# 5 Lagrangian Points of Circular Orbits

Arthur Xu

October 2020

## 1 Introduction

In Celestial Mechanics, there are these points called Lagrangian Points. If we consider a simple 2 body system, such as a sun and a planet in our solar system, we know that the planet orbits around the sun as the sun provides a strong gravitational force. The planet technically provides a gravitational force, however it is so small that when considering the sun, we neglect the movement of the sun when calculating orbits of planets. The gravitational force of the planets is the motivation behind this idea of Lagrangian Points. There are 5 such points in space relative to a sun-planet system that allows for objects (of negligible mass) placed at these points to orbit with the planet. This means that it has the same period as the planet, yet it has a different radius of orbit.

The five points are described as follows. L1, L2 and L3 are all located on the line of the sun-planet system. L1 is in between the sun and the planet, which means that the force of gravitation from the planet weakens the net gravitational force. L2 is located further away from the Sun compared to the planet causing the planet to increase the net gravitational force. L3 is located on the opposite side of the Sun and Earth so the sun again increases the net gravitational force (but not by as much compared to L2 because the distance is twice as much). L4 and L5 are actually located along the orbit of the planet. In those sections, I will prove the pre-established theory that L4/L5, the planet, and the Sun form an equilateral triangle.

For all sections, the following variables are used.

$$\begin{aligned}
M_s &: \text{Mass of Sun} \\
m_p &: \text{Mass of planet} \\
m_l &: \text{Mass of object located at Lagrange Point} \\
r_{sp} &: \text{Distance from Sun to planet} \\
r_{sl} &: \text{Distance from Sun to Lagrange Point 1} \\
r_{pl} &: \text{Distance from planet to Lagrange Point 1} \\
v_l &: \text{Speed of object at Lagrange Point 1}
\end{aligned}$$

The constraint that all these problems work with is that the net gravitational force is equal to the the Centripetal force. So we have:

$$\begin{aligned}
\vec{F_s} + \vec{F_p} &= \vec{F_c} \\
\vec{F_s} + \vec{F_p} &= \frac{m_l v_l^2}{r_{sl}}
\end{aligned} \tag{1}$$

The next step is to find an expression for $v_l^2$. We can simply rewrite it as the circumference over the period.

$$\begin{aligned}
v_l &= \frac{2\pi r_{sl}}{T_l} \\
v_l^2 &= \frac{4\pi^2 r_{sl}^2}{T_l^2}
\end{aligned} \tag{2}$$

$T_l$ is still unknown. However, due to the definition of the Lagrangian point, we know that it has the same period as the planet – which is something that we can calculate using Keplar's third law.

$$\begin{aligned} T_l^2 &= T_p^2 \\ &= \left(\frac{4\pi^2}{GM_s}\right) r_{sp}^3 \end{aligned} \tag{3}$$

Now let's substitute (3) into (2).

$$\begin{aligned} v_l^2 &= \frac{4\pi^2 r_{sl}^2}{T_l^2} \\ &= \frac{4\pi^2 r_{sl}^2}{\left(\frac{4\pi^2}{GM_s}\right) r_{sp}^3} \\ &= \frac{GM_s r_{sl}^2}{r_{sp}^3} \end{aligned} \tag{4}$$

Finally, substituting (4) into (1) yields the following equation which will be used in 5 sections.

$$\begin{aligned} \vec{F}_s + \vec{F}_p &= \frac{GM_s m_l r_{sl}^2}{r_{sl} r_{sp}^3} \\ &= \frac{GM_s m_l r_{sl}}{r_{sp}^3} \end{aligned} \tag{5}$$

In this equation note that the centripetal force had a positive sign. This means that along our axis, forces directed toward the centre are positive and forces directed away from the centre are negative.

## 2 Lagrangian Point 1

L1 is located in between the planet and the Sun, thus we have the following equations for forces. The planet pulls L1 away from the centre while the Sun pulls L1 toward the centre.

$$\vec{F}_s = \frac{GM_s m_l}{r_{sl}^2} \tag{6}$$

$$\vec{F}_p = -\frac{Gm_p m_l}{r_{pl}^2} \tag{7}$$

Substituting (6) and (7) into (5) yields:

$$\begin{aligned} \frac{GM_s m_l}{r_{sl}^2} - \frac{GM_p m_l}{r_{pl}^2} &= \frac{GM_s m_l r_{sl}}{r_{sp}^3} \\ \frac{M_s}{r_{sl}^2} - \frac{M_p}{r_{pl}^2} &= \frac{M_s r_{sl}}{r_{sp}^3} \end{aligned} \tag{8}$$

