than one dimension. If he decides that the behavior may be generated in a space of more than one dimension, he is faced with deciding what kind of a distance function to employ.

On this latter point, the use of a Euclidean distance function was almost universal until very recently. Perhaps the earliest thoughts of alternative distance functions occur in a paper of H. M. Johnson's (1935) and these ideas are more fully developed in the conjunctive and disjunctive models of the Coombs-Kao nonmetric factor analysis monograph. Another, perhaps very significant, alternative distance function is suggested by Householder and Landahl (1945) and has been picked up by Attneave (1950) and applied to the area of perception. Once the barrier of convention is broken down, one may expect a great variety of possible distance functions to be developed, each corresponding to a theory about how complex behavior is generated. Distance functions conceived of in the context of data in one quadrant suggest the construction of equivalent models for other quadrants.

To illustrate the kind of decisions an experimenter makes in analyzing data, suppose he has the pair comparison preferences of each of a number of individuals over a set of alternatives. Such behavior has been used here to illustrate data in Quadrant Ia, which, when analyzed by the unfolding technique, leads to a joint genotypic space with both individuals and stimuli located in it and the dimensions of this space correspond to the latent attributes generating the individuals' preferences.

One may recognize here, however, that the individual is making a pair comparison between distances, a distance being the distance between the point corresponding to him and the point corresponding to the alternative. The experimenter may decide then, that the *distance* is the stimulus and each individual's behavior may be interpreted as an order relation on a pair of stimuli which are these distances. This *distance* for each alternative is how much the individual dislikes each alternative, i.e., the further the stimulus point from the ideal point the less it is liked. The experimenter then, may decide that he wishes to construct a stimulus scale for the alternatives, representing a measure of their preferability. Consequently, he maps the behavior into Quadrant IIIa and scales the alternatives, say by the law of comparative judgment. One obtains, then, a scale with only the alternatives on it ranging from most to least preferred, representing an amalgamation of the individuals' preferences. The interpretation of such a scale and its relation to the results obtained by analyzing the same behavior as Quadrant Ia data are discussed in Coombs (1952, 1954b).

This serves to introduce the more general case in which behavior mapped into Quadrants I and IV may always be so interpreted as to be mapped into Quadrant III. This comes about in this way. In Quadrants I and IV, a comparison is being made between distances—in Quadrant I it is a distance between an individual and a stimulus (or more generally, a distance between points in distinct sets) and in Quadrant IV, it is a distance between stimuli (or more generally a distance between points in the same set.) If, however, the experimenter chooses to regard these *distances* as the *stimuli,* then the behavior maps into Quadrant III, with Ia and IVa going into IIIa, and Ib and IVb going into IIIb.

That this commonly occurs with Ia data in the construction of scales of preference has already been discussed above. This also occurs in multidimensional psychophysics as a preliminary step in which a scale of the *distances* between pairs of stimuli is first constructed and then these distances are analyzed by the methods of multidimensional psychophysics to recover a space in which the original stimuli may lie.

These distances between stimuli, however, may be initially obtained by other methods, such as rating scales (Quadrant IIb) or the method of categorical judgments (Quadrant IIa). These are the approaches used by Ekman (1954) and Mellinger (1956) respectively.

The psychophysical methods represent models which apply to data in more than one quadrant. They apply to data which fall into Quadrant IIa or IIIa, depending on whether the experimenter conceives of the behavior as reflecting a measure of the individual (as in threshold determination) or as reflecting measures of the stimuli. These are respectively IIa and IIIa data.

Other Response Measures

The point has been made that psychological observations are interpreted as relations on points or in equivalent terms, as distance measurements. The psychological observations which have been used for illustration have been for the most part judgmental responses but these are by no means the only kind of observations that are made nor the only kind of distance measurement. Other kinds of response measures which are used to generate data are observations of inconsistency of response, latency, amplitude, and confusion errors.

These response characteristics or measures may also be interpreted as relations on points and hence yield psychological data to which psychological measurement models may be applied. There appears to be an interesting difference between the psychological data obtained from such response measures as these and that obtained from judgmental responses. The former seem to be interpretable as information only about the *absolute* distances between pairs of points whereas the latter may yield information about either *absolute* or *algebraic* distances. The more often an individual confuses two stimuli, the longer it takes him to choose between them, the less the distance between their respective points in the psychological space. These response measures do not appear to have information in them as to which stimulus is on which side of the other, i.e. the algebraic distance. Models which make use of such data have been generally concerned with the kind of transformation that should be made to reflect a measure of psychological magnitude. Thus, there are the models of Thurstone (1927) and Luce (1959) for the transformation of inconsistency into psychological distance. Both apply to data mapped into Quadrant IIIa and yield measures of the stimuli. Less has been done in a formal way to construct models for the other kinds of response measures. The difficulty is that there is little experimental literature which is of aid in suggesting the kinds of assumptions one can make for transforming the response measure into a psychological distance. What is necessary is some intensive experimental work on these fundamental aspects of psychological measurement.

