Exercice 6

1) $f_a(x) = \exp(-a x^2)$

$\hat{f}_a(\xi) = \int_{\mathbb{R}} \exp(-a x^2) \exp(-i x \xi)\, dx$ → compliqué à calculer

Equation diff vérifiée par f_a

$f'_a(x) = -2ax \exp(-ax^2) = -2ax f_a(x) \Rightarrow \boxed{f'_a(x) + 2a x f_a(x) = 0}$

↳ dérivation des fcts composées

2) $\widehat{f'_a} + 2a \widehat{x f_a(x)} = 0$

$\hat{f}'(\xi) = \int_{\mathbb{R}} f'(x) \exp(-i \xi x)\, dx \overset{IPP}{=} (i\xi) \int f(x) \exp(-i\xi x) = (i\xi) f(x)$ (cours)

$\widehat{f'_a} = (i\xi)\hat{f}_a$

Autre formule $\dfrac{d \hat{f}(\xi)}{d\xi} = -i \widehat{x f(x)}\, \xi$

$\widehat{f'_a} + 2ai\,\widehat{(-ix f_a(x))} = 0 \qquad -i \times i = -(-1) = 1$

$(i\xi) \hat{f}_a(\xi) + 2ia \dfrac{d \hat{f}(\xi)}{d\xi} = 0$

$\boxed{\dfrac{d}{d\xi}[\hat{f}_a(\xi)] + \dfrac{1}{2a} \xi \hat{f}_a(\xi) = 0}$ éq du type $y(\xi) + \dfrac{1}{2a} y'(\xi) = 0$

$\Rightarrow \hat{f}_a(\xi) = C \exp\left(-\dfrac{\xi^2}{4a}\right) \qquad C = \hat{f}_a(0)$

Rmq La transformée de Fourier d'une gaussienne est une Gaussienne de coeff $\left(\dfrac{1}{4a}\right)$

$\hat{f}_a(0) = \int_{\mathbb{R}} \exp(-ax^2)\, dx$

On sait $\int_{\mathbb{R}} \exp -x^2\, dx = \sqrt{\pi}$ Par changement de variable $u = \sqrt{a}\, x \quad du = \sqrt{a}\, dx$

$\hat{f}_a(0) = \dfrac{1}{\sqrt{a}} \int \exp -u^2\, du = \sqrt{\dfrac{\pi}{a}}$

$\hat{f}_a(\xi) = \sqrt{\dfrac{\pi}{a}} \exp\left(-\dfrac{\xi^2}{4a}\right)$

21/03/17 Test 1

Exo 1 a) $y' = y \cos t$ (linéaire) $\dfrac{y'}{y} = \cos t \qquad \ln \dfrac{y(t)}{y_0} = \int \cos t\, dt = \sin t$

$y(t) = y_0 e^{\sin t} = e^{\sin t}$

b) $y'(t) = y(t) + \cos(t)$ (variation cst^{te}) $\quad u' = u \quad u = e^t \qquad y = c e^t$

$y' = c' e^t + \cancel{c e^t} = \cancel{y} + \cos t$

$c' = \cos t\, e^{-t} = Re(e^{it - t})$

$c(t) = Re \int_0^t e^{t(i-1)} + c(0) = Re\left(\dfrac{e^{t(i-1)}}{i-1}\right) + c(0)$

$\dfrac{1}{i-1} e^{t(i-1)} = \dfrac{1}{2}(i+1) e^{t(i-1)} = \dfrac{1}{2} e^{-t}(\cos t + i \sin t)(i+1) = \dfrac{1}{2} e^{-t}[\cos t - \sin t]$

$y = (\dfrac{1}{2} e^{-t}[\cos t - \sin t] + c) e^t$

c) $y' = \dfrac{\exp(-y)^2}{y} \qquad y' = f(y)$ méthode séparation des variables

$\dfrac{y'}{f(y)} = 1 \qquad \dfrac{dy}{f(y)} = dt \qquad \int \dfrac{dy}{f(y)} = t \qquad \dfrac{dy}{dt} y \exp y^2 = 1$

$y \exp y^2\, dy = dt \qquad \int_1^{y(t)} y \exp y^2\, dy = t$

$\frac{d}{dy}\left(\frac{1}{2}\exp y^2\right) \Rightarrow \frac{1}{2}\left[\exp y^2(t) - e\right] = t$

$= \exp y^2(t) = 2t + e$

$y^2(t) = \ln(2t+e) \qquad y(t) = \sqrt{\ln(2t+e)}$

d) Variation de la constante

Eq homogène $u'(t) = \frac{u(t)}{t} \qquad \frac{du}{dt} = \frac{u(t)}{t} \Rightarrow \int \frac{du}{u} = \frac{dt}{t} \qquad \ln \frac{u(t)}{u(t_0)} = \ln t$

