Differentiating Under the Integral Sign

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1 Introduction

This technique of integration a method used to evaluate definite integrals when we can't find the anti-derivative analytically. Suppose we have a definite integral,

$$I = \int_a^b f(x)\, dx$$

Where we can't find the anti-derivative of $f(x)$. The trick is to introduce a parameter (I'll use t) within $f(x)$ to make it a function of 2 variables. As a result, I is also variable based on the value of t. We then have,

$$I(t) = \int_a^b f(x,t)\, dx$$

Furthermore, we also want to make sure that at some value $t = t_0$, $I(t_0) = I = \int_a^b f(x)\, dx$.

The next step is differentiate both sides with respect to the introduced parameter, which in our case is t. Thus,

$$I'(t) = \frac{d}{dt} \int_a^b f(x,t)\, dx$$

Leibniz's rule of differentiating under the integral sign allows us to bring the derivative to inside the integral sign (as the name suggests). However, we have to convert it to a partial derivative as the integrand is a function of both x and t. Thus,

$$I'(t) = \int_a^b \frac{\partial}{\partial t} \left[f(x,t) \right] dx$$

When performing this technique, it is imperative that the choice of "location" of the parameter allows the partial derivative of the function to have its anti-derivative determined analytically. Let's suppose that this is the case and that the antiderivative is $F(x,t)$. Then by the fundamental theorem of calculus we have,

$$I'(t) = F(b,t) - F(a,t)$$

Since a and b are constants, F is simply a function of t. We can then take the indefinite integral to find a function for $I(t)$. Again, a good choice of t will allow for this integral to be determinable analytically. After the indefinite integral is found, and initial condition is needed to actually determine the function $I(t)$. This will often be determined by substituting special cases of t in the paramaterized function to cancel a lot of terms.

2 A Putnam Problem

The goal is to simply evaluate the following definite integral,

$$I = \int_0^1 \frac{\ln(x+1)}{x^2+1} dx$$

I've plugged this integral into various algebra engines (such as WolframAlpha, Maple, Integral-Calculator) and they were all unable to produce an anti-derivative. This is a scenario where the technique of differentiating under the integral sign may come in handy. Let's introduce the parameter t such that,

$$I(t) = \int_0^1 \frac{\ln(tx+1)}{x^2+1} dx$$

Thus, we note that the definite integral we are seeking to evaluate is equal to $I(1)$. We also see that if $t = 0$, the integrand will become 0 ($\ln 1 = 0$) and thus the entire definite integral will evaluate to 0. Then we also know that $I(0) = 0$. This is the initial condition that we will use in the end of the problem.

We will continue with the process and differentiate both sides with respect to t.

$$I'(t) = \frac{d}{dt} \int_0^1 \frac{\ln(tx+1)}{x^2+1} dx$$

Bring the derivative into the integral sign and convert to a partial derivative

$$= \int_0^1 \frac{\partial}{\partial t} \left[\frac{\ln(tx+1)}{x^2+1} \right] dx$$

Bring out the x terms as they are constant relative to t

$$= \int_0^1 \frac{1}{x^2+1} \frac{\partial}{\partial t} \left[\ln(tx+1) \right] dx$$

Differentiate with respect to t

$$= \int_0^1 \frac{1}{x^2+1} \frac{1}{tx+1} (x) \, dx$$

$$= \int_0^1 \frac{x}{(x^2+1)(tx+1)} \tag{1}$$

This has to be expanded using partial fraction decomposition. We have,

$$\frac{x}{(x^2+1)(tx+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{tx+1} \tag{2}$$

$$= \frac{Atx^2 + Ax + Btx + B + Cx^2 + C}{(x^2+1)(tx+1)}$$

$$= \frac{(At+C)x^2 + (A+Bt)x + B + C}{(x^2+1)(tx+1)}$$

We then have a system of 3 equations

$$At + C = 0 \tag{3}$$

$$A + Bt = 1 \tag{4}$$

$$B + C = 0 \tag{5}$$

From, (3) and (5), we can see that $At = B$. Substituting into (4), we have

$$\begin{aligned} A + (At)\,t &= 1 \\ A\left(1+t^2\right) &= 1 \\ A &= \frac{1}{1+t^2} \end{aligned}$$

Therefore,

$$B = \frac{t}{1+t^2}$$

And

$$C = \frac{-t}{1+t^2}$$

Substituting into (2) and then into (1), we have

$$I'(t) = \int_0^1 \left(\frac{\frac{1}{1+t^2}x + \frac{t}{1+t^2}}{x^2+1} - \frac{\frac{t}{1+t^2}}{tx+1} \right) dx$$

Factor out the $\frac{1}{1+t^2}$

$$\begin{aligned} &= \frac{1}{1+t^2} \int_0^1 \left(\frac{x+t}{x^2+1} - \frac{t}{tx+1} \right) dx \\ &= \frac{1}{1+t^2} \int_0^1 \left(\frac{x}{x^2+1} + \frac{t}{x^2+1} - \frac{t}{tx+1} \right) dx \end{aligned}$$