Relation to Older Theory of Data

The first explicit form of the theory of data (Coombs, 1953) constructed four quadrants on the basis of behavior being interpreted as relative or irrelative and as Task *A* or Task *B*. In one of the four quadrants so generated, the one corresponding to single stimulus data, there was a further dichotomy based on whether the stimuli were monotone or nonmonotone (also referred to as cumulative and noncumulative). The mapping between the old and new form of the theory is indicated in Table 1.

TABLE 1

Relation Between Old and New Theory of Data

Old	New
Quadrant I	Quadrant Ia
No Equivalent	Quadrant Ib
Quadrant IIa, Quadrant III	Quadrant IIa
Quadrant IIb, Quadrant III	Quadrant IIb
Quadrant IV	Quadrant IIIa, IIIb, IVa, IVb

The old Quadrant III is imbedded in the new Quadrants IIa and IIb and the old Quadrant IV has been differentiated into four octants. The old Quadrant III would include all methods involving the evaluation of stimuli, one at a time with respect to an attribute, i.e., rating scale methods, category scaling, and magnitude estimation. As such methods may be thought of as yielding data which are relations on pairs of points from distinct sets (a point for each stimulus and a point for each response alternative) they satisfy the formal models used by data in Quadrants IIa and IIb.

Summary

An abstract theory of psychological data has been constructed for the purpose of organizing and systematizing the domain of psychological methodology. It is asserted that from the point of view of psychological measurement theories all behavioral observations satisfy, at the simplest level, each of three dichotomies, generating eight classes called octants which were organized into four quadrants. Any behavioral observations when mapped into data involve accepting a miniature behavioral theory implicit in the method used to analyze the data. All of the various kinds of data were illustrated and some of the interrelations within and between quadrants were pointed out.

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(Received August 10, 1959)

A BRIEF GLOSSARY

Individuals = undefined.
Stimuli = undefined.
Behavior = any potentially observable relation among individuals and stimuli.
Raw Data = observed relations among real world objects, for example, pass-fail, preferential choice, yes-no, accept-reject, amplitude, latency, inconsistency, etc.
Data = raw data mapped into relations between points.
A Psychological Space = an abstract space in which lie points corresponding to objects in the real world (stimuli and/or individuals) and the relations among points reflect the observed relations among real world objects.
A Psychological Measurement Model = a set of assumptions from which is derived a calculus to construct a psychological space from the data matrix.
A Method of Collecting Data = the rules and the lore for arriving at raw data.
A Behavior Theory = the mapping from raw data into data.

APPENDIX

The following sets are given:

$$\begin{aligned} D &= \{1,2,\cdots d\cdots r\} \\ H &= \{1,2,\cdots h\cdots t\} \\ I &= \{1,2,\cdots i\cdots m\} \\ j &= \{1,2,\cdots j,k,l,\cdots n\} \end{aligned}$$

Axiom 1. There exists distinct sets $K^{(d)}$, $d \in D$, where each $K^{(d)}$ is a segment of the real line.

Definition. Let $K = \{x | x = (x^{(1)},x^{(2)}, \cdots x^{(d)}, \cdots x^{(r)}\}$ where $x^{(d)} \in K^{(d)}$, in which the elements x are vectors in r-dimensional space. Let, in addition,

$$\begin{array}{ll} C \subseteq K, & C = \{c_i | i \in I\} \\ Q \subseteq K, & Q = \{q_i | j \in J\} \\ A = C \times Q, & A = \{(c_i,q_j)\} \\ A_i \subseteq A, & A_i = \{(c_i,q_j) | i \text{ fixed}\} \\ B = Q \times Q, & B = \{(q_j,q_k)\} \end{array}$$

The sets A, A_i, and B are sets whose elements are pairs of points. We will also have a need for sets whose elements consist of pairs of such pairs of points, in particular we construct the sets:

$$A \times A, \quad A_i \times A_i, \quad B \times B$$

Axiom 2. There exists a function p on $K \times K$ into the real line, which satisfies the following conditions for a distance function:

$$\begin{aligned} &|p(a,b)| = |p(b,a)| \\ &p(a,b) \;= O \Longleftrightarrow a = b \\ &|p(a,b)| \leq |p(a,c)| + |p(b,c)| \end{aligned}$$

Axiom 3. Given two vectors differing only in one component, the sign of p is determined by that one component.

Axiom 4. Every measurement model satisfies the following three conditions:

1. A relation exists on a pair of points, or a pair of pairs of points.
2. The elements of the pair of points are drawn from two distinct sets as in A or from one set as in B.
3. The relation is either a proximity relation (O) or an order relation ($>$).

