A generalisation of the replication models approach to price gas storages connected to two hubs

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This paper generalises the valuation of natural gas storage contracts to two hubs through the replication model approach. We are using a portfolio of forwards for the firm injections and withdrawals and spread options to add optionality. We consider injections and withdrawals to and from a gas storage connected to two interconnected hubs.

1 Notation

The parameters of the model are the following:

- independent withdrawal days are indexed by $t = 1...n$;
- the initial energy in store is S_0 and the final energy in store must lie in $\left[S_{\text{end}}^{-}, S_{\text{end}}^{+}\right]$;
- position on the firm withdrawal strategy k exercised on day t of value $V_k^{\mathrm{W}}(t)$ (euros) is $W_k(t) \in [0,1]$ and the associated withdrawn energy is $E_k^{\mathrm{W}}(t)$ (mwh) and position on the firm injection strategy k exercised on day t of energy $E_k^{\mathrm{I}}(t)$ (mwh) of value $V_{\mathrm{k}}^{\mathrm{I}}(t)$ is $I_{\mathrm{k}}(t) \in [0,1]$ with $k \in \{1,2,3,4\}$ and $t \in \{1,...,n\}$;
- position on the option that substitutes $V_k(t)$ by $V_l(u)$ of value $C\big(V_k(t), V_l(u)\big)$ is $O\big(V_k(t), V_l(u)\big) \in [0,1]$ with $k, l \in \{1,2,3,4\}$ and $t, u \in \{1,...,n\}$;
- inventory level constraints: the nominal capacity is S_{MAX}. The minimum energy in store at the end of period t is $S_{\text{min}}(t)$. The maximum energy in store at the end of period t is $S_{\text{max}}(t)$;
- time dependent flow costs from hub h_k to the storage facility is $K_{h_k \to \text{SF}}(t)$;
- injection and withdrawal costs: the discounted fixed injection cost in period t is $K_{\mathrm{I}}^{0}(t)$. The injection fuel charge is a percentage $K_{\mathrm{I}}^{1}(t)$ of the market price. The discounted fixed withdrawal cost is $K_{\mathrm{W}}^{0}(t)$. The withdrawal fuel charge is $K_{\mathrm{W}}^{1}(t)$;
- injection and withdrawal rates: compressors' nominal capacities in period t are $f_{\mathrm{I}}(t)$ for injection and $f_{\mathrm{W}}(t)$ for withdrawal. Given current energy in store S, the level dependent injection reduction factor is $g_{\mathrm{I}}(t,S)$; the corresponding level dependent withdrawal reduction factor is $g_{\mathrm{W}}(t,S)$;
- time dependent reserved transport capacities from the storage facility to hub k is $Tr_{\text{SF}\to h_k}(t)$ and from hub k to the storage facility $Tr_{h_k \to \text{SF}}(t)$;

Costs are in the same unit as forward prices. Withdrawal capacity is in energy units per day. Days are indexed from 1 to the number or periods n.

2 Natural gas price model

We denote by $F_h(t_i, t_j)$ the forward price for delivery day t_j observed at hub h on date t_i.

$$F_h(t_i, t_j) = F_h(0, t_j) \exp\left(-\frac{1}{2}\sigma(t_i, t_j)^2 t_i + \sigma(t_i, t_j) W(t_i, m)\right)$$

where

- m is the month containing period t_j
- $\sigma(t_i, t_j)$ is the volatility for period t_j and expiry date t_i. It depends on the at-the-money monthly volatility $\sigma(t_i, m)$ for expiry on t_i and on the forward volatility $\sigma_{fwd}(m)$. If the exercise date t_i is either five days or less before the start date of t_j, or in the same month as t_j,

$$\sigma(t_i, t_j) = \sqrt{\frac{\sigma(t_i(m), m)^2 t_i(m) + \sigma_{fwd}(m)^2 \Delta t_i}{t_i(m) + \Delta t_i}}$$

  where $t_i(m)$ is the last day of the month preceding m and Δt_i is a time constant representing 15 days. By convention, $\sigma(t_i, m)$ is the standard expiry volatility when t_i is past the standard expiry date. If the exercise date t_i is at once more than five days before the start date of t_j and in an earlier month,

$$\sigma(t_i, t_j) = \sigma(t_i, m)$$

- $W(., m)$ is a standard Brownian motion. We denote by $\rho(m_1; m_2)$ the time spread correlation between the standard Brownian motions $W(., m_1)$ and $W(., m_2)$ associated to months m_1 and m_2. It does not depend on the exercise date

We recall that volatilities $\sigma(t_i, m)$ are given by swaption coefficients applied to standard expiry volatilities $\sigma(t_i(m), m)$.

3 Firm strategies

In this model we optimise the firm withdrawal strategies and the firm injection strategies, which are built similarly as in the previous model.

Here again, for simplicity we assume that withdrawal costs are included in the transport cost on the edge from the storage facility to the hub. A more realistic representation of the costs included in the strategies can be found in [Appendix A].

3.1 Firm withdrawals

On each day we have the four following (non mutually exclusive) firm withdrawal strategies

- The strategy withdrawing from the storage facility to sell at hub h_1 and on day t:

$$V_1^{\mathrm{W}}(t) = Tr_{\mathrm{SF}\to h_1}(t)\left[F_1(t) - K_{\mathrm{SF}\to h_1}(t)\right] + \min\left\{Tr_{\mathrm{SF}\to h_2}(t), Tr_{h_2\to h_1}(t)\right\}\left[F_1(t) - K_{\mathrm{SF}\to h_2}(t) - K_{h_2\to h_1}(t)\right]$$

- The strategy withdrawing from the storage facility to sell at hub h_1 and buying at hub h_2 to sell at hub h_1 on day t

$$\begin{aligned} V_2^{\mathrm{W}}(t) = {} & Tr_{\mathrm{SF}\to h_1}(t)\left[F_1(t) - K_{\mathrm{SF}\to h_1}(t)\right] + \min\left\{Tr_{\mathrm{SF}\to h_2}(t), Tr_{h_2\to h_1}(t)\right\}\left[F_1(t) - K_{\mathrm{SF}\to h_2}(t) - K_{h_2\to h_1}(t)\right] \\ & + \left(Tr_{h_2\to h_1}(t) - Tr_{\mathrm{SF}\to h_2}(t)\right)^+\left[F_1(t) - F_2(t) - K_{h_2\to h_1}(t)\right] \end{aligned}$$

- The strategy withdrawing from the storage facility to sell at hub h_2 and on day t:

$$V_3^{\mathrm{W}}(t) = Tr_{\mathrm{SF}\to h_2}(t)\,[F_2(t) - K_{\mathrm{SF}\to h_2}(t)] + \min\left\{Tr_{\mathrm{SF}\to h_1}(t), Tr_{h_1\to h_2}(t)\right\}\left[F_2(t) - K_{\mathrm{SF}\to h_1}(t) - K_{h_1\to h_2}(t)\right]$$

- The strategy withdrawing from the storage facility to sell at hub h_2 and buying at hub h_1 to sell at hub h_2 on day t

$$\begin{aligned} V_4^{\mathrm{W}}(t) = {} & Tr_{\mathrm{SF}\to h_2}(t)\,[F_2(t) - K_{\mathrm{SF}\to h_2}(t)] + \min\left\{Tr_{\mathrm{SF}\to h_1}(t), Tr_{h_1\to h_2}(t)\right\}\left[F_2(t) - K_{\mathrm{SF}\to h_1}(t) - K_{h_1\to h_2}(t)\right] \\ & + (Tr_{h_1\to h_2}(t) - Tr_{\mathrm{SF}\to h_1}(t))^+[F_2(t) - F_1(t) - K_{h_1\to h_2}(t)] \end{aligned}$$

3.2 Firm injections

Similarly as with firm withdrawing strategies we get the following four injection strategies

- a strategy where we inject from hub h_1 and use other edges to maximize the send in from h_1

$$V_1^{\mathrm{I}}(t) = Tr_{h_1\to\mathrm{SF}}(t)\,[F_1(t) + K_{h_1\to\mathrm{SF}}(t)] + \min\{Tr_{h_2\to\mathrm{SF}}(t), Tr_{h_1\to h_2}(t)\}[F_1(t) + K_{h_1\to h_2}(t) + K_{h_2\to S_F}(t)]$$

- the same as the first strategy but where we can buy from h_2 and inject into the storage facility

$$\begin{aligned} V_2^{\mathrm{I}}(t) = {} & Tr_{h_1\to\mathrm{SF}}(t)\,[F_1(t) + K_{h_1\to\mathrm{SF}}(t)] + \min\{Tr_{h_2\to\mathrm{SF}}(t), Tr_{h_1\to h_2}(t)\}[F_1(t) + K_{h_1\to h_2}(t) + K_{h_2\to S_F}(t)] \\ & + (Tr_{h_2\to\mathrm{SF}}(t) - Tr_{h_1\to h_2}(t))^+[F_2(t) + K_{h_2\to\mathrm{SF}}(t)] \end{aligned}$$