All of these terms are fairly straight forward integrals with respect to x, which was the goal of differentiating under the integral sign. We can then simply find the anti-derivatives and use the Fundamental Theorem of Calculus to find $I'(t)$

$$\begin{aligned} &= \frac{1}{1+t^2} \left[\frac{1}{2}\ln\left(x^2+1\right) + t\arctan(x) - \ln(tx+1) \Big|_{x=0}^{x=1} \right] \\ &= \frac{1}{1+t^2} \left[\frac{1}{2}\ln\left(1^2+1\right) + t\arctan(1) - \ln(t(1)+1) - \frac{1}{2}\ln\left(0^2+1\right) - t\arctan(0) + \ln(t(0)+1) \right] \\ &= \frac{1}{1+t^2} \left[\frac{1}{2}\ln(2) + \frac{\pi}{4}t - \ln(t+1) \right] \end{aligned}$$

For reasons that we will soon see, I will integrate both sides with respect to t and setting the bounds of integration to be from 0 to 1 to yield $I(1)$

$$\begin{aligned} I(1) &= \int_0^1 \frac{1}{1+t^2} \left[\frac{1}{2}\ln(2) + \frac{\pi}{4}t - \ln(t+1) \right] dt \\ &= \frac{1}{2}\ln(2) \int_0^1 \frac{1}{1+t^2} dt + \int_0^1 \frac{\pi}{4}\frac{t}{1+t^2} dt - \int_0^1 \frac{\ln(t+1)}{1+t^2} dt \end{aligned}$$

The final term is simply $I(1)$ as exchanging the x's for t's doesn't change the integral. Therefore, we have,

$$\begin{aligned}
I(1) &= \frac{1}{2}\ln(2)\int_0^1 \frac{1}{1+t^2}dt + \int_0^1 \frac{\pi}{4}\frac{t}{1+t^2}dt - I(1) \\
2I(1) &= \frac{1}{2}\ln(2)\left[\arctan(t)\Big|_{t=0}^{t=1}\right] + \frac{\pi}{4}\left[\frac{1}{2}\ln\left(1+t^2\right)\Big|_{t=0}^{t=1}\right] \\
2I(1) &= \frac{1}{2}\ln(2)\left[\frac{\pi}{4}\right] + \frac{\pi}{4}\left[\frac{1}{2}\ln(2)\right] \\
2I(1) &= \frac{\pi}{4}\ln(2) \\
I(1) &= \frac{\pi}{8}\ln(2)
\end{aligned}$$

At the start the choice of parameter meant that the definite integral we were seeking to evaluate was equal to $I(1)$. Therefore,

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \frac{\pi}{8}\ln(2)$$

3 The Dirichlet Integral

The Dirichlet Integral is as follows,

$$I = \int_0^\infty \frac{\sin(x)}{x}dx$$

The indefinite integral cannot be found, so our first step is to introduce a parameter as follows,

$$I(t) = \int_0^\infty e^{-tx}\frac{\sin(x)}{x}dx$$

This integral converges for $t \geq 0$. The choice of parameter means that the value of the definite integral is equal to $I(0)$. Furthermore, we should note that as t approaches ∞, the value of $I(t)$ is 0 because the integrand will simply be 0. We proceed by differentiating under the integral sign to find $I'(t)$.

$$I'(t) = \frac{d}{dt}\int_0^\infty e^{-tx}\frac{\sin(x)}{x}dx$$

Bringing the derivative to inside the integral and convert it to a partial derivative,

$$= \int_0^\infty \frac{\partial}{\partial t}\left[e^{-tx}\frac{\sin(x)}{x}\right]dx$$

Because we are differentiating with respect to t, the $\sin(x)$ and x terms can be brought out.

$$\begin{aligned}
&= \int_0^\infty \frac{\sin(x)}{x}\frac{\partial}{\partial t}\left[e^{-tx}\right]dx \\
&= \int_0^\infty \frac{\sin(x)}{x}\left(-xe^{-tx}\right)dx \\
&= -\int_0^\infty \sin(x)e^{-tx}dx
\end{aligned}$$

I'll introduce the variable a so we can convert infinity into a limit

$$= \lim_{a\to\infty}\left[-\int_0^a \sin(x)e^{-tx}dx\right] \tag{6}$$

I will now find the anti-derivative using integration by parts and then I'll substitute it into (6) and use the Fundamental Theorem of Calculus. We set up the integration by parts as follows.

$$\begin{aligned} u &= \sin(x) & dv &= e^{-tx} \\ du &= \cos(x)dx & v &= -\frac{1}{t}e^{-tx} \end{aligned}$$

We then start our integration by parts.

$$\begin{aligned} \int udv &= uv - \int vdu \\ \int \sin(x)e^{-tx}dx &= \frac{-e^{-tx}\sin(x)}{t} + \int -\frac{1}{t}e^{-tx}\cos(x)dx \\ \int \sin(x)e^{-tx}dx &= \frac{-e^{-tx}\sin(x)}{t} - \frac{1}{t}\int e^{-tx}\cos(x)dx \end{aligned}$$

We have to apply integration by parts again for the right integral. We set it up as follows.