However, we still have two unknowns in (8) ($r_{sl}$ and $r_{pl}$). A simple solution exists however. We note that $r_{sl} + r_{pl} = r_{sp}$, where the distance from the Sun to the planet is a known value. This equation is true because the distance from the Sun to L1 plus the distance from the planet to L1 should lead to the distance from the Sun to the planet (L1 is in between the planet and the Sun). Thus we now have two unknowns and two equations allowing us to solve for both $r_{sl}$ and $r_{pl}$. Let's substitute $r_{pl} = r_{sp} - r_{sl}$ into (8).

$$\frac{M_s}{r_{sl}^2} - \frac{m_p}{(r_{sp} - r_{sl})^2} = \frac{M_s r_{sl}}{r_{sp}^3} \tag{9}$$

We have finally reduced our equation to have one unknown: $r_{sl}$ (the distance from the sun to a planet can be assumed to be known). Before we continue, I would like to rename our variables to reduce the clutter due to subscripts. The mass of the sun will become $M$ instead of $M_s$. The mass of the planet will become $m$ instead of $m_p$. The distance from the Sun to the planet will become $r$ instead of $r_{sp}$. The distance from the Sun to the Lagrange Point is $x$ instead of $r_{sl}$. Thus, we are now solving for $x$. (9) can now be rewritten as follows.

$$\frac{M}{x^2} - \frac{m}{(r-x)^2} = \frac{Mx}{r^3} \tag{10}$$

To continue with our algebra. We need to find the LCM of $x^2$, $(r-x)^2$, and $r^3$, which is $r^3x^2(r-x)^2$. Then we multiply both sides of (10) so that all fractions have that denominator. We can then cancel out the denominators and end up with a polynomial function of $x$.

$$\begin{gathered}\left(\frac{M}{x^2}\right)\left(\frac{r^3(r-x)^2}{r^3(r-x)^2}\right) - \left(\frac{m}{(r-x)^2}\right)\left(\frac{r^3x^2}{r^3x^2}\right) = \left(\frac{Mx}{r^3}\right)\left(\frac{x^2(r-x)^2}{x^2(r-x)^2}\right)\\ \frac{Mr^3(r-x)^2}{r^3x^2(r-x)^2} - \frac{mr^3x^2}{r^3x^2(r-x)^2} = \frac{Mx^3(r-x)^2}{r^3x^2(r-x)^2}\\ Mr^3(r-x)^2 - mr^3x^2 = Mx^3(r-x)^2\end{gathered}$$

We then expand and end up with a quintic expression for $x$

$$\begin{gathered}Mr^3\left(r^2-2rx+x^2\right) - mr^3x^2 = Mx^3\left(r^2-2rx+x^2\right)\\ Mr^5 - 2Mr^4x + Mr^3x^2 - mr^3x^2 = Mx^3r^2 - 2Mx^4r + Mx^5\end{gathered}$$

Moving everything to one side yields the following equation.

$$-Mx^5 + 2Mrx^4 - Mr^2x^3 + \left(Mr^3 - mr^3\right)x^2 - 2Mr^4x + Mr^5 = 0$$

Thus, $x$ is the solution to the general quintic.

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \tag{11}$$

Where:

$$\begin{aligned}a &= -M\\ b &= 2Mr\\ c &= -Mr^2\\ d &= Mr^3 - mr^3\\ e &= -2Mr^4\\ f &= Mr^5\end{aligned}$$

# 3 Lagrangian Point 2

L2 is located exterior to both the planet and the Sun, thus we have the following equations for forces. Both the Sun and the planet pull L2 toward the centre.

$$\vec{F_s} = \frac{GM_sm_l}{r_{sl}^2} \tag{12}$$

$$\vec{F_p} = \frac{Gm_pm_l}{r_{pl}^2} \tag{13}$$

Substituting (12) and (13) into (5) yields:

$$\begin{aligned} \frac{GM_s m_l}{r_{sl}^2} + \frac{GM_p m_l}{r_{pl}^2} &= \frac{GM_s m_l r_{sl}}{r_{sp}^3} \\ \frac{M_s}{r_{sl}^2} + \frac{M_p}{r_{pl}^2} &= \frac{M_s r_{sl}}{r_{sp}^3} \end{aligned} \tag{14}$$

However, we still have two unknowns in (14) ($r_{sl}$ and $r_{pl}$). A simple solution exists however. We note that $r_{sl} = r_{pl} + r_{sp}$, where the distance from the Sun to the planet is a known value. This equation is true because the distance from the Sun to the planet plus the distance from the planet to L2 should lead to the distance from the Sun to L2 (the planet is now in between L2 and the Sun). Thus we now have two unknowns and two equations allowing us to solve for both $r_{sl}$ and $r_{pl}$. Let's substitute $r_{pl} = r_{sl} - r_{sp}$ into (14).

