The Power of the Tangent Addition Formula

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1 A Quick Derivation

To derive the tangent addition formula, the cosine and sine addition formulas must first be derived first.

Deriving Cosine Addition Formula

Consider a triangle OAB on the cartesian plane, where A and B are points on the unit circle:

- O is at (0,0).
- A is a point above the x axis and OA makes an angle α with the x-axis.
  - Thus, A has coordinates $(\cos(\alpha), \sin(\alpha))$.
- B is a point below the x-axis and OB makes an angle β with the x-axis.
  - Thus, B has coordinates $(\cos(\beta), -\sin(\beta))$.

Next, rotate OAB about the origin such that B $\rightarrow$ B' and B' is on the x-axis (i.e rotate OAB β degrees in the CCW direction). The coordinates of the new triangle OA'B' are:

- O is (0,0).
- A' is $(\cos(\alpha+\beta), \sin(\alpha+\beta))$.
- B' is (1,0) because the triangle was initially on the unit circle and B has moved to the x-axis.

Since the rotation of the triangle will not change the length of AB, using the equality $\overline{AB} = \overline{A'B'}$ and the distance formula will yield the cosine double angle formula.

$$\overline{AB} = \overline{A'B'}$$
$$\sqrt{(A_x - B_x)^2 + (A_y - B_y)^2} = \sqrt{(A'_x - B'_x)^2 + \left(A'_y - B'_y\right)^2}$$

Removing the square root of both sides and substituting in the coordinates,

$$(\cos(\alpha) - \cos(\beta))^2 + (\sin(\alpha) + \sin(\beta))^2 = (\cos(\alpha+\beta) - 1)^2 + (\sin(\alpha+\beta) - 0)^2$$
$$\cos^2(\alpha) - 2\cos(\alpha)\cos(\beta) + \cos^2(\beta) + \sin^2(\alpha) + 2\sin(\alpha)\sin(\beta) + \sin^2(\beta) = \cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta)$$

Applying the Pythagorean theorem of trigonometry to simplify,

$$2 - 2\cos(\alpha)\cos(\beta) + 2\sin(\alpha)\sin(\beta) = 2 - 2\cos(\alpha+\beta)$$

Rearranging and dividing both sides by 2 yields the cosine double angle formula.

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{1}$$

Deriving Sine Addition Formula

We can simply use the identity that $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$. Thus,

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2} - \alpha - \beta\right) \\
&= \cos\left[\left(\frac{\pi}{2} - \alpha\right) + (-\beta)\right] \\
&= \cos\left(\frac{\pi}{2} - \alpha\right)\cos(-\beta) - \sin\left(\frac{\pi}{2} - \alpha\right)\sin(-\beta)
\end{aligned}$$

Using the fact that cosine is even and sin is odd, and also using the above identity yields the sine addition formula

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{2}$$

Deriving Tangent Addition Formula

The Tangent addition formula is easily derived using the tangent function's definition and then substituting the sine/cosine addition formulas: (1) and (2).

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}
\end{aligned}$$

Dividing the numerator and denominator by $\cos(\alpha)\cos(\beta)$,

$$\begin{aligned}
&= \frac{\dfrac{\sin(\alpha)}{\cos(\alpha)} + \dfrac{\sin(\beta)}{\cos(\beta)}}{1 - \dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\
&= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}
\end{aligned} \tag{3}$$

2 Problem 1

Problem: Consider the following function

$$f(x) = \frac{x + \sqrt{3}}{1 - \sqrt{3}x}$$

Define $f_n(x) = f(x)$ composed of itself n times. So, $f_2(x) = f(f(x))$ and $f_3(x) = f(f(f(x)))$. Evaluate $f_{2021}(2021)$ and express it in the following form: $\frac{p + q\sqrt{r}}{s + t\sqrt{u}}$, where r and u are square-free and p, q, r, s, t, u are all integers. This problem is from the 2021 CTMC.

Solution:

Since this is a contest problem, it may seem natural to simply compose the function of itself multiple times expecting to find a cycle. This of course does work, but I don't find it as elegant as the following solution.

We start by making the following observation which comes directly out of the tangent addition formula (because $\tan(60^\circ) = \sqrt{3}$):

$$x = \tan(\theta) \implies f(x) = \tan(\theta + 60^\circ) \tag{4}$$

For $f_1(2021)$, we simply have $\theta = \arctan(2021)$, and $f_1(2021) = \tan(\arctan(2021) + 60^\circ)$.

