Interesting Lines and Planes Problem

Arthur Xu

January 2021

1 Problem

Let l_1 be the following line in 3-d space in symmetric form:

$$x = y = z + 2 \tag{1}$$

Let l_2 be the following line in 3-d space in symmetric form:

$$\frac{x}{-2} = \frac{y+3}{1} = \frac{z}{3} \tag{2}$$

We want to find the equation of a line l_3 that passes through the point $(0, 1, 2)$ and intersects both l_1 and l_2 (not necessarily at the same point). This problem was taken out of Algebra and Geometry by Dunkley, Gilbert, Anderson, Crippin, Davdison, Rachich, and Zirzitto. My friend, George Paraschiv, and I solved this problem using 2 different methods, both of which are presented in this paper.

2 The Simple Solution doesn't Work

The first instinct and seemingly simple solution is to find the point of intersection of l_1 and l_2. Then we can construct l_3 as the line that crosses through the point of intersection and $(0, 1, 2)$. Then, l_3 would cross through both lines and the point.

It turns out that l_1 and l_2 are skew lines, so that method won't work as there won't be a point of intersection. The proof is as follows. First we assume that l_1 does intersect with l_2. This means that there exists some t that when substituted into l_1's parametric equations, the (x, y, z) coordinates satisfy all the equalities in l_2's symmetric equations. Converting (1) into parametric form, we have:

$$\begin{aligned} t &= x & t &= y & t &= z + 2 \\ x &= t & y &= t & z &= t - 2 \end{aligned} \tag{3}$$

Substituting (3) into (2) we have,

$$\frac{t}{-2} = \frac{t+3}{1} = \frac{t-2}{3} \tag{4}$$

If l_1 intersects l_2 at a point, then the same value of t will satisfy both equalities. Solving for t when equating the first and second expression,

$$\begin{aligned} \frac{t}{-2} &= \frac{t+3}{1} \\ t &= -2t - 6 \\ 3t &= -6 \\ t &= -2 \end{aligned}$$

Solving for t when equating the second and third expression,

$$\begin{aligned}
\frac{t+3}{1} &= \frac{t-2}{3} \\
3t+9 &= t-2 \\
2t &= -11 \\
t &= \frac{-11}{2}
\end{aligned}$$

This contradicts our assumption as different values of t solves each equality, and thus there is no common t that satisfies (2). As a result, we know that l_1 doesn't intersect l_2. It can also be noted that l_1 and l_2 are not parallel since $\overrightarrow{m_1} = (1,1,1)$ and $\overrightarrow{m_2} = (-2,1,3)$, which are evidently not scalar multiples of each other. As a result, we can conclude that l_1 and l_2 are skew lines.

3 Line Intersection Solution

To start off this solution, I used the general vector equation of a line to produce a general form for l_3. The problem states that l_3 passes through $(0,1,2)$ and thus $\overrightarrow{r_0} = (0,1,2)$. We then have,

$$\overrightarrow{r_3} = (0,1,2) + t\left(m_x, y_0 m_x, z_0 m_x\right); t \in \mathbb{R} \tag{5}$$

As opposed to writing the direction vector as having a distinct x, y and z coordinates, I wrote the y coordinates and z coordinates as scalar multiples of the x-coordinate. This is because direction vectors aren't unique for each line, a scalar multiple of a direction vector is the same direction vector. The defining part of a direction vector is the $x : y : z$ ratio. Without loss of generality, I'll just let $m_x = 1$ for ease of calculation (I do recognize that this will not work if the x coordinate of the direction vector is 0. However, if that were the case, the system of equations I will later solve will have no solutions, and thus I can conclude that $m_x = 0$). Then from (5),

$$\overrightarrow{r_3} = (0,1,2) + t\left(1, y_0, z_0\right); t \in \mathbb{R} \tag{6}$$

Consider the components to find the parametric form

$$(x,y,z) = (t, 1 + y_0 t, 2 + z_0 t); t \in \mathbb{R} \tag{7}$$

If we want this l_3 to intersect with both l_1 and l_2, this means that for some $t = t_1$, (7) will satisfy all the equalities in (1) and for some $t = t_2$ (which may or may not equal t_1), (7) will satisfy all the equalities in (2). By observing this approach, we see that we will end up with 4 unknowns and 4 equations, allowing us to solve for y_0 and z_0.