Axiom 5. To each triple (h,i,j) and to each quadruple (hi,jk) corresponds a subset $D' = D'(h,i,j)$ or $D'' = D''(hi,jk)$ of D, that is, $D' \subseteq D$, $D'' \subseteq D$. D' or D'', as the case may be, will be called *the set of relevant dimensions.*

Definition.

1. q_{hij} is the projection of the vector q_j in the set of relevant dimensions, D' or D''.
2. c_{hij} is the projection of the vector c_i in the set of relevant dimensions, D'.
3. $p_{hij} = p(c_{hij},q_{hij})$ is the image of the ordered pair (c_{hij},q_{hij}), (the "distance" between the pair of points) in the set of relevant dimensions, D'.
4. $p_{hi,jk} = p(q_{hij},q_{hik})$ is the image of the function p in the set of relevant dimensions D''.

Definitions of the information in the behavior mapped into the various quadrants (see Fig. 1).

Quadrant Ia

$$|p_{hij}| - |p_{hik}| \leq O \Leftrightarrow j \gtrdot k$$

where: $j \gtrdot k$ signifies responses of the form "j preferred to k."

Quadrant Ib

$$||p_{hij}| - |p_{hik}|| \leq \epsilon_{hi,jk} \Leftrightarrow j\ \dot{M}\ k$$

where: ϵ is a nonnegative number and $j\ \dot{M}\ K$ signifies responses of the form "I cannot choose between j and k" or "I do not prefer one more than the other." (The symbol $\dot{M}$ is used to signify "*matches* in preference.")

Quadrant I with an intermediate category of responses

$$|p_{hij}| - |p_{hik}| < -\epsilon_{hi,jk} \Leftrightarrow j \gtrdot k$$
$$||p_{hij}| - |p_{hik}|| \leq \epsilon_{hi,jk} \Leftrightarrow j\ \dot{M}\ k$$
$$|p_{hij}| - |p_{hik}| > \epsilon_{hi,jk} \Leftrightarrow k \gtrdot j$$

Quadrant IIa

$$p_{hij} \geq O \Leftrightarrow i > j$$

where: $i > j$ signifies responses of the form the individual, i, passes, accepts, etc., the stimulus j. More generally, an element i of one set dominates an element j of another set.

Quadrant IIb

$$|p_{hij}| \leq \epsilon_{hij} \Leftrightarrow i\ \dot{M}\ j$$

where: $i\ \dot{M}\ j$ signifies responses of the form the individual i says yes, agrees, endorses, etc., the stimulus j. More generally, an element i of one set is matched with an element j of another set.

Quadrant II with an intermediate category of responses

$$p_{hij} > \epsilon_{hij} \Leftrightarrow i > j$$
$$|p_{hij}| \leq \epsilon_{hij} \Leftrightarrow i\ \dot{M}\ j$$
$$p_{hij} < \epsilon_{hij} \Leftrightarrow j > i$$

Quadrant IIIa

$$p_{hi,jk} \geq O \Leftrightarrow j > k$$

where: $j > k$ signifies responses of the form "j is greater than k."

Quadrant IIIb

$$|p_{hi,jk}| \leq \epsilon_{hi,jk} \Leftrightarrow j\ M\ k$$

where: $j\ M\ k$ signifies responses of the form "these stimuli j and k are not different, they match."

Quadrant III with an intermediate category of response

$$p_{hi,jk} > \epsilon_{hi,jk} \Leftrightarrow j > k$$
$$|p_{hi,jk}| \leq \epsilon_{hi,jk} \Leftrightarrow j\ M\ k$$
$$p_{hi,jk} < -\epsilon_{hi,jk} \Leftrightarrow k > j$$

Quadrant IVa

$$|p_{hi,jk}| - |p_{hi,lm}| \leq O \Leftrightarrow (j,k) < (l,m)$$

where: $(j,k) < (l,m)$ signifies responses of the form "the pair of stimuli (j,k) are more alike than the pair of stimuli (l,m)."

Quadrant IVb

$$||p_{hi,jk}| - |p_{hi,lm}|| \leq \epsilon_{hi,jk,lm} \Leftrightarrow (j,k)\ M\ (l,m)$$

where: $(j,k)\ M\ (l,m)$ signifies responses of the form "the pair of stimuli (j,k) are no more different than are the pair of stimuli (l,m)."

$$|p_{hi,jk}| - |p_{hi,lm}| < -\epsilon_{hi,jk,lm} \Leftrightarrow (j,k) < (l,m)$$
$$||p_{hi,jk}| - |p_{hi,lm}|| \leq \epsilon_{hi,jk,lm} \Leftrightarrow (j,k)\ M\ (l,m)$$
$$|p_{hi,jk}| - |p_{hi,lm}| > \epsilon_{hi,jk,lm} \Leftrightarrow (l,m) < (j,k)$$