My first MIT Integration Bee Integral

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1 The problem

The integral at hand is as follows,

$$\int_0^{\frac{\pi}{4}} \frac{\sin(x)+\cos(x)}{9+16\sin(2x)}dx \tag{1}$$

This integral is from the 2006 MIT Integration Bee where students were given 4 minutes to evaluate integrals. I'll just say that I took 20 minutes.

2 Solution using substitution

The first step is what seems to be the most natural u-substitution. We're going to set,

$$u = 9+16\sin(2x) \tag{2}$$

As a result,

$$du = 32\cos(2x)dx$$

Using trigonometric identities we can expand $\cos(2x)$

$$du = 32\left(\cos^2(x) - \sin^2(x)\right)dx$$

By difference of squares,

$$du = 32\left(\cos(x)-\sin(x)\right)\left(\cos(x)+\sin(x)\right)dx$$

The "plus factor" is part of our initial integral, which is the point of the u-substitution. We can rewrite the equality for direct substitution into the integral

$$\left(\cos(x)+\sin(x)\right)dx = \frac{1}{32\left(\cos(x)-\sin(x)\right)}du \tag{3}$$

The question now is how do we express the "negative factor" as a function of u since it isn't part of the initial integral. Or as blackpenredpen likes to say, we need to bring this into the u-world. We can do some manipulation as follows,

$$\begin{aligned}
\left(\cos(x)-\sin(x)\right)^2 &= \cos^2(x) - 2\sin(x)\cos(x) + \sin^2(x) \\
\left(\cos(x)-\sin(x)\right)^2 &= 1-\sin(2x) \\
\left|\cos(x)-\sin(x)\right| &= \sqrt{1-\sin(2x)}
\end{aligned} \tag{4}$$

What's nice is that in our integral we are going from $x = 0$ to $x = \frac{\pi}{4}$. On this interval, $\cos(x) - \sin(x)$ is always positive. So we can simply omit the absolute value. Our initial u-substitution can be rearranged to show that,

$$\sin(2x) = \frac{1}{16}(u - 9) \tag{5}$$

Substituting (5) into (4) yields,

$$\begin{aligned} \cos(x) - \sin(x) &= \sqrt{1 - \left(\frac{1}{16}(u-9)\right)} \\ &= \sqrt{\frac{25}{16} - \frac{u}{16}} \\ &= \frac{1}{4}\sqrt{25 - u} \end{aligned} \tag{6}$$

Substituting (6) into (3) yields, the following expression which can be directly substituted into the initial integral to complete the u-sub.

$$\begin{aligned} (\cos(x) + \sin(x))\, dx &= \frac{1}{32\left(\frac{1}{4}\sqrt{25-u}\right)} du \\ &= \frac{1}{8\sqrt{25-u}} \end{aligned} \tag{7}$$

The final step is to consider the bounds of integration. At $x = 0$, $u = 9$. And at $x = \frac{\pi}{4}$, $u = 25$. Therefore substituting (2) and (7) into the initial integral (1), we get

$$\int_0^{\frac{\pi}{4}} \frac{\sin(x) + \cos(x)}{9 + 16\sin(2x)} dx = \frac{1}{8}\int_9^{25} \frac{1}{u\sqrt{25-u}} du \tag{8}$$

Another substitution is needed to evaluate (8). It's a pretty common integral, I'll just make the substitution that I've done in the past, but literally almost any substitution will work.

$$\begin{aligned} \text{Let } t &= \sqrt{25-u} \\ dt &= \frac{-1}{2\sqrt{25-u}} du \\ -2dt &= \frac{1}{\sqrt{25-u}} du \end{aligned}$$

Furthermore,

$$\begin{aligned} t &= \sqrt{25-u} \\ u &= 25 - t^2 \end{aligned}$$

For the bounds of integration, when $u = 9$, $t = 4$. When $u = 25, t = 0$. Applying the substitution into (8), we have

$$\begin{aligned} \frac{1}{8}\int_9^{25} \frac{1}{u\sqrt{25-u}} du &= \frac{1}{8}\int_4^0 \frac{-2}{25-t^2} dt \\ &= \frac{1}{4}\int_0^4 \frac{1}{25-t^2} dt \end{aligned} \tag{9}$$

(9) has a formula to find the antiderivative which is derived using partial fraction decomposition, I'll show the whole process.

