Spring Break Integrals

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Integral 1

Evaluate:

$$\int_0^{\pi} \frac{(x+1)\sin(x)}{3+\cos^2(x)}dx \tag{1}$$

Solution:

Let's start by making the substitution $u = \pi - x$, which implies $x = \pi - u$. As a result, $du = -dx$.

For the bounds of integration, when $x = 0$, $u = \pi$ and when $x = \pi$, $u = 0$.

After making the substitutions, note what happens to the sine and cosine terms:

$$\begin{aligned} \sin(x) &= \sin(\pi - u) \\ &= \sin(u) \end{aligned} \tag{2}$$

$$\begin{aligned} \cos^2(x) &= (\cos(\pi - u))^2 \\ &= (-\cos(u))^2 \\ &= \cos^2(u) \end{aligned} \tag{3}$$

Substituting into (2) and (3) into (1) yields:

$$\int_0^{\pi} \frac{(x+1)\sin(x)}{3+\cos^2(x)}dx = \int_{\pi}^{0} \frac{(\pi - u + 1)\sin(u)}{3+\cos^2(u)}(-du)$$

The negative of the differential can be cancelled with the switching of the bounds of integration

$$= \int_0^{\pi} \frac{(\pi - u + 1)\sin(u)}{3+\cos^2(u)}du$$

Split this integral such that a second term of the initial integral pops up. This was the purpose of the initial substitution.

$$\begin{aligned} \int_0^{\pi} \frac{(x+1)\sin(x)}{3+\cos^2(x)}dx &= \int_0^{\pi} \frac{(\pi+2)\sin(u)}{3+\cos^2(u)}du - \int_0^{\pi} \frac{(u+1)\sin(u)}{3+\cos^2(u)}du \\ I &= \int_0^{\pi} \frac{(\pi+2)\sin(u)}{3+\cos^2(u)}du - I \\ 2I &= (\pi+2)\int_0^{\pi} \frac{\sin(u)}{3+\cos^2(u)}du \end{aligned} \tag{4}$$

This simplified integral becomes straight forward after the substitution $t = \cos(u)$. As a result, $dt = -\sin(u)\,du$.

For the bounds of integration, when $u = 0$, $t = 1$. When $u = \pi$, $t = -1$. Substituting into (4),

$$2I = (\pi + 2)\int_{1}^{-1} \frac{-1}{3 + t^2} dt$$

Cancel the negative 1 by swapping the bounds of integration. Also, divide both sides by 2,

$$\begin{aligned}
I &= \left(\frac{\pi + 2}{2}\right)\int_{-1}^{1} \frac{1}{3 + t^2} dt \\
&= \left(\frac{\pi + 2}{2}\right)\int_{-1}^{1} \frac{1}{3\left(1 + \left(\frac{1}{\sqrt{3}}\right)^2\right)} dt \\
&= \left(\frac{\pi + 2}{6}\right)\int_{-1}^{1} \frac{1}{1 + \left(\frac{1}{\sqrt{3}}\right)^2} dt
\end{aligned}$$

This is just inverse tangent after another quick substitution. Using the Fundamental Theorem of Calculus:

$$\begin{aligned}
I &= \left(\frac{\pi + 2}{6}\right)\left(\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}t\right)\Bigg|_{t=-1}^{t=1}\right) \\
&= \left(\frac{\pi + 2}{2\sqrt{3}}\right)\left(\arctan\left(\frac{1}{\sqrt{3}}t\right) - \arctan\left(\frac{-1}{\sqrt{3}}t\right)\right) \\
&= \left(\frac{\pi + 2}{2\sqrt{3}}\right)\left(\left(\frac{\pi}{6}\right) - \left(-\frac{\pi}{6}\right)\right) \\
&= \frac{\pi(\pi + 2)}{6\sqrt{3}}
\end{aligned}$$

Integral 2

Evaluate:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{x\arctan(x)}{1 + e^{\tan(x)}} dx \tag{5}$$

Solution:

Note that the integrand mainly contains odd functions. In addition, the upper bound is the negative of the lower bound. A reasonable substitution is $x = -t$ (which implies $dx = -dt$). This also causes the bounds of integration to be flipped. Applying the substitution into (5),

