Maximizing Surface Area of Cylinder inscribed in Sphere

Arthur Xu

October 2020

1 Problem

What is the maximum surface area of a cylinder inscribed in a sphere with radius R. Furthermore, find the dimensions of the cylinder: h and r

2 Differentiating with respect to r

The first approach to this problem is to represent half the height of the cylinder as a function of r. It is important to note that h is only half the actual height of the cylinder. Using the Pythagorean theorem we get:

$$h = \left(R^2 - r^2\right)^{\frac{1}{2}} \tag{1}$$

If we consider that h is half the height of the cylinder, we get the following equation for the surface area of the cylinder:

$$SA = 2\pi r^2 + 4\pi rh \tag{2}$$

We can then substitute (1) into (2) to get the following equation. We can note that the surface area is now a function of R ad r. Since we're maximizing the cylinder within a sphere, we can set R to be constant making the surface area strictly a function of r.

$$SA(r) = 2\pi r^2 + 4\pi r\left(R^2 - r^2\right)^{\frac{1}{2}} \tag{3}$$

To maximize the surface area, we will differentiate with respect to r, equate it with 0 to see what value of r will produce the maximum. Differentiating with respect to r yields:

$$\begin{aligned} SA'(r) &= 4\pi r + 4\pi\left(R^2 - r^2\right)^{\frac{1}{2}} - \frac{1}{2}\left(R^2 - r^2\right)^{-\frac{1}{2}}(2r)(4\pi r) \\ &= 4\pi\Big(r + \left(R^2 - r^2\right)^{\frac{1}{2}} - r^2\left(R^2 - r^2\right)^{-\frac{1}{2}}\Big) \\ &= 4\pi\left(R^2 - r^2\right)^{-\frac{1}{2}}\Big(r\left(R^2 - r^2\right)^{\frac{1}{2}} + \left(R^2 - r^2\right) - r^2\Big) \\ &= 4\pi\left(R^2 - r^2\right)^{-\frac{1}{2}}\Big(r\left(R^2 - r^2\right)^{\frac{1}{2}} + R^2 - 2r^2\Big) \end{aligned} \tag{4}$$

When we set (4) to be 0, we can remove the 4π term. Furthermore, we don't need to consider setting $\left(R^2 - r^2\right)^{\frac{1}{2}}$ in the denominator to 0 because it yields $r = R$ which implies that $h = 0$ which isn't a possible solution. Therefore setting (4) to 0 yields:

$$\begin{aligned} r\left(R^2 - r^2\right)^{\frac{1}{2}} + R^2 - 2r^2 &= 0 \\ r\left(R^2 - r^2\right)^{\frac{1}{2}} &= 2r^2 - R^2 \end{aligned}$$

Square both sides

$$\begin{aligned} r^2\left(R^2-r^2\right) &= 4r^4-4r^2R^2+R^4 \\ 0 &= 5r^4-5r^2R^2+R^4 \end{aligned} \tag{5}$$

Using the quadratic formula we get

$$\begin{aligned} r^2 &= \frac{5R^2 \pm \sqrt{\left(-5R^2\right)^2-4\,(5)\left(R^4\right)}}{10} \\ &= \frac{5R^2 \pm \sqrt{25R^4-20R^4}}{10} \\ &= \frac{5R^2 \pm R^2\sqrt{5}}{10} \\ &= R^2\left(\frac{5\pm\sqrt{5}}{10}\right) \end{aligned}$$

We now have two roots, one of which is extraneous due to the squaring of both sides of our equation. We must substitute these back into (5) to confirm which one yields 0 and which one doesn't.

$$r_1 = R\sqrt{\frac{5+\sqrt{5}}{10}} \qquad\qquad r_2 = R\sqrt{\frac{5-\sqrt{5}}{10}}$$

Substituting r_1 yields:

$$\begin{aligned} &= R\sqrt{\frac{5+\sqrt{5}}{10}}\left(R^2-\left(R\sqrt{\frac{5+\sqrt{5}}{10}}\right)^2\right)^{\frac{1}{2}}+R^2-2\left(R\sqrt{\frac{5+\sqrt{5}}{10}}\right)^2 \\ &= R\sqrt{\frac{5+\sqrt{5}}{10}}\left(R^2-R^2\left(\frac{5+\sqrt{5}}{10}\right)\right)^{\frac{1}{2}}+R^2-2R^2\left(\frac{5+\sqrt{5}}{10}\right) \\ &= R\sqrt{\frac{5+\sqrt{5}}{10}}\left(R^2\left(\frac{5-\sqrt{5}}{10}\right)\right)^{\frac{1}{2}}+R^2\left(1-\frac{5+\sqrt{5}}{5}\right) \\ &= \left(R\sqrt{\frac{5+\sqrt{5}}{10}}\right)\left(R\sqrt{\frac{5-\sqrt{5}}{10}}\right)+R^2\left(-\frac{\sqrt{5}}{5}\right) \\ &= R^2\sqrt{\frac{\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)}{100}}-\frac{\sqrt{5}}{5}R^2 \\ &= R^2\left(\sqrt{\frac{25-5}{100}}-\frac{\sqrt{5}}{5}\right) \\ &= R^2\left(\frac{\sqrt{5}}{5}-\frac{\sqrt{5}}{5}\right) \\ &= 0 \end{aligned}$$

Thus, r_1 isn't the extraneous root and is a solution. This technically shows that r_2 is an extraneous root, but we'll substitute it in anyways. Substituting r_2 yields:

$$
\begin{aligned}
&= R\sqrt{\frac{5-\sqrt{5}}{10}}\left(R^2-\left(R\sqrt{\frac{5-\sqrt{5}}{10}}\right)^2\right)^{\frac{1}{2}}+R^2-2\left(R\sqrt{\frac{5-\sqrt{5}}{10}}\right)^2 \\
&= R\sqrt{\frac{5-\sqrt{5}}{10}}\left(R^2-R^2\left(\frac{5-\sqrt{5}}{10}\right)\right)^{\frac{1}{2}}+R^2-2R^2\left(\frac{5-\sqrt{5}}{10}\right) \\
&= R\sqrt{\frac{5-\sqrt{5}}{10}}\left(R^2\left(\frac{5+\sqrt{5}}{10}\right)\right)^{\frac{1}{2}}+R^2\left(1-\frac{5-\sqrt{5}}{5}\right) \\
&= \left(R\sqrt{\frac{5-\sqrt{5}}{10}}\right)\left(R\sqrt{\frac{5+\sqrt{5}}{10}}\right)+R^2\left(\frac{\sqrt{5}}{5}\right) \\
&= R^2\sqrt{\frac{\left(5-\sqrt{5}\right)\left(5+\sqrt{5}\right)}{100}}-\frac{\sqrt{5}}{5}R^2 \\
&= R^2\left(\sqrt{\frac{25-5}{100}}+\frac{\sqrt{5}}{5}\right) \\
&= R^2\left(\frac{\sqrt{5}}{5}+\frac{\sqrt{5}}{5}\right) \\
&= \frac{2\sqrt{5}}{5}R^2 \\
&\neq 0
\end{aligned}
$$

Thus, it has been shown that r_2 isn't a solution and that our solution for r is r_1 which equals $R\sqrt{\frac{5+\sqrt{5}}{10}}$ Now we substitute this back into (1) and (3) to find the height and surface area. Substituting into (1) yields:

$$
\begin{aligned}
h &= \left(R^2-r^2\right)^{\frac{1}{2}} \\
&= \left(R^2-\left(R\sqrt{\frac{5+\sqrt{5}}{10}}\right)^2\right)^{\frac{1}{2}} \\
&= \left(R^2-R^2\left(\frac{5+\sqrt{5}}{10}\right)\right)^{\frac{1}{2}} \\
&= \left(R^2\left(1-\frac{5+\sqrt{5}}{10}\right)\right)^{\frac{1}{2}} \\
&= R\sqrt{\frac{5-\sqrt{5}}{10}}
\end{aligned}
$$

The variable h was defined as half of the height at the start of the solution, so the actual height of the cylinder would be twice the value calculated. Substituting r and h into (2) yields:

$$\begin{aligned}
SA &= 2\pi r^2 + 4\pi rh \\
&= 2\pi\left(R\sqrt{\frac{5+\sqrt{5}}{10}}\right)^2 + 4\pi\left(R\sqrt{\frac{5+\sqrt{5}}{10}}\right)\left(R\sqrt{\frac{5-\sqrt{5}}{10}}\right) \\
&= 2\pi R^2\left(\frac{5+\sqrt{5}}{10}\right) + 4\pi R^2\sqrt{\frac{\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)}{100}} \\
&= 2\pi R^2\left(\frac{5+\sqrt{5}}{10}\right) + \frac{4\pi R^2}{10}\sqrt{25-5} \\
&= \pi R^2\left(\frac{5+\sqrt{5}}{5}\right) + \frac{2}{5}\pi R^2\left(2\sqrt{5}\right) \\
&= \pi R^2\left(\frac{5+\sqrt{5}+4\sqrt{5}}{5}\right) \\
&= \pi R^2\left(1+\sqrt{5}\right)
\end{aligned}$$