- a strategy where we inject from hub h_2 and use other edges to maximize the send in from h_2

$$V_3^{\mathrm{I}}(t) = Tr_{h_2\to\mathrm{SF}}(t)\,[F_2(t) + K_{h_2\to\mathrm{SF}}(t)] + \min\{Tr_{h_1\to\mathrm{SF}}(t), Tr_{h_2\to h_1}(t)\}[F_2(t) + K_{h_2\to h_1}(t) + K_{h_1\to S_F}(t)]$$

- the same as the first strategy but where we can buy from h_1 and inject into the storage facility

$$\begin{aligned} V_4^{\mathrm{I}}(t) = {} & Tr_{h_2\to\mathrm{SF}}(t)\,[F_2(t) + K_{h_2\to\mathrm{SF}}(t)] + \min\{Tr_{h_1\to\mathrm{SF}}(t), Tr_{h_2\to h_1}(t)\}[F_2(t) + K_{h_2\to h_1}(t) + K_{h_1\to S_F}(t)] \\ & + (Tr_{h_1\to\mathrm{SF}}(t) - Tr_{h_2\to h_1}(t))^+[F_1(t) + K_{h_1\to\mathrm{SF}}(t)] \end{aligned}$$

4 Options

4.1 Types of options

We distinguish three types of options:

- Options to switch a withdrawal strategy k on day t to another withdrawal strategy l on day u

$$C_{\mathrm{W}}(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)) = \mathbb{E}\left[\left(V_l^{\mathrm{W}}(u) - V_k^{\mathrm{W}}(t)\right)^+\right]$$

- Options to switch a injection strategy k on day t to another injection strategy l on day u

$$C_{\mathrm{I}}(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)) = \mathbb{E}\left[\left(V_l^{\mathrm{I}}(u) - V_k^{\mathrm{I}}(t)\right)^+\right]$$

- Options to switch an injection strategy k on day t to a withdrawal strategy l on day u

$$C_{\mathrm{IW}}(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)) = \mathbb{E}\left[\left(V_l^{\mathrm{W}}(u) - V_k^{\mathrm{I}}(t)\right)^+\right]$$

4.2 Valuation

Similarly to the first model, we optimize firm injections and withdrawals to the two hubs and positions on options. We denote by $C\big(V_k(t), V_l(u)\big)$ the option that substitutes strategy $V_l(u)$ to strategy $V_k(t)$. The value of such an option is

$$\begin{aligned} C\big(V_k(t), V_l(u)\big) &= \mathbb{E}\left[\big(V_l(u) - V_k(t)\big)^+\right] \\ &= \mathbb{E}\left[\big(-\alpha K(t) + \beta F_1(t) + \zeta F_2(t) + \gamma F_1(u) + \theta F_2(u)\big)^+\right], \end{aligned}$$

where $K(t) = K^0_{\mathrm{W}}(t) + K^1_{\mathrm{W}}(t) + K^0_{\mathrm{I}}(t) + K^1_{\mathrm{I}}(t) + \sum_{l,k} K^0_{h_l \to h_k} + \sum_{l,k} K^1_{h_l \to h_k}$

5 Transport Capacities

We define $E^H_k(t)$ the energy withdrawn from the storage facility at time t by strategy k that flows to the hub h_H (whether it is to be sold at hub h_H or to be sold at the second hub). The transport capacity constraint writes

$$\forall (t, H) : \sum_k W_k(t) E^H_k(t) + \sum_{k,l,u} O_{\mathrm{W}}(V_l(u), V_k(t)) E^H_k(t) - \sum_{k,l,u} O_{\mathrm{W}}(V_l(t), V_k(u)) E^H_k(t) < Tr_{\mathrm{SF} \to h_H}(t)$$

$$\forall (t, H) : \sum_k I_k(t) E^H_k(t) + \sum_{k,l,u} O_{\mathrm{I}}(V_l(u), V_k(t)) E^H_k(t) - \sum_{k,l,u} O_{\mathrm{I}}(V_l(t), V_k(u)) E^H_k(t) < Tr_{h_H \to \mathrm{SF}}(t)$$

6 Bounds on energy in store

Denoting

$$E_k(t) = E^{\mathrm{W}}_k(t) + E^{\mathrm{I}}_k(t)$$

and

$$O(V_k(t), V_l(u)) = O_{\mathrm{W}}(V_k(t), V_l(u)) + O_{\mathrm{I}}(V_k(t), V_l(u)) + O_{\mathrm{IW}}(V_k(t), V_l(u)),$$

The net algebraic energy withdrawn until t is

$$\begin{aligned} &\sum_{t_2 \le t} \Big[\sum_k W_k(t_2) E^{\mathrm{W}}_k(t_2) - \sum_k I_k(t_2) E^{\mathrm{I}}_k(t_2) \\ &+ \sum_{t_1 \le t} \sum_{k,l} O(V_k(t_1), V_l(t_2))(E_l(t_2) - E_k(t_1)) + \sum_{t_1 > t} \sum_{k,l} O(V_k(t_1), V_l(t_2)) E_l(t_2) \\ &- \sum_{t_1 \le t} \sum_{k,l} O(V_k(t_2), V_l(t_1))(E_k(t_2) - E_l(t_1)) - \sum_{t_1 > t} \sum_{k,l} O(V_k(t_2), V_l(t_1)) E_k(t_2) \Big] \\ &= \sum_{t_2 \le t} \Big[\sum_k W_k(t_2) E^{\mathrm{W}}_k(t_2) - \sum_k I_k(t_2) E^{\mathrm{I}}_k(t_2) \\ &+ \sum_{t_1 \le t} \sum_{k,l} O\left(V_k(t_1), V_l(t_2)\right) \left(E_l(t_2) - E_k(t_1)\right)^+ - \sum_{t_1 \le t} \sum_{k,l} O\left(V_k(t_1), V_l(t_2)\right) \left(E_k(t_1) - E_l(t_2)\right)^+ \\ &+ \sum_{t_1 > t} \sum_{k,l} O\left(V_k(t_1), V_l(t_2)\right) E_l(t_2) \\ &- \sum_{t_1 \le t} \sum_{k,l} O\left(V_k(t_2), V_l(t_1)\right) \left(E_k(t_2) - E_l(t_1)\right)^+ + \sum_{t_1 \le t} \sum_{k,l} O\left(V_k(t_2), V_l(t_1)\right) \left(E_l(t_1) - E_k(t_2)\right)^+ \\ &- \sum_{t_1 > t} \sum_{k,l} O\left(V_k(t_2), V_l(t_1)\right) E_k(t_2) \Big] \end{aligned}$$

To get the lower bound on energy withdrawn we drop the positive optional contributions

$$\sum_{t_2 \le t} \Big[\sum_k W_k(t_2) E_k^{\mathrm{W}}(t_2) - \sum_k I_k(t_2) E_k^{\mathrm{I}}(t_2)$$
$$- \sum_{t_1 \le t} \sum_{k,l} O\left(V_k(t_1), V_l(t_2)\right) \left(E_k(t_1) - E_l(t_2)\right)^+ - \sum_{t_1 \le t} \sum_{k,l} O\left(V_k(t_2), V_l(t_1)\right) \left(E_k(t_2) - E_l(t_1)\right)^+$$
$$- \sum_{t_1 > t} \sum_{k,l} O\left(V_k(t_2), V_l(t_1)\right) E_k(t_2) \Big]$$

To get the upper bound on energy withdrawn we drop the negative optional contributions

$$\sum_{t_2 \le t} \Big[\sum_k W_k(t_2) E_k^{\mathrm{W}}(t_2) - \sum_k I_k(t_2) E_k^{\mathrm{I}}(t_2)$$
$$+ \sum_{t_1 \le t} \sum_{k,l} O\left(V_k(t_1), V_l(t_2)\right) \left(E_l(t_2) - E_k(t_1)\right)^+ + \sum_{t_1 > t} \sum_{k,l} O\left(V_k(t_1), V_l(t_2)\right) E_l(t_2)$$
$$+ \sum_{t_1 \le t} \sum_{k,l} O\left(V_k(t_2), V_l(t_1)\right) \left(E_l(t_1) - E_k(t_2)\right)^+ \Big]$$

A lower bound on energy in store at the end of period t is

$$S_{\mathrm{LB}}(t) = S_0 + \sum_{t_2 \le t} \Big[\sum_k I_k(t_2) E_k^{\mathrm{I}}(t_2) - \sum_k W_k(t_2) E_k^{\mathrm{W}}(t_2)$$
$$- \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_l(t_2) - E_k(t_1)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_l(t_1) - E_k(t_2)\right)^+ \Big)$$
$$- \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) E_l(t_2) \Big) \Big]$$

and an upper bound on energy in store at the end of period t is

$$S_{\mathrm{UB}}(t) = S_0 + \sum_{t_2 \le t} \Big[\sum_k I_k(t_2) E_k^{\mathrm{I}}(t_2) - \sum_k W_k(t_2) E_k^{\mathrm{W}}(t_2)$$
$$+ \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_k(t_1) - E_l(t_2)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_k(t_2) - E_l(t_1)\right)^+ \Big)$$
$$+ \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_2), V_l(t_1)\right) E_k(t_2) \Big) \Big]$$

A recursion formula for bounds on energy in store can be found in [Appendix B].

