Another Interesting Complex Number Problem

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1 Problem

Simplify,

$$\prod_{n=1}^{\infty}\left(\frac{1+i\cot\left(\dfrac{n\pi}{2n+1}\right)}{1-i\cot\left(\dfrac{n\pi}{2n+1}\right)}\right)^{\frac{1}{n}} \tag{1}$$

To a form $\left(\dfrac{p}{q}\right)^{\pi i}$ where p and q are relatively prime integers. I solved this problem in the 2021 CHMMC run by Cal Tech university.

2 Solution

For simplicity, let $\theta = \dfrac{n\pi}{2n+1}$ and $b = \cot(\theta)$. First, consider the inside of the product without the exponent, it can be written as,

$$\frac{1+i\cot\left(\dfrac{n\pi}{2n+1}\right)}{1-i\cot\left(\dfrac{n\pi}{2n+1}\right)} = \frac{1+bi}{1-bi}$$

Simplifying by multiplying the numerator and denominator by the conjugate,

$$\begin{aligned} &= \frac{(1+bi)^2}{(1-bi)(1+bi)} \\ &= \frac{1+2bi+b^2i^2}{1+b^2} \\ &= \frac{1-b^2+2bi}{1+b^2} \end{aligned} \tag{2}$$

Substituting the definition of b into (2)

$$= \frac{1-\cot^2(\theta)+2\cot(\theta)i}{1+\cot^2(\theta)}$$

By Pythagorean theorem of trigonometry, the denominator becomes $\csc^2(\theta)$

$$= \frac{1-\cot^2(\theta)+2\cot(\theta)i}{\csc^2(\theta)}$$

Dividing each term in the numerator by the denominator and reduce everything to sines and cosines to simplify

$$\begin{aligned}
&= \sin^2(\theta) - \frac{\cos^2(\theta)}{\sin^2(\theta)}\sin^2(\theta) + 2\frac{\cos(\theta)i}{\sin(\theta)}\sin^2(\theta) \\
&= \sin^2(\theta) - \cos^2(\theta) + 2\sin(\theta)\cos(\theta)i
\end{aligned}$$

Applying double angle formula, while noting that the first 2 terms collect to $-cos(2\theta)$

$$= -\cos(2\theta)) + \sin(2\theta)i$$

Euler's formula cannot be directly applied as the cosine term is negative. To fix this we want to shift the angle such that both cosine and sine are both positive. One way to think about it is that the point in the complex plane that is represented is in Q2 with an angle of 2θ from the negative x-axis. So this can be rewritten as follows,

$$\begin{aligned}
&= \cos(\pi-2\theta) + \sin(\pi-2\theta)i \\
&= \exp\left[(\pi-2\theta)i\right] \qquad (3)
\end{aligned}$$

Substituting the definition of θ into (3)

$$\begin{aligned}
&= \exp\left[\left(\pi - \frac{2n\pi}{2n+1}\right)i\right] \\
&= \exp\left[\left(\frac{(2n+1)\pi - 2n\pi}{2n+1}\right)i\right] \\
&= \exp\left(\frac{\pi i}{2n+1}\right) \qquad (4)
\end{aligned}$$

Substituting (4) back into the original product (1),

$$\begin{aligned}
\prod_{n=1}^{\infty}\left(\frac{1+i\cot\left(\frac{n\pi}{2n+1}\right)}{1-i\cot\left(\frac{n\pi}{2n+1}\right)}\right)^{\frac{1}{n}} &= \prod_{n=1}^{\infty}\left(\exp\left(\frac{\pi i}{2n+1}\right)^{\frac{1}{n}}\right) \\
&= \prod_{n=1}^{\infty}\left(\exp\left(\frac{\pi i}{n(2n+1)}\right)\right)
\end{aligned}$$

Since the product of exponents can be written as the base to the power of the sum of the exponents we have,

$$\begin{aligned}
&= \exp\left(\sum_{n=1}^{\infty}\left(\frac{\pi i}{n(2n+1)}\right)\right) \\
&= \exp\left(\pi i\sum_{n=1}^{\infty}\left(\frac{1}{n(2n+1)}\right)\right) \qquad (5)
\end{aligned}$$

To evaluate the sum, first note that the partial fraction decomposition is as follows,

$$\frac{1}{n(2n+1)} = \frac{1}{n} - \frac{2}{2n+1}$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{1}{n\left(2n+1\right)}\right) &= 2\left(\frac{1}{2n}-\frac{1}{2n+1}\right)\\
&= 2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\dots\right)\\
&= 2\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}\dots\right)
\end{aligned}$$

Note that from the Taylor Series of the logarithm function (see Section 3), it is known that:

$$\ln\left(2\right) = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$$

To evaluate the sum, let S be the sum that we seek, i.e $S = \frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\dots$. Hence,

$$\begin{aligned}
\ln\left(2\right) &= 1-S\\
S &= 1-\ln\left(2\right)
\end{aligned}$$

Substituting back into the sum,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n\left(2n+1\right)}\right) = 2\left(1-\ln\left(2\right)\right)$$

And one last final substitution into (5),

$$\begin{aligned}
\exp\left(\pi i\sum_{n=1}^{\infty}\left(\frac{1}{n\left(2n+1\right)}\right)\right) &= \exp\left(2\pi i\left(1-\ln\left(2\right)\right)\right)\\
&= \exp\left(\pi i\left(2-\ln\left(4\right)\right)\right)\\
&= \left(\frac{e^{2}}{e^{\ln\left(4\right)}}\right)^{\pi i}\\
&= \frac{e^{2\pi i}}{4^{\pi i}}\\
&= \frac{1}{4^{\pi i}}\\
&= \left(\frac{1}{4}\right)^{\pi i}
\end{aligned}$$

Thus, we have reduced the product to the form $\left(\frac{p}{q}\right)^{\pi i}$ where the relatively prime integers p and q are 1 and 4 respectively.