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{x\arctan(x)}{1 + e^{\tan(x)}} dx = \int_{\sqrt{3}}^{-\sqrt{3}} \frac{(-t)\arctan(-t)}{1 + e^{\tan(-t)}}(-dt)$$

Bringing the negative out of inverse tan and tan because they are odd functions:

$$= \int_{\sqrt{3}}^{-\sqrt{3}} \frac{(-t)(-\arctan(t))}{1 + e^{-\tan(t)}}(-dt)$$

There are 4 negative signs if we also include the switching of the bounds of integration. Cancelling them in pairs yields:

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{t \arctan(t)}{1 + e^{-\tan(t)}} dt$$

Multiply the numerator and denominator by $e^{\tan(t)}$. Note that this makes the denominator the same as the original integrand.

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{t \arctan(t)\, e^{\tan(t)}}{e^{\tan(t)} + 1} dt$$

Add and subtract 1 as follows:

$$\begin{aligned} &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{t \arctan(t) \left(e^{\tan(t)} + 1 - 1\right)}{e^{\tan(t)} + 1} dt \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{t \arctan(t) \left(e^{\tan(t)} + 1\right)}{e^{\tan(t)} + 1} dt - \int_{-\sqrt{3}}^{\sqrt{3}} \frac{t \arctan(t)}{e^{\tan(t)} + 1} dt \end{aligned} \tag{6}$$

The second integral in (6) is the same as the original integral, (5). The first integral simplifies by cancelling out the numerator with the denominator.

$$\begin{aligned} I &= \int_{-\sqrt{3}}^{\sqrt{3}} t \arctan(t)\, dt - I \\ 2I &= \int_{-\sqrt{3}}^{\sqrt{3}} t \arctan(t)\, dt \end{aligned} \tag{7}$$

This integral can be evaluated by parts as follows:

$$\begin{aligned} u &= \arctan(t) & dv &= t dt \\ du &= \frac{1}{1+t^2} dt & v &= \frac{1}{2} t^2 \end{aligned}$$

Integrating:

$$\begin{aligned} 2I &= uv\Big|_{t=-\sqrt{3}}^{t=\sqrt{3}} - \int_{-\sqrt{3}}^{\sqrt{3}} v du \\ I &= \frac{1}{2}\left[\frac{x^2 \arctan(x)}{2}\Big|_{t=-\sqrt{3}}^{t=\sqrt{3}} - \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{2} t^2 \frac{1}{1+t^2} dt\right] \\ &= \frac{1}{2}\left[\frac{\left(\sqrt{3}\right)^2 \arctan\left(\sqrt{3}\right)}{2} - \frac{\left(-\sqrt{3}\right)^2 \arctan\left(-\sqrt{3}\right)}{2} - \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{t^2}{1+t^2} dt\right] \end{aligned}$$

Evaluating the constant term, and adding and subtracting 1 to the numerator of the integral:

$$= \frac{1}{2}\left[\left(\frac{3}{2}\right)\left(\frac{\pi}{3}\right) - \left(\frac{3}{2}\right)\left(\frac{-\pi}{3}\right) - \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{t^2+1-1}{1+t^2} dt\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\pi - \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} 1 - \frac{1}{1+t^2}dt\right] \\
&= \frac{1}{2}\left[\pi - \frac{1}{2}\left(t - \arctan(t)\Big|_{t=-\sqrt{3}}^{t=\sqrt{3}}\right)\right] \\
&= \frac{1}{2}\left[\pi - \frac{1}{2}\left(\sqrt{3} - \arctan\left(\sqrt{3}\right) - \left(-\sqrt{3}\right) + \arctan\left(-\sqrt{3}\right)\right)\right] \\
&= \frac{1}{2}\left[\pi - \frac{1}{2}\left(2\sqrt{3} - \frac{2\pi}{3}\right)\right] \\
&= \frac{1}{2}\left(\frac{4\pi}{3} - \sqrt{3}\right) \\
&= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}
\end{aligned}
$$

Integral 3

Evaluate:

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} x \ln\left(1 + e^{x\sqrt{1-x^2}}\right) dx \tag{8}$$

