

The marginal problem for sets of desirable gamble sets

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Compatibility

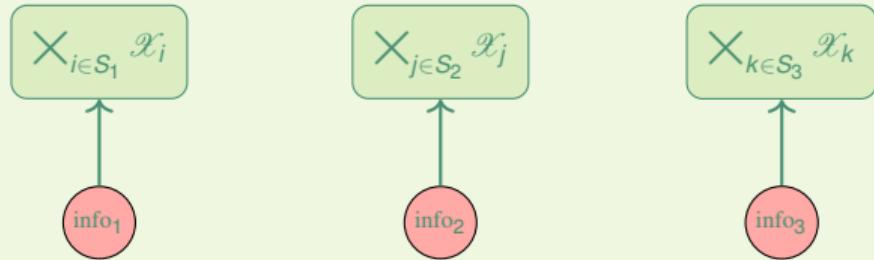
$$\times_{i \in S_1} \mathcal{X}_i$$

$$\times_{j \in S_2} \mathcal{X}_j$$

$$\times_{k \in S_3} \mathcal{X}_k$$

Cartesian domains

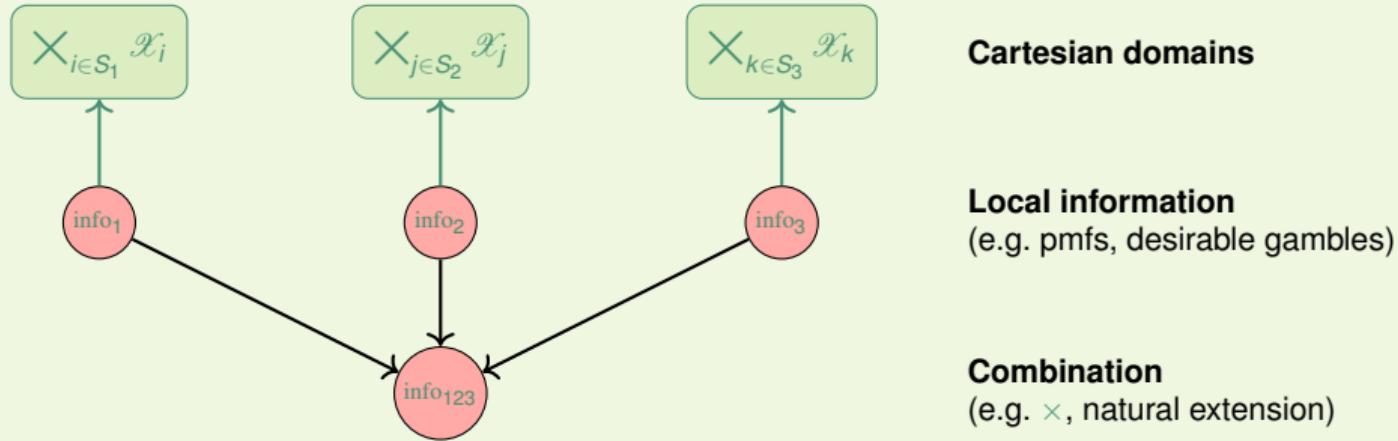
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Cartesian domains

Local information
(e.g. pmfs, desirable gambles)