First let's consider substituting (7) into (1) with $t = t_1$.

$$t_1 = 1 + t_1 y_0 = 2 + t_1 z_0 + 2$$

By equating the first and second expression we have,

$$\begin{aligned}
t_1 &= 1 + t_1 y_0 \\
t_1\left(1 - y_0\right) &= 1 \\
t_1 &= \frac{1}{1 - y_0}
\end{aligned} \tag{8}$$

By equating the first and third expression we have,

$$\begin{aligned}
t_1 &= 4 + t_1 z_0 \\
t_1\left(1 - z_0\right) &= 4 \\
t_1 &= \frac{4}{1 - z_0}
\end{aligned} \tag{9}$$

Then we can equate (8) and (9) to get an expression in terms of y_0 and z_0.

$$\begin{aligned}\frac{1}{1-y_0} &= \frac{4}{1-z_0}\\ 1-z_0 &= 4-4y_0\\ 4y_0-z_0 &= 3 \qquad (10)\end{aligned}$$

Now let's substitute (7) into (2) with $t = t_2$

$$\frac{t_2}{-2} = 1 + t_2 y_0 + 3 = \frac{2 + t_2 z_0}{3}$$

By equating the first and second expression we have,

$$\begin{aligned}\frac{t_2}{-2} &= 4 + t_2 y_0\\ t_2 &= -8 - 2t_2 y_0\\ t_2\left(1+2y_0\right) &= -8\\ t_2 &= \frac{-8}{1+2y_0} \qquad (11)\end{aligned}$$

By equating the first and third expression we have,

$$\begin{aligned}\frac{t_2}{-2} &= \frac{2+t_2 z_0}{3}\\ 3t_2 &= -4 - 2t_2 z_0\\ t_2\left(3+2z_0\right) &= -4\\ t_2 &= \frac{-4}{3+2z_0} \qquad (12)\end{aligned}$$

Then we can equate (11) and (12) to get a second expression in terms of y_0 and z_0.

$$\begin{aligned}\frac{-8}{1+2y_0} &= \frac{-4}{3+2z_0}\\ -24-16z_0 &= -4-8y_0\\ 8y_0-16z_0 &= 20\\ 2y_0-4z_0 &= 5 \qquad (13)\end{aligned}$$

Note that (10) and (13) form a system of 2 equations in terms of y_0 and z_0. From (10), we have $z_0 = 4y_0 - 3$. Substituting into (13),

$$\begin{aligned}2y_0 - 4\left(4y_0-3\right) &= 5\\ 2y_0 - 16y_0 + 12 &= 5\\ -14y_0 &= -7\\ y_0 &= \frac{1}{2}\end{aligned}$$

Substituting back into (10),

$$\begin{aligned}z_0 &= 4y_0 - 3\\ z_0 &= 4\left(\frac{1}{2}\right) - 3\\ z_0 &= -1\end{aligned}$$

Finally, we can substitute our results back into (6), to have an equation for l_3.

$$\overrightarrow{r_3} = (0,1,2) + t\left(1, \frac{1}{2}, -1\right); t \in \mathbb{R}$$

Multiply the direction vector by 2 to make them all integers.

$$\overrightarrow{r_3} = (0,1,2) + t\left(2,1,-2\right); t \in \mathbb{R} \qquad (14)$$

4 Plane Intersection Solution

Consider a plane π_1 that contains l_1 and the point $(0,1,2)$. And consider a plane π_2 that contains l_2 and the point $(0,1,2)$. Assuming that π_1 is not parallel to π_2, we can make a few observations. Firstly, $(0,1,2)$ is a shared point amongst the 2 planes, so that point is contained by the line of intersection. Furthermore, if the line of intersection is not parallel to l_1 or l_2, then the line of intersection will intersect both of those lines as they are on the same plane. Thus, we can conclude that the line of intersection of π_1 and π_2 is the l_3 that we are seeking.