Solution:

Again, I'll start with the substitution $u = -x$ which means $du = -dx$. Like the second integral, the bounds of integration are just negatives of each other, so this substitution simply switch the bounds. Substituting into (8)

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} x \ln\left(1 + e^{x\sqrt{1-x^2}}\right) dx = \int_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} -u \ln\left(1 + e^{-u\sqrt{1-(-u)^2}}\right) (-du)$$

Cancel one of the negatives (from the u or du) by swapping the bounds of integration:

$$= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} -u \ln\left(1 + e^{-u\sqrt{1-u^2}}\right) du$$

Rewrite the inside of the logarithm

$$
\begin{aligned}
&= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} -u \ln\left(1 + \frac{1}{e^{u\sqrt{1-u^2}}}\right) du \\
&= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} -u \ln\left(\frac{e^{u\sqrt{1-u^2}} + 1}{e^{u\sqrt{1-u^2}}}\right) du
\end{aligned}
$$

The logarithm can be split using logarithm rules which allows the integral to be split into 2, one of which is the original integral

$$\begin{aligned}\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} x\ln\left(1+e^{x\sqrt{1-x^2}}\right)dx &= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} -u\left(\ln\left(e^{u\sqrt{1-u^2}}+1\right)-\ln\left(e^{u\sqrt{1-u^2}}\right)\right)du \\ I &= -\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} -u\ln\left(e^{u\sqrt{1-u^2}}+1\right)du + \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} u\ln\left(e^{u\sqrt{1-u^2}}\right)du \qquad (9)\end{aligned}$$

The left integral in (9) is the same as the original integral just with a dummy variable subbed in. The right integral can be evaluated as the logarithm cancels the exponent.

$$\begin{aligned}I &= -I + \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} u\left(u\sqrt{1-u^2}\right)du \\ 2I &= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} u^2\sqrt{1-u^2}du \qquad (10)\end{aligned}$$

This integral can be evaluated using a trig substitution of $u = \sin(\theta)$ which means $du = \cos(\theta)\,d\theta$. The indices of integration are translated to $\theta = -\frac{\pi}{4}$ and $\frac{\pi}{4}$. Substituting into (10):

$$\begin{aligned}2I &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2(\theta)\sqrt{1-\sin^2(\theta)}\cos(\theta)\,d\theta \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2(\theta)\cos^2(\theta)\,d\theta \qquad (11)\end{aligned}$$

This integral must be evaluated by converting the sines and cosines to linear degree. This can be done using the following 2 trig identities (which come from the double angle formula of cos).

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \qquad (12)$$

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \qquad (13)$$

Substituting (12) and (13) into the integrand of (11):

$$\begin{aligned}\sin^2(\theta)\cos^2(\theta) &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right) \\ &= \frac{1-\cos^2(2\theta)}{4}\end{aligned}$$

Substituting (13) again but this time for $\cos^2(2\theta)$

$$\begin{aligned}\sin^2(\theta)\cos^2(\theta) &= \frac{1}{4}\left(1-\frac{1+\cos(4\theta)}{2}\right) \\ &= \frac{1}{4}\left(\frac{1}{2}-\frac{\cos(4\theta)}{2}\right) \\ &= \frac{1-\cos(4\theta)}{8} \qquad (14)\end{aligned}$$

Substituting (14) into (11),

$$\begin{aligned}
2I &= \frac{1}{8}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-\cos(4\theta))d\theta \\
I &= \frac{1}{16}\left[\theta - \frac{1}{4}\sin(4\theta)\Big|_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}}\right] \\
&= \frac{1}{16}\left[\frac{\pi}{4} - \frac{1}{4}\sin(\pi) - \left(-\frac{\pi}{4} - \frac{1}{4}\sin(-\pi)\right)\right] \\
&= \frac{1}{16}\left(\frac{\pi}{2}\right) \\
&= \frac{\pi}{32}
\end{aligned}$$

Integral 4

Evaluate:

$$\int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan(x))}{2+\sin(2x)+\cos(2x)}dx \tag{15}$$

Solution:

Start by making the substitution $u = \frac{\pi}{4} - x$, which implies $x = \frac{\pi}{4} - u$. As a result, $du = -dx$. This substitution allows the bounds of integration to be swapped. This substitution is useful because of what occurs to the trigonometric terms. First, observe sine and cosine:

$$\begin{aligned}
\sin(2x) &= \sin\left(2\left(\frac{\pi}{4} - u\right)\right) \\
&= \sin\left(\frac{\pi}{2} - 2u\right) \\
&= \cos(2u)
\end{aligned} \tag{16}$$

$$\begin{aligned}
\cos(2x) &= \cos\left(2\left(\frac{\pi}{4} - u\right)\right) \\
&= \cos\left(\frac{\pi}{2} - 2u\right) \\
&= \sin(2u)
\end{aligned} \tag{17}$$

We have to use the double angle formula for tangent to observe what changes occur in the numerator. It turns out nice because $\tan\left(\frac{\pi}{4}\right) = 1$

$$\begin{aligned}
\ln(1+\tan(x)) &= \ln\left(1+\tan\left(\frac{\pi}{4} - u\right)\right) \\
&= \ln\left(1 + \frac{\tan\left(\frac{\pi}{4}\right) + \tan(-u)}{1 - \tan\left(\frac{\pi}{4}\right)\tan(-u)}\right) \\
&= \ln\left(1 + \frac{1-\tan(u)}{1+\tan(u)}\right) \\
&= \ln\left(\frac{1+\tan(u)+1-\tan(u)}{1+\tan(u)}\right) \\
&= \ln(2) - \ln(1+\tan(u))
\end{aligned} \tag{18}$$

Substituting (16), (17) and (18) into (15) while considering the differential and the bounds of integration leads to:

$$\int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan(x))}{2+\sin(2x)+\cos(2x)}dx = \int_{\frac{\pi}{4}}^{0} \frac{\ln(2) - \ln(1+\tan(u))}{2+\cos(2u)+\sin(2u)}(-du)$$

Cancel the negative by swapping the bounds of integration. The integral can then be split such that the original integral pops occurs on the right side.

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} \frac{\ln(2) - \ln(1+\tan(u))}{2+\cos(2u)+\sin(2u)} du \\
I &= \int_0^{\frac{\pi}{4}} \frac{\ln(2)}{2+\sin(2u)+\cos(2u)} du - \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan(u))}{2+\sin(2u)+\cos(2u)} du \\
I &= \ln(2) \int_0^{\frac{\pi}{4}} \frac{1}{2+\sin(2u)+\cos(2u)} du - I \\
2I &= \ln(2) \int_0^{\frac{\pi}{4}} \frac{1}{2+\sin(2u)+\cos(2u)} du
\end{aligned}$$

Just to make the integral a bit neater, I'll make the substitution $v = 2u$ which implies $\frac{1}{2}dv = du$. This also results in the top bound to double to $\frac{\pi}{2}$.

$$\begin{aligned}
2I &= \ln(2) \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin(v)+\cos(v)} \frac{1}{2} dv \\
I &= \frac{\ln(2)}{4} \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin(v)+\cos(v)} dv
\end{aligned}$$

This is a rational expression of sines and cosines, so a Weierstrass substitution of $t = \tan\left(\frac{v}{2}\right)$ will solve the integral. In Weierstrass substitution, $\sin(v) = \frac{2t}{1+t^2}$, $\cos(v) = \frac{1-t^2}{1+t^2}$, and $dv = \frac{2}{1+t^2}dt$. Also note that applying this substitution leads the bounds of integration to go from 0 to 1.

$$\begin{aligned}
I &= \frac{\ln(2)}{4} \int_0^1 \frac{1}{2+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt \\
&= \frac{\ln(2)}{2} \int_0^1 \frac{1}{2+2t^2+2t+1-t^2} dt \\
&= \frac{\ln(2)}{2} \int_0^1 \frac{1}{t^2+2t+3} dt \\
&= \frac{\ln(2)}{2} \int_0^1 \frac{1}{(t+1)^2+2} dt \\
&= \frac{\ln(2)}{2} \int_0^1 \frac{1}{2\left(\left(\frac{t+1}{\sqrt{2}}\right)^2+1\right)} dt \\
&= \frac{\ln(2)}{4} \int_0^1 \frac{1}{\left(\frac{t+1}{\sqrt{2}}\right)^2+1} dt \\
&= \frac{\ln(2)}{4} \left[\sqrt{2} \arctan\left(\frac{t+1}{\sqrt{2}}\right) \Bigg|_{t=0}^{t=1} \right] \\
&= \frac{\sqrt{2}\ln(2)}{4} \left(\arctan\left(\sqrt{2}\right) - \arctan\left(\frac{1}{\sqrt{2}}\right) \right)
\end{aligned}$$

