Angular Collision

Arthur Xu

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1 Problem

Four thins uniform rods each of mass M and length $d = 0.5$m, are rigidly connected at one end to a vertical axis to form a turnstile. The angle between the rods is $90°$. The turnstile rotates clockwise around the axle without friction when viewed from above. The axle is mounted to the floor. The initial angular velocity of the turnstile is -2 rads^{-1} where the negative sign indicates the clockwise rotation. a mud ball of mass $m = \frac{1}{3}M$ and initial speed $v_i = 12$ms^{-1} is thrown in the plane of the turnstile arms and hits the end of one arm at an angle of $30°$ to the longitudinal axis of the arm. The mud ball sticks to the turnstile arm. What is the final angular velocity of the ball-turnstile system.

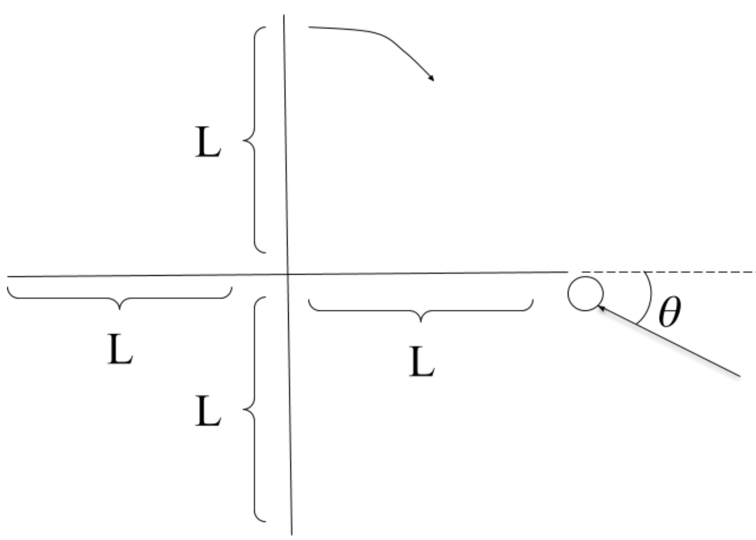

Figure 1: Turnstile is moving clockwise and the mud ball's initial velocity is directed in a way that will cause counter clockwise angular velocity

2 Conservation of Angular Momentum

In this problem, there are no external forces acting on the system. Thus, the conservation of angular momentum applies. Initially, the 2 components (ball and turnstile) are separate entities and each have their own angular momentum and velocity which will be denoted as a subscript t and b. After the collision, they stick together as a system and has a singular angular momentum and velocity which will be denoted as a subscript sys. Using the conservation of angular momentum, we can then produce the following equation.

Note that angular momentum and velocity are technically vectors, but they are all in one dimension (CW or CCW), so we're just going to omit the vector symbols.

$$\begin{aligned} L_i &= L_f \\ L_{b_i} + L_{t_i} &= L_{sys} \\ I_b\omega_{b_i} + I_t\omega_{t_i} &= I_{sys}\omega_{sys} \end{aligned} \tag{1}$$

3 Rotational Inertia

There are two structures that we need to find the rotational inertia of. Firstly, the ball can be simply treated as a particle. Thus we have,

$$\begin{aligned} I_b &= m_b r_b^2 \\ I_b &= \frac{M}{3}d^2 \\ I_b &= \frac{M}{3}0.5^2 \\ I_b &= \frac{M}{12} \end{aligned} \tag{2}$$