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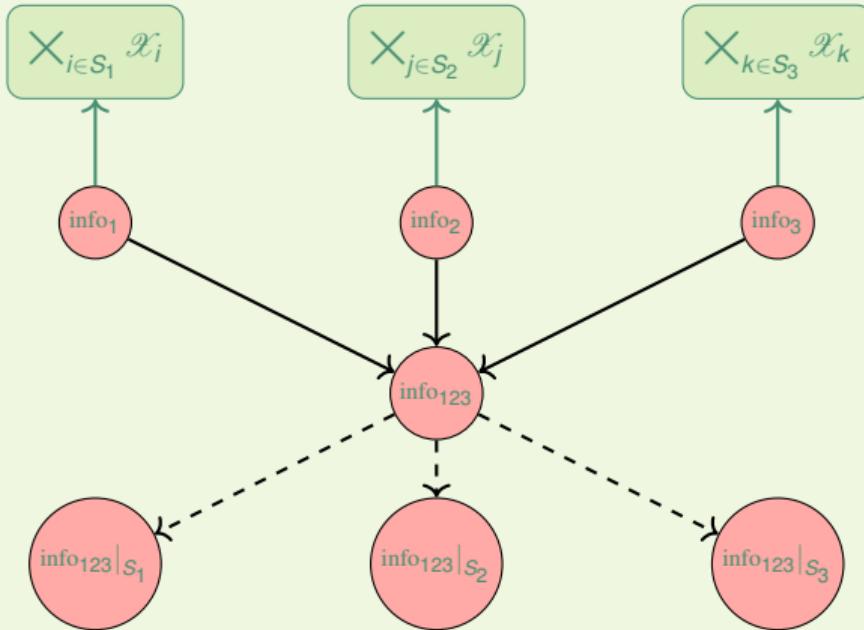


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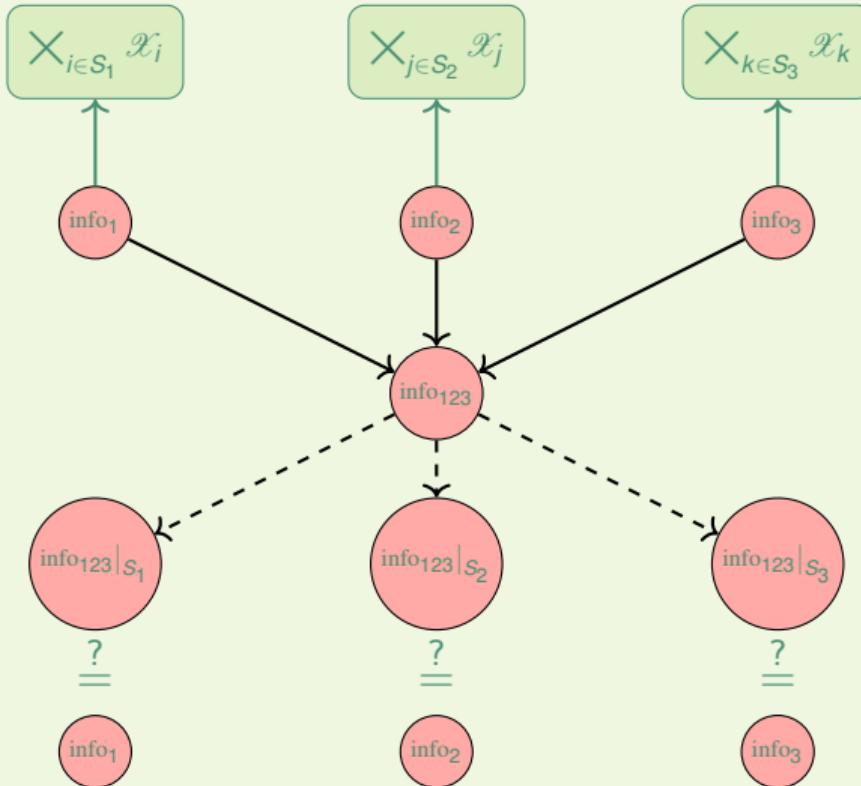
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Marginalization
(projected back to smaller scopes)

Compatibility



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Local information
(e.g. pmfs, desirable gambles)

Combination
(e.g. \times , natural extension)

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Question:
Do we recover the original information?

Sets of Desirable Gamble Sets

Want to express that f is desirable
or g is desirable.