$$\frac{M_s}{r_{sl}^2} + \frac{m_p}{(r_{sl} - r_{sp})^2} = \frac{M_s r_{sl}}{r_{sp}^3} \tag{15}$$

We have finally reduced our equation to have one unknown: $r_{sl}$ (the distance from the sun to a planet can be assumed to be known). Before we continue, I would like to rename our variables to reduce the clutter due to subscripts. The mass of the sun will become $M$ instead of $M_s$. The mass of the planet will become $m$ instead of $m_p$. The distance from the Sun to the planet will become $r$ instead of $r_{sp}$. The distance from the Sun to the Lagrange Point is $x$ instead of $r_{sl}$. Thus, we are now solving for $x$. (15) can now be rewritten as follows.

$$\frac{M}{x^2} + \frac{m}{(x-r)^2} = \frac{Mx}{r^3} \tag{16}$$

To continue with our algebra. We need to find the LCM of $x^2$, $(x-r)^2$, and $r^3$, which is $r^3 x^2 (x-r)^2$. Then we multiply both sides of (16) so that all fractions have that denominator. We can then cancel out the denominators and end up with a polynomial function of $x$.

$$\begin{aligned} \left(\frac{M}{x^2}\right)\left(\frac{r^3 (x-r)^2}{r^3 (x-r)^2}\right) + \left(\frac{m}{(x-r)^2}\right)\left(\frac{r^3 x^2}{r^3 x^2}\right) &= \left(\frac{Mx}{r^3}\right)\left(\frac{x^2 (x-r)^2}{x^2 (x-r)^2}\right) \\ \frac{Mr^3 (x-r)^2}{r^3 x^2 (x-r)^2} + \frac{mr^3 x^2}{r^3 x^2 (x-r)^2} &= \frac{Mx^3 (x-r)^2}{r^3 x^2 (x-r)^2} \\ Mr^3 (x-r)^2 + mr^3 x^2 &= Mx^3 (x-r)^2 \end{aligned}$$

We then expand and end up with a quintic expression for $x$

$$\begin{aligned} Mr^3 \left(x^2 - 2rx + r^2\right) + mr^3 x^2 &= Mx^3 \left(x^2 - 2rx + r^2\right) \\ Mr^3 x^2 - 2Mr^4 x + Mr^5 + mr^3 x^2 &= Mx^5 - 2Mx^4 r + Mx^3 r^2 \end{aligned}$$

Moving everything to one side yields the following equation.

$$-Mx^5 + 2Mrx^4 - Mr^2 x^3 + \left(Mr^3 + mr^3\right) x^2 - 2Mr^4 x + Mr^5 = 0$$

Thus, $x$ (distance from sun to L2) is the solution to the general quintic.

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \tag{17}$$

Where:

$$\begin{aligned} a &= -M \\ b &= 2Mr \\ c &= -Mr^2 \\ d &= Mr^3 + mr^3 \\ e &= -2Mr^4 \\ f &= Mr^5 \end{aligned}$$

# 4 Lagrangian Point 3

L3 is on the opposite side of the sun relative to the planet. As a result, the planet's gravitational pull is toward the centre of the system. Thus we have the following equations for forces.

$$\vec{F_s} = \frac{GM_s m_l}{r_{sl}^2} \tag{18}$$

$$\vec{F_p} = \frac{Gm_p m_l}{r_{pl}^2} \tag{19}$$

Substituting (18) and (19) into (5) yields:

$$\begin{aligned} \frac{GM_s m_l}{r_{sl}^2} + \frac{GM_p m_l}{r_{pl}^2} &= \frac{GM_s m_l r_{sl}}{r_{sp}^3} \\ \frac{M_s}{r_{sl}^2} + \frac{M_p}{r_{pl}^2} &= \frac{M_s r_{sl}}{r_{sp}^3} \end{aligned} \tag{20}$$

However, we still have two unknowns in (20) ($r_{sl}$ and $r_{pl}$). A simple solution exists however. We note that $r_{pl} = r_{sl} + r_{sp}$, where the distance from the Sun to the planet is a known value. This equation is true because the distance from the Sun to the planet plus the distance from the sun to L3 should lead to the distance from the planet to L3 (the sun is in between the planet and L3). Thus we now have two unknowns and two equations allowing us to solve for both $r_{sl}$ and $r_{pl}$. Let's substitute $r_{pl} = r_{sl} + r_{sp}$ into (20).

