Exo 1

θ = proportion de votant pour C (inconnue)

1) θ est estimé par $\hat\theta$ = proportion de votant pour C dans l'echantillon

$$= \frac{516}{1000}$$

2) $$IC_{95\%}(\theta) = \left[\hat\theta \pm 1{,}96\sqrt{\frac{\hat\theta(1-\hat\theta)}{n}}\right] = [0{,}485 \ ; \ 0{,}547]$$

0,5 $\in$ l'$IC_{95\%}(\theta)$ donc semble difficile de prévoir s'il va gagner

3) $$e = 1{,}96\sqrt{\frac{\hat\theta(1-\hat\theta)}{n}} = 0{,}031$$

4) n tq $e = 0{,}01$ $\Rightarrow$ $n_{max} = \left(\frac{1{,}96}{2e}\right)^2 = 9604$ car $\theta \to \theta(1-\theta)$ est maximum pour $\theta = \frac{1}{2}$ et vaut $\frac{1}{4}$

Exo 2

1) $$E(X_i) = \int_0^\theta \frac{4x^4}{\theta^4} = \frac{4}{\theta^4}\left[\frac{x^5}{5}\right]_0^\theta = \frac{4}{5}\theta$$

$$\boxed{\hat\theta_1 = \frac{5}{4}\bar X}$$

2) $E(\hat\theta_1) = \frac{5}{4}E(\bar X) = \frac{5}{4}E(X_i) = \theta$ Donc $\hat\theta_1$ est sans biais

$$R(\hat\theta_1, \theta) = \text{biais}^2 + \text{var}(\hat\theta_1)$$
$$= \text{Var}\left(\frac{5}{4}\bar X\right)$$
$$= \frac{25}{16}\,\frac{\text{Var}(X_i)}{n}$$

avec $\text{Var}(X_i) = E(X_i^2) - E(X_i)^2$

$$E(X_i^2) = \int_0^4 x^2\,\frac{4x^3}{\theta^4}\,dx = \frac{4}{\theta^4}\left[\frac{x^6}{6}\right]_0^\theta = \frac{2\theta^2}{3}$$

$$\Leftrightarrow \text{var}(X_i) = \frac{2\theta^2}{3} - \frac{16}{25}\theta^2 = \frac{50-48}{75}\theta^2 = \frac{2\theta^2}{75}$$

$$\Rightarrow R(\hat\theta_1, \theta) = \frac{25}{16}\times\frac{2\theta^2}{75n} = \frac{2\theta^2}{48n} = \frac{\theta^2}{24n}$$

3) $E(X_i^2) = \frac{2\theta^2}{3} \Rightarrow \theta = \sqrt{\frac{3}{2} E(X_i^2)} \Rightarrow \hat{\theta}_2 = \sqrt{\frac{3}{2} \frac{1}{m} \Sigma X_i^2}$

4) critère biais / variance donc calcul du risque

EXO 3

1) $E(X) = \frac{\sqrt{\pi}}{2} \theta^{3/2} \Rightarrow \theta^{3/2} = \frac{2}{\sqrt{\pi}} E(X) \Rightarrow \boxed{\hat{\theta}_1 = \left(\frac{2}{\sqrt{\pi}} \bar{X}\right)^{2/3}}$

2) $L = \prod_{i=1}^{m} f_\theta(x_i) = \prod_{i=1}^{m} \frac{2 x_i}{\theta^3} e^{-\frac{x_i^2}{\theta^3}}$ si $x_i \geq 0 \ \forall i$

or $P(X_i > 0) = \int_0^\infty f_\theta(x) = 1$

$= \left(\frac{2}{\theta^3}\right)^n \prod_{i=1}^{m} x_i \ e^{-\frac{x_i^2}{\theta^3}}$

$\ell = \ln(2^n) + \Sigma \ln(x_i) - 3m \ln\theta - \frac{1}{\theta^3} \sum_{i=1}^{m} x_i^2$ $\quad \theta^{-3} \to -3\theta$

$\frac{\partial \ell}{\partial \theta} = -\frac{3m}{\theta} + 3\left(\sum_{i=1}^{m} x_i^2\right) \times \frac{1}{\theta^4}$

$\geqq 0$

$\Leftrightarrow \frac{\Sigma x_i^2}{\theta^4} \geqq \frac{n}{\theta} \Leftrightarrow \theta^3 \leqq \frac{\Sigma x_i^2}{n} \Rightarrow \boxed{\hat{\theta}_2 = \left(\frac{1}{n} \Sigma X_i^2\right)^{1/3}}$

On vérifie que $\hat{\theta}_2$ est bien un maximum.

3) Consistance ?