7 Injection and withdrawal ratchet constraints

Given energy in store S at the beginning of period t, the net daily withdrawal rate must not exceed the compressors' capacity $f_{\mathrm{W}}(t) g_{\mathrm{W}}(t, S)$ and the daily injection rate must not exceed $f_{\mathrm{I}}(t) g_{\mathrm{I}}(t, S)$. The reduction factors $g_{\mathrm{W}}(t, S)$ and $g_{\mathrm{I}}(t, S)$ are continuous piecewise linear functions of S. The ratchet constraints are formulated in three different ways depending on their shape:

- we use the formulation of section 7.1 for periods t where $g_{\mathrm{W}}(t, S)$ and $f_{\mathrm{W}}(t)$ are constant

- the formulation of section 7.2 for periods t where $g_{\mathrm{W}}(t,S)$ is a concave function of S
- the formulation of section 7.4 for periods t where $g_{\mathrm{W}}(t,S)$ is a monotonic function of S

The constraint for withdrawals can be written as

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big)\right) E_k^{\mathrm{W}}(t) \leq f_{\mathrm{W}}(t) g_{\mathrm{W}}(t, S(t-1))$$

Similarly for $g_{\mathrm{I}}(t,S)$, the constraint can be written as

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O\big(V_k(t), V_l(u)\big)\right) E_k^{\mathrm{I}}(t) \leq f_{\mathrm{I}}(t) g_{\mathrm{I}}(t, S(t-1))$$

We assume that $g_{\mathrm{W}}(t,S)$ is defined by a set of n_{W} points $(s_{\mathrm{W},k}, g_{\mathrm{W},k}(t))$. Without loss of generality, the abscissa $s_{\mathrm{W},k}$ are the same for every period and sorted in increasing order.
They are given in energy units: S_{MAX} multiplied by percentages of nominal capacity. Similarly for injection constraints, we have .

7.1 Constant injection and withdrawal rates

When $f_{\mathrm{W}}(t)$, $f_{\mathrm{I}}(t)$ and $g_{\mathrm{W}}(t,S)$, $g_{\mathrm{I}}(t,S)$ are constant the constraints translate to

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big)\right) E_k^{\mathrm{W}}(t) \leq MaxSendOut$$

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O\big(V_k(t), V_l(u)\big)\right) E_k^{\mathrm{I}}(t) \leq MaxSendIn$$

7.2 Concave injection and withdrawal rates

We calculate the slopes

$$\alpha_{\mathrm{W},k}(t) = \frac{g_{\mathrm{W},k+1}(t) - g_{\mathrm{W},k}(t)}{s_{\mathrm{W},k+1} - s_{\mathrm{W},k}}$$

of function $g_{\mathrm{W}}(t,S)$ on the $n_{\mathrm{W}} - 1$ intervals $[s_{\mathrm{W},k}, s_{\mathrm{W},k+1}]$. If it is concave, then we have

$$g_{\mathrm{W}}(t,s) = \min_{k \leq n_{\mathrm{W}}-1} \{\alpha_{\mathrm{W},k}(t)(S - s_{\mathrm{W},k}) + g_{\mathrm{W},k}(t)\}$$

for any S. At the beginning of day t, we have $S \in [S_{\mathrm{LB}}(t-1), S_{\mathrm{UB}}(t-1)]$, then for the daily withdrawal rate in period t to be less than $f_{\mathrm{W}}(t) g_{\mathrm{W}}(t,S)$, it is sufficient that the following $n_{\mathrm{W}} - 1$ constraints are met:

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big)\right) E_k^{\mathrm{W}}(t) \leq f_{\mathrm{W}}(t) \left(\frac{g_{\mathrm{W,k+1}}(t) - g_{\mathrm{W,k}}(t)}{s_{\mathrm{W,k+1}} - s_{\mathrm{W,k}}} \big(S_{\mathrm{UB}}(t-1) - s_{\mathrm{W,k}}\big) + g_{\mathrm{W,k}}(t)\right)$$

where $g_{\mathrm{W}}(t,S)$ is decreasing;

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big)\right) E_k^{\mathrm{W}}(t) \leq f_{\mathrm{W}}(t) \left(\frac{g_{\mathrm{W,k+1}}(t) - g_{\mathrm{W,k}}(t)}{s_{\mathrm{W,k+1}} - s_{\mathrm{W,k}}} \big(S_{\mathrm{LB}}(t-1) - s_{\mathrm{W,k}}\big) + g_{\mathrm{W,k}}(t)\right)$$

where $g_{\mathrm{W}}(t, S)$ is increasing;

this yields to

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big)\right) E_k^{\mathrm{W}}(t) \le f_{\mathrm{W}}(t) \left(\frac{g_{\mathrm{W,k+1}}(t) - g_{\mathrm{W,k}}(t)}{s_{\mathrm{W,k+1}} - s_{\mathrm{W,k}}}\big(S - s_{\mathrm{W,k}}\big) + g_{\mathrm{W,k}}(t)\right)$$

For concave injection rates we calculate the slopes:

$$\alpha_{\mathrm{I},k}(t) = \frac{g_{\mathrm{I},k+1}(t) - g_{\mathrm{I},k}(t)}{s_{\mathrm{I},k+1} - s_{\mathrm{I},k}}$$

For the daily injection rate on day t to be less than $f_{\mathrm{I}}(t) g_{\mathrm{I}}(t, S)$, if the slope $\alpha_{\mathrm{I},K}(t)$ is negative we need

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O\big(V_k(t), V_l(u)\big)\right) E_k^{\mathrm{I}}(t) \le f_{\mathrm{I}}(t) \left(\frac{g_{\mathrm{I,k+1}}(t) - g_{\mathrm{I,k}}(t)}{s_{\mathrm{I,k+1}} - s_{\mathrm{I,k}}}\big(S_{\mathrm{UB}}(t-1) - s_{\mathrm{I,k}}\big) + g_{\mathrm{I,k}}(t)\right)$$

For intervals where the slope is positive we need

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O\big(V_k(t), V_l(u)\big)\right) E_k^{\mathrm{I}}(t) \le f_{\mathrm{I}}(t) \left(\frac{g_{\mathrm{I,k+1}}(t) - g_{\mathrm{I,k}}(t)}{s_{\mathrm{I,k+1}} - s_{\mathrm{I,k}}}\big(S_{\mathrm{LB}}(t-1) - s_{\mathrm{I,k}}\big) + g_{\mathrm{I,k}}(t)\right)$$

7.3 Alternative representation of continuous piecewise linear functions

The generic approach presented in this section does not rely on concavity but on monotonicity. Consider a function g that interpolates linearly a set of n points (s_k, r_k) such that $s_1 < s_2 < ... < s_n$. Assume first that g is increasing. Two variables $s \in [s_1, s_n]$ and r satisfy the equality $r = g(s)$ if and only if there exists $n-1$ weights $X_k \in [0,1]$ and $n-2$ binary variables Y_k meeting the following conditions:

$$s = s_n + \sum_{k \le n-1} X_k(s_{k+1} - s_k)$$

$$r = r_n + \sum_{k \le n-1} X_k(r_{k+1} - r_k)$$

and $X_{k+1} < Y_k < X_k,\ \forall\, 1 < k < n-2$. It follows from these inequalities that if $X_k > 0$, then $Y_{k-1} = 1$ and in turn $X_\ell = 1,\ \forall\, \ell < k$. If $X_k < 1$, then $Y_k = 0$ and $X_\ell = 0,\ \forall\, \ell > k$.

As a consequence, there is at most one weight X_k which is neither zero nor one. If such a weight exists, then $s \in]s_k, s_{k+1}[$. When function g is decreasing, weights $X_k \in [0,1]$ and binary variables Y_k solve:

$$s = s_n + \sum_{k \le n-1} X_k(s_k - s_{k+1})$$

$$r = r_n + \sum_{k \le n-1} X_k(r_k - r_{k+1})$$

and $X_k \le Y_k \le X_{k+1},\ \forall\, 1 \le k \le n-2$. From these inequalities: if $X_k > 0$, then $X_\ell = 1,\ \forall\, \ell > k$; if $X_k < 1$, then $X_\ell = 0,\ \forall\, \ell < k$. Observe that all relationships between variables are linear.