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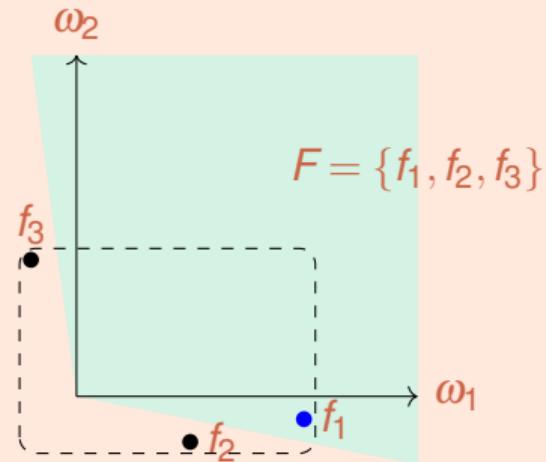
In other words, in the set $\{f, g\}$, at least one is desirable.

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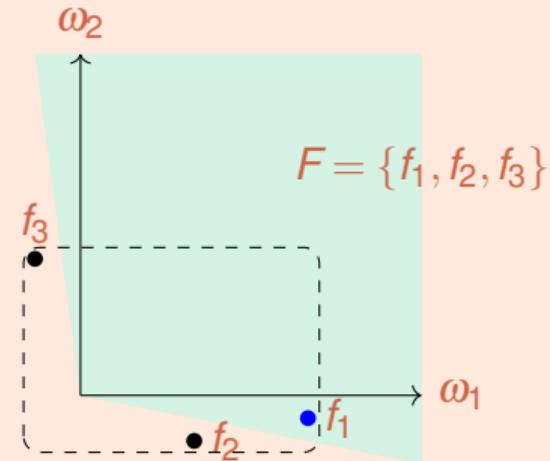


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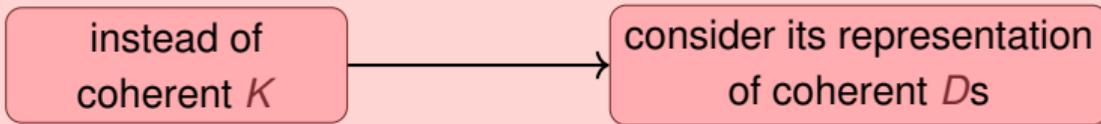
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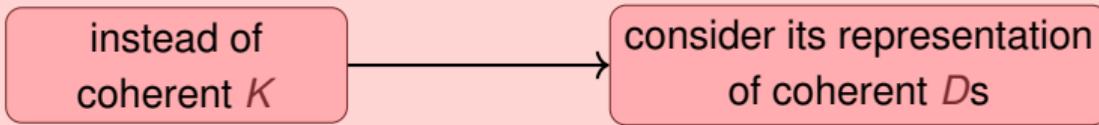
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Representations



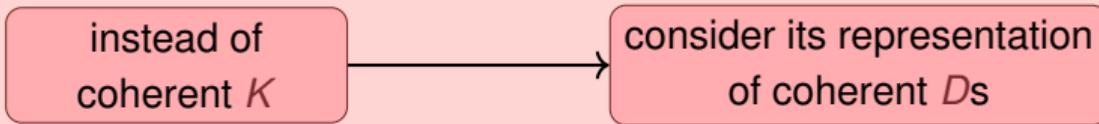
Representations



$$K := \bigcap_{D \in \mathcal{D}} K_D = \bigcap_{D \in \mathcal{D}} \{\text{gamble sets } F : F \cap D \neq \emptyset\}$$

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- ⊕ *largest representation:* $\mathcal{D}_K := \{\text{coherent } Ds : K \subseteq K_D\}$
- ⊕ *finite representation:* there is a finite \mathcal{D} representing K .

Representation (continued)