$$\frac{M_s}{r_{sl}^2} + \frac{m_p}{(r_{sl} + r_{sp})^2} = \frac{M_s r_{sl}}{r_{sp}^3} \tag{21}$$

We have finally reduced our equation to have one unknown: $r_{sl}$ (the distance from the sun to a planet can be assumed to be known). Before we continue, I would like to rename our variables to reduce the clutter due to subscripts. The mass of the sun will become $M$ instead of $M_s$. The mass of the planet will become $m$ instead of $m_p$. The distance from the Sun to the planet will become $r$ instead of $r_{sp}$. The distance from the Sun to the Lagrange Point is $x$ instead of $r_{sl}$. Thus, we are now solving for $x$. (21) can now be rewritten as follows.

$$\frac{M}{x^2} + \frac{m}{(x+r)^2} = \frac{Mx}{r^3} \tag{22}$$

To continue with our algebra. We need to find the LCM of $x^2$, $(x+r)^2$, and $r^3$, which is $r^3 x^2 (x+r)^2$. Then we multiply both sides of (22) so that all fractions have that denominator. We can then cancel out the denominators and end up with a polynomial function of $x$.

$$\begin{aligned} \left(\frac{M}{x^2}\right)\left(\frac{r^3 (x+r)^2}{r^3 (x+r)^2}\right) + \left(\frac{m}{(x+r)^2}\right)\left(\frac{r^3 x^2}{r^3 x^2}\right) &= \left(\frac{Mx}{r^3}\right)\left(\frac{x^2 (x+r)^2}{x^2 (x+r)^2}\right) \\ \frac{Mr^3 (x+r)^2}{r^3 x^2 (x+r)^2} + \frac{mr^3 x^2}{r^3 x^2 (x+r)^2} &= \frac{Mx^3 (x+r)^2}{r^3 x^2 (x+r)^2} \\ Mr^3 (x+r)^2 + mr^3 x^2 &= Mx^3 (x+r)^2 \end{aligned}$$

We then expand and end up with a quintic expression for $x$

$$\begin{aligned} Mr^3 \left(x^2 + 2rx + r^2\right) + mr^3 x^2 &= Mx^3 \left(x^2 + 2rx + r^2\right) \\ Mr^3 x^2 + 2Mr^4 x + Mr^5 + mr^3 x^2 &= Mx^5 + 2Mx^4 r + Mx^3 r^2 \end{aligned}$$

Moving everything to one side yields the following equation.

$$-Mx^5 - 2Mrx^4 - Mr^2 x^3 + \left(Mr^3 + mr^3\right) x^2 + 2Mr^4 x + Mr^5 = 0$$

Thus, $x$ (distance from sun to L3) is the solution to the general quintic.

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \tag{23}$$

Where:

$$\begin{aligned}
a &= -M \\
b &= -2Mr \\
c &= -Mr^2 \\
d &= Mr^3 + mr^3 \\
e &= 2Mr^4 \\
f &= Mr^5
\end{aligned}$$

# 5 Results for Lagrangian Points 1, 2, and 3

For all the first three Lagrangian Points, we see that we need to find the roots of a quintic. This can be calculated with fairly high accuracy using Newton's method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{24}$$

Once $x_{n+1} = x_n$ for all decimals that can be produced, a solution is approximated. This was done in Microsoft Excel for all 3 Lagrange Points for all 8 planets in the solar system. The initial guesses were always the distance from the planet to the Sun or $r$ because the distance from the sun to Lagrangian nodes are relatively close to the distance from the sun to the planet. The results are as follows.

| Planet | Average Orbital Distance (m) | L1(m) | L2(m) | L3(m) |
|---|---|---|---|---|
| Mercury | $5.79 \times 10^{10}$ | $5.767\,97 \times 10^{10}$ | $5.812\,09 \times 10^{10}$ | $5.79 \times 10^{10}$ |
| Venus | $1.082 \times 10^{11}$ | $1.071\,92 \times 10^{11}$ | $1.092\,14 \times 10^{11}$ | $1.082 \times 10^{11}$ |
| Earth | $1.496 \times 10^{11}$ | $1.481\,09 \times 10^{11}$ | $1.511\,01 \times 10^{11}$ | $1.496 \times 10^{11}$ |
| Mars | $2.279 \times 10^{11}$ | $2.268\,18 \times 10^{11}$ | $2.289\,86 \times 10^{11}$ | $2.279 \times 10^{11}$ |
| Jupiter | $7.7834 \times 10^{11}$ | $7.264\,44 \times 10^{11}$ | $8.326\,52 \times 10^{11}$ | $7.784\,02 \times 10^{11}$ |
| Saturn | $1.4335 \times 10^{12}$ | $1.369\,06 \times 10^{12}$ | $1.499\,93 \times 10^{12}$ | $1.433\,53 \times 10^{12}$ |
| Uranus | $2.8725 \times 10^{12}$ | $2.803\,06 \times 10^{12}$ | $2.943\,08 \times 10^{12}$ | $2.872\,51 \times 10^{12}$ |
| Neptune | $4.4951 \times 10^{12}$ | $4.380\,31 \times 10^{12}$ | $4.611\,88 \times 10^{12}$ | $4.495\,12 \times 10^{12}$ |