7.4 Monotonic injection and withdrawal rates

Assume that $g_{\mathrm{W}}(t,S)$ is an increasing function of S. Then,

$$g_{\mathrm{W}}(t,S) \leq g_{\mathrm{W}}(t,S_{\mathrm{LB}}(t-1))$$

For every day t but the first, we introduce $n_{\mathrm{W}}-1$ weights $X_{\mathrm{W},k}(t)$ and $n_{\mathrm{W}}-2$ binary variables $Y_{\mathrm{W},k}(t)$ subject to $X_{\mathrm{W},k}(t) \geq Y_{\mathrm{W},k}(t) \geq X_{\mathrm{W},k+1}(t)$. We use these variables to link

$$S_{\mathrm{LB}}(t-1) = s_{\mathrm{W},1} + \sum_{k \leq n_{\mathrm{W}}-1} X_{\mathrm{W},k}(t)(s_{\mathrm{W},k+1} - s_{\mathrm{W},k})$$

and the corresponding reduction factor on day t

$$g_{\mathrm{W}}(t,S_{\mathrm{LB}}(t-1)) = g_{\mathrm{W},n_{\mathrm{W}}}(t) + \sum_{k \leq n_{\mathrm{W}}-1} X_{\mathrm{W},k}(t)(g_{\mathrm{W},k+1}(t) - g_{\mathrm{W},k}(t))$$

For the withdrawal rate on day t to be less than $f_{\mathrm{W}}(t)g_{\mathrm{W}}(t,S)$ we need

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_l^{\mathrm{W}}(u), V_k^{\mathrm{W}}(t)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{I}}(t)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{W}}(t)\big)\right) E_k^{\mathrm{W}}(t)$$
$$\leq f_{\mathrm{W}}(t)\left(g_{\mathrm{W},1}(t) + \sum_k X_{\mathrm{W},k}(t)\left(g_{\mathrm{W},k+1}(t) - g_{\mathrm{W},k}(t)\right)\right);$$

For decreasing withdrawal rates, $X_{\mathrm{W},k+1}(t) \geq Y_{\mathrm{W},k}(t) \geq X_{\mathrm{W},k}(t)$,

$$S_{\mathrm{UB}}(t-1) = s_{\mathrm{W},n_{\mathrm{W}}} + \sum_{k \leq n_{\mathrm{W}}-1} X_{\mathrm{W},k}(t)(s_{\mathrm{W},k} - s_{\mathrm{W},k+1})$$

and

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_l^{\mathrm{W}}(u), V_k^{\mathrm{W}}(t)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{I}}(t)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{W}}(t)\big)\right) E_k^{\mathrm{W}}(t)$$
$$\leq f_{\mathrm{W}}(t)\left(g_{\mathrm{W},n_{\mathrm{W}}}(t) + \sum_k X_{\mathrm{W},k}(t)\left(g_{\mathrm{W},k}(t) - g_{\mathrm{W},k+1}(t)\right)\right);$$

For increasing injection rates, we introduce weights $X_{\mathrm{I},k}(t)$ and binary variables $Y_{\mathrm{I}k}(t)$ subject to $X_{\mathrm{I},k}(t) \geq Y_{\mathrm{I},k}(t) \geq X_{\mathrm{I},k+1}(t)$ and

$$S_{\mathrm{LB}}(t-1) = s_{\mathrm{I},1} + \sum_{k \leq n_{\mathrm{I}}-1} X_{\mathrm{I},k}(t)(s_{\mathrm{I},k+1} - s_{\mathrm{I},k})$$

For the injection rate on day t to be less than $f_{\mathrm{I}}(t)g_{\mathrm{I}}(t,S)$ we need

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big)\right) E_k^{\mathrm{I}}(t)$$
$$\leq f_{\mathrm{I}}(t)\left(g_{\mathrm{I},1}(t) + \sum_k X_{\mathrm{I},k}(t)\left(g_{\mathrm{I},k+1}(t) - g_{\mathrm{I},k}(t)\right)\right);$$

For decreasing injection rates on day t, $X_{\mathrm{I},k+1}(t) \geq Y_{\mathrm{I},k}(t) \geq X_{\mathrm{I},k}(t)$,

$$S_{\mathrm{UB}}(t-1) = s_{\mathrm{I},n_{\mathrm{I}}} + \sum_{k \leq n_{\mathrm{I}}-1} X_{\mathrm{I},k}(t)(s_{\mathrm{I},k} - s_{\mathrm{I},k+1})$$

and

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big) \right) E_k^{\mathrm{I}}(t)$$
$$\leq f_{\mathrm{I}}(t) \left(g_{\mathrm{I},n_{\mathrm{I}}}(t) + \sum_k X_{\mathrm{I},k}(t) \left(g_{\mathrm{I},k}(t) - g_{\mathrm{I},k+1}(t) \right) \right);$$

Because of binary variables, this formulation is not as tractable as that of section 7.2 which is therefore preferred if injection and withdrawal reduction factors are concave.

8 Optimization Model

We assume mutually exclusive injections and withdrawals. The cost of injecting one unit of energy on day u and to withdraw it from and to every hub h_k is the random variable

$$\begin{aligned}
\kappa(t,u) = &\sum_k Tr_{\mathrm{SF}\to h_k}(u) \left[F_k(t,u) K_{\mathrm{W}}^1(u) + K_{\mathrm{W}}^0(u)\right] + \sum_k Tr_{h_k \to \mathrm{SF}}(u) \left[F_k(t,u) K_{\mathrm{I}}^1(u) + K_{\mathrm{I}}^0(u)\right] \\
&+ \sum_{l \neq k} \min\{Tr_{\mathrm{SF}\to h_l}(u), Tr_{h_l \to h_k}(u)\} \left[F_k(t,u) K_{\mathrm{W}}^1(u) + K_{\mathrm{W}}^0(u)\right] \\
&+ \sum_{l \neq k} \min\{Tr_{h_l \to \mathrm{SF}}(u), Tr_{h_k \to h_l}(u)\} \left[F_k(t,u) K_{\mathrm{I}}^1(u) + K_{\mathrm{I}}^0(u)\right] \\
&+ \sum_{l \neq k} (Tr_{h_l \to \mathrm{SF}}(u) - Tr_{h_k \to h_l}(u))^+ [F_l(t,u) K_{\mathrm{I}}^1(u) + K_{\mathrm{I}}^0(u)]
\end{aligned}$$

- To ensure that firm injection strategies and firm withdrawal strategies remain mutually exclusive in our linear model, we restrict optimization to options $C_{\mathrm{I}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{I}}(v_2)\big)$ such that

$$\begin{aligned}
\mathbb{E}\Big[\kappa(t,v_2) \mathbb{1}\Big(&\sum_k Tr_{h_k \to \mathrm{SF}}(v_2) \left[F_k(t,v_2)\left(1 + K_{\mathrm{I}}^1(v_2)\right) + K_{\mathrm{I}}^0(v_2)\right] \\
&+ \sum_{l \neq k} \min\{Tr_{h_l \to \mathrm{SF}}(v_2), Tr_{h_k \to h_l}(v_2)\} \left[F_k(t,v_2)\left(1 + K_{\mathrm{I}}^1(v_2)\right) + K_{\mathrm{I}}^0(v_2)\right] \\
&+ \sum_{l \neq k} (Tr_{h_l \to \mathrm{SF}}(v_2) - Tr_{h_k \to h_l}(v_2))^+ [F_l(t,v_2)\left(1 + K_{\mathrm{I}}^1(v_2)\right) + K_{\mathrm{I}}^0(v_2)] \\
&> \sum_k Tr_{h_k \to \mathrm{SF}}(v_1) \left[F_k(t,v_1)\left(1 + K_{\mathrm{I}}^1(v_1)\right) + K_{\mathrm{I}}^0(v_1)\right] \\
&+ \sum_{l \neq k} \min\{Tr_{h_l \to \mathrm{SF}}(v_1), Tr_{h_k \to h_l}(v_1)\} \left[F_k(t,v_1)\left(1 + K_{\mathrm{I}}^1(v_1)\right) + K_{\mathrm{I}}^0(v_1)\right] \\
&+ \sum_{l \neq k} (Tr_{h_l \to \mathrm{SF}}(v_1) - Tr_{h_k \to h_l}(v_1))^+ [F_l(t,v_1)\left(1 + K_{\mathrm{I}}^1(v_1)\right) + K_{\mathrm{I}}^0(v_1)]\Big)\Big] \leq \frac{1}{2}\kappa(0,v_2)
\end{aligned} \tag{1}$$

and to options $C_{\mathrm{W}}\big(V_k^{\mathrm{W}}(v_1), V_l^{\mathrm{W}}(v_2)\big)$ such that

$$\begin{aligned}
&\mathbb{E}\Big[\kappa(t,v_1)\mathbb{1}\Big(\sum_k Tr_{\mathrm{SF}\to h_k}(v_2)\left[F_k(t,v_2)\left(1-K^1_{\mathrm{W}}(v_2)\right)-K^0_{\mathrm{W}}(v_2)\right]\\
&+\sum_{l\neq k}\min\{Tr_{\mathrm{SF}\to h_l}(v_2), Tr_{h_l\to h_k}(v_2)\}\left[F_k(t,v_2)\left(1-K^1_{\mathrm{W}}(v_2)\right)-K^0_{\mathrm{W}}(v_2)\right]\\
&>\sum_k Tr_{\mathrm{SF}\to h_k}(v_1)\left[F_k(t,v_1)\left(1-K^1_{\mathrm{W}}(v_1)\right)-K^0_{\mathrm{W}}(v_1)\right]\\
&+\sum_{l\neq k}\min\{Tr_{\mathrm{SF}\to h_l}(v_1), Tr_{h_l\to h_k}(v_1)\}\left[F_k(t,v_1)\left(1-K^1_{\mathrm{W}}(v_1)\right)-K^0_{\mathrm{W}}(v_1)\right]\Big)\Big]\leq\frac{1}{2}\kappa(0,v_1)
\end{aligned}\tag{2}$$