To solve this problem, define a value θ_n such that

$$\tan(\theta_n) = f_n(2021) \tag{5}$$

For example, $\theta_1 = \arctan(2021) + 60^\circ$. With this idea in mind, we can find θ_{2021}. Let's start by considering the function itself.

$$f_{n+1}(2021) = f(f_n(2021))$$

From what we established in (5)

$$\tan(\theta_{n+1}) = f(\tan(\theta_n))$$

Then, using (4), since we have a nested tangent on the RHS.

$$\begin{aligned} \tan(\theta_{n+1}) &= \tan(\theta_n + 60^\circ) \\ \theta_{n+1} &= \theta_n + 60^\circ \end{aligned}$$

Since $\theta_1 = \arctan(2021) + 60^\circ$, we have $\theta_{2021} = \arctan(2021) + 2021 \cdot 60^\circ$. Since we will be taking the tangent of this angle, we only care about its modulo 360, which is 300 (i.e $2021 \cdot 60 \equiv 300 \pmod{360}$). Finally solving,

$$\begin{aligned} f_{2021}(2021) &= \tan(\theta_{2021}) \\ &= \tan(\arctan(2021) + 2021 \cdot 60^\circ) \\ &= \tan(\arctan(2021) + 300^\circ) \end{aligned}$$

Applying tangent addition formula,

$$\begin{aligned} &= \frac{\tan(\arctan(2021)) + \tan(300^\circ)}{1 - \tan(\arctan(2021))\tan(300^\circ)} \\ &= \frac{2021 - \sqrt{3}}{1 + 2021\sqrt{3}} \end{aligned}$$

Note that this solution was written with a bit more rigour in mind. In fact, after the first observation, it is rather intuitive that each subsequent angle just increases by 60 degrees.

3 Problem 2

Problem:

The first 2 terms of a sequence are $a_1 = 1$ and $a_2 = \dfrac{1}{\sqrt{3}}$. For all a_n, $n \geq 1$

$$a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}} \tag{6}$$

What is a_{2009}?

This was problem 25 from the 2009 AMC 12A and the first ever problem 25 that I solved during my practice.

Solution:

Our solution starts very similarly to the previous question. We start by defining a sequence of θ_n such that,

$$a_n = \tan(\theta_n) \tag{7}$$

Substituting (7) into (6) allows for a really nice simplification because of the tangent addition formula.

$$\begin{aligned}
a_{n+2} &= \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}} \\
\tan(\theta_{n+2}) &= \frac{\tan(\theta_n) + \tan(\theta_{n+1})}{1 - \tan(\theta_n)\tan(\theta_{n+1})} \\
\tan(\theta_{n+2}) &= \tan(\theta_n + \theta_{n+1}) \\
\theta_{n+2} &= \theta_n + \theta_{n+1}
\end{aligned} \tag{8}$$

All that remains to do is find θ_{2009} using this much simpler recursive identity. We begin by noting that $\theta_1 = 45°$ because $\tan(45°) = 1 = a_1$ and $\theta_2 = 30°$ because $\tan(30°) = \dfrac{1}{\sqrt{3}} = a_2$.

2 more ideas need to be considered to make the calculation a bit more efficient. First because we are dealing with angles, and the tangent function is periodic over 180 degrees, we only need to consider the angles mod 180. Secondly, we can make note that θ_1 and θ_2 are both multiples of 15, and so all subsequent terms will also be multiples of 15 since we are only summing them up. We can then use another sequence $b_n = \frac{\theta_n}{15}$ to simplify the arithmetic. And since $180 = (12)(15)$, we only care about the terms in b_n mod 12. We can simply continue basic arithmetic until the sequence is periodic (we know it's periodic because there's a finite number of consecutive pairs that can occur in the sequence).

term number (n)	b_n	θ_n
1	3	45
2	2	30
3	5	75
4	7	105
5	0	0
6	7	105
7	7	105
8	2	30
9	9	135
10	11	165
11	8	120
12	7	105
13	3	45
14	10	150
15	1	15
16	11	165
17	0	0
18	11	165
19	11	165
20	10	150
21	9	135
22	7	105
23	4	60
24	11	165
25	3	45
26	2	30

Since $\theta_{25} = \theta_1$ and $\theta_{26} = \theta_2$, we know there's a period of 24 terms. So to find θ_{2009}, we simply need to find 2009 mod 24, which is 17.

$$\begin{aligned} a_{2009} &= \tan(\theta_{2009}) \\ &= \tan(\theta_{17}) \\ &= \tan(0^\circ) \\ &= 0 \end{aligned}$$