Notice that for the interior planets, the L3 value has the "same" value as the average orbital distance. The mass of the planet compared to the Sun is so minuscule that Microsoft Excel wouldn't output the extra decimal places when outputting the results.

# 6 Lagrangian Point 4 and 5

Before starting this section I want to emphasize that this is a proof and less so of a derivation. We seek to prove that the triangle formed by the Sun, planet and Lagrange point is an equilateral triangle. A corollary of this is that it shows that the Lagrangian points 4 and 5 are located on the orbit of the planet. This is because an equilateral triangle would mean that the distance from the Sun to the planet will be equal to the distance from the Sun to the Lagrangian point. There isn't a need for a different set of proofs for L4 and L5 since one is simply a vertical reflection of the other.

The L4 and L5 points are much more complicated than the previous ones. The first adjustment that must be made is that the three bodies are actually orbiting around their centre of mass. In truth, the mass of the Sun is so large compared to any planet, that the centre of mass is extremely close to the Sun anyways. We cannot make this approximation in this case however because, the result that is yielded shows that Lagrangian Node is located on the planet. I suppose this is true that if an object were to be placed on the planet, it does comply to the rules of a Lagrangian Node. However, we are not looking for this.

Instead we have to consider that all three masses are orbiting around their centre of mass. We have a triangle but with a 4th point of interest, the centre of mass which is labelled C. C will be placed on the segment SP (or it's on the line of the planet and Sun) because any object placed at a Lagrangian Point will have negligeable mass. We also now have more angles that we will eventually deal with. Every variable that will be used in this section is shown in the following diagram.

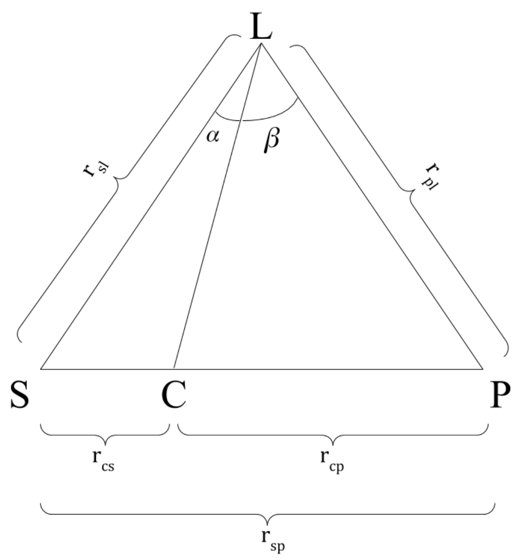


Figure 1: A diagram which shows all of the variables used in this section

First let's define the distances $r_{cp}$ and $r_{cs}$ using the definition of the centre of mass and taking the Sun to be at the origin.

$$\begin{aligned} r_{cs} &= \frac{r_s M_s + r_p m_p}{M_s + m_p} \\ &= \frac{m_p}{M_s + m_p} r_{sp} \end{aligned} \tag{25}$$

We can then use (25) to find $r_{cp}$

$$\begin{aligned} r_{cp} &= r_{sp} - r_{cs} \\ &= r_{sp}\left(1 - \frac{m_p}{M_s + m_p}\right) \\ &= \frac{M_s}{M_s + m_p} r_{sp} \end{aligned} \tag{26}$$

This proof has several constraints and several unknowns. Let's start to list all of our constraints. The first constraint that is common with the previous parts is that the period of the Lagrange Point is the same as the period of the planet. We have to be careful and note that the resulting equation will be different as we now have to consider the orbits around the centre of mass.

$$\begin{aligned}
T_l &= T_p \\
\frac{2\pi r_{cl}}{v_l} &= \frac{2\pi r_{cp}}{v_p} \\
\frac{r_{cl}}{v_l} &= \frac{r_{cp}}{v_p}
\end{aligned}$$

We can then substitute (26) in to get

$$\frac{r_{cl}}{v_l} = \frac{r_{sp} M_s}{v_p \left(M_s + m_p\right)} \tag{27}$$

The second constraint is that the net gravitational force acting on the planet (which is only the gravitation of the sun) is equal to the centripetal force of the planet. Since they both have inward radial directions, we can simply consider the magnitudes. We must be careful here as the gravitational force is dependent on the distance between the Sun and the planet whereas the centripetal force is dependent on the distance between the planet and the point of orbit (the centre of mass).