- We show in [Appendix C] that if conditions (1) and (2) are met then it cannot be optimal to have both $\sum_k W_k(t) > 0$ and $\sum_k I_k(t) > 0$

9 Model Summary

Decision variables are:

- $W_k(t) \in [0,1]$ the position on strategy of value $V_k(t)$ (euros) that withdraws energy $E_k(t)$ (Mwh) from the storage
- $O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) \in [0,1]$ the position on options that meet condition (2), of value $C_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big)$ that allow to report a proportion $O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) \in [0,1]$ of the quantity $I_k(t)$, i.e. a proportion of $E_k^{\mathrm{I}}(t)$ allocated to strategy $V_k^{\mathrm{I}}(t)$.
- $O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) \in [0,1]$ the position on options that meet condition (1), of value $C_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big)$ that allow to report a proportion $O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) \in [0,1]$ of the quantity $W_k(t)$, i.e. a proportion of $E_k^{\mathrm{W}}(t)$ allocated to strategy $V_k^{\mathrm{W}}(t)$.
- $O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big) \in [0,1]$ the position on option of value $C_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V\mathrm{W}_l(u)\big)$ that allow to report a proportion $O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big) \in [0,1]$ of the quantity $I_k(t)$, i.e. a proportion of $E_k^{\mathrm{I}}(t)$ allocated to strategy $V_k^{\mathrm{I}}(t)$ for energy $E_k^{\mathrm{W}}(t)$ on $V_k^{\mathrm{W}}(t)$.
- $S_{\mathrm{UB}}(t)$ upper bounds on energy in store
- $S_{\mathrm{LB}}(t)$ lower bounds on energy in store

And we have:

$$\max\Big\{\sum W_k(t)V_k^{\mathrm{W}}(t) + \sum I_k(t)V_k^{\mathrm{I}}(t) + \sum O(V_k(t), V_l(u))\, C(V_k(t), V_l(u))\Big\}$$

subject to the following constraints:

- the sum of withdrawal strategies k on day t shifted to strategies l on day u cannot exceed $W_k(t)$ (in Mwh)

$$\forall(k,t):\ \sum_{l,\,u} O\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big)E_k^{\mathrm{W}}(t) < W_k(t)E_k^{\mathrm{W}}(t) \tag{3}$$

- the sum of injection strategies k on day t shifted to strategies l on day u cannot exceed $I_k(t)$ (in Mwh)

$$\forall(k,t):\ \sum_{l,\,u} O\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big)E_k^{\mathrm{I}}(t) < I_k(t)E_k^{\mathrm{I}}(t) \tag{4}$$

- recursion formula on lower bound on energy in store (in Mwh):

$$
\begin{aligned}
\forall t: S_{\text{LB}}(t) = S_{\text{LB}}(t-1) + Ship(t) - \sum_k W_k(t) E_k(t) \\
+ \sum_{t_2 \le t} O\left(V_k(t), V_l(t_2)\right)\left(E_l(t_2) - E_k(t_1)\right)^+ - \sum_{t_1 \le t} O\left(V_k(t_1), V_l(t)\right)\left(E_l(t_2) - E_k(t_1)\right)^+ \\
+ \sum_{t_2 \le t} O\left(V_k(t), V_l(t_1)\right)\left(E_l(t_1) - E_k(t_2)\right)^+ - \sum_{t_1 \le t} O\left(V_k(t_2), V_l(t)\right)\left(E_l(t_1) - E_k(t_2)\right)^+ \\
+ \sum_{t_2 \le t} O\left(V_k(t), V_l(t_2)\right) E_l(t_2) - \sum_{t_1 > t} O\left(V_k(t_1), V_l(t)\right) E_l(t_2)
\end{aligned}
$$

- the energy in store at the end of period t must be more than $S_{\text{min}}(t)$:

$$
S_{\text{LB}}(t) \ge S_{\text{min}}(t)
$$

- recursion formula on upper bound on energy in store (in Mwh):

$$
\begin{aligned}
\forall t: S_{\text{UB}}(t) = S_{\text{UB}}(t-1) + Ship(t) - \sum_k W_k(t) E_k(t) \\
- \sum_{t_2 \le t} O\left(V_k(t_1), V_l(t)\right)\left(E_k(t_1) - E_l(t_2)\right)^+ + \sum_{t_1 \le t} O\left(V_k(t), V_l(t_2)\right)\left(E_k(t_1) - E_l(t_2)\right)^+ \\
- \sum_{t_2 \le t} O\left(V_k(t_2), V_l(t)\right)\left(E_k(t_2) - E_l(t_1)\right)^+ + \sum_{t_1 \le t} O\left(V_k(t), V_l(t_1)\right)\left(E_k(t_2) - E_l(t_1)\right)^+ \\
- \sum_{t_2 \le t} O\left(V_k(t_2), V_l(t)\right) E_k(t_2) + \sum_{t_1 > t} O\left(V_k(t), V_l(t_1)\right) E_k(t_2)
\end{aligned}
$$

- the energy in store at the end of period t must be less than $S_{\text{max}}(t)$:

$$
S_{\text{UB}}(t) \le S_{\text{max}}(t)
$$

- the energy flowed through an edge must be less than its capacity:

$$
\forall (t,H): \sum_k W_k(t) E_k^H(t) + \sum_{k,l,u} O_{\text{W}}(V_l(u), V_k(t)) E_k^H(t) - \sum_{k,l,u} O_{\text{W}}(V_l(t), V_k(u)) E_k^H(t) < Tr_{\text{SF} \to h_H}(t)
$$

$$
\forall (t,H): \sum_k I_k(t) E_k^H(t) + \sum_{k,l,u} O_{\text{I}}(V_l(u), V_k(t)) E_k^H(t) - \sum_{k,l,u} O_{\text{I}}(V_l(t), V_k(u)) E_k^H(t) < Tr_{h_H \to \text{SF}}(t)
$$

- daily constant withdrawal rate constraint (in Mwh)

$$
\left(\sum_k W_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big) \right) E_k(t) \le MaxSendOut(t)
$$

- properties of weights $X_{\text{W},k}(t) \in [0,1]$ and associated binary variables $Y_{\text{W},k}$

$$
X_{\text{W},k}(t) > Y_{\text{W},k}(t)
$$

$$
Y_{\text{W},k}(t) > X_{\text{W},k+1}(t)
$$

$$
s_{\text{W},1} + \sum_k X_{\text{W},k}(t)(S_{\text{W},k+1} - S_{\text{W},k}) = S_{\text{LB}}(t-1)
$$

- if $g_{\mathrm{W}}(p,S)$ is monotonic, the daily withdrawal rate must be less than the compressors capacity $f_{\mathrm{W}}(t)g_{\mathrm{W}}(t,S_{\mathrm{LB}}(t-1))$:

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_l^{\mathrm{W}}(u), V_k^{\mathrm{W}}(t)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{I}}(t)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{W}}(t)\big)\right) E_k(t)$$
$$\leq f_{\mathrm{W}}(t)\left(g_{\mathrm{W},1}(t) + \sum_k X_{\mathrm{W},k}(t)\left(g_{\mathrm{W},k+1}(t) - g_{\mathrm{W},k}(t)\right)\right);$$

- if $g_{\mathrm{W}}(p,S)$ is concave, the daily withdrawal rate must be less than the compressors capacity $f_{\mathrm{W}}(t)g_{\mathrm{W}}(t,S_{\mathrm{LB}}(t-1))$:

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_l^{\mathrm{W}}(u), V_k^{\mathrm{W}}(t)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{I}}(t)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{W}}(t)\big)\right) E_k(t)$$
$$\leq f_{\mathrm{W}}(t)\left(\frac{g_{\mathrm{W,k+1}}(t) - g_{\mathrm{W,k}}(t)}{s_{\mathrm{W},k+1} - s_{\mathrm{W},k}}\left(S_{\mathrm{UB}}(t-1) - s_{\mathrm{W},k}\right) + g_{\mathrm{W},k}(t)\right);$$

  for all intervals $[s_{\mathrm{W},k}, s_{\mathrm{W},k+1}]$ where $g_{\mathrm{W}}(t,S)$ is decreasing and

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_l^{\mathrm{W}}(u), V_k^{\mathrm{W}}(t)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{I}}(t)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{W}}(t)\big)\right) E_k(t)$$
$$\leq f_{\mathrm{W}}(t)\left(\frac{g_{\mathrm{W,k+1}}(t) - g_{\mathrm{W,k}}(t)}{s_{\mathrm{W},k+1} - s_{\mathrm{W},k}}\left(S_{\mathrm{LB}}(t-1) - s_{\mathrm{W},k}\right) + g_{\mathrm{W},k}(t)\right);$$

  for all intervals where $g_{\mathrm{W}}(t,S)$ is increasing;