$$\begin{aligned}\frac{1}{25-t^2} &= \frac{1}{(5-t)(5+t)} \\ &= \frac{A}{5-t} + \frac{B}{5+t} \\ &= \frac{5A + At + 5B - Bt}{(5-t)(5+t)}\end{aligned} \tag{10}$$

Therefore,

$$A - B = 0 \tag{11}$$

And,

$$5A + 5B = 1 \tag{12}$$

From (11), $A = B$, substituting that into (12), shows the following,

$$\begin{aligned}10A &= 1 \\ A = B &= \frac{1}{10}\end{aligned} \tag{13}$$

Substituting (13), into (10) and finally into (9), we have

$$\begin{aligned}\frac{1}{4}\int_0^4 \frac{1}{25-t^2}dt &= \frac{1}{4}\int_0^4 \frac{1}{10(5-t)} + \frac{1}{10(5+t)}dt \\ &= \frac{1}{40}\int_0^4 \frac{1}{5-t} + \frac{1}{5+t}dt \\ &= \frac{1}{40}\left[-\ln(5-t) + \ln(5+t)\Big|_{t=0}^{t=4}\right] \\ &= \frac{1}{40}\left[\ln\left(\frac{5+t}{5-t}\right)\Big|_{t=0}^{t=4}\right] \\ &= \frac{1}{40}\left[\ln\left(\frac{5+4}{5-4}\right) - \ln\left(\frac{5+0}{5-0}\right)\right] \\ &= \frac{1}{40}\left[\ln(9) - \ln(1)\right] \\ &= \frac{1}{20}\ln(3)\end{aligned}$$

3 The Anti-derivative

In this section, I'm going to use the work in Section 2 to find the anti-derivative. Then I'll evaluate the definite integral using the anti-derivative and then I'll differentiate to show that the equation found is indeed the anti-derivative.

From (9) and (8) (the same work applies, we just aren't considering the bounds of integration anymore), we have

$$\int \frac{\sin(x)+\cos(x)}{9+16\sin(2x)}dx = \frac{1}{8}\int \frac{1}{u\sqrt{25-u}}du = -\frac{1}{4}\int \frac{1}{25-t^2}dt$$
$$= -\frac{1}{40}\ln\left|\frac{5+t}{5-t}\right| + C$$
$$= \frac{1}{40}\ln\left|\frac{5-t}{5+t}\right| + C$$

We can the substitute our expression for t and then our expression for u

$$\int \frac{\sin(x)+\cos(x)}{9+16\sin(2x)}dx = \frac{1}{40}\ln\left|\frac{5-t}{5+t}\right| + C = \frac{1}{40}\ln\left|\frac{5-\sqrt{25-u}}{5+\sqrt{25-u}}\right| + C$$
$$= \frac{1}{40}\ln\left|\frac{5-\sqrt{25-(9+16\sin(2x))}}{5+\sqrt{25-(9+16\sin(2x))}}\right| + C$$
$$= \frac{1}{40}\ln\left|\frac{5-\sqrt{16(1-\sin(2x))}}{5+\sqrt{16(1-\sin(2x))}}\right| + C$$
$$= \frac{1}{40}\ln\left|\frac{5-4\sqrt{(1-\sin(2x))}}{5+4\sqrt{(1-\sin(2x))}}\right| + C$$

We can then use the Fundamental Theorem of Calculus to evaluate the integral from 0 to $\frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}} \frac{\sin(x)+\cos(x)}{9+16\sin(2x)}dx = \left[\frac{1}{40}\ln\left|\frac{5-4\sqrt{1-\sin(2x)}}{5+4\sqrt{1-\sin(2x)}}\right|\right]\Bigg|_{x=0}^{x=\frac{\pi}{4}}$$
$$= \frac{1}{40}\left[\ln\left|\frac{5-4\sqrt{1-\sin\left(2\left(\frac{\pi}{4}\right)\right)}}{5+4\sqrt{1-\sin\left(2\left(\frac{\pi}{4}\right)\right)}}\right| - \ln\left|\frac{5-4\sqrt{1-\sin(2(0))}}{5+4\sqrt{1-\sin(2(0))}}\right|\right]$$
$$= \frac{1}{40}\left[\ln\left|\frac{5-4\sqrt{1-1}}{5+4\sqrt{1-1}}\right| - \ln\left|\frac{5-4\sqrt{1-0}}{5+4\sqrt{1-0}}\right|\right]$$
$$= \frac{1}{40}\left[\ln\left|\frac{5}{5}\right| - \ln\left|\frac{5-4}{5+4}\right|\right]$$
$$= -\frac{1}{40}\ln\left|\frac{1}{9}\right|$$
$$= \frac{1}{20}\ln(3)$$

Which is the same answer that we achieved in section 2.

