An Interesting Complex Number Proof

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1 Problem

a. If z_0, z_1, z_2, $\cdots$, z_{n-1} are the nth roots of 1, prove that,

$$z_0 + z_1 + z_2 + \cdots + z_{n-1} = 0$$

b. Also, prove for $n \geq 2$

$$1 + \cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \cdots + \cos\left(\frac{(n-1)\,2\pi}{n}\right) = 0$$

And

$$\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \cdots + \sin\left(\frac{(n-1)\,2\pi}{n}\right) = 0$$

This problem was taken out of Algebra and Geometry by Dunkley, Gilbert, Anderson, Crippin, Davdison, Rachich, and Zirzitto. I solved this problem with my good buddies George Paraschiv and Eric Du.

2 Part (a) using Vieta's Formulas

If we are trying to find the nth roots of 1, we are then dealing with the following polynomial

$$\begin{aligned} z^n &= 1 \\ z^n - 1 &= 0 \end{aligned} \tag{1}$$

According to Veita's Formulas, if we have a polynomial of degree n as follows,

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$$

Then, the sum of the roots is equal to the negative of the coefficient in the $n-1$ term divided by the coefficient of the n term.

$$\sum_{k=0}^{n-1} x_k = \frac{-a_{n-1}}{a_n} \tag{2}$$

Where x_k denotes each root. In our polynomial (1), a_{n-1} is 0 as there is no x^{n-1} term. Thus by (2), we thus have that the sum of the roots of the polynomial is 0. This proves part (a).

3 Part (a) using Complex Numbers

We can rewrite (1) by saying $1 = e^{2k\pi i}$ where k is an integer. Then, we have

$$z^n = e^{2k\pi i} \tag{3}$$

Since the polynomial has degree n, there are n roots and thus, k ranges from 0 to $n-1$ (inclusive). Hence by taking the nth root of both sides in (3),

$$z_k = e^{\frac{2k\pi}{n} i} \tag{4}$$

Now let's let S represent the sum of all the roots,

$$S = \sum_{k=0}^{n-1} z_k$$

Substitute (4)

$$S = \sum_{k=0}^{n-1} e^{\frac{2k\pi}{n} i}$$

Multiply both sides by $e^{\frac{2\pi}{n} i}$

$$Se^{\frac{2\pi}{n} i} = e^{\frac{2\pi}{n} i} \sum_{k=0}^{n-1} e^{\frac{2k\pi}{n} i}$$

We can bring this term into the sum as it is not dependent on k

$$Se^{\frac{2\pi}{n} i} = \sum_{k=0}^{n-1} e^{\frac{2\pi}{n} i} e^{\frac{2k\pi}{n} i}$$

$$Se^{\frac{2\pi}{n} i} = \sum_{k=0}^{n-1} e^{\frac{2\pi}{n}(k+1)i}$$

Adjusting the indices of the sum

$$Se^{\frac{2\pi}{n} i} = \sum_{k=1}^{n} e^{\frac{2k\pi}{n} i}$$

What's neat here is that $e^{\frac{2k\pi}{n} i}$ has the same value when $k = 0$ and $k = n$ because $e^{2\pi i} = e^0$. As a result, we can simply replace the $k = n$ term in the sum to a $k = 0$ term and then the indices of the sum revert back to $k = 0$ and $k = n - 1$.

$$Se^{\frac{2\pi}{n} i} = \sum_{k=0}^{n-1} e^{\frac{2k\pi}{n} i}$$

The right side is simply S.

$$Se^{\frac{2\pi}{n} i} = S$$

This can only be true when $S = 0$, and thus we have proved that the sum of the nth roots of 1 is 0. An important note is that S is not necessarily 0 if $n = 1$ as $e^{2\pi i} = 1$. This is expected since if $n = 1$, there is only one root, and the first root of 1 is not 0. As a result, S will not be 0. This is true for $n \geq 2$, which is important for part b).