- properties of weights $X_{\mathrm{I},k}(t) \in [0,1]$ and associated binary variables $Y_{\mathrm{I},k}$

$$X_{\mathrm{I},k+1}(t) > Y_{\mathrm{I},k}$$
$$Y_{\mathrm{I},k}(t) > X_{\mathrm{I},k+1}$$
$$s_{\mathrm{I},1} + \sum_k X_{\mathrm{I},k}(t)(S_{\mathrm{I},k} - S_{\mathrm{I},k+1}) = S_{\mathrm{UB}}(t-1)$$

- if $g_{\mathrm{I}}(p,S)$ is monotonic, the daily injection rate must be less than the compressors capacity $f_{\mathrm{I}}(t)g_{\mathrm{I}}(t,S_{\mathrm{UB}}(t-1))$:

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big)\right) E_k(t)$$
$$\leq f_{\mathrm{I}}(t)\left(g_{\mathrm{I},n_{\mathrm{I}}}(t) + \sum_k X_{\mathrm{I},k}(t)\left(g_{\mathrm{I},k}(t) - g_{\mathrm{I},k+1}(t)\right)\right);$$

- if $g_{\mathrm{I}}(p,S)$ is concave, the daily injection rate must be less than the compressors capacity $f_{\mathrm{I}}(t)g_{\mathrm{I}}(t,S_{\mathrm{UB}}(t-1))$:

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big)\right) E_k(t)$$
$$\leq f_{\mathrm{I}}(t)\left(\frac{g_{\mathrm{I,k+1}}(t) - g_{\mathrm{I,k}}(t)}{s_{\mathrm{I},k+1} - s_{\mathrm{I},k}}\left(S_{\mathrm{UB}}(t-1) - s_{\mathrm{I},k}\right) + g_{\mathrm{I},k}(t)\right);$$

for all intervals $[s_{\mathrm{I},k}, s_{\mathrm{I},k+1}]$ where $g_{\mathrm{I}}(t,S)$ is decreasing and

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big)\right) E_k(t)$$
$$\leq f_{\mathrm{I}}(t)\left(\frac{g_{\mathrm{I,k+1}}(t) - g_{\mathrm{I,k}}(t)}{s_{\mathrm{I},k+1} - s_{\mathrm{I},k}}\left(S_{\mathrm{LB}}(t-1) - s_{\mathrm{I},k}\right) + g_{\mathrm{I},k}(t)\right);$$

for all intervals where $g_{\mathrm{I}}(t,S)$ is increasing;

Appendix A. A more realistic representation of costs

We include fixed and proportional withdrawal costs to get a more realistic representation of the costs within the strategies.

A.1 Firm withdrawals

On each day we have the four following (non mutually exclusive) firm withdrawal strategies

- The strategy withdrawing from the storage facility to sell at hub h_1 and on day t:

$$\begin{aligned}
V_1^{\mathrm{W}}(t) &= Tr_{\mathrm{SF}\to h_1}(t)\left[F_1(t)\left(1-K_{\mathrm{W}}^1(t)-K_{\mathrm{SF}\to h_1}^1(t)\right)-K_{\mathrm{W}}^0(t)-K_{\mathrm{SF}\to h_1}^0(t)\right]\\
&+\min\left\{Tr_{\mathrm{SF}\to h_2}(t),Tr_{h_2\to h_1}(t)\right\}\left[F_1(t)\left(1-K_{\mathrm{W}}^1(t)-K_{\mathrm{SF}\to h_2}^1(t)-K_{h_2\to h_1}^1(t)\right)\right.\\
&\left.-K_{\mathrm{W}}^0(t)-K_{\mathrm{SF}\to h_2}^0(t)-K_{h_2\to h_1}^0(t)\right]
\end{aligned}$$

- The strategy withdrawing from the storage facility to sell at hub h_1 and buying at hub h_2 to sell at hub h_1 on day t

$$\begin{aligned}
V_2^{\mathrm{W}}(t) &= Tr_{\mathrm{SF}\to h_1}(t)\left[F_1(t)\left(1-K_{\mathrm{W}}^1(t)-K_{\mathrm{SF}\to h_1}^1(t)\right)-K_{\mathrm{W}}^0(t)-K_{\mathrm{SF}\to h_1}^0(t)\right]\\
&+\min\left\{Tr_{\mathrm{SF}\to h_2}(t),Tr_{h_2\to h_1}(t)\right\}\left[F_1(t)\left(1-K_{\mathrm{W}}^1(t)-K_{\mathrm{SF}\to h_2}^1(t)-K_{h_2\to h_1}^1(t)\right)\right.\\
&\left.-K_{\mathrm{W}}^0(t)-K_{\mathrm{SF}\to h_2}^0(t)-K_{h_2\to h_1}^0(t)\right]\\
&+(Tr_{h_2\to h_1}(t)-Tr_{\mathrm{SF}\to h_2}(t))^+[F_1(t)(1-K_{h_2\to h_1}^1(t))-F_2(t)(1+K_{h_2\to h_1}^1(t))-K_{h_2\to h_1}^0(t)]
\end{aligned}$$

- The strategy withdrawing from the storage facility to sell at hub h_2 and on day t:

$$\begin{aligned}
V_3^{\mathrm{W}}(t) &= Tr_{\mathrm{SF}\to h_2}(t)\left[F_2(t)\left(1-K_{\mathrm{W}}^1(t)-K_{\mathrm{SF}\to h_2}^1(t)\right)-K_{\mathrm{W}}^0(t)-K_{\mathrm{SF}\to h_2}^0(t)\right]\\
&+\min\left\{Tr_{\mathrm{SF}\to h_1}(t),Tr_{h_1\to h_2}(t)\right\}\left[F_2(t)\left(1-K_{\mathrm{W}}^1(t)-K_{\mathrm{SF}\to h_1}^1(t)-K_{h_1\to h_2}^1(t)\right)\right.\\
&\left.-K_{\mathrm{W}}^0(t)-K_{\mathrm{SF}\to h_1}^0(t)-K_{h_1\to h_2}^0(t)\right]
\end{aligned}$$

- The strategy withdrawing from the storage facility to sell at hub h_2 and buying at hub h_1 to sell at hub h_2 on day t

$$\begin{aligned}
V_4^{\mathrm{W}}(t) &= Tr_{\mathrm{SF}\to h_2}(t)\left[F_2(t)\left(1-K_{\mathrm{W}}^1(t)-K_{\mathrm{SF}\to h_2}^1(t)\right)-K_{\mathrm{W}}^0(t)-K_{\mathrm{SF}\to h_2}^0(t)\right]\\
&+\min\left\{Tr_{\mathrm{SF}\to h_1}(t),Tr_{h_1\to h_2}(t)\right\}\left[F_2(t)\left(1-K_{\mathrm{W}}^1(t)-K_{\mathrm{SF}\to h_1}^1(t)-K_{h_1\to h_2}^1(t)\right)\right.\\
&\left.-K_{\mathrm{W}}^0(t)-K_{\mathrm{SF}\to h_1}^0(t)-K_{h_1\to h_2}^0(t)\right]\\
&+(Tr_{h_1\to h_2}(t)-Tr_{\mathrm{SF}\to h_1}(t))^+[F_2(t)(1-K_{h_1\to h_2}^1(t))-F_1(t)(1+K_{h_1\to h_2}^1(t))-K_{h_1\to h_2}^0(t)]
\end{aligned}$$

A.2 Firm injections

Similarly as with firm withdrawing strategies we get the following four strategies

- a strategy where we inject from hub h_1 and use other edges to maximize the send in from h_1

$$\begin{aligned}
V_1^{\mathrm{I}}(t) &= Tr_{h_1\to\mathrm{SF}}(t)\left[F_1(t)(1+K_{\mathrm{I}}^1(t)+K_{h_1\to\mathrm{SF}}^1(t))+K_{\mathrm{I}}^0(t)+K_{h_1\to\mathrm{SF}}^0(t)\right]\\
&+\min\{Tr_{h_2\to\mathrm{SF}}(t),Tr_{h_1\to h_2}(t)\}[F_1(t)(1+K_{\mathrm{I}}^1(t)+K_{h_1\to h_2}^1(t)+K_{h_2\to S_F}^1(t))\\
&+K_{\mathrm{I}}^0(t)+K_{h_1\to h_2}^0(t)+K_{h_2\to\mathrm{SF}}^0(t)]
\end{aligned}$$