- $\bar{X} \xrightarrow[\text{LfGN}]{P} E(X_i) = \frac{\sqrt{\pi}}{2} \theta^{3/2} \Rightarrow \hat{\theta}_1 = \left(\frac{2}{\sqrt{\pi}} \bar{X}\right)^{2/3} \xrightarrow{P} g\left(\frac{\sqrt{\pi}}{2} \theta^{3/2}\right)$

car $E(X_i) < \infty$

car $g(x) = \left(\frac{2}{\sqrt{\pi}} x\right)^{2/3}$ continue sur $x > 0$ (et $\bar{X} = \frac{1}{n} \Sigma X_i$ avec $X_i >$ ps

Donc $\hat{\theta}_1 \xrightarrow{P} \left(\frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \theta^{3/2}\right)^{2/3} = \theta$

- $\frac{1}{m} \Sigma X_i^2 \xrightarrow[\text{LfGN}]{P} E(X_i^2) = \theta^3 \Rightarrow \hat{\theta}_2 \xrightarrow{P} g(\theta^3) = \theta$

car $E(X_i^2) < \infty$

car $g(x) = x^{1/3}$ continue sur $x > 0$

Donc $\hat{\theta}_2 \xrightarrow{P} \theta$

Asympt normal ?

- TCL : $\sqrt{n}\left(\overline{X} - \frac{\sqrt{\pi}}{2}\theta^{3/2}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \underbrace{var(X_i)})$

$$\underset{=}{} E(X_i^2) - E(X_i)^2 = \theta^3 - \left[\frac{\sqrt{\pi}}{2}\theta^{3/2}\right]^2$$
$$= \theta^3 - \frac{\pi}{4}\theta^3$$

delta method avec $g(x) = \left(\frac{2}{\sqrt{\pi}}x\right)^{2/3}$ derivable en $\frac{\sqrt{\pi}}{2}\theta^{3/2} > 0$

$$g'(x) = \frac{2}{3}\left(\frac{2}{\sqrt{\pi}}x\right)^{2/3-1} \times \frac{2}{\sqrt{\pi}} = \frac{4}{3\sqrt{\pi}}\left(\frac{2}{\sqrt{\pi}}x\right)^{2/3-1}$$

$$g'\left(\frac{\sqrt{\pi}}{2}\theta^{3/2}\right) = \frac{4}{3\sqrt{\pi}}\left(\frac{2}{\sqrt{\pi}}\right)^{-1/3}\left(\frac{\sqrt{\pi}}{2}\theta^{3/2}\right)^{-1/3}$$

$$= \frac{4}{3\sqrt{\pi}}\theta^{-1/2} = \frac{4}{3\sqrt{\theta\pi}}$$

$$\Rightarrow \sqrt{n}\left(\hat{\theta}_1 - \theta\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \boxed{\left(\frac{4}{3\sqrt{\theta\pi}}\right)^2 \theta^3\left(1-\frac{\pi}{4}\right)}\right)$$

$$\frac{16}{9\pi}\left(1-\frac{\pi}{4}\right)\theta^2 \simeq 0{,}12\,\theta^2$$

- TCL $\sqrt{n}\left(\frac{1}{n}\sum X_i^2 - \theta^3\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \underbrace{Var(X_i^2)})$

$$E(X_i^4) - E(X_i^2)^2 = 2\theta^6 - (\theta^3)^2 = \theta^6$$

delta-method avec $g(x) = x^{1/3}$ derivable en $\theta^3 > 0$

$$g'(x) = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3}$$

$$g'(\theta^3) = \frac{1}{3}(\theta^3)^{-2/3} = \frac{1}{3\theta^2}$$

$$\Rightarrow \sqrt{n}\left(\hat{\theta}_2 - \theta\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{9\theta^4} \times \theta^6 = \frac{\theta^2}{9}\right) \simeq 0{,}11\,\theta^2$$

Donc l'EMV est legerement meilleur car de variance asymptotique + faible

4) $\sqrt{n}\left(\hat{\theta}_2 - \theta\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\theta^2}{9}\right)$

$\Rightarrow \dfrac{\sqrt{n}\left(\hat{\theta}_2 - \theta\right)}{\frac{\theta}{3}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$

et $\hat{\theta}_2 \xrightarrow{P} \theta$

$\overset{\text{Slutsky}}{\Longrightarrow} \dfrac{\sqrt{n}\left(\hat{\theta}_2 - \theta\right)}{\frac{\hat{\theta}_2}{3}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$

$$\Rightarrow \boxed{IC_{99\%}(\theta) = \left[\hat{\theta}_2 \pm q\,\frac{\hat{\theta}_2}{3\sqrt{n}}\right]} \quad \text{avec } q = 2{,}578$$