First let's find the Cartesian equations of the planes to allow us to find the line of intersection. First let's find the vector form of (1) and (2).

I already found the parametric form of (1) in section 2 to be (3). Converting to vector form is as follows,

$$\begin{aligned} (x,y,z) &= (t,t,t-2) \\ \overrightarrow{r_1} &= (0,0,-2) + t(1,1,1); t \in \mathbb{R} \end{aligned} \tag{15}$$

Converting (2) into parametric form we have:

$$t = \frac{x}{-2} \qquad t = \frac{y+3}{1} \qquad t = \frac{z}{3}$$

$$x = -2t \qquad y = t-3 \qquad z = 3t$$

Using components to find vector form,

$$\begin{aligned} (x,y,z) &= (-2t, t-3, 3t) \\ \overrightarrow{r_2} &= (0,-3,0) + t\,(-2,1,3)\,; t \in \mathbb{R} \end{aligned} \tag{16}$$

To find the Cartesian equation of each, we need to find the normal for each plane. Since we know that for each plane it **contains** the corresponding line in addition to the point $(0,1,2)$, we have 2 vectors that are parallel to the plain: the direction vector of the line, and the vector between $P(0,1,2)$ and r_0 in each vector equation (i.e $\overrightarrow{r_0P}$). The normal vector is then the cross product of these 2 vectors.

For π_1, we have $P\,(0,1,2)$, $r_0 = (0,0,-2)$ and $\overrightarrow{m} = (1,1,1)$. Hence,

$$\begin{aligned} \overrightarrow{n} &= \overrightarrow{m} \times \overrightarrow{r_0P} \\ &= (1,1,1) \times [(0,1,2) - (0,0,-2)] \\ &= (1,1,1) \times (0,1,4) \\ &= [(1)\,(4) - (1)\,(1)\,, (1)\,(0) - (1)\,(4)\,, (1)\,(1) - (1)\,(0)] \\ &= (3,-4,1) \end{aligned}$$

To find the Cartesian form after finding the normal vector, we say that $\overrightarrow{P_1P} \cdot \overrightarrow{n} = 0$ where P is the general point (x,y,z) and P_1 is a point on the plane, which in this case we can use $(0,1,2)$. So, the Cartesian equation of π_1 is as follows,

$$\begin{aligned} \overrightarrow{P_1P} \cdot \overrightarrow{n} &= 0 \\ (x, y-1, z-2) \cdot (3,-4,1) &= 0 \\ 3x - 4y + 4 + z - 2 &= 0 \\ 3x - 4y + z &= -2 \end{aligned} \tag{17}$$

We use the same process for π_2 with $P(0,1,2)$, $r_0 = (0,-3,0)$ and $\overrightarrow{m} = (-2,1,3)$.

$$\begin{aligned}
\overrightarrow{n} &= \overrightarrow{m} \times \overrightarrow{r_0 P} \\
&= (-2,1,3) \times [(0,1,2) - (0,-3,0)] \\
&= (-2,1,3) \times (0,4,2) \\
&= [(1)(2) - (3)(4), (3)(0) - (-2)(2), (-2)(4) - (1)(0)] \\
&= (-10,4,-8)
\end{aligned}$$

For this method, we can multiply or divide the normal by any scalar. To reduce to integers, I'll divide by -2.

$$= (5,-2,4)$$

Then, the Cartesian equation of π_2 is,

$$\begin{aligned}
\overrightarrow{P_1 P} \cdot \overrightarrow{n} &= 0 \\
(x, y-1, z-2) \cdot (5,-2,4) &= 0 \\
5x - 2y + 2 + 4z - 8 &= 0 \\
5x - 2y + 4z &= 6 \qquad (18)
\end{aligned}$$

To solve for the intersection of the 2 planes, I will produce the matrix for (17) and (18). Then I will reduce it to row-echelon form.