Integral 5

Evaluate:

$$\int_0^{2\pi} \frac{x+\tan(\sin(x))}{2+\cos(x)}dx \tag{19}$$

Solution:

Similarly to the first integral, start by making the substitution $u = 2\pi - x$, which implies $x = 2\pi - u$. As a result, $du = -dx$.

For the bounds of integration, when $x = 0$, $u = 2\pi$ and when $x = 2\pi$, $u = 0$.

After making the substitutions, note what happens to the sine and cosine terms:

$$\begin{aligned}\sin(x) &= \sin(2\pi - u)\\ &= \sin(-u)\\ &= -\sin(u)\end{aligned} \tag{20}$$

$$\begin{aligned}\cos(x) &= \cos(2\pi - u)\\ &= \cos(-u)\\ &= \cos(u)\end{aligned} \tag{21}$$

Substituting into (20) and (21) along with the substitution into (19) yields:

$$\int_0^{2\pi} \frac{x+\tan(\sin(x))}{2+\cos(x)}dx = \int_{2\pi}^{0} \frac{2\pi - u + \tan(-\sin(u))}{2+\cos(u)}(-du)$$

Cancel the negative of the differential by swapping the bounds of integration. Also, bring the negative out of tan because it is odd

$$= \int_0^{2\pi} \frac{2\pi - u - \tan(\sin(u))}{2+\cos(u)}du \tag{22}$$

Splitting the integral in (22) allows the original integral to show up.

$$\begin{aligned}\int_0^{2\pi} \frac{x+\tan(\sin(x))}{2+\cos(x)}dx &= \int_0^{2\pi} \frac{2\pi}{2+\cos(u)}du - \int_0^{2\pi} \frac{u+\tan(\sin(u))}{2+\cos(u)}du\\ I &= 2\pi\int_0^{2\pi} \frac{1}{2+\cos(u)}du - I\\ 2I &= 2\pi\int_0^{2\pi} \frac{1}{2+\cos(u)}du\\ I &= \pi\int_0^{2\pi} \frac{1}{2+\cos(u)}du\end{aligned}$$

To evaluate this integral, I intend to use the Weierstrass substitution of $v = \tan\left(\frac{u}{2}\right)$. However, $\tan\left(\frac{u}{2}\right)$ has an asymptote at $u = \pi$. As a result, we need to split this integral first.

$$I = \pi\left[\int_0^{\pi} \frac{1}{2+\cos(u)}du + \int_{\pi}^{2\pi} \frac{1}{2+\cos(u)}du\right]$$

In Weierstrass substitution, $\cos(u) = \dfrac{1-v^2}{1+v^2}$ and $du = \dfrac{2}{1+v^2}dv$. Furthermore, at $u = 0$ and 2π, $v = 0$. When u approaches π from the left, v approaches positive infinity and when u approaches π from the right, v approaches negative infinity. Substituting yields:

$$\begin{aligned}
I &= \pi\left[\int_0^\infty \frac{1}{2+\frac{1-v^2}{1+v^2}}\frac{2}{1+v^2}dv + \int_{-\infty}^0 \frac{1}{2+\frac{1-v^2}{1+v^2}}\frac{2}{1+v^2}dv\right] \\
&= \pi\left[\int_{-\infty}^\infty \frac{2}{2+2v^2+1-v^2}dv\right] \\
&= 2\pi\left[\int_{-\infty}^\infty \frac{1}{v^2+3}dv\right]
\end{aligned}$$