- the same as the first strategy but where we can buy from h_2 and inject into the storage facility

$$\begin{aligned}V_2^{\mathrm{I}}(t) = & Tr_{h_1\to\mathrm{SF}}(t)\left[F_1(t)(1+K_{\mathrm{I}}^1(t)+K_{h_1\to\mathrm{SF}}^1(t))+K_{\mathrm{I}}^0(t)+K_{h_1\to\mathrm{SF}}^0(t)\right]\\ & +\min\{Tr_{h_2\to\mathrm{SF}}(t),Tr_{h_1\to h_2}(t)\}[F_1(t)(1+K_{\mathrm{I}}^1(t)+K_{h_1\to h_2}^1(t)+K_{h_2\to S_F}^1(t))\\ & +K_{\mathrm{I}}^0(t)+K_{h_1\to h_2}^0(t)+K_{h_2\to\mathrm{SF}}^0(t)]\\ & +(Tr_{h_2\to\mathrm{SF}}(t)-Tr_{h_1\to h_2}(t))^+[F_2(t)(1+K_{\mathrm{I}}^1(t)+K_{h_2\to\mathrm{SF}}^1(t))+K_{h_2\to\mathrm{SF}}^0(t)+K_{\mathrm{I}}^0(t)]\end{aligned}$$

- a strategy where we inject from hub h_2 and use other edges to maximize the send in from h_2

$$\begin{aligned}V_3^{\mathrm{I}}(t) = & Tr_{h_2\to\mathrm{SF}}(t)\left[F_2(t)(1+K_{\mathrm{I}}^1(t)+K_{h_2\to\mathrm{SF}}^1(t))+K_{\mathrm{I}}^0(t)+K_{h_2\to\mathrm{SF}}^0(t)\right]\\ & +\min\{Tr_{h_1\to\mathrm{SF}}(t),Tr_{h_2\to h_1}(t)\}[F_2(t)(1+K_{\mathrm{I}}^1(t)+K_{h_2\to h_1}^1(t)+K_{h_1\to S_F}^1(t))\\ & +K_{\mathrm{I}}^0(t)+K_{h_2\to h_1}^0(t)+K_{h_1\to\mathrm{SF}}^0(t)]\end{aligned}$$

- the same as the first strategy but where we can buy from h_1 and inject into the storage facility

$$\begin{aligned}V_4^{\mathrm{I}}(t) = & Tr_{h_2\to\mathrm{SF}}(t)\left[F_2(t)(1+K_{\mathrm{I}}^1(t)+K_{h_2\to\mathrm{SF}}^1(t))+K_{\mathrm{I}}^0(t)+K_{h_2\to\mathrm{SF}}^0(t)\right]\\ & +\min\{Tr_{h_1\to\mathrm{SF}}(t),Tr_{h_2\to h_1}(t)\}[F_2(t)(1+K_{\mathrm{I}}^1(t)+K_{h_2\to h_1}^1(t)+K_{h_1\to S_F}^1(t))\\ & +K_{\mathrm{I}}^0(t)+K_{h_2\to h_1}^0(t)+K_{h_1\to\mathrm{SF}}^0(t)]\\ & +(Tr_{h_1\to\mathrm{SF}}(t)-Tr_{h_2\to h_1}(t))^+[F_1(t)(1+K_{\mathrm{I}}^1(t)+K_{h_1\to\mathrm{SF}}^1(t))+K_{h_1\to\mathrm{SF}}^0(t)+K_{\mathrm{I}}^0(t)]\end{aligned}$$

Appendix B. Recursion formula for bounds on energy in store

Similarly in the model with injections and withdrawals we get the following recursion formulas. Setting $S_{\mathrm{LB}}(0) = S_{\mathrm{UB}}(0) = S_0$, $S_{\mathrm{UB}}(t)$ and $S_{\mathrm{LB}}(t)$ can be obtained by recursion

$$\begin{aligned}S_{\mathrm{LB}}(t) = & S_{\mathrm{LB}}(t-1)+\sum_k I_k(t)E_k^{\mathrm{I}}(t)-\sum_k W_k(t)E_k^{\mathrm{W}}(t)+\sum_{t_2\le t}\Big[\\ & -\sum_{t_1\le t}\sum_{k,l}\Big(O\left(V_k(t_1),V_l(t_2)\right)\left(E_l(t_2)-E_k(t_1)\right)^+ + O\left(V_k(t_2),V_l(t_1)\right)\left(E_l(t_1)-E_k(t_2)\right)^+\Big)\\ & -\sum_{t_1>t}\sum_{k,l}\Big(O\left(V_k(t_1),V_l(t_2)\right)E_l(t_2)\Big)\Big]\\ = & S_{\mathrm{LB}}(t-1)+\sum_k I_k(t)E_k^{\mathrm{I}}(t)-\sum_k W_k(t)E_k^{\mathrm{W}}(t)\\ & +\sum_{t_2\le t}O\left(V_k(t),V_l(t_2)\right)\left(E_l(t_2)-E_k(t)\right)^+ -\sum_{t_1\le t}O\left(V_k(t_1),V_l(t)\right)\left(E_l(t)-E_k(t_1)\right)^+\\ & +\sum_{t_2\le t}O\left(V_k(t_2),V_l(t)\right)\left(E_l(t)-E_k(t_2)\right)^+ -\sum_{t_1\le t}O\left(V_k(t),V_l(t_1)\right)\left(E_l(t_1)-E_k(t)\right)^+\\ & +\sum_{t_2\le t}O\left(V_k(t),V_l(t_2)\right)E_l(t_2)-\sum_{t_1>t}O\left(V_k(t_1),V_l(t)\right)E_l(t)\end{aligned}$$

and

$$
\begin{aligned}
S_{\mathrm{UB}}(t) = S_{\mathrm{UB}}(t-1) + \sum_k I_k(t) E_k^{\mathrm{I}}(t) - \sum_k W_k(t) E_k^{\mathrm{W}}(t) + \sum_{t_2 \le t} \Big[\\
+ \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_k(t_1) - E_l(t_2)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_k(t_2) - E_l(t_1)\right)^+ \Big) \\
+ \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_2), V_l(t_1)\right) E_k(t_2) \Big) \Big] \\
= S_{\mathrm{UB}}(t-1) + \sum_k I_k(t) E_k^{\mathrm{I}}(t) - \sum_k W_k(t) E_k(t) \\
- \sum_{t_2 \le t} O\left(V_k(t), V_l(t_2)\right) \left(E_k(t) - E_l(t_2)\right)^+ + \sum_{t_1 \le t} O\left(V_k(t_1), V_l(t)\right) \left(E_k(t_1) - E_l(t)\right)^+ \\
- \sum_{t_2 \le t} O\left(V_k(t_2), V_l(t)\right) \left(E_k(t_2) - E_l(t)\right)^+ + \sum_{t_1 \le t} O\left(V_k(t), V_l(t_1)\right) \left(E_k(t) - E_l(t_1)\right)^+ \\
- \sum_{t_2 \le t} O\left(V_k(t_2), V_l(t)\right) E_k(t_2) + \sum_{t_1 > t} O\left(V_k(t), V_l(t_1)\right) E_k(t)
\end{aligned}
$$

Appendix C. Mutually exclusive injections and withdrawals

- Observing that

  $$\mathbb{E}\left[X^+\right] - \mathbb{E}\left[Y^+\right] > \mathbb{E}\left[(X - Y)\, \mathbb{1}_{Y>0}\right]$$

  and denoting

$$
\begin{aligned}
X = &\Big(\sum_k Tr_{\mathrm{SF}\to h_k}(u) \left[F_k(t,u)\left(1 - K_{\mathrm{W}}^1(u)\right) - K_{\mathrm{W}}^0(u)\right] \\
&+ \sum_{l \ne k} \min\{Tr_{\mathrm{SF}\to h_l}(u), Tr_{h_l \to h_k}(u)\} \left[F_k(t,u)\left(1 - K_{\mathrm{W}}^1(u)\right) - K_{\mathrm{W}}^0(u)\right] \Big) \\
&- \Big(\sum_k Tr_{h_k \to \mathrm{SF}}(v_1) \left[F_k(t,v_1)\left(1 + K_{\mathrm{I}}^1(v_1)\right) + K_{\mathrm{I}}^0(v_1)\right] \\
&+ \sum_{l \ne k} \min\{Tr_{h_l \to \mathrm{SF}}(v_1), Tr_{h_k \to h_l}(v_1)\} \left[F_k(t,v_1)\left(1 + K_{\mathrm{I}}^1(v_1)\right) + K_{\mathrm{I}}^0(v_1)\right] \Big)
\end{aligned}
$$

$$
\begin{aligned}
Y = &\Big(\sum_k Tr_{h_k \to \mathrm{SF}}(u) \left[F_k(t,u)\left(1 + K_{\mathrm{I}}^1(u)\right) + K_{\mathrm{I}}^0(u)\right] \\
&+ \sum_{l \ne k} \min\{Tr_{h_l \to \mathrm{SF}}(u), Tr_{h_k \to h_l}(u)\} \left[F_k(t,u)\left(1 + K_{\mathrm{I}}^1(u)\right) + K_{\mathrm{I}}^0(u)\right] \Big) \\
&- \Big(\sum_k Tr_{h_k \to \mathrm{SF}}(v_1) \left[F_k(t,v_1)\left(1 + K_{\mathrm{I}}^1(v_1)\right) + K_{\mathrm{I}}^0(v_1)\right] \\
&+ \sum_{l \ne k} \min\{Tr_{h_l \to \mathrm{SF}}(v_1), Tr_{h_k \to h_l}(v_1)\} \left[F_k(t,v_1)\left(1 + K_{\mathrm{I}}^1(v_1)\right) + K_{\mathrm{I}}^0(v_1)\right] \Big)
\end{aligned}
$$