$$\begin{aligned}
F_{c_p} &= F_{g_p} \\
\frac{m_p v_p^2}{r_{cp}} &= \frac{G M_s m_p}{r_{sp}^2} \\
\frac{v_p^2}{r_{cp}} &= \frac{G M_s}{r_{sp}^2}
\end{aligned}$$

Again, we're going to substitute (26) in which yields

$$\begin{aligned}
\frac{v_p^2}{r_{cp}} &= \frac{G M_s}{r_{sp}^2} \\
\frac{v_p^2}{\frac{M_s}{M_s+m_p} r_{sp}} &= \frac{G M_s}{r_{sp}^2} \\
\frac{v_p^2 \left(M_s + m_p\right)}{M_s} &= \frac{G M_s}{r_{sp}}
\end{aligned} \tag{28}$$

The third constraint is that the centripetal force of the Lagrangian Point is equal to the sum of the gravitational force applied by the planet and the Sun. However, we have to note that neither gravitational forces are radial. Thus, we need to find the radial components. To do so we're going to assign angles. Let $\alpha = \angle SLC$ and $\beta = \angle PLC$. Or in other words $\alpha$ is the angle between the Sun and centre of mass while $\beta$ is the angle between the planet and the centre of mass. Looking at the diagram we can see that the cosine of both angles will yield the respective radial forces (toward the centre of mass). Thus, we have the following equation.

$$\begin{aligned}
F_{c_l} &= F_{g_l} \\
\frac{m_l v_l^2}{r_{cl}} &= \frac{G M_s m_l}{r_{sl}^2} \cos\left(\alpha\right) + \frac{G m_p m_l}{r_{pl}^2} \cos\left(\beta\right) \\
\frac{v_l^2}{r_{cl}} &= G\left(\frac{M_s}{r_{sl}^2} \cos\left(\alpha\right) + \frac{m_p}{r_{pl}^2} \cos\left(\beta\right)\right)
\end{aligned} \tag{29}$$

The final constraint also comes from the definition of a Lagrangian Point, but perhaps a less obvious consequence. Because the Lagrangian Point orbits with the planet, this results in the Lagrangian point being in equilibrium relative to the orbit (this is another way to justify (29)). This also means that the tangential components of the gravitational pulls equal each other. We can see that to get the tangential components from the magnitudes by using the sine of the angles. Thus we have the following equation.

$$\begin{aligned} \frac{GM_s m_l}{r_{sl}^2}\sin(\alpha) &= \frac{Gm_p m_l}{r_{pl}^2}\sin(\beta) \\ \frac{M_s}{r_{sl}^2}\sin(\alpha) &= \frac{m_p}{r_{pl}^2}\sin(\beta) \end{aligned} \tag{30}$$

These are all the constraints and now we can start solving. The first step in our process is to obtain to expressions for $v_p^2$ and to equate them. From (27) we have,

$$\begin{aligned} \frac{r_{cl}}{v_l} &= \frac{r_{sp}M_s}{v_p(M_s+m_p)} \\ v_p &= \frac{r_{sp}}{r_{cl}}\frac{M_s}{M_s+m_p}v_l \\ v_p^2 &= \frac{r_{sp}^2}{r_{cl}^2}\frac{M_s^2}{(M_s+m_p)^2}v_l^2 \end{aligned} \tag{31}$$

From (28) we have,

$$\begin{aligned} \frac{v_p^2(M_s+m_p)}{M_s} &= \frac{GM_s}{r_{sp}} \\ v_p^2 &= \frac{GM_s^2}{r_{sp}(M_s+m_p)} \end{aligned} \tag{32}$$

We can then equate (31) and (32) to get the following equality,

$$\begin{aligned} \frac{r_{sp}^2}{r_{cl}^2}\frac{M_s^2}{(M_s+m_p)^2}v_l^2 &= \frac{GM_s^2}{r_{sp}(M_s+m_p)} \\ \frac{v_l^2}{r_{cl}} &= \frac{G(M_s+m_p)r_{cl}}{r_{sp}^3} \end{aligned} \tag{33}$$

During simplification, one $r_{cl}$ term was multiplied into the RHS whereas the second one was left in the denominator. This was done because it allows (33) to be directly equated with (29).