Now let's take the derivative. It's extremely messy, so I'll start by only applying the chain rule once.

$$\frac{d}{dx}\left[\frac{1}{40}\ln\left|\frac{5-4\sqrt{1-\sin(2x)}}{5+4\sqrt{1-\sin(2x)}}\right|\right] = \left[\frac{1}{40}\frac{5+4\sqrt{1-\sin(2x)}}{5-4\sqrt{1-\sin(2x)}}\right]\frac{d}{dx}\left[\frac{5-4\sqrt{1-\sin(2x)}}{5+4\sqrt{1-\sin(2x)}}\right] \tag{14}$$

We can then find the derivative of the chain using quotient rule.

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{5-4\sqrt{1-\sin(2x)}}{5+4\sqrt{1-\sin(2x)}}\right] &= \frac{\frac{-4(-\cos(2x))(2)}{2\sqrt{1-\sin(2x)}}\left(5+4\sqrt{1-\sin(2x)}\right)-\frac{4(-\cos(2x))(2)}{2\sqrt{1-\sin(2x)}}\left(5-4\sqrt{1-\sin(2x)}\right)}{\left(5+4\sqrt{1-\sin(2x)}\right)^2} \\
&= \frac{\frac{4\cos(2x)}{\sqrt{1-\sin(2x)}}\left(5+4\sqrt{1-\sin(2x)}\right)+\frac{4\cos(2x)}{\sqrt{1-\sin(2x)}}\left(5-4\sqrt{1-\sin(2x)}\right)}{\left(5+\sqrt{1-\sin(2x)}\right)^2} \\
&= \frac{\frac{4\cos(2x)}{\sqrt{1-\sin(2x)}}\left(5+4\sqrt{1-\sin(2x)}+5-4\sqrt{1-\sin(2x)}\right)}{\left(5+4\sqrt{1-\sin(2x)}\right)^2} \\
&= \frac{40\cos(2x)}{\sqrt{1-\sin(2x)}\left(5+4\sqrt{1-\sin(2x)}\right)^2} \qquad (15)
\end{aligned}
$$

Substituting (15) into (14) yields,

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{1}{40}\ln\left|\frac{5-4\sqrt{1-\sin(2x)}}{5+4\sqrt{1-\sin(2x)}}\right|\right] &= \left[\frac{1}{40}\frac{5+4\sqrt{1-\sin(2x)}}{5-4\sqrt{1-\sin(2x)}}\right]\left[\frac{40\cos(2x)}{\sqrt{1-\sin(2x)}\left(5+4\sqrt{1-\sin(2x)}\right)^2}\right] \\
&= \frac{\cos(2x)}{\sqrt{1-\sin(2x)}}\frac{1}{\left(5-4\sqrt{1-\sin(2x)}\right)\left(5+4\sqrt{1-\sin(2x)}\right)} \\
&= \frac{\cos(2x)}{\sqrt{1-\sin(2x)}}\frac{1}{25-16\left(1-\sin(2x)\right)} \\
&= \frac{\cos(2x)}{\sqrt{1-\sin(2x)}}\frac{1}{9+16\sin(2x)} \qquad (16)
\end{aligned}
$$

From (4), we know $\sqrt{1-\sin(2x)} = \cos(x) - \sin(x)$ and thus by the cosine double angle formula and difference of squares, $\frac{\cos(2x)}{\sqrt{1-\sin(2x)}} = \sin(x) + \cos(x)$. Then we can directly substitute which shows that the derivative of our antiderivative is indeed the integrand. This means that we performed the integral correctly.

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{1}{40}\ln\left|\frac{5-4\sqrt{1-\sin(2x)}}{5+4\sqrt{1-\sin(2x)}}\right|\right] &= \left(\sin(x)+\cos(x)\right)\frac{1}{9+16\sin(2x)} \\
&= \frac{\sin(x)+\cos(x)}{9+16\sin(2x)}
\end{aligned}
$$