Electric Field of a Hydrogen Atom

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1 Problem

A hydrogen atom can be considered as having a central point-like proton charge $+q$ and an electron of negative charge $-q$, where q is the fundamental charge. Instead of considering the Bohr-Ruthorford diagram or perhaps the spdf orbitals, we're going to model the charge of the electron based on a charge volume density:

$$\rho\left(r\right) = Ae^{\frac{-2r}{a_o}} \tag{1}$$

In (1), ρ is the charge density. r is the radial distance from the centre of the atom. a_o is called the Bohr radius which is approximately 0.53×10^{-10}. A is a constant that we seek to determine. Furthermore, we seek to find the electric field produced by the atom at the Bohr radius.

2 An Important Integral

An integral that pops up in the solution is the following.

$$\int_0^\infty x^2 e^{-kx}\,dx$$

We will evaluate this integral by first finding it's antiderivative and then evaluating it at both infinity and 0. The method used for finding the antiderivative is integration by parts.

$$u = x^2 \qquad dv = e^{-kx}$$
$$du = 2xdx \qquad v = -\frac{1}{k}e^{kx}$$

We then start our integration by parts.

$$\int udv = uv - \int vdu$$
$$\int x^2 e^{-kx}\,dx = -\frac{x^2 e^{-kx}}{k} - \int \frac{2}{k} - xe^{-kx}dx$$
$$\int x^2 e^{-kx}\,dx = -\frac{x^2 e^{-kx}}{k} + \frac{2}{k}\int xe^{-kx}dx$$

We then set up integration by parts for the right integral.

$$u = x \qquad dv = e^{-kx}$$
$$du = dx \qquad v = -\frac{1}{k}e^{kx}$$

$$\begin{aligned}
\int x^2 e^{-kx}\,dx &= -\frac{x^2e^{-kx}}{k} + \frac{2}{k}\left(-\frac{xe^{-kx}}{k} - \int -\frac{1}{k}e^{-kx}dx\right)\\
&= -\frac{x^2e^{-kx}}{k} + \frac{2}{k}\left(-\frac{xe^{-kx}}{k} + \int \frac{1}{k}e^{-kx}dx\right)\\
&= -\frac{x^2e^{-kx}}{k} + \frac{2}{k}\left(-\frac{xe^{-kx}}{k} - \frac{1}{k^2}e^{-kx}\right)\\
&= -\frac{x^2e^{-kx}}{k} - \frac{2xe^{-kx}}{k^2} - \frac{2e^{-kx}}{k^3} + C\\
&= -e^{-kx}\left(\frac{x^2}{k} + \frac{2x}{k^2} + \frac{2}{k^3}\right) + C
\end{aligned}$$

Now let's consider the bounds of integration. For a more rigorous procedure, we're going to take a limit because we are working with infinities.

$$\begin{aligned}
\int_0^\infty x^2e^{-kx}dx &= \lim_{t\to\infty}\int_0^t x^2e^{-kx}dx\\
&= \lim_{t\to\infty}\left(-\frac{x^2e^{-kx}}{k} - \frac{2xe^{-kx}}{k^2} - \frac{2e^{-kx}}{k^3}\Bigg|_{x=0}^{x=t}\right)\\
&= \lim_{t\to\infty}\left(-\frac{t^2e^{-kt}}{k} - \frac{2te^{-kt}}{k^2} - \frac{2e^{-kt}}{k^3}\right) - \left(-0 - 0 - \frac{2e^0}{k^3}\right)\\
&= 0 + \frac{2}{k^3}\\
&= \frac{2}{k^3}
\end{aligned} \tag{2}$$

3 Finding the value of A

We first seek to find the value of the constant A. We do this by considering that the sum of all the negative charge produced by the atom should add up to $-q$. To find the value, we will consider an infinitesimally small shell of charge. We then want to sum up every one of these shells (all the way to infinity). We can represent this with the following integral

$$-q = \int_0^\infty dq \tag{3}$$

Here, dq is the amount of charge that each shell contains. We can find the value of dq for a given radial distance from the centre by considering equation given for charge density. By definition of charge density we have

$$\begin{aligned}
\rho &= \frac{dq}{dV}\\
dq &= \rho dV\\
dq &= Ae^{\frac{-2r}{a_o}}dV
\end{aligned} \tag{4}$$

We can find dV by using it's definition

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ dV &= 4\pi r^2 dr \end{aligned} \tag{5}$$

