2017 AMC 12B Logarithm Problem

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1 Problem

Two real numbers x and y are chosen randomly with even distribution on the interval (0,1). What is the probability that,

$$\lfloor \log_2 (x) \rfloor = \lfloor \log_2 (y) \rfloor \tag{1}$$

This was problem 20 from the 2017 AMC 12B. I also want to solve the general case when we have base n where $n >= 2$. Note that n doesn't even have to be an integer. So what is the probability that,

$$\lfloor \log_n (x) \rfloor = \lfloor \log_n (y) \rfloor \tag{2}$$

2 My solution for base 2

I started by seeing when $\log (x)$ crosses each integer. We know that $\log (x) = 0$ if, $x = 1$. Moving leftwards on the x-axis towards 0, $\log (x)$first reaches -1 at 2^{-1} or $\frac{1}{2}$. It then reaches -2 at 2^{-2} or $\frac{1}{4}$. It reaches -3 at 2^{-3} or $\frac{1}{8}$. In general, for every positive integer k, $\log (x)$ will reach $-k$ at 2^{-k} or $\frac{1}{2^k}$. This process will go until negative infinity as we expect since $\log_2 (x)$ approaches negative infinity as x approaches 0.

Due to the continuity of the logarithm function, when we round down using the floor function we end up with a piecewise function.

$$\begin{cases} -1, & x \in \left[\frac{1}{2}, 1\right) \\ -2, & x \in \left[\frac{1}{4}, \frac{1}{2}\right) \\ -3, & x \in \left[\frac{1}{8}, \frac{1}{4}\right) \\ \dots \end{cases}$$

Note that the domains of the intervals are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ...and it continues to halve each time. The probabilities that both of the randomly chosen numbers are within the same domain is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$, $\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$, $\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)$, and so forth. Because these are all independent events, we can add up all the individual probabilities to get the total probability (P) that the 2 values are the same (i.e this is the probability that (1) is true).

$$\begin{aligned} P &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{8}\right)\left(\frac{1}{8}\right) + \dots \\ &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \dots \\ &= \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots \end{aligned} \tag{3}$$

Note that P is simply a geometric series where the starting value (a) is $\frac{1}{4}$ and the common ratio (r) is $\frac{1}{4}$. We know that an infinite sum of a geometric series is

$$S = \frac{a}{1-r} \tag{4}$$

Applying (4) to (3), we get

$$\begin{aligned} P &= \frac{\frac{1}{4}}{1-\frac{1}{4}} \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} \\ &= \frac{3}{4} \end{aligned}$$

3 The General Case

Using the same logic as Section 2, I can produce the following table for the piecewise function. Recall the part that is used to determine the probability as used in Section 2 is the "length" of the domain.

$\log_n(x)$	Domain of x	"Length" of domain
-1	$x \in \left(\frac{1}{n}, 1\right)$	$\frac{n-1}{n}$
-2	$x \in \left(\frac{1}{n^2}, \frac{1}{n}\right)$	$\frac{n-1}{n^2}$
-3	$x \in \left(\frac{1}{n^3}, \frac{1}{n^2}\right)$	$\frac{n-1}{n^3}$
...	...	...
-k	$x \in \left(\frac{1}{n^k}, \frac{1}{n^{k-1}}\right)$	$\frac{n-1}{n^k}$

Again, we have to sum up the squares of the lengths of the domains. This is because each length is the probability that one number will lie within that domain and we want both of the randomly generated numbers to lie within the same domain. We then have the following expression for probability for P_n to denote that it's for base n.

$$\begin{aligned} P_n &= \left(\frac{n-1}{n}\right)^2 + \left(\frac{n-1}{n^2}\right)^2 + \left(\frac{n-1}{n^3}\right)^2 + \dots \\ &= \frac{(n-1)^2}{n^2} + \frac{(n-1)^2}{n^4} + \frac{(n-1)^2}{n^6} + \dots \end{aligned} \tag{5}$$

Similarly to Section 2, we can apply (4) to (5) to get. In this case, $a = \frac{(n-1)^2}{n^2}$ and $r = \frac{1}{n^2}$. It's important to note that (4) converges only when $|r| < 1$. In our case, this would mean that the probability that we end up with is only defined for $n > 1$.

$$\begin{aligned} P_n &= \frac{\frac{(n-1)^2}{n^2}}{1-\frac{1}{n^2}} \\ &= \frac{\frac{(n-1)^2}{n^2}}{\frac{n^2-1}{n^2}} \\ &= \frac{(n-1)^2}{n^2-1} \\ &= \frac{(n-1)^2}{(n-1)(n+1)} \\ &= \frac{n-1}{n+1} \end{aligned}$$

It's nice to note that $\lim_{n \to \infty} P_n = 1$ as we would expect. This is because if the base gets extremely large, almost all the values will floor to -1. This would mean that almost always the 2 numbers generated will have the same value.

4 The Computer Program

I ended up writing a computer program for this problem to verify my answer. It is simply a probability simulator that runs through the scenarios thousands of times. If you found this paper in a google drive folder, the code for this problem is found on my GitHub (@ArthurXu17). If you found this paper in my GitHub repository "Math Simulations", the corresponding program is titled Logarithms.java.