For the turnstile, we remember that its rotational inertia is the sum of the rotational inertias of all its parts. In this case we're going to treat it as 2 rods each of mass $2M$ and length 1 rotating about their centre. I will then proceed to derive an equation for the rotational inertia of a rod spinning around it's centre. By definition,

$$I = \int r^2 dm \tag{3}$$

This integral basically means that we're summing all the dm's that this object contains. To find dm by considering the linear density (μ) of the rod in the differential form,

$$\begin{aligned} \mu &= \frac{dm}{dr} \\ dm &= \mu dr \end{aligned} \tag{4}$$

For future reference, we can also find the linear density by considering the rod as a whole

$$\mu = \frac{m}{L} \tag{5}$$

In(3), the variable r is the distance of each dm from the point of rotation. As a result, for the integral we expect the indices of integration to be 0 to $\frac{L}{2}$ and we multiply the whole integral by 2. We then substitute (4), and we have an integral that is fairly straight forward. Note that the linear density is a constant as it doesn't change if the distance changes.

$$\begin{aligned} I &= 2\int_0^{\frac{L}{2}} r^2 \left(\mu dr\right) \\ &= 2\mu \int_0^{\frac{L}{2}} r^2 dr \\ &= 2\mu \left(\left. \frac{r^3}{3} \right|_0^{\frac{L}{2}} \right) \\ &= \frac{2\mu L^3}{\left(2^3\right)\left(3\right)} \\ &= \frac{\mu L^3}{12} \end{aligned}$$

Substituting in (5) we get

$$I = \frac{mL^2}{12} \tag{6}$$

This is the general equation for one rod rotating about its centre. In our case, we need to double this as we have 2 rods. Furthermore, we'll substitute that $L = 2d = 1$ and $m = 2M$. We then have,

$$\begin{aligned} I_t &= 2\left(\frac{(2M)\left(1^2\right)}{12}\right) \\ &= \frac{M}{3} \end{aligned} \tag{7}$$

After the collision, the system simply a total rotational inertia which is the sum of its parts (the ball and turnstile can now be viewed as one object). So we can simply sum (2) and (7) to find I_{sys}.

4 Solution

In (1), the only variable that is missing is ω_{b_i} as ω_{sys} is the value we are trying to solve for. The ball has a speed, we simply need to convert the speed into angular velocity. Angular velocity is related to **tangential** velocity as follows,

$$\omega_b = \frac{v_{b_\perp}}{r} \tag{8}$$

Due to the geometry of the problem, we can use basic vector mathematics to deduce the tangential component of the velocity as follows,

$$v_{b_\perp} = v_{b_i} \sin(\theta) \tag{9}$$

We can then substitute (9) into (8) while considering that $r = d = 0.5$ as the radius of the circle that the ball is travelling on at the instant right before the collision is simply the length of a rod. By observing the diagram, we can also observe that the tangential component will cause a counter clockwise angular velocity. So, we don't need to add a negative sign. Therefore,

$$\begin{aligned} \omega_b &= \frac{v_{b_i}\sin(\theta)}{d} \\ &= \frac{12\sin(30^\circ)}{0.5} \\ &= 12 \text{ rads}^{-1} \end{aligned} \tag{10}$$

By sheer coincidence, the angular velocity and initial speed happen to have the same value. (1) can be rearranged to have ω_{sys} be expressed solely as a function of known values.

$$\begin{aligned} \omega_{sys} &= \frac{I_b\omega_{b_i} + I_t\omega_{t_i}}{I_{sys}} \\ &= \frac{I_b\omega_{b_i} + I_t\omega_{t_i}}{I_b + I_t} \end{aligned}$$

We can then substitute in our values noticing that the M's all cancel out.

$$\begin{aligned} \omega_{sys} &= \frac{\left(\frac{M}{12}\right)(12) + \left(\frac{M}{3}\right)(-0.8)}{\frac{M}{12} + \frac{M}{3}} \\ &= \frac{1 - \frac{0.8}{3}}{\frac{1}{12} + \frac{1}{3}} \\ &= +0.8 \text{ rads}^{-1} \end{aligned}$$

The positive sign indicates that the system starts moving at 0.8 rads^{-1} in the counter clockwise direction as clockwise was established to be negative in the problem. So we can see that the ball not only slows the turnstile, but the mass (and thus rotational inertia) was significant enough to cause the turnstile in the opposite direction.