$$\begin{aligned} \frac{G(M_s+m_p)r_{cl}}{r_{sp}^3} &= G\left(\frac{M_s}{r_{sl}^2}\cos(\alpha) + \frac{m_p}{r_{pl}^2}\cos(\beta)\right) \\ \frac{(M_s+m_p)r_{cl}}{r_{sp}^3} &= \frac{M_s}{r_{sl}^2}\cos(\alpha) + \frac{m_p}{r_{pl}^2}\cos(\beta) \end{aligned} \tag{34}$$

There are a couple substitutions we're going to make before moving on. First we want to rearrange (30) for $m_p$ with the intent of substituting the result into the RHS ONLY. Rearranging yields,

$$\begin{aligned} \frac{M_s}{r_{sl}^2}\sin(\alpha) &= \frac{m_p}{r_{pl}^2}\sin(\beta) \\ m_p &= M_s\frac{r_{pl}^2}{r_{sl}^2}\frac{\sin(\alpha)}{\sin(\beta)} \end{aligned} \tag{35}$$

For the LHS, we're going to replace one of the $r_{sp}$ with (26) and leaving the rest as $r_{sp}^2$. Rearranging (26) yields,

$$\begin{aligned} r_{cp} &= \frac{M_s}{M_s+m_p}r_{sp} \\ r_{sp} &= \frac{M_s+m_p}{M_s}r_{cp} \end{aligned} \tag{36}$$

Substituting (36) into the LHS in the manner described above yields,

$$\begin{aligned}\frac{(M_s + m_p)\, r_{cl}}{r_{sp}^3} &= \frac{(M_s + m_p)\, r_{cl}}{r_{sp}^2 \left(\frac{M_s + m_p}{M_s} r_{cp}\right)} \\ &= \frac{M_s r_{cl}}{r_{sp}^2 r_{cp}}\end{aligned} \tag{37}$$

Finally, substituting (35) and (37) into (34) yields

$$\begin{aligned}\frac{M_s r_{cl}}{r_{sp}^2 r_{cp}} &= \frac{M_s}{r_{sl}^2} \cos(\alpha) + \frac{\cos(\beta)}{r_{pl}^2} M_s \frac{r_{pl}^2}{r_{sl}^2} \frac{\sin(\alpha)}{\sin(\beta)} \\ \frac{r_{cl}}{r_{sp}^2 r_{cp}} &= \frac{1}{r_{sl}^2} \left(\cos(\alpha) + \frac{\cos(\beta)\sin(\alpha)}{\sin(\beta)}\right)\end{aligned}$$

We can then factor out the $\frac{1}{\sin(\beta)}$ because this allows us to use the double angle formula of sines.

$$\begin{aligned}\frac{r_{cl}}{r_{sp}^2 r_{cp}} &= \frac{1}{r_{sl}^2 \sin(\beta)} \left(\cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha)\right) \\ \frac{r_{cl}}{r_{sp}^2 r_{cp}} &= \frac{1}{r_{sl}^2 \sin(\beta)} \sin(\alpha + \beta)\end{aligned} \tag{38}$$

Now we can use law of sines to find an expression for the sines. First let's take a look at triangle $PLS$. The law of sines states the following relationship:

$$\begin{aligned}\frac{\sin(\alpha + \beta)}{r_{sp}} &= \frac{\sin(\angle P)}{r_{sl}} \\ \sin(\angle P) &= \frac{r_{sl} \sin(\alpha + \beta)}{r_{sp}}\end{aligned} \tag{39}$$

Now taking a look at triangle $PLC$. The law of sines states the following relationship:

$$\begin{aligned}\frac{\sin(\beta)}{r_{cp}} &= \frac{\sin(\angle P)}{r_{cl}} \\ \sin(\angle P) &= \frac{r_{cl} \sin(\beta)}{r_{cp}}\end{aligned} \tag{40}$$

We notice that $\angle P$ in both (39) and (40) are the same. Thus we can equate the 2 equations which yields the following expression.

$$\begin{aligned}\frac{r_{sl} \sin(\alpha + \beta)}{r_{sp}} &= \frac{r_{cl} \sin(\beta)}{r_{cp}} \\ \frac{\sin(\alpha + \beta)}{\sin(\beta)} &= \frac{r_{sp} r_{cl}}{r_{sl} r_{cp}}\end{aligned} \tag{41}$$

Substituting (41) into (38) will show that $r_{sl} = r_{sp}$, which is our first step to showing that triangle $PLS$ is equilateral.

$$\begin{aligned}\frac{r_{cl}}{r_{sp}^2 r_{cp}} &= \frac{1}{r_{sl}^2} \frac{r_{sp} r_{cl}}{r_{sl} r_{cp}} \\ \frac{1}{r_{sp}^3} &= \frac{1}{r_{sl}^3} \\ r_{sp} &= r_{sl}\end{aligned}$$

An extremely similar method is used to show that $r_{sp} = r_{pl}$, which in turn shows that the Sun, planet and Lagrange point form an equilateral triangle completing the proof.