Consider some K , its largest representation \mathcal{D}_K , some $D_1, D_2 \in \mathcal{D}_K$ with $D_1 \subseteq D_2$:

$$\implies K_{D_1} \subseteq K_{D_2}$$

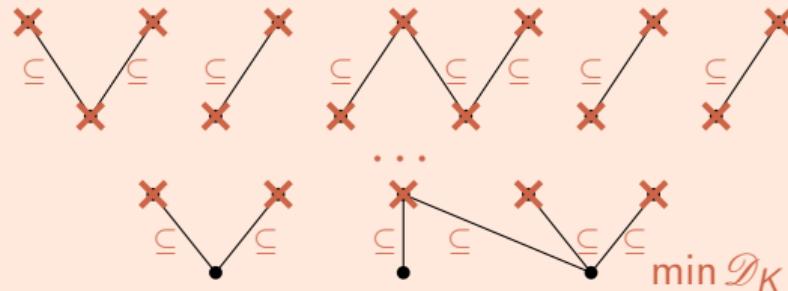
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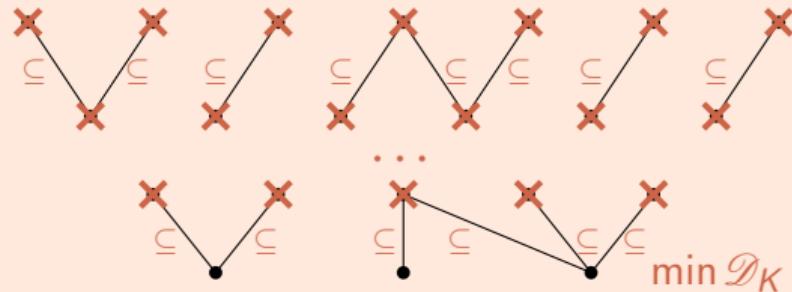


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Continuing this way, we obtain *the set of minimal elements*:

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which satisfies

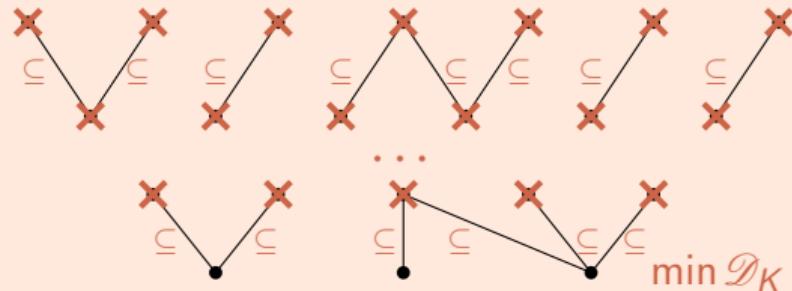
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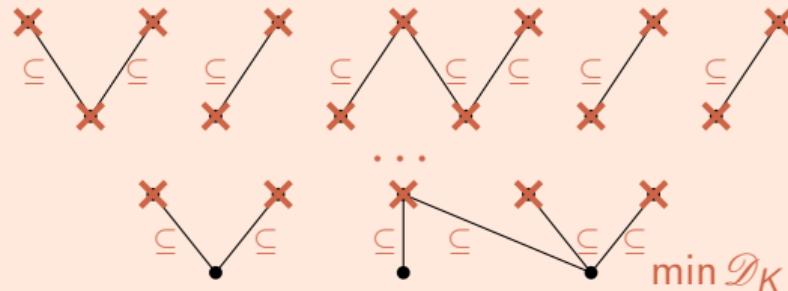
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① Is it defined only for the largest representations?

+ It might be empty for other representations than the largest.

Pairwise Compatibility

Consider two sets:

$$K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1}), \quad K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2}).$$

Definition

Sets of desirable gamble sets K_1 and K_2 are said to be *pairwise compatible* if:

$$\text{Marg}_{S_1 \cap S_2} K_1 = \text{Marg}_{S_1 \cap S_2} K_2.$$

In other words, their marginals agree on the common domain.