By substituting (5) into (4), we then have

$$dq = 4\pi A r^2 e^{\frac{-2r}{a_o}} dr \tag{6}$$

Substituting (6) into (3) we get the following integral

$$\begin{aligned} -q &= \int_0^\infty 4\pi A r^2 e^{\frac{-2r}{a_o}} dr \\ -q &= 4\pi A \int_0^\infty r^2 e^{\frac{-2r}{a_o}} dr \end{aligned} \tag{7}$$

(7) is in the same form as (2) where $k = \frac{2}{a_o}$. Substituting the result, we get our value for A.

$$\begin{aligned} -q &= 4\pi A \frac{2}{\left(\frac{2}{a_o}\right)^3} \\ -q &= \frac{8\pi a_o^3 A}{8} \\ A &= \frac{-q}{\pi a_o^3} \end{aligned}$$

Substituting in the values, we get that $A \approx -2.6135 \times 10^{11}$ Cm^{-3}.

4 Finding the Electric Field at the Bohr Radius

To do this we will consider Gauss' law which relates the charge enclosed by a Gaussian Surface and the net electric flux through the surface.

$$q_{enc} = \varepsilon_0 \oint \vec{E} \cdot d\vec{A} \tag{8}$$

The Gaussian surface of choice is a sphere whose centre is the proton and whose radius is equal to the Bohr radius. This allows us to find the electric field at the Bohr radius.

The enclosed charge is that of the proton and the portion of the electron charge that is enclosed. The charge of proton is simply $+q$. A very similar process to Section 3 can be used to find the enclosed charge of the electron. Instead of summing up all the shells to infinity, we only need to sum up those within the radius of the Gaussian Surface (Which in this case is a_o). We then have the following

$$\begin{aligned} q_{enc} &= q_p + q_e \\ &= +q + 4\pi A \int_0^{a_o} r^2 e^{\frac{-2r}{a_o}} dr \\ &= q - 4\pi \frac{q}{\pi a_o^3} \int_0^{a_o} r^2 e^{\frac{-2r}{a_o}} dr \\ &= q - \frac{4q}{a_o^3} \int_0^{a_o} r^2 e^{\frac{-2r}{a_o}} dr \end{aligned} \tag{9}$$

We can evaluate the integral in (9) by considering the same anti-derivative that we found in Section 2.

$$
\begin{aligned}
q_{enc} &= q - \frac{4q}{a_o^3}\left[-e^{-\frac{2}{a_o}x}\left(\frac{x^2}{\frac{2}{a_o}} + \frac{2x}{\left(\frac{2}{a_o}\right)^2} + \frac{2}{\left(\frac{2}{a_o}\right)^3}\right)\Bigg|_{r=0}^{r=a_o}\right] \\
&= q - \frac{4q}{a_o^3}\left[-e^{-\frac{2}{a_o}x}\left(\frac{a_0x^2}{2} + \frac{a_o^2x}{2} + \frac{a_o^3}{4}\right)\Bigg|_{r=0}^{r=a_o}\right] \\
&= q - \frac{4q}{a_o^3}\left[-e^{-\frac{2}{a_o}a_o}\left(\frac{a_o^3}{2} + \frac{a_o^3}{2} + \frac{a_o^3}{4}\right) + e^0\left(0 + 0 + \frac{a_o^3}{4}\right)\right] \\
&= q - \frac{4q}{a_o^3}\left[\left(-e^{-2}\right)\frac{5a_o^3}{4} + \frac{a_o^3}{4}\right] \\
&= q - \frac{4q}{a_o^3}\left[\left(\frac{1-5e^{-2}}{4}\right)a_o^3\right] \\
&= q - q\left(1-5e^{-2}\right) \\
&= \frac{5q}{e^2}
\end{aligned} \tag{10}
$$

For the ring integral, the choice of our Gaussian surface removes the use of advanced calculus. Since the electric field lines are radiating radially, the choice of a concentric sphere means that $\vec{E} \parallel \vec{A}$ which means that $\Phi = EA$. The area can be determined using elementary geometry where the radius of the sphere is the Bohr radius, and so we have

$$
\oint \vec{E}\cdot d\vec{A} = E\left(4\pi a_o^2\right) \tag{11}
$$

Substituting (10) and (11) into (8), we can rearrange to find magnitude of the electric field.

$$
\begin{aligned}
\frac{5q}{e^2} &= \varepsilon_0 E\left(4\pi a_o^2\right) \\
E &= \frac{5q}{4\pi\varepsilon_0 e^2 a_o^2} \\
&\approx 2.90\times 10^{11}\ \mathrm{NC}^{-1}
\end{aligned}
$$