We have

$$\begin{aligned}\sum_{k,l} C_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{W}}(u)\big) - \sum_{k,l} C_{\mathrm{I}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{I}}(u)\big) &= \mathbb{E}\left[X^+\right] - \mathbb{E}\left[Y^+\right] \\ &> \mathbb{E}\left[(X-Y)\,\mathbb{1}_{Y>0}\right] \\ &= -\,\mathbb{E}\left[\kappa(t,u)\mathbb{1}_{Y>0}\right]\end{aligned} \tag{5}$$

if options $C_{\mathrm{I}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{I}}(u)\big)$ meet condition (1) , i.e. if

$$\mathbb{E}\left[\kappa(t,u)\mathbb{1}_{Y>0}\right] \leq \frac{1}{2}\kappa(0,u)$$

then

$$\sum_{k,l} C_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{W}}(u)\big) - \sum_{k,l} C_{\mathrm{I}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{I}}(u)\big) \geq -\frac{1}{2}\kappa(0,u)$$

It is similar for options $C_{\mathrm{W}}\big(V_k^{\mathrm{W}}(u), V_l^{\mathrm{W}}(v_2)\big)$ meeting condition (2)

$$\sum_{k,l} C_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(u), V_l^{\mathrm{W}}(v_2)\big) - \sum_{k,l} C_{\mathrm{W}}\big(V_k^{\mathrm{W}}(u), V_l^{\mathrm{W}}(v_2)\big) \geq -\frac{1}{2}\kappa(0,u)$$

where

$$\begin{aligned}Y = \Big(&\sum_k Tr_{\mathrm{SF}\to h_k}(v_2)\left[F_k(t,v_2)\left(1-K_{\mathrm{W}}^1(v_2)\right) - K_{\mathrm{W}}^0(v_2)\right] + \sum_{l\neq k}\min\{Tr_{\mathrm{SF}\to h_l}(v_2), Tr_{h_l\to h_k}(v_2)\} \\ &\left[F_k(t,v_2)\left(1-K_{\mathrm{W}}^1(v_2)\right) - K_{\mathrm{W}}^0(v_2)\right]\Big) - \Big(\sum_k Tr_{\mathrm{SF}\to h_k}(u)\left[F_k(t,u)\left(1-K_{\mathrm{W}}^1(u)\right) - K_{\mathrm{W}}^0(u)\right] \\ &+ \sum_{l\neq k}\min\{Tr_{\mathrm{SF}\to h_l}(u), Tr_{h_l\to h_k}(u)\}\left[F_k(t,u)\left(1-K_{\mathrm{W}}^1(u)\right) - K_{\mathrm{W}}^0(u)\right]\Big)\end{aligned}$$

- Considering an optimal portfolio $\mathcal{P}$ of present value PV and assuming that $\sum_k I_k(u) > 0$ and $\sum_k W_k(u) > 0$ we show that constraints (3) and (4) are active. Let $\mathcal{P}$ be

$$\mathcal{P} = \Big\{\sum_k I_k(u), \sum_k W_k(u), \sum_{k,l} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{W}}(v_2)\big), \sum_{k,l} O_{\mathrm{W}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{W}}(v_2)big\big), \sum_{k,l} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{W}}(v_2)\big)\Big\}$$

if neither (3) or (4) are active, let $\epsilon > 0$, we define a new portfolio $\mathcal{P}^+$ with the same positions as $\mathcal{P}$ except the firm positions

$$\sum_k I_k(u)^+ = \sum_k I_k(u) - \epsilon,\ \sum_k W_k(u)^+ = \sum_k W_k(u) - \epsilon$$

We choose ϵ small enough so that all model's constraints are met and we have

$$PV^+ - PV = \epsilon\kappa\,(0,u) > 0$$

which is inconsistent with $\mathcal{P}$'s optimality. If only (4) is met, there exists v_1 such that $\sum_{k,l} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{I}}(u)\big) > 0$ and $\mathcal{P}^+$ has same positions as $\mathcal{P}$ except for positions

$$\sum_k I_k(u)^+ = \sum_k I_k(u) - \epsilon,\ \sum_k W_k(u)^+ = \sum_k W_k(u) - \epsilon$$

$$\sum_{k,l} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{W}}(u)\big)^+ = \sum_{k,l} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{W}}(u)\big) - \epsilon,\ \sum_{k,l} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{W}}(u)\big)^+ = \sum_{k,l} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{W}}(u)\big) + \epsilon$$

Combined with (5),

$$\begin{aligned}
\frac{PV^+ - PV}{\epsilon} &= \kappa\left(0,u\right) + \sum_{k,l} C_{\text{IW}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big) - \sum_{k,l} C_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{I}}(u)\big)\\
&> \kappa\left(0,u\right) - \mathbb{E}\left[\kappa(t,u)\mathbb{1}_{Y>0}\right]\\
&= F(p)\Big(\sum_k Tr_{\text{SF}\to h_k}(u) K_{\text{W}}^1(u) + \sum_k Tr_{h_k\to\text{SF}}(u) K_{\text{I}}^1(u) + \sum_{l\neq k} \min\{Tr_{\text{SF}\to h_l}(u), Tr_{h_l\to h_k}(u)\} K_{\text{W}}^1(u)\\
&+ \sum_{l\neq k} \min\{Tr_{h_l\to\text{SF}}(u), Tr_{h_k\to h_l}(u)\} K_{\text{I}}^1(u)\Big)\left(1 - \mathbb{E}\left[\frac{F_k(t,u)}{F_k(t)}\mathbb{1}_{Y>0}\right]\right)\\
&+ \Big(\sum_k Tr_{\text{SF}\to h_k}(u) K_{\text{W}}^0(u) + \sum_k Tr_{h_k\to\text{SF}}(u) K_{\text{I}}^0(u) + \sum_{l\neq k} \min\{Tr_{\text{SF}\to h_l}(u), Tr_{h_l\to h_k}(u)\} K_{\text{W}}^0(u)\\
&+ \sum_{l\neq k} \min\{Tr_{h_l\to\text{SF}}(u), Tr_{h_k\to h_l}(u)\} K_{\text{I}}^0(u)\Big)\big(1 - \mathbb{E}\left[\mathbb{1}_{Y>0}\right]\big)\\
&< 1
\end{aligned}$$

Then $PV^+ > PV$ which is inconsistent with $\mathcal{P}$'s optimality.

The argument is similar when only (3) is met.

- Now, assuming (3) and (4) are met and still assuming $\sum_k I_k(u) > 0$, $\sum_k W_k(u) > 0$, there exists at least one day v_1 such that $\sum_{k,l} O_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{I}}(u)\big) > 0$ and one day v_2 such that $\sum_{k,l} O_{\text{W}}\big(V_k^{\text{W}}(u), V_l^{\text{W}}(v_2)\big) > 0$. We define $\mathcal{P}^+$ as a portfolio whose positions are the same as $\mathcal{P}$ except for positions

$$\sum_k I_k(u)^+ = \sum_k I_k(u) - \epsilon,\ \sum_k W_k(u)^+ = \sum_k W_k(u) - \epsilon$$

$$\sum_{k,l} O_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{I}}(u)\big)^+ = \sum_{k,l} O_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{I}}(u)\big) - \epsilon,\ \sum_{k,l} O_{\text{IW}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big)^+ = \sum_{k,l} O_{\text{IW}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big) + \epsilon$$

$$\sum_{k,l} O_{\text{W}}\big(V_k^{\text{W}}(u), V_l^{\text{W}}(v_2)\big)^+ = \sum_{k,l} O_{\text{W}}\big(V_k^{\text{W}}(u), V_l^{\text{W}}(v_2)\big) - \epsilon,\ \sum_{k,l} O_{\text{IW}}\big(V_k^{\text{I}}(u), V_l^{\text{W}}(v_2)\big)^+ = \sum_{k,l} O_{\text{IW}}\big(V_k^{\text{I}}(u), V_l^{\text{W}}(v_2)\big) + \epsilon$$

Choosing ϵ small enough such that all model's conditions are met, we have, under constraints (3) and (4)

$$\begin{aligned}
PV^+ - PV = \epsilon\Big(\kappa\left(0, uahbn\right) &+ \sum_{k,l} C_{\text{IW}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big) - \sum_{k,l} C_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{I}}(u)\big)\\
&+ \sum_{k,l} C_{\text{IW}}\big(V_k^{\text{I}}(u), V_l^{\text{W}}(v_2)\big) - \sum_{k,l} C_{\text{W}}\big(V_k^{\text{W}}(u), V_l^{\text{W}}(v_2)\big)\Big) > 0
\end{aligned}$$

which is inconsistent with $\mathcal{P}$'s optimality.

We conclude that if I and W are optimal we cannot have $\sum_k I_k(u) > 0$ and $\sum_k W_k(u) > 0$.