The process is still the same all the way until (34). The difference is that we are going to rearrange (30) for $M_s$ as opposed to $m_p$. Furthermore, on the left hand side, we're going to represent one term of $r_{sp}$ as a function of $r_{cs}$ as opposed to $r_{cl}$. Rearranging (30) for $M_s$ yields,

$$\begin{aligned} \frac{M_s}{r_{sl}^2}\sin(\alpha) &= \frac{m_p}{r_{pl}^2}\sin(\beta) \\ M_S &= M_p\frac{r_{sl}^2}{r_{pl}^2}\frac{\sin(\beta)}{\sin(\alpha)} \end{aligned} \tag{42}$$

For the LHS we rearrange (25) to show,

$$\begin{aligned} r_{cs} &= \frac{m_p}{M_s+m_p}r_{sp} \\ r_{sp} &= \frac{M_s+m_p}{m_p}r_{cs} \end{aligned} \tag{43}$$

Substituting (43) into the LHS in the manner described above yields,

$$\begin{aligned} \frac{(M_s+m_p)\,r_{cl}}{r_{sp}^3} &= \frac{(M_s+m_p)\,r_{cl}}{r_{sp}^2\left(\frac{M_s+m_p}{m_p}r_{cs}\right)} \\ &= \frac{m_p r_{cl}}{r_{sp}^2 r_{cs}} \end{aligned} \tag{44}$$

With similar logic as before we substitute (42) and (44) back into (34). This yields:

$$\begin{aligned} \frac{m_p r_{cl}}{r_{sp}^2 r_{cs}} &= \frac{m_p}{r_{pl}^2}\cos(\beta) + \frac{\cos(\alpha)}{r_{sl}^2}m_p\frac{r_{sl}^2}{r_{pl}^2}\frac{\sin(\beta)}{\sin(\alpha)} \\ \frac{r_{cl}}{r_{sp}^2 r_{cs}} &= \frac{1}{r_{pl}^2}\left(\cos(\beta) + \frac{\cos(\alpha)\sin(\beta)}{\sin(\alpha)}\right) \\ \frac{r_{cl}}{r_{sp}^2 r_{cs}} &= \frac{1}{r_{pl}^2\sin(\alpha)}\left(\cos(\beta)\sin(\alpha)+\cos(\alpha)\sin(\beta)\right) \\ \frac{r_{cl}}{r_{sp}^2 r_{cs}} &= \frac{1}{r_{pl}^2\sin(\alpha)}\sin(\alpha+\beta) \end{aligned} \tag{45}$$

Now we use law of sines again. First let's take a look at triangle $PLS$. In the first half, we looked to equate $\angle P$. This time around, we look to equate $\angle S$ The law of sines states the following relationship:

$$\begin{aligned} \frac{\sin(\alpha+\beta)}{r_{sp}} &= \frac{\sin(\angle S)}{r_{pl}} \\ \sin(\angle S) &= \frac{r_{pl}\sin(\alpha+\beta)}{r_{sp}} \end{aligned} \tag{46}$$

Now taking a look at triangle $CLS$. The law of sines states the following relationship:

$$\begin{aligned} \frac{\sin(\alpha)}{r_{cs}} &= \frac{\sin(\angle S)}{r_{cl}} \\ \sin(\angle S) &= \frac{r_{cl}\sin(\alpha)}{r_{cs}} \end{aligned} \tag{47}$$

We notice that $\angle S$ in both (46) and (47) are the same. Thus we can equate the 2 equations which yields the following expression.

$$\begin{aligned} \frac{r_{sl}\sin(\alpha+\beta)}{r_{sp}} &= \frac{r_{cl}\sin(\alpha)}{r_{cs}} \\ \frac{\sin(\alpha+\beta)}{\sin(\alpha)} &= \frac{r_{sp}r_{cl}}{r_{sl}r_{cs}} \end{aligned} \tag{48}$$

Substituting (48) into (45) will show that $r_{pl} = r_{sp}$ which equals $r_{sl}$, which completes our proof for an equilateral triangle.

$$\begin{aligned}\frac{r_{cl}}{r_{sp}^2 r_{cs}} &= \frac{1}{r_{pl}^2}\frac{r_{sp}r_{cl}}{r_{pl}r_{cs}}\\ \frac{1}{r_{sp}^3} &= \frac{1}{r_{pl}^3}\\ r_{pl} &= r_{sp} = r_{sl}\end{aligned}$$

■