$$\begin{aligned}
&\left[\begin{array}{ccc|c} 3 & -4 & 1 & -2 \\ 5 & -2 & 4 & 6 \end{array}\right] \\
&R_2 - \frac{5}{3}R_1 \\
&= \left[\begin{array}{ccc|c} 3 & -4 & 1 & -2 \\ 0 & \frac{14}{3} & \frac{7}{3} & \frac{28}{3} \end{array}\right] \\
&\frac{3}{7}R_2 \\
&= \left[\begin{array}{ccc|c} 3 & -4 & 1 & -2 \\ 0 & 2 & 1 & 4 \end{array}\right] \qquad (19)
\end{aligned}$$

From (19), we have $2y + z = 4$. Because we know we will have a line of intersection, I'll let z be the parameter t. Hence,

$$\begin{aligned}
2y + t &= 4 \\
2y &= 4 - t \\
y &= 2 - \frac{1}{2}t
\end{aligned}$$

Back substituting into the first row or (17),

$$\begin{aligned}
3x - 4y + z &= -2 \\
3x - 4\left(2 - \frac{1}{2}t\right) + t &= -2 \\
3x - 8 + 2t + t &= -2 \\
3x &= -3t + 6 \\
x &= 2 - t
\end{aligned}$$

Thus the line of intersection of the 2 planes, or as mentioned before l_3, is represented by the parametric equation,

$$(x,y,z) = \left(2 - t, 2 - \frac{1}{2}t, t\right) \qquad (20)$$

Converting to vector

$$\overrightarrow{r_3} = (2,2,0) + t\left(-1, -\frac{1}{2}, 1\right); t \in \mathbb{R}$$

Similarly to Section 3, I'm going to multiply the direction vector by -2 to get integers

$$\overrightarrow{r_3} = (2,2,0) + t\,(2,1,-2)\,; t \in \mathbb{R} \tag{21}$$

5 Checking the Solution

First I will simply show that the 2 lines represent the same line. In section 3, I ended up with (14), which is

$$\overrightarrow{r_3} = (0,1,2) + t\,(2,1,-2)\,; t \in \mathbb{R}$$

And in Section 4, I ended up with (21), which is

$$\overrightarrow{r_3} = (2,2,0) + t\,(2,1,-2)\,; t \in \mathbb{R}$$

Evidently, they have the same direction vector, so they are parallel. I just need to show that they share a point, which implies they are coincident and represent the same line. It is evident that the former passes through $(0,1,2)$ as that is the $\overrightarrow{r_0}$. For the latter to pass through, the same t has to satisfy all 3 coordinate points. We can see that this is true when $t = -1$, and thus the 2 equations represent the same line.

$$\begin{aligned}\overrightarrow{r_3} &= (2,2,0) + (-1)\,(2,1,-2)\\ &= (2,2,0) - (2,1,-2)\\ &= (0,1,2)\end{aligned}$$

In addition, we already showed that the line passes through $(0,1,2)$. Now I'll show that it intersects with both l_1 and l_2. Converting (14) to parametric we have,

$$\begin{aligned}x &= 2t\\ y &= 1+t\\ z &= 2-2t\end{aligned}$$

First I'll substitute into the symmetric equation of line 1 (1),

$$2t = 1 + t = 2 - 2t + 2$$

By equating the first and second expression we have,

$$\begin{aligned}2t &= 1+t\\ t &= 1\end{aligned}$$

By equating the first and third expression we have,

$$\begin{aligned}2t &= 4-2t\\ 4t &= 4\\ t &= 1\end{aligned}$$

Since the same value of t was obtained, we see that l_3 intersects with l_1. Let's do the same process after substituting into the symmetric equation of line 2 (2),

$$\frac{2t}{-2} = \frac{1+t+3}{1} = \frac{2-2t}{3}$$

By equating the first and second expression we have,

$$\begin{aligned} -t &= 4 + t \\ 2t &= -4 \\ t &= -2 \end{aligned}$$

By equating the first and third expression we have,

$$\begin{aligned} -t &= \frac{2 - 2t}{3} \\ -3t &= 2 - 2t \\ t &= -2 \end{aligned}$$

Since the same value of t was obtained, we see that l_3 intersects with l_2.

And thus, we have shown that $\overrightarrow{r_3} = (0, 1, 2) + t\,(2, 1, -2)\,; t \in \mathbb{R}$ is a line that intersects l_1, l_2, and passes through the point $(0, 1, 2)$.