This exact integrand was found in Integral 1 after the substitution in (4). Using the same antiderivative yields the following:

$$\begin{aligned}
I &= 2\pi \lim_{a\to\infty}\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}v\right)\Bigg|_{v=-a}^{v=a}\right] \\
&= \frac{2\pi}{\sqrt{3}}\left[\lim_{a\to\infty}\arctan\left(\frac{1}{\sqrt{3}}a\right) - \lim_{a\to-\infty}\arctan\left(\frac{1}{\sqrt{3}}a\right)\right] \\
&= \left(\frac{2\pi}{\sqrt{3}}\right)\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\
&= \frac{2\pi^2}{\sqrt{3}}
\end{aligned}$$

Integral 6

Evaluate:

$$\int_2^3 \ln\left(\sqrt{x+1}-\sqrt{x-1}\right)dx \tag{23}$$

Solution:

This integral turns out to be much simpler than the other 5 integrals. It is the only one that algebra engines can solve. The method may seem a bit unnatural but all this integral requires is an integration by parts.

$$\begin{aligned}
u &= \ln\left(\sqrt{x+1}-\sqrt{x-1}\right) & dv &= dx \\
du &= \frac{\frac{1}{2}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{x-1}}\right)}{\sqrt{x+1}-\sqrt{x-1}}dx & v &= x
\end{aligned}$$

Thus,

$$\int udv = uv - \int vdu$$

$$\int \ln\left(\sqrt{x+1}-\sqrt{x-1}\right)dx = x\ln\left(\sqrt{x+1}-\sqrt{x-1}\right) - \int x\frac{\frac{1}{2}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{x-1}}\right)}{\sqrt{x+1}-\sqrt{x-1}}dx$$

Bringing the numerator into a single fraction allows for a cancellation.

$$
\begin{aligned}
&= x\ln\left(\sqrt{x+1}-\sqrt{x-1}\right) - \frac{1}{2}\int x\frac{\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x+1}\sqrt{x-1}}}{\sqrt{x+1}-\sqrt{x-1}}dx \\
&= x\ln\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{2}\int \frac{x}{\sqrt{x+1}\sqrt{x-1}}dx \\
&= x\ln\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{2}\int \frac{x}{x^2-1}dx
\end{aligned}
$$

This integral can be evaluated using the substitution $t = x^2 - 1$. Then, $dt = 2xdx$ which cancels nicely with the numerator.

$$
\begin{aligned}
&= x\ln\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{2}\int \frac{\frac{1}{2}dv}{\sqrt{v}} \\
&= x\ln\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{4}\int \frac{1}{\sqrt{v}}dv \\
&= x\ln\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{4}\int \frac{1}{\sqrt{v}}dv \\
&= x\ln\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{2}\sqrt{v} + C \\
&= x\ln\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{2}\sqrt{x^2-1} + C
\end{aligned}
$$

Now applying the Fundamental Theorem of Calculus to evaluate the definite integral.

$$
\begin{aligned}
\int_2^3 \ln\left(\sqrt{x+1}-\sqrt{x-1}\right)dx &= x\ln\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{2}\sqrt{x^2-1}\Bigg|_{x=2}^{x=3} \\
&= 3\ln\left(\sqrt{3+1}-\sqrt{3-1}\right) + \frac{1}{2}\sqrt{3^2-1} - 2\ln\left(\sqrt{2+1}-\sqrt{2-1}\right) - \frac{1}{2}\sqrt{2^2-1} \\
&= 3\ln\left(2-\sqrt{2}\right) - 2\ln\left(\sqrt{3}-1\right) + \frac{2\sqrt{2}-\sqrt{3}}{2} \\
&= \ln\left(\frac{\left(2-\sqrt{2}\right)^3}{\left(\sqrt{3}-1\right)^2}\right) + \frac{2\sqrt{2}-\sqrt{3}}{2} \\
&= \ln\left(\frac{8-3\cdot4\sqrt{2}+3\cdot2\cdot2-2\sqrt{2}}{3-2\sqrt{3}+1}\right) + \frac{2\sqrt{2}-\sqrt{3}}{2} \\
&= \ln\left(\frac{20-14\sqrt{2}}{4-2\sqrt{3}}\right) + \frac{2\sqrt{2}-\sqrt{3}}{2}
\end{aligned}
$$