Investigation on Binomial Theorem

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1 Introduction

Binomial Theorem is simply a way to expand binomials raised to a power of an integer (both negative and positive). For positive numbers, it is most popularly represented as follows,

$$(a+b)^n = \sum_{k=0}^{n} \left[\binom{n}{k} a^{n-k} b^k \right]$$

Technically, the top limit of integration doesn't have to be n, it can theoretically go to infinity. This is because for all $k > n$, $\binom{n}{k} = 0$ (and as a result $\binom{n}{k} a^{n-k} b^k = 0$. It can be expressed as follows, but the limit usually capped at n as it is all that is necessary.

$$(a+b)^n = \sum_{k=0}^{\infty} \left[\binom{n}{k} a^{n-k} b^k \right]$$

This form is also true when $n < 0$ and as a result is extremely useful if we don't want to have a denominator.

2 An Intuitive Proof

We can rewrite $(a+b)^n$ by simply removing the power and repeating multiplication n times.

$$(a+b)^n = (a+b)(a+b)\ldots\text{n times}\ldots(a+b)(a+b) \tag{1}$$

If we were to distribute this manually, we will have to pick a or b from each term. As a result, the sum of their powers in all terms will have to be n. This is expressed in binomial theorem as each term will have $a^{n-k}b^k$ where k is some natural number less than or equal to n. Suppose I wanted to create the term $a^{n-k}b^k$. The amount of ways in which this can be created in the expansion will yield the coefficient. If we consider the term literally, this means that we have to choose k amount of 'b's (and as a result $n-k$ amount of 'a's) out of n choices. Well, that is simply $\binom{n}{k}$. This thought process is done for all $k \leq n$. It isn't possible for the expansion to contain an exponent greater than n as that would imply greater than n terms. Furthermore, we can't have an exponent less than 0. Thus we simply represent the expansion by summing up all the terms for $0 \leq k \leq n$. Therefore, we end up with binomial theorem,

$$\sum_{k=0}^{n} \left[\binom{n}{k} a^{n-k} b^k \right] = (a+b)^n$$

3 Proof by Induction for Natural Numbers

Using i for the limits of summation as opposed to k, we seek to prove that,

$$\sum_{i=0}^{n}\left[\binom{n}{i}a^{n-i}b^{i}\right]=(a+b)^{n} \tag{2}$$

Step 1: Base case (n = 1)

$$\begin{aligned}LS&=\sum_{i=0}^{1}\left[\binom{1}{i}a^{1-i}b^{i}\right]\\&=\binom{1}{0}a^{1-0}b^{0}+\binom{1}{1}a^{1-1}b^{1}\\&=a+b\end{aligned}$$

$$\begin{aligned}RS&=(a+b)^{1}\\&=a+b\end{aligned}$$

Since $LS=RS$, the base case was chosen to be true.

Step 2: Induction Hypothesis

Assume that the statement (2) is true for $n=k$. Therefore we have,

$$\sum_{i=0}^{k}\left[\binom{k}{i}a^{k-i}b^{i}\right]=(a+b)^{k} \tag{3}$$

Step 3: Prove that it is true for $n=k+1$

Therefore, we need to show,

$$\sum_{i=0}^{k+1}\left[\binom{k+1}{i}a^{k+1-i}b^{i}\right]=(a+b)^{k+1} \tag{4}$$

First I'm going to bring the extra constant a out of the sum

$$\sum_{i=0}^{k+1}\left[\binom{k+1}{i}a^{k+1-i}b^{i}\right]=a\left[\sum_{i=0}^{k+1}\left[\binom{k+1}{i}a^{k-i}b^{i}\right]\right]$$

Next, we're going to bring the $i=k+1$ and $i=0$ indices out of the sum

$$\begin{aligned}&=a\left[\sum_{i=1}^{k}\left[\binom{k+1}{i}a^{k-i}b^{i}\right]+\binom{k+1}{0}a^{k-0}b^{0}+\binom{k+1}{k+1}a^{k-(k+1)}b^{k+1}\right]\\&=a\left[\sum_{i=1}^{k}\left[\binom{k+1}{i}a^{k-i}b^{i}\right]+a^{k}+a^{-1}b^{k+1}\right]\end{aligned}$$