Thus it has been shown that maximum surface area of a cylinder inscribed in a sphere of radius R is $\pi R^2\left(1+\sqrt{5}\right)$. This achieved when the cylinder has a radius of $R\sqrt{\frac{5+\sqrt{5}}{10}}$ and a height of $2R\sqrt{\frac{5-\sqrt{5}}{10}}$

3 Differentiating with respect to θ

Another approach to this problem involves a less obvious approach, but actually yields much simpler algebra. If we take the same triangle used in section 2 which established (1), we can use basic trigonometry to yield the following relationships:

$$r = R\cos\theta \qquad (6) \qquad\qquad h = R\sin\theta \qquad (7)$$

We then substitute (6) and (7) into (2) to get the surface area of the cylinder as a function of θ

$$\begin{aligned}
SA(\theta) &= 2\pi r^2 + 4\pi rh \\
&= 2\pi\left(R\cos\theta\right)^2 + 4\pi\left(R\cos\theta\right)\left(R\sin\theta\right) \\
&= 2\pi R^2\cos^2\theta + 2\pi R^2\sin 2\theta \\
&= 2\pi R^2\left(\cos^2\theta + \sin 2\theta\right)
\end{aligned} \tag{8}$$

Differentiating with respect to θ yields

$$\begin{aligned}
SA'(\theta) &= 2\pi R^2\left(2\cos\theta\left(-\sin\theta\right) + 2\cos 2\theta\right) \\
&= 2\pi R^2\left(-\sin 2\theta + 2\cos 2\theta\right)
\end{aligned} \tag{9}$$

The algebra to set (9) to 0 is relatively straight forward compared to (4). Some basic trig will do the job

$$\begin{aligned}
-\sin 2\theta + 2\cos 2\theta &= 0 \\
2\cos 2\theta &= \sin 2\theta \\
\tan 2\theta &= 2 \\
2\theta &= \arctan 2 \\
\theta &= \frac{\arctan 2}{2}
\end{aligned}$$

We can simply get approximate answers by finding the decimal approximation and then substituting into our definitions of h and r. However, I want to find exact answers so that I can confirm that this yields the EXACT same result as the above solutions. To proceed I will let $\alpha = 2\theta$ so the value of α that maximizes surface area is $\arctan 2$ We then want to find r and h in terms of α. I use half-angle formulas to expand the definitions.

$$\begin{aligned} r &= R\cos\theta \\ &= R\cos\frac{\alpha}{2} \\ &= R\sqrt{\frac{1+\cos\alpha}{2}} \end{aligned} \tag{10}$$

$$\begin{aligned} h &= R\sin\theta \\ &= R\sin\frac{\alpha}{2} \\ &= R\sqrt{\frac{1-\cos\alpha}{2}} \end{aligned} \tag{11}$$

The question now becomes, if $\alpha = \arctan 2$, what is $\cos(\arctan 2)$? The idea behind the exact solution is that $\arctan 2$ is an angle in a right angle triangle where the opposite side length is 2 and the adjacent side length is 1. We can then conclude using the Pythagorean Theorem that the hypotenuse is $\sqrt{5}$. If that's the case, we get the following value:

$$\begin{aligned} \cos(\arctan 2) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{1}{\sqrt{5}} \end{aligned}$$

We can substitute this into (10) and (11). It may initially appear different than the answer achieved in section 2, however a bit of algebraic manipulation will do the job.

$$\begin{aligned} r &= R\sqrt{\frac{1+\cos\alpha}{2}} \\ &= R\sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}} \\ &= R\sqrt{\frac{1+\frac{\sqrt{5}}{5}}{2}} \\ &= R\sqrt{\frac{5+\sqrt{5}}{10}} \end{aligned}$$

$$\begin{aligned} h &= R\sqrt{\frac{1-\cos\alpha}{2}} \\ &= R\sqrt{\frac{1-\frac{1}{\sqrt{5}}}{2}} \\ &= R\sqrt{\frac{1-\frac{\sqrt{5}}{5}}{2}} \\ &= R\sqrt{\frac{5-\sqrt{5}}{10}} \end{aligned}$$

As shown in Section 2, the maximum surface area of the cylinder if the radius is $R\sqrt{\frac{5+\sqrt{5}}{10}}$ and half the height is $R\sqrt{\frac{5-\sqrt{5}}{10}}$ is $\pi R^2\left(1+\sqrt{5}\right)$