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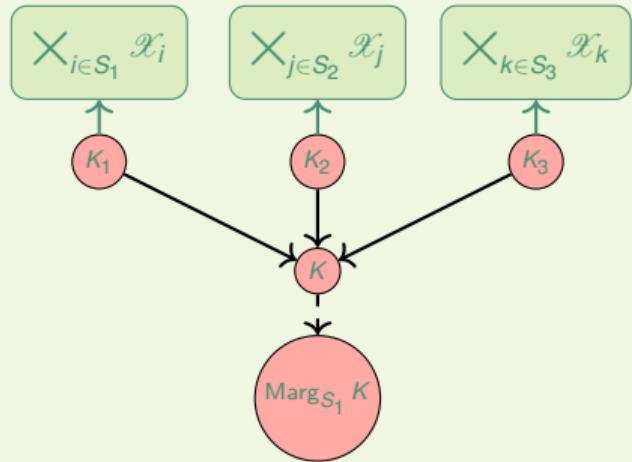
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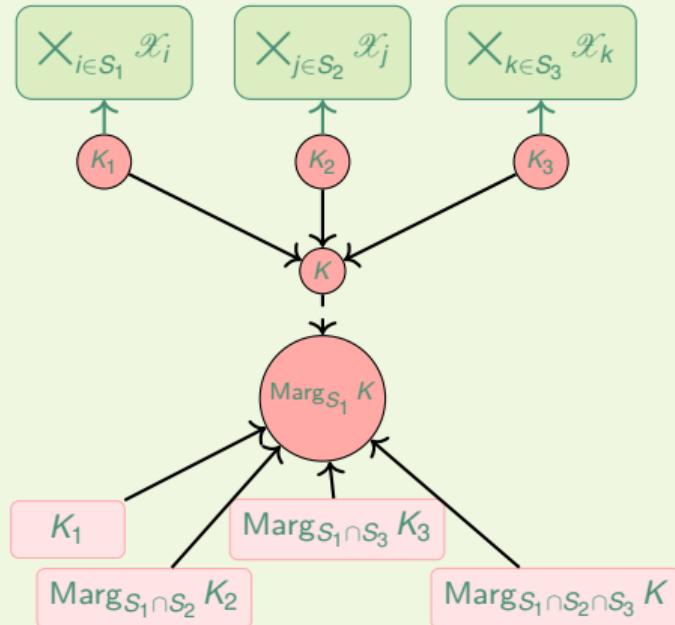
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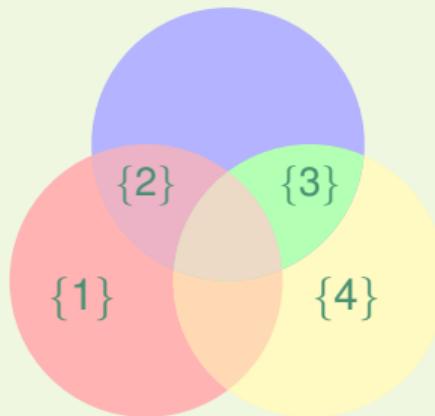
② What if there is some information inside the intersection of more than two domains?

RIP (Running Intersection Property)

$$(\forall \ell \in \{2, \dots, m\})(\exists i^* < \ell) S_\ell \cap S_{i^*} = S_\ell \cap \bigcup_{i < \ell} S_i$$

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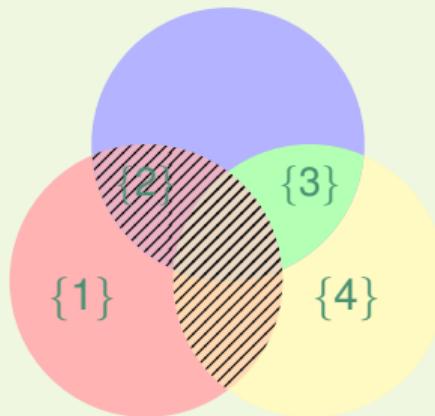
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Satisfies RIP
For some order(s)

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Compatibility for sets of desirable gamble sets