$u(t) = t$

Ensuite variation de la c^{te}

Exo 2 Méthode des caractéristiques $\frac{dx(t)}{dt} = x^3(t) \qquad \frac{dx}{dx^3} = dt$

$\int \frac{dx}{x^3} = \int dt \qquad \frac{dx}{x^3} = d\left(-\frac{1}{2x^2}\right)$

$\left[-\frac{1}{2x^2(t)}\right]_t^{t^2} = \left(t\right)_t^{t^2} = -\frac{1}{2x^2(t^2)} + \frac{1}{2x^2(t)} = (t^2 - t)$

$x(t) = \left(\sqrt{2(t^2-t) + \frac{1}{x^2(t^2)}}\right)^{-1}$

$x(0) = \left(\sqrt{2t - \frac{1}{x_0^2}}\right)^{-1} = \frac{1}{\sqrt{2t^2 + \frac{1}{x_0^2}}} \qquad u(x^2, t^2) = u(x(t), t) \quad \forall t \in [0, t^2]$

$= u(x(0), 0) = u_0(x(0))$

$= \sin\left(\frac{1}{\sqrt{2t^2 \frac{1}{x_0^2}}}\right)$

Exo 3

$u_t + u_{xxxx} = 0$

1) unicité ; méthode énergie On multiplie par u

$u_t u + u_{xxxx} u = 0$ On intègre sur une période à t posé

$\frac{d}{dt}\int_0^{2\pi} \frac{1}{2}|u|^2(x,t)\,dx + \int_0^{2\pi} u_{xxxx}(x,t)\, u(x,t)\,dx = 0$

$\frac{d}{dt}\left(\frac{1}{2}\int_0^{2\pi} |u|^2(x,t)\,dx\right) + \int_0^{2\pi} |u_{xx}|^2(x,t)\,dx = 0$

$\Rightarrow \frac{d}{dt}\left(\int_0^{2\pi} |u|^2(x,t)\,dx\right) \leq 0 \qquad$ si $u_0 = 0 \quad u(x,t) = 0 \quad \forall (x,t)$

2) On cherche une solution 2π périodique par rapport à x

$u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}(k,t)\exp(ikx)\,dx \qquad \hat{u}(k,t) = \frac{1}{2\pi}\int_0^{2\pi} u(x,t)\exp -ikx\,dx$

$\widehat{u_{xxxx}}(k,t) = (ik)^4\,\hat{u}(k,t) = k^4\,\hat{u}(k,t) \qquad \boxed{\frac{d}{dt}\hat{u}(k,t) = -k^4\,\hat{u}(k,t)}$

$\hat{u}(k,t) = \hat{u}(k,0)\exp(-k^2 t)$

$\hat{u}(k,t) = \hat{u}_0(k)\exp -k^4 t$

$u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_0(k)\exp(k^4 t)\exp ikx$

3) $u_0(x) = (\sin(x))^2$

$\sin x = \frac{1}{2i}(\exp ix - \exp -ix)$

$\sin^2 x = -\frac{1}{-4}(\exp 2ix + \exp -2ix - 2) \qquad \hat{u}_0(2) = -\frac{1}{4}$

$\hat{u}_0(2) = -\frac{1}{4}$

$\hat{u}_0(0) = \frac{1}{2}$

Les autres sont nuls

$u(x,t) = -\frac{1}{4}\exp(-4t)\exp 2ix - \frac{1}{4}\exp(-4t)\exp -2i + \frac{1}{2}$

4) Eq diff pour le coef de Fourier

$u_t + u_{xxxx} = f \qquad \frac{d}{dt}\hat{u}(k,t) = -k^4\,\hat{u}(k,t) + \hat{f}(k,t)$

$\hat{u}(k,t) = \int_0^t \exp(-k^4(t-s))\,\hat{f}(k,s)\,ds \qquad \hat{u}(k,t) = 0 \qquad$ Ici $f(x,t) = t(\cos x)^3$

On développe $(\cos x)^3$

$$(\cos x)^3 = \frac{1}{8}(\exp ix + \exp -ix)^3 = \frac{1}{8}(\exp 3ix + \exp -3ix + 3\exp ix + 3\exp -ix)$$

