Centre of Mass of a Draining Soda Can

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1 Introduction

Suppose we have a cylindrical can of height h. Initially, it is completely full of soda (of uniform density). The soda then drains ideally meaning that it is always shaped as a cylinder as well. We want to find the position of the centre of mass of the system as a function of the height of the soda left in the can. We then want to see when the centre of mass is at it's lowest. This is pretty fascinating since if we think about it, when the can is full of soda, the centre of mass is at the centre of the can. The same can be said when the can is empty. However, initially, as soda drains out, it is intuitive that the centre of mass is lowering. The question now is, at what point does it start increasing – which is perhaps the less intuitive part of this problem. The variables we can work with are the height of the can, the mass of the can, and the mass of the soda when the can is full.

This problem is adapted from Fundamentals of Physics 9th edition by Halliday, Resnick and Walker.

2 Deriving a function

We will set up a y axis that is parallel to the height of the can/soda. By symmetry, we know that the centre of mass of both the can and soda will be on the y axis (this set's the x and z-axis to be constant). The only component of the centre of mass that changes is the y-component. We then consider then find the centre of mass of the system by considering the centre of masses of each individual portion. In the following equation, subscripts of c indicate the can and s are for the soda.

$$y_{com} = \frac{1}{m_c + m_s}\left(m_c y_{com,c} + m_s y_{com,s}\right) \tag{1}$$

Let's take a look at each variable in(1) and try to make each component strictly a function of the height of the soda or of any given constants. For simplicity, we're going to denote the height of the soda as y to indicate that it's variable. Let's first observe the centre of mass components. Because the can does not change shape as the soda drains (at least I wouldn't hope so), we know that the centre of mass of the can is simply half of the height of the can.

$$y_{com,c} = \frac{h_c}{2} \tag{2}$$

With similar logic, the centre of mass of the soda is half of it's instantaneous height (don't forget that this height is changing). Because this is changing, we're going to done this as y_s

$$y_{com,s} = \frac{y}{2} \tag{3}$$

The mass of the can is given. We just need to find an expression for the mass of the soda. Because the problem states that the density is uniform, we can determine an expression of the mass of the soda using the definition of linear density (μ). By definition, we have:

$$\begin{aligned}\mu &= \frac{m}{L} \\ m &= \frac{\mu}{L}\end{aligned} \tag{4}$$

Now, how do we apply what we know to determine an expression for mass?. Length will simply be the instantaneous height of the soda. For linear density, we can take into consideration the initial condition as the density can't change during the draining. This is useful as we are given the mass of the soda when the can is full. We then have,

$$\mu = \frac{m_{s,i}}{h_c} \tag{5}$$

Substituting (5) and the fact that $L = y$, into (4), we have

$$m_s = \frac{m_{s,i}}{h_c} y \tag{6}$$

Now, we can substitute (2), (3), and (6) into (1) to represent y_{com} solely as a function of y

$$\begin{aligned}y_{com}(y) &= \frac{1}{m_c + \frac{m_{s,i}}{h_c} y}\left(m_c \frac{h_c}{2} + \left(\frac{m_{s,i}}{h_c} y\right)\frac{y}{2}\right) \\ &= \frac{1}{m_c + \frac{m_{s,i}}{h_c} y}\left(\frac{m_c h_c}{2} + \frac{m_{s,i}}{2h_c} y^2\right)\end{aligned} \tag{7}$$

3 Finding the minimum

This problem is a pretty straight forward calculus problem actually. All we have to do is simply take the derivative as set it to 0.

To make the algebra a bit more organized, we will rewrite $\frac{m_{x,i}}{h_c}$ as μ without the subscript as it's obvious that it's that of the soda. When achieve a solution, we will then re-substitute the values back in. Thus we'll rewrite (7) as follows,

$$\begin{aligned}y_{com}(y) &= \frac{1}{m_c + \mu y}\left(\frac{m_c h_c}{2} + \frac{\mu}{2} y^2\right) \\ &= \frac{m_c h_c + \mu y^2}{2(m_c + \mu y)}\end{aligned}$$

We can then simply take the derivative and set it equal to 0.

$$\begin{aligned}y'_{com}(y) &= \frac{1}{2}\left[\frac{2\mu y (m_c + \mu y) - \mu (m_c h_c + \mu y^2)}{(m_c + \mu y)^2}\right] \\ &= \frac{1}{2(m_c + \mu y)^2}\left(2\mu y m_c + 2\mu^2 y^2 - \mu m_c h_c - \mu^2 y^2\right) \\ &= \frac{1}{2(m_c + \mu y)^2}\left(\mu^2 y^2 + 2\mu m_c y - \mu m_c h_c\right) \\ &= \frac{\mu}{2(m_c + \mu y)^2}\left(\mu y^2 + 2 m_c y - m_c h_c\right)\end{aligned}$$

Now we need to set it to 0. We only need to consider the quadratic as is is the only factor that will force the derivative to be 0. We don't need to consider when the denominator is 0 because we are dealing with a rational function, and that will simply tell us the vertical asymptote. We can simply use the quadratic formula to produce our solution.

$$\begin{aligned} y &= \frac{-2m_c \pm \sqrt{4m_c^2 - 4\mu\left(-m_c h_c\right)}}{2\mu} \\ &= \frac{-2m_c \pm 2\sqrt{m_c^2 + \mu m_c h_c}}{2\mu} \\ &= \frac{-m_c \pm \sqrt{m_c\left(m_c + \mu h_c\right)}}{\mu} \end{aligned}$$

However, the height of the soda remaining, must be positive. Since the denominator is positive, we only consider the positive square root. Therefore,

$$y = \frac{-m_c + \sqrt{m_c\left(m_c + \mu h_c\right)}}{\mu} \tag{8}$$

We finally substitute the definition of density into (8) to get an answer purely as a function of given values.

$$\begin{aligned} y &= \frac{-m_c + \sqrt{m_c\left(m_c + \frac{m_{s,i}}{h_c} h_c\right)}}{\frac{m_{s,i}}{h_c}} \\ &= \frac{h_c\left(-m_c + \sqrt{m_c\left(m_c + m_{s,i}\right)}\right)}{m_{s,i}} \end{aligned}$$