A generalization of Kirk's formula

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The price of an option is given by

$$\begin{aligned} C(V_k(t), V_l(u)) &= \mathbb{E}\left[\left(V_l(u) - V_k(t)\right)^+\right] \\ &= \mathbb{E}\left[\left(\zeta_1 F_1(t) + \zeta_2 F_1(u) + \zeta_3 F_2(t) + \zeta_4 F_2(u) - \alpha K\right)^+\right] \end{aligned} \tag{1}$$

Denoting $F_1 = F_1(t)$, $F_2 = F_1(u)$,$F_3 = F_2(t)$,$F_4 = F_1(u)$, we have

$$\begin{aligned} &= \mathbb{E}\left[\left(\zeta_1 F_1 - \alpha K - \sum_{i=2}^{4} \zeta_i F_i\right)^+\right] \\ &= \zeta_1 F_1 N(d_1) - \left(\sum_{i=2}^{4} \zeta_i F_i + \alpha K\right) N(d_2) \end{aligned} \tag{2}$$

where

$$d_1 = \frac{1}{\bar{\sigma}_- \sqrt{\tau}} \left(\ln \left(\frac{\zeta_1 F_1}{\sum_{i=2}^{4} \zeta_i F_i + \alpha K} \right) + \frac{1}{2} \bar{\sigma}_-^2 \tau \right)$$

$$d_2 = d_1 - \bar{\sigma}_- \sqrt{\tau}$$

$$\bar{\sigma}_- = \sqrt{\sigma_1^2 - 2\tilde{\rho}\sigma_1 \bar{\sigma}_+ + \bar{\sigma}_+^2}$$

$$\bar{\sigma}_+ = \tilde{\sigma}_+ \left(\frac{\sum_{i=2}^{4} \zeta_i F_i}{\sum_{i=2}^{4} \zeta_i F_i + \alpha K} \right)$$

$$\tilde{\sigma}_+ = \frac{\sqrt{\sum_{i,j=2}^{4} \zeta_i F_i \zeta_j F_j \sigma_j \sigma_i \rho_{ji}}}{\sum_{i=2}^{4} \zeta_i F_i}$$

$$\tilde{\rho} = \frac{1}{\tilde{\sigma}_+} \left(\frac{\sum_{i,j=2}^{4} \zeta_i F_i \sigma_i \rho_{i1}}{\sum_{i,j=2}^{4} \zeta_i F_i} \right)$$