Floating-point arithmetic and error analysis (AFAE)

Tutorial n° 3 - Error analysis, conditioning

Exercise 1 (Summation)**.** Let $p_i \in \mathbb{F}, 1 \leq i \leq n$ be a sequence of n floating-point numbers.

1. Show that the condition number of the computation of the summation satisfies

$$\operatorname{cond}(\sum_{i=1}^{n} p_i) = \frac{\sum_{i=1}^{n} |p_i|}{|\sum_{i=1}^{n} p_i|}.$$

   We recall that by definition

$$\operatorname{cond}(\sum_{i=1}^{n} p_i) := \lim_{\varepsilon \to 0} \sup \left\{ \left| \frac{\sum_{i=1}^{n} \widetilde{p}_i - \sum_{i=1}^{n} p_i}{\varepsilon \sum_{i=1}^{n} p_i} \right| : |\widetilde{p}_i - p_i| \leq \varepsilon |p_i| \text{ for } i = 1, \dots, n \right\}.$$

2. Show that the recursive summation algorithm is *backward-stable.*
3. Derive a bound on the relative error for the summation.
4. Redo all the questions for the dot product.

Exercise 2 (Polynomial evaluation)**.** Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree n with floating-point coefficients.

1. Recall the formula for the condition number $\operatorname{cond}(p, x)$ of the polynomial evaluation of p in x.
2. Show that the Horner scheme for polynomial evaluation is *backward-stable.*
3. Derive a bound on the relative error for the polynomial evaluation.
4. Given a polynomial $q(x) = \sum_{i=0}^{n} b_i x^i$, we define the distance $d(p, q) = \max_i \{|a_i - b_i|/|a_i|\}$. Show that given p and z,

$$\min\{d(p, q) : q(z) = 0\} = 1/\operatorname{cond}(p, z).$$

Exercise 3 (Roots of polynomials)**.** Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree n with floating-point coefficients and α a simple root ($p(\alpha) = 0$ and $p'(\alpha) \neq 0$).

1. We define the condition number of the simple root α by

$$K(p, \alpha) := \lim_{\varepsilon \to 0} \sup_{|\Delta a_i| \leq \varepsilon |a_i|} \left\{ \frac{|\Delta \alpha|}{\varepsilon |\alpha|} \right\}.$$

   Show that

$$K(p, \alpha) = \frac{\widetilde{p}(|\alpha|)}{|\alpha||p'(\alpha)|},$$

   with $\widetilde{p}(x) := \sum_{i=0}^{n} |a_i| x^i$.
2. When is a simple root ill-conditioned?

Exercise 4 (Conditioning of the inverse of a matrix)**.** In the sequel, we will use the Euclidean $\|\cdot\|$. We define the condition number of the computation of the inverse of a matrix by

$$\kappa(A) := \lim_{\varepsilon\to 0} \sup_{\|\Delta A\|\le\varepsilon\|A\|} \left(\frac{\|(A+\Delta A)^{-1} - A^{-1}\|}{\varepsilon\|A^{-1}\|} \right).$$

1. Show that $\kappa(A) = \|A\|\|A^{-1}\|$.

2. We define the *distance to singularity* of a matrix A by

$$\operatorname{dist}(A) := \min\left\{ \frac{\|\Delta A\|}{\|A\|} : A + \Delta A \text{ singular} \right\}.$$

Show that $\operatorname{dist}(A) = \kappa(A)^{-1}$.

3. Express $\kappa(A)$ in terms of the singular values of A.