**Pairwise
Compatibility**

+

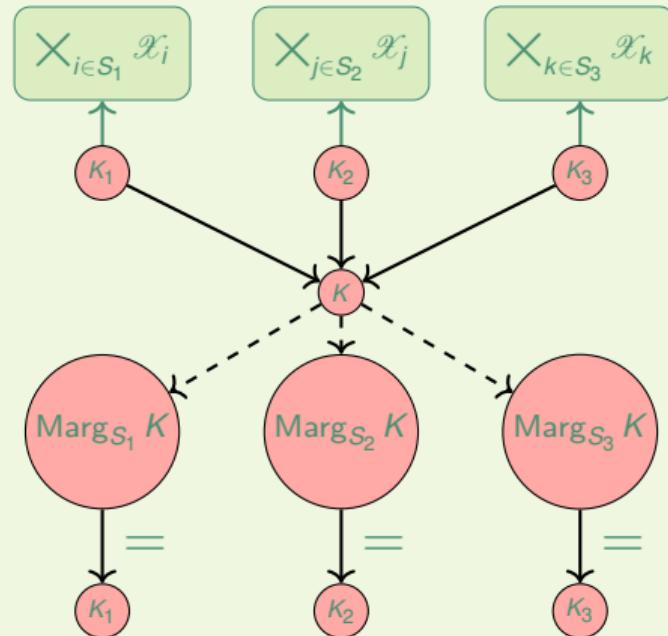
RIP

+

**Finite
Representations**



Compatibility



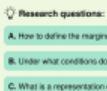
The marginal problem for sets of desirable gamble sets

1 Problem statement

Given a finite number of belief assessments on overlapping domains, e.g. priors $p_1(X_1, X_2)$ and $p_2(X_2, X_3)$.

When can we retrieve the original information from their joint assessment?

This question is known as the **marginal problem**. Here, we investigate it in the context of sets of desirable gamble sets.



2 Sets of desirable gambles

Possibility space: Consider $n \in \mathbb{N}$, disjoint variables X_1, \dots, X_n taking values in finite possibility spaces $\mathcal{X}_1, \dots, \mathcal{X}_n$. Let $\mathcal{X} := \{1, \dots, n\}$ be the global index set. Beliefs about X_i are expressed using gambles on \mathcal{X} -set $\mathcal{X}_i \subseteq \mathcal{X}$. For any subset $I \subseteq \mathcal{X}$, the type of uncertain variable X_I takes values in $\mathcal{X}_I := \bigcup_{i \in I} \mathcal{X}_i$.

Gambles: A gamble f is a real-valued function on \mathcal{X} , regarded as a risky transaction: once the outcome $x \in \mathcal{X}$ is revealed, the agent receives $f(x)$ units of linear utility, which may be negative.

We collect all gambles f that the agent finds desirable in her set of desirable gambles D .

We denote the set of all gambles as $\mathcal{L}(\mathcal{X})$, and all positive gambles as $\mathcal{L}_{\geq 0}(\mathcal{X})$.

Coherence axioms: A set of desirable gambles D is coherent if for all gambles f and g and all real $\lambda > 0$:



We collect all coherent sets of desirable gambles in $\overline{\mathcal{D}}(\mathcal{X})$.

Marginalization: For any set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X})$ and any $S \subseteq \mathcal{X}$, its S -marginal $\text{marg}_S D \subseteq \mathcal{L}(\mathcal{X}_S)$ is defined as

$$\text{marg}_S D := D \cap \mathcal{L}(\mathcal{X}_S).$$

4 Representations

Representation: A set of desirable gamble sets $K \subseteq \mathcal{D}(\mathcal{X})$ is coherent if and only if there is a non-empty set of coherent sets of desirable gambles $\mathcal{D} \subseteq \overline{\mathcal{D}}(\mathcal{X})$ such that

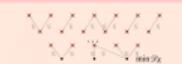
$$K = K_{\mathcal{D}} := \bigcap_{D \in \mathcal{D}} K_D = \bigcap_{D \in \mathcal{D}} \{F \in \mathcal{L}(\mathcal{X}) : F \cap D \neq \emptyset\}.$$

We then say \mathcal{D} is a representation of K , and K is represented by \mathcal{D} . Moreover, K 's largest representation is $\mathcal{D}_K := \{D : K \subseteq K_D\}$.