$\hat{f}(3,0) = \frac{3}{8}$ $\quad \hat{f}(1,0) = \frac{3\cdot 0}{8}$

$\hat{f}(-3,0) = \frac{3}{8}$ $\quad \hat{f}(-1,0) = \frac{3\cdot 0}{8}$

$$\hat{u}(3,t) = \frac{1}{8}\int_0^t \exp(-81(t-s))\,s\,ds = \frac{\exp(-81t)}{8}\int_0^t \exp(81s)\,s\,ds$$

$$\int_0^t \exp(+81s)\,s\,ds \overset{ipp}{=} \left[\frac{\exp(81)s}{81}s\right]_0^t + \frac{1}{81}\int_0^t \exp(-81s)\,ds$$

$$= -\exp(-81t) + \frac{+1}{(81)^2}\left[\exp(-81t)-1\right] = \frac{\exp(-81t)}{8}\left[\frac{t\exp(-81t)}{81} + \frac{1}{81}\left[\frac{\exp(-81t)}{81} - 1\right]\right]$$

Exo 6

$f_a(x) = \exp(-a|x|) \quad \forall x \in \mathbb{R},\ a \in \mathbb{C}$

$$\hat{f}_a(\xi) = \int_{\mathbb{R}} \exp(-a|x|)\exp(-i\xi x)\,dx = \int_0^{+\infty} \exp(-a-i\xi)x\,dx + \int_{-\infty}^0 \exp(a-i\xi)x\,dx$$

$$= \left[\frac{\exp(-a-i\xi)x}{-a-i\xi}\right]_0^{+\infty} + \left[\frac{\exp(a-i\xi)x}{(a-i\xi)}\right]_{-\infty}^0 = \frac{1}{a+i\xi} + \frac{1}{a-i\xi} = \frac{(a-i\xi)+(a+i\xi)}{a^2+\xi^2}$$

$$\hat{f}_a(\xi) = \frac{2a}{a^2+\xi^2}$$

2) On utilise parseval $\qquad a = 1$

$$\int_{\mathbb{R}} f_g^2(x)\,dx = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\xi)|^2\,d\xi = \frac{1}{2\pi}\int \frac{4}{(1+\xi)^2}\,d\xi = \frac{2I}{\pi}$$

$$I = \frac{\pi}{2}\int_{\mathbb{R}} \exp(-2|x|)\,dx = \pi\left(\int_0^{+\infty} \exp -2x\,dx\right) = \pi\left(\frac{\exp -2x}{-2}\right)_0^{+\infty} = \frac{\pi}{2}$$

28/03/17

$\boxed{\text{TD4}}$

$T \in \mathcal{D}'(\Omega) \qquad T : \varphi \mapsto \langle T, \varphi\rangle,\ \forall \varphi \in \mathcal{D}(\Omega)$

⇒ Linéaire

⇒ Continue au sens suivant $\forall K \subset \Omega$ compact $\exists\, d_K, C_K > 0$

$$|\langle T,\varphi\rangle| \leq C_K \sup_{\substack{x\in K\\ |\alpha|\leq d_K}} \{|\partial^\alpha \varphi(x)|\} \quad \forall \varphi \in \mathcal{D}(\Omega) \quad \operatorname{supp}\varphi \subset K$$

↳ On identifie la fct de $L^1_{loc}(\Omega)$ à des distributions

$$\langle f,\varphi\rangle = \int_\Omega f(x)\varphi(x)\,dx$$

Exercice 1

1) T n'est pas linéaire ⇒ (ce n'est pas une distribution)

2) $T(\varphi) = \int_1^2 \varphi(x)\,dx$ $\quad T$ linéaire $\qquad T(\varphi) = \int_{\mathbb{R}} \mathbb{1}_{[1,2]}\varphi(x)\,dx$

Oui $\qquad$ "$T = \mathbb{1}_{[1,2]}$"

3) pas linéaire ⇒ ce n'est pas distribution

$T(\varphi) = \exp(\varphi(x))$

$T(\varphi - \varphi) \neq T(\varphi) + T(-\varphi)$

4) $T(\varphi) = \sum_{n\in\mathbb{N}} \frac{1}{n}\left[\varphi\left(\frac{1}{n}\right) - \varphi(0)\right]$ $\qquad T$ linéaire (on a que des opérations finies)

bien définie ? $\quad$ Théorème d'accroissement finis $\quad |\varphi(\frac{1}{n}) - \varphi(0)| \leq \frac{1}{n}\|\varphi'\|_\infty$

$$\Rightarrow |T(\varphi)| \leq \sum_{n=1}^{+\infty} \frac{1}{n^2}\|\varphi'\|_\infty \qquad \|T(\varphi)\| \leq C_0\|\varphi'\|_\infty \qquad C_0 = \sum_{n=1}^{+\infty}\frac{1}{n^2} < +\infty$$

$$\leq \left[\sum_{n=1}^{+\infty}\frac{1}{n^2}\right]\|\varphi'\|_\infty \qquad \text{Continuité} \quad (d_K = 1) \quad C_K = C_0 \quad \forall K \subset \mathbb{R}$$