3 Logarithm Taylor Series

To find a Taylor Series representation of $\ln\left(x\right)$, we are going to consider $\ln\left(1+x\right)$ as it is defined for $x=0$ and as a result, we will see that it will yield a very simple series as we can find the Taylor Series approximation at $x_0 = 0$ as opposed to somewhere else. Then by the definition of a Taylor Series,

$$f\left(x\right) = \sum_{k=0}^{\infty}\frac{f^{(k)}\left(x_0\right)}{k!}x^{k} \tag{6}$$

For $k = 0$, we have $f(0) = \ln(1+0) = 0$. For $k \geq 1$, we can notice a pattern by taking the derivative a few times. Note $f^k(0)$ denotes the kth derivative evaluated at 0 (which is our x_0.

$$
\begin{aligned}
f^1(0) &= \left.\frac{1}{1+x}\right|_{x=0} \\
&= 1 \\
f^2(0) &= \left.\frac{-1}{(1+x)^2}\right|_{x=0} \\
&= -1 \\
f^3(0) &= \left.\frac{(2)(1)}{(1+x)^3}\right|_{x=0} \\
&= 2 \\
f^4(0) &= \left.\frac{(-3)(2)(1)}{(1+x)^4}\right|_{x=0} \\
&= -6 \\
f^5(0) &= \left.\frac{(4)(3)(2)(1)}{(1+x)^5}\right|_{x=0} \\
&= 24
\end{aligned}
$$

In general, it can be observed that,

$$f^{(k)}(0) = (-1)^{k-1}(k-1)! \tag{7}$$

Substituting (7) into (6),

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} x^k \\
&= \sum_{k=0}^{\infty} (-1)^{k-1} \frac{(k-1)!}{k!} x^k \\
&= \sum_{k=0}^{\infty} (-1)^{k-1} \frac{x^k}{k}
\end{aligned}
$$

And thus,

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{x^k}{k} \tag{8}$$

Because this is a Taylor Series, we have to determine it's radius of convergence. I will do so by using the ratio test. For the ratio test, we need to find the interval of x where,

$$
\begin{gathered}
\lim_{k \to \infty} \rho < 1 \\
\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| < 1
\end{gathered} \tag{9}
$$

Note if $\rho < 1$, it is guaranteed that the series converges, if it is exactly 1, it may or may not converge. To evaluate this limit, it is as follows,

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^k\,\dfrac{x^{k+1}}{k+1}}{(-1)^{k-1}\,\dfrac{x^k}{k}}\right|$$

We can ignore the -1 exponents because of the absolute value

$$\begin{aligned}
&= \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k\,\dfrac{k}{k+1}}\right| \\
&= \lim_{k\to\infty}|x|\left|\frac{k}{k+1}\right| \\
&= |x|
\end{aligned}$$

So, $\rho < 1$, when $|x| < 1$. This cannot be concluded as the whole radius of convergence as we have to manually check the cases where $\rho = 1$. In this case, this is when $x = \pm 1$.

First checking $x = -1$,

$$\begin{aligned}
\sum_{k=0}^{\infty}(-1)^{k-1}\frac{x^k}{k} &= \sum_{k=0}^{\infty}(-1)^{k-1}\frac{(-1)^k}{k} \\
&= \sum_{k=0}^{\infty}(-1)^{2k-1}\frac{1}{k} \\
&= -\frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots
\end{aligned}$$

This diverges as it is simply the negative of the harmonic series, which diverges.

Checking $x = 1$,

$$\begin{aligned}
\sum_{k=0}^{\infty}(-1)^{k-1}\frac{x^k}{k} &= \sum_{k=0}^{\infty}(-1)^{k-1}\frac{(1)^k}{k} \\
&= \sum_{k=0}^{\infty}(-1)^{k-1}\frac{1}{k} \\
&= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots
\end{aligned}$$

This is an alternating series and we can use the alternating series test to see if it converges. For the alternating series test, there are 2 conditions that are met if the series converges,

$$a_1 \geq a_2 \geq a_3 \geq a_k \geq a_{k+1} \dots \tag{10}$$

$$\lim_{k\to\infty} a_k = 0 \tag{11}$$

It is evident that (10) is satisfied since $\dfrac{1}{k} \geq \dfrac{1}{k+1}$, for all $k > 0$. (11) is also satisfied because $\lim_{k\to\infty}\dfrac{1}{k} = 0$. And thus the Taylor Series converges at $x = 1$ by the alternating series test.

Thus, it can be concluded that $\ln(x+1)$ can be represented in the Taylor Series shown in (8) and has a radius of convergence of $x \in (-1, 1]$. And thus $\ln(2)$ can be evaluated by plugging in $x = 1$.