4 Proving Part (b) using Part (a)

In Section 3 we showed that the following is true for $n \geq 2$.

$$S = \sum_{k=0}^{n-1} e^{\frac{2k\pi}{n}i} = 0 \tag{5}$$

We can then apply Euler's formula into this result.

$$\begin{aligned} S = \sum_{k=0}^{n-1} e^{\frac{2k\pi}{n}i} &= \sum_{k=0}^{n-1} \left[\text{cis}\left(\frac{2k\pi}{n} \right) \right] \\ &= \sum_{k=0}^{n-1} \left[\cos\left(\frac{2k\pi}{n} \right) + i \sin\left(\frac{2k\pi}{n} \right) \right] \end{aligned}$$

We can then split the sum into the cos and sin terms

$$= \sum_{k=0}^{n-1} \left[\cos\left(\frac{2k\pi}{n} \right) \right] + \sum_{k=0}^{n-1} \left[i \sin\left(\frac{2k\pi}{n} \right) \right]$$

Bring i out of the sigma

$$= \sum_{k=0}^{n-1} \left[\cos\left(\frac{2k\pi}{n} \right) \right] + i \sum_{k=0}^{n-1} \left[\sin\left(\frac{2k\pi}{n} \right) \right] \tag{6}$$

Remember that (6) is equal to S which by (5) is equal to 0. As a result, both the real parts and the imaginary parts have to be equal to 0. In (6) the sum with cosine is the real part and the sum with sine is the imaginary part. By expanding the sums, the 2 equations we needed to show for part (b) arise.

For the real part,

$$\begin{aligned} \sum_{k=0}^{n-1} \left[\cos\left(\frac{2k\pi}{n} \right) \right] &= 0 \\ \cos\left(\frac{2\,(0)\,\pi}{n} \right) + \cos\left(\frac{2\,(1)\,\pi}{n} \right) + \cos\left(\frac{2\,(2)\,\pi}{n} \right) + \cdots + \cos\left(\frac{2\,(n-1)\,\pi}{n} \right) &= 0 \\ 1 + \cos\left(\frac{2\pi}{n} \right) + \cos\left(\frac{4\pi}{n} \right) + \cdots + \cos\left(\frac{2\,(n-1)\,\pi}{n} \right) &= 0 \end{aligned}$$

For the imaginary part,

$$\begin{aligned} \sum_{k=0}^{n-1} \left[\sin\left(\frac{2k\pi}{n} \right) \right] &= 0 \\ \sin\left(\frac{2\,(0)\,\pi}{n} \right) + \sin\left(\frac{2\,(1)\,\pi}{n} \right) + \sin\left(\frac{2\,(2)\,\pi}{n} \right) + \cdots + \sin\left(\frac{2\,(n-1)\,\pi}{n} \right) &= 0 \\ \sin\left(\frac{2\pi}{n} \right) + \sin\left(\frac{4\pi}{n} \right) + \cdots + \sin\left(\frac{2\,(n-1)\,\pi}{n} \right) &= 0 \end{aligned}$$

5 An Intuitive Geometric Approach using Vectors

An interesting point to note is that the proofs in section 2 and 3 are true for the roots of any complex number, not just 1. There is actually something really cool happening behind the scenes. If we were trying to find the nth roots of a complex number, we would so as follows by using de Movire's Theorem. z_k are the kth roots and $r \operatorname{cis}(\theta)$ is the complex number we are trying to find the roots of. We also need to add $2k\pi$ to the angle where k is an integer as all those angles are the same.

$$\begin{aligned} z_k^n &= r \operatorname{cis}(\theta + 2k\pi) \\ z_k &= r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right) \end{aligned}$$

It's important to remember that $k \in [0, n-1]$ as for $k \geq n$ we reach the same roots as we have before. There are 2 things that we can note about the roots z_k. They all have the same modulus of $r^{\frac{1}{n}}$. Furthermore, they are all equally spaced by an angle of $\frac{2\pi}{n}$.

What I am about to is extremely mathematically unrigorous but at the same time very intuitive. We can consider a complex number as being a 2-D vector where the real part is the "x-component" and the imaginary part is the "y-component". We can then add vectors by using the head-to-tail method. As mentioned above, the "vectors" we are dealing with all have the same magnitude and are evenly-spaced by $\frac{2\pi}{n}$. This is one way of defining a regular polygon, i.e all the vectors form a regular polygon. Well, if that's the case, head-to-tail addition forces the sum of those vectors to be 0 as the last vector's head will be at the first vector's tail. So, we've just shown that the sum of the roots of any complex numbers is 0. The proof of part a) is implied as 1 is a complex number in the form of $1 \operatorname{cis}(\theta)$, which in turn proves part b) as shown in Section 3.