Marginalization: For any representation $\mathcal{D} \subseteq \overline{\mathcal{D}}(\mathcal{X})$ and any $S \subseteq \mathcal{X}$, its S -marginal $\text{marg}_S \mathcal{D} \subseteq \overline{\mathcal{D}}(\mathcal{X}_S)$ is defined as

$$\text{marg}_S \mathcal{D} := \{\text{marg}_S D : D \in \mathcal{D}\}.$$

Finite representation: For any coherent set of desirable gamble sets $K \subseteq \mathcal{D}(\mathcal{X})$ there is a finite representation if there is a finite subset $\mathcal{D} \subseteq \overline{\mathcal{D}}(\mathcal{X})$ that represents K .



Minimal elements: For every representation $\mathcal{D} \subseteq \overline{\mathcal{D}}(\mathcal{X})$, the set $\min \mathcal{D} := \{D \in \mathcal{D} : \forall D' \in \mathcal{D} \setminus D, D \cap D' = \emptyset\} = D$ contains \mathcal{D} 's minimal elements.

Minimal elements representation: For any coherent set of desirable gamble sets K , we have that $\min \mathcal{D}_K = K$, so the poset $(\mathcal{D}_K, \subseteq)$ has minimal elements. Moreover, $\mathcal{D}_K = \min \mathcal{D}_K$. As a consequence $K = \text{marg}_{\mathcal{D}_K} K$, so \mathcal{D}_K represents K .

Results:

- ✓ The marginal $\text{marg}_S K$ is finite and represented by $\text{marg}_S \mathcal{D}_K$.
- ✓ For K with a finite representation $\text{marg}_S \mathcal{D}_K$ is a representation of K .
- ✓ For K with a finite representation \mathcal{D} , $\text{marg}_S \mathcal{D}$ is a representation of K .
- ✓ K has a finite representation if and only if $\min \mathcal{D}_K$ is finite.
- ✓ For any two K_1 and K_2 with finite representations $K_1 \sqsubseteq K_2$ if and only if $\min \mathcal{D}_1 \sqsubseteq \min \mathcal{D}_2$.

5 Marginal problem

Pairwise compatibility: Two coherent sets of desirable gamble sets $K_1 \subseteq \mathcal{D}(\mathcal{X}_1)$ and $K_2 \subseteq \mathcal{D}(\mathcal{X}_2)$ are pairwise compatible if any two of them are pairwise compatible.

$$\text{marg}_{\mathcal{D}_1 \cup \mathcal{D}_2} K_1 = \text{marg}_{\mathcal{D}_2 \cup \mathcal{D}_1} K_2$$

The as coherent $K_1 \subseteq \mathcal{D}(\mathcal{X}_1)$, $I \subseteq \{1, \dots, n\}$ are pairwise compatible if any two of them are pairwise compatible.

$$\text{marg}_{\mathcal{D}_1 \cup \mathcal{D}_2} \mathcal{D}_K = \text{marg}_{\mathcal{D}_2 \cup \mathcal{D}_1} \mathcal{D}_K$$



A. Compatibility: The as coherent $K_i \subseteq \mathcal{D}(\mathcal{X}_{I_i})$, $I_i \subseteq \{1, \dots, n\}$ are called compatible if there is a i -pairwise compatible with each other, so that $\text{marg}_{\mathcal{D}_1 \cup \mathcal{D}_2} K_1 = K_2$.

✓ Compatible K is the natural extension $\text{marg}_{\{1, \dots, n\}} K$.

B. Marginal result: Two coherent $K_i \subseteq \mathcal{D}(\mathcal{X}_{I_i})$, $I_i \subseteq \{1, \dots, n\}$ with finite representations are compatible if S_1, \dots, S_m satisfy $\forall j, K_1, \dots, K_m$ are pairwise compatible.

For coherent and compatible $\{K_i\}_{i=1}^m$, their natural extension K is represented by

$$\mathcal{D} := \{c_{1j} (J_1 \cup \dots \cup J_m) : J_1 \in \mathcal{D}_1, \dots, J_m \in \mathcal{D}_m\}$$

For questions, please come to my poster!