We will expand the choose notation in the sum using Pascal's identity

$$\begin{aligned}&=a\left[\sum_{i=1}^{k}\left(\left[\binom{k}{i}+\binom{k}{i-1}\right]a^{k-i}b^{i}\right)+a^{k}+a^{-1}b^{k+1}\right]\\&=a\left[\sum_{i=1}^{k}\left[\binom{k}{i}a^{k-i}b^{i}\right]+\sum_{i=1}^{k}\left[\binom{k}{i-1}a^{k-i}b^{i}\right]+a^{k}+a^{-1}b^{k+1}\right]\end{aligned}$$

First I'll tackle the left sum. It is extremely similar to our induction hypothesis. Note that if $i = 1$, the summand will be a^k. We already have an a^k and we can "plug it in" to the sum and change the indices of the sum.

$$= a\left[\sum_{i=0}^{k}\left[\binom{k}{i}a^{k-i}b^i\right] + \sum_{i=1}^{k}\left[\binom{k}{i-1}a^{k-i}b^i\right] + a^{-1}b^{k+1}\right]$$

Substitute our induction hypothesis (3)

$$= a\left[(a+b)^k + \sum_{i=1}^{k}\left[\binom{k}{i-1}a^{k-i}b^i\right] + a^{-1}b^{k+1}\right]$$

I will adjust the indices of the sum so that the choose notation will be $\binom{k}{i}$ to match our induction hypothesis

$$\begin{aligned}
&= a\left[(a+b)^k + \sum_{i=0}^{k-1}\left[\binom{k}{(i+1)-1}a^{k-(i+1)}b^{i+1}\right] + a^{-1}b^{k+1}\right] \\
&= a\left[(a+b)^k + \sum_{i=0}^{k-1}\left[\binom{k}{i}a^{k-i-1}b^{i+1}\right] + a^{-1}b^{k+1}\right]
\end{aligned}$$

Take out $a^{-1}b$ from the sum to make the summand match that of the induction hypothesis

$$= a\left[(a+b)^k + a^{-1}b\left(\sum_{i=0}^{k-1}\left[\binom{k}{i}a^{k-i}b^i\right]\right) + a^{-1}b^{k+1}\right]$$

Add and subtract $\binom{k}{k}a^{k-k}b^k$

$$= a\left[(a+b)^k + a^{-1}b\left(\sum_{i=0}^{k-1}\left[\binom{k}{i}a^{k-i}b^i\right] + \binom{k}{k}a^{k-k}b^k - \binom{k}{k}a^{k-k}b^k\right) + a^{-1}b^{k+1}\right]$$

Bring the positive term into the sum and simplify the negative term

$$= a\left[(a+b)^k + a^{-1}b\left(\sum_{i=0}^{k}\left[\binom{k}{i}a^{k-i}b^i\right] - b^k\right) + a^{-1}b^{k+1}\right]$$

Substitute the induction hypothesis again and simplify

$$\begin{aligned}
&= a\left[(a+b)^k + a^{-1}b\left((a+b)^k - b^k\right) + a^{-1}b^{k+1}\right] \\
&= a\left[(a+b)^k + a^{-1}b\,(a+b)^k - a^{-1}b^{k+1} + a^{-1}b^{k+1}\right] \\
&= a\left[(a+b)^k + a^{-1}b\,(a+b)^k\right] \\
&= a\,(a+b)^k + b\,(a+b)^k \\
&= (a+b)^k\,(a+b) \\
&= (a+b)^{k+1}
\end{aligned}$$

This matches (4) which is the equation that we needed to show to prove the statement for $n = k + 1$.

The statement was proven to be true for $n = k + 1$ if it is also true for $n = k$ and the base case was also shown to be true. Therefore (2) (the binomial theorem for natural numbers) was proven by the process of mathematical induction. ■