$$\begin{aligned} u &= \cos(x) & dv &= e^{-tx} \\ du &= -\sin(x)dx & v &= -\frac{1}{t}e^{-tx} \end{aligned}$$

Therefore,

$$\begin{aligned} \int \sin(x)e^{-tx}dx &= \frac{-e^{-tx}\sin(x)}{t} - \frac{1}{t}\left[\frac{-e^{-tx}\cos(x)}{t} + \int(-\sin(x))\left(-\frac{1}{t}e^{-tx}\right)dx\right] \\ \int \sin(x)e^{-tx}dx &= \frac{-e^{-tx}\sin(x)}{t} - \frac{1}{t}\left[\frac{-e^{-tx}\cos(x)}{t} + \frac{1}{t}\int \sin(x)e^{-tx}dx\right] \end{aligned}$$

Distributing

$$\int \sin(x)e^{-tx}dx = \frac{-e^{-tx}\sin(x)}{t} - \frac{-e^{-tx}\cos(x)}{t^2} - \frac{1}{t^2}\int \sin(x)e^{-tx}dx$$

Notice that the initial integral pops up again, and we can move it over to the left side.

$$\begin{aligned} \left(1+\frac{1}{t^2}\right)\int \sin(x)e^{-tx}dx &= \frac{-te^{-tx}\sin(x) - e^{-tx}\cos(x)}{t^2} \\ \left(\frac{t^2+1}{t^2}\right)\int \sin(x)e^{-tx}dx &= \frac{-e^{-tx}\left(t\sin(x)+\cos(x)\right)}{t^2} \\ \int \sin(x)e^{-tx}dx &= \frac{-e^{-tx}\left(t\sin(x)+\cos(x)\right)}{t^2+1} \end{aligned} \tag{7}$$

Substituting (7) into (6) after considering the Fundamental Theorem of Calculus we get,

$$\begin{aligned} I'(t) &= \lim_{a\to\infty}\left[-\frac{-e^{-tx}\left(t\sin(x)+\cos(x)\right)}{t^2+1}\Bigg|_{x=0}^{x=a}\right] \\ &= \lim_{a\to\infty}\left[\frac{e^{-tx}\left(t\sin(x)+\cos(x)\right)}{t^2+1}\Bigg|_{x=0}^{x=a}\right] \\ &= \lim_{a\to\infty}\left(\frac{e^{-ta}t\sin(a)}{t^2+1}\right) + \lim_{a\to\infty}\left(\frac{e^{-ta}\cos(a)}{t^2+1}\right) - \frac{e^0\left(t\sin(0)+\cos(0)\right)}{t^2+1} \end{aligned}$$

Bringing the t terms out of the limits

$$= \frac{t}{t^2+1} \lim_{a\to\infty} \left(e^{-ta}\sin(a)\right) + \frac{1}{t^2+1} \lim_{a\to\infty} \left(e^{-ta}\cos(a)\right) - \frac{1}{t^2+1} \tag{8}$$

Squeeze theorem can be used to show that both limits in the expression evaluate to 0. We know that for all values of a the absolute value of sine and cosine are always going to be less than 1.

$$-1 \leq \sin(a) \leq 1 \qquad\qquad -1 \leq \cos(a) \leq 1$$

We can then multiply all sides of the inequality by e^{-ta}

$$-e^{-ta} \leq e^{-ta}\sin(a) \leq e^{-ta} \qquad\qquad -e^{-ta} \leq e^{-ta}\cos(a) \leq e^{-ta}$$

Take the limit as $a \to \infty$ on all sides

$$\begin{aligned}\lim_{a\to\infty} -e^{-ta} \leq \lim_{a\to\infty} e^{-ta}\sin(a) \leq \lim_{a\to\infty} \leq e^{-ta} \\ 0 \leq \lim_{a\to\infty} e^{-ta}\sin(a) \leq 0\end{aligned} \qquad\qquad \begin{aligned}\lim_{a\to\infty} -e^{-ta} \leq \lim_{a\to\infty} e^{-ta}\cos(a) \leq \lim_{a\to\infty} e^{-ta} \\ 0 \leq \lim_{a\to\infty} e^{-ta}\cos(a) \leq 0\end{aligned}$$

And thus by Squeeze Theorem, both limits are 0. We can substitute these into (8).

$$\begin{aligned} I'(t) &= \frac{t}{t^2+1}(0) + \frac{1}{t^2+1}(0) - \frac{1}{t^2+1} \\ &= \frac{-1}{t^2+1} \end{aligned}$$

Integrating, we have

$$I(t) = -\arctan(t) + C$$

As mentioned above, the limit of $I(t)$ when $t \to \infty$ is 0. Taking the limit of both sides as $t \to \infty$ we have,

$$\begin{aligned} 0 &= -\frac{\pi}{2} + C \\ C &= \frac{\pi}{2} \end{aligned}$$

Therefore,

$$I(t) = -\arctan(t) + \frac{\pi}{2}$$

We seek $I(0)$

$$\begin{aligned} I(0) &= -\arctan(0) + \frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

At the start the choice of parameter meant that the definite integral we were seeking to evaluate was equal to $I(0)$. Therefore,

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$