Understanding Analysis Practice

Solutions for some exercise problems

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Contents

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.9

Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?
Solution $f^{-1}(A) = [-2, 2], f^{-1}(B) = [-1, 1]$.
$f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$.
$f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$.

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.
Solution We show that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$.
($\Rightarrow$) If $x \in g^{-1}(A \cap B)$, then by definition $g(x) \in A \cap B$. By the definition of intersection, $g(x) \in A$ and $g(x) \in B$. Then $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, $x \in g^{-1}(A) \cap g^{-1}(B)$.
($\Leftarrow$) Apply the abov procedure backward.
The proof of $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ is similar, just change the intersection into union, and into or.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.
Solution A real number i is the *greatest lower bound* of a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

  (i) i is a lower bound of A;

  (ii) if b is any lower bound for A, then $i \geq b$.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.
Solution State: *Assume $i \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then, $i = \inf A$ if and only if , for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $i + \epsilon > a$.*
Proof.
($\Rightarrow$) Assume $i = \inf A$, and $i + \epsilon > i$. $i + \epsilon$ is not a lower bound of A based on the criteria (ii) above. Then there must be some element $a \in A$ such that $i + \epsilon > a$.
($\Leftarrow$) Assume i is a lower bound such that for all $\epsilon > 0$, $i + \epsilon$ is not a lower bound of A. Let $b = i + \epsilon$, then this implies that if $i < b$, then b is not a lower bound of A. This is the contrapositive statement of the criteria (ii) above. □

Exercise 1.3.6

Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.
Solution Since $s = \sup A$ and $t = \sup B$, $\forall (a \in A, b \in B), s \geq a, t \geq b$. Then $\forall a + b \in A + B, s + t \geq a + b$, $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.
Solution $\forall a + b \in A + B, u \geq a + b, u - a \geq b$. Fix $a \in A$, $u - a$ is a upper bound of B. Since $t = \sup B$, $t \leq u - a$.

(c) Finally, show $\sup(A+B) = s+t$.
Solution (a) has shown that $s+t$ is an upper bound for $A+B$, we only need to show that it's the least. That is if u be an arbitrary upper bound for $A+B$, then $s+t \leq u$.
From (b) we have $\forall a \in A, t \leq u-a$. $s = \sup A$, then $\forall a \in A, s \geq a$, $t \leq u-a \leq u-s$. Thus $s+t \leq u$.

(d) Construct another proof of this same fact using Lemma 1.3.8.
Solution We need to show that for every choice of $\epsilon > 0$, there exists an element $a+b \in A+B$ satisfying $s+t-\epsilon < a+b$.
Given the choice of $\epsilon > 0$, since s and t are least upper bounds, we apply Lemma 1.3.8 to them:
$\forall \epsilon_1 > 0, \epsilon_2 > 0, \exists a \in A, b \in B$, s.t. $s - \epsilon_1 < a, t - \epsilon_2 < b$. Taking $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$ and adding two inequalities finish the proof.

Exercise 1.3.11

Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.
Solution True. $\forall a \in A, a \leq \sup A$, $\forall b \in B, b \leq \sup B$. Since $A \subseteq B$, $\forall a \in A, a \in B$. Then $\forall a \in A, a \leq \sup B$, $\sup B$ is an upper bound for A. By the definition of $\sup A$, $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
Solution True. $\forall c \in (\sup A, \inf B), (\forall a \in A, b \in B, a < c < b)$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.
Solution False. If $c = \sup A = \inf B$, then $\forall a \in A, b \in B, a < c < b$, and $\sup A = \inf B$.

1.4 Consequences of Completeness

Exercise 1.4.2

Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}, s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.
Solution Proof by contradiction.
If $s > \sup A$, then let $\epsilon = s - \sup A > 0$, there must exist $n \in \mathbf{N}$ s.t. $n > \frac{1}{\epsilon}$. $s - \frac{1}{n} > s - \epsilon = \sup A$. Then $s - \frac{1}{n}$ is an upper bound of A, contradiction. If $s < \sup A$, then similarly, there exists n contradicts to $s + \frac{1}{n}$ being an upper bound. $\square$

Exercise 1.4.8

Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.
Solution Let $A = \{1 - \frac{1}{2n} : n \in \mathbf{N}\}, B = \{1 - \frac{1}{2n-1} : n \in \mathbf{N}\}$, then $A \cap B = \emptyset$. $\sup A = \sup B = 1, 1 \notin A$ and $1 \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.
Solution Let $J_n = (-\frac{1}{n}, \frac{1}{n}), n \in \mathbf{N}$, then $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n = \{0\}$.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in R : x \geq a\}$.)
Solution Let $L_n = [n, \infty), n \in \mathbf{N}$, then $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.
Solution Impossible. Let $K_n = \bigcap_{m=1}^{n} I_m$, then for each $n \in \mathbf{N}$, K_n is a closed interval and $K_n \subseteq K_{n+1}$ since $K_{n+1} = K_n \cap I_{n+1}$.

By the **Nested Interval Property** (Theorem 1.4.1), $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. Well, $\bigcap_{n=1}^{\infty} K_n = \bigcap_{n=1}^{\infty}(\bigcap_{m=1}^{n} I_m) = \bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Chapter 2

Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.2

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.
Solution *Proof* For any $n \in \mathbf{N}$,

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{5(2n+1) - 2(5n+4)}{5(5n+4)}\right| = \left| - \frac{3}{5(5n+4)}\right| = \frac{3}{25n+20} < \frac{25}{25n} = \frac{1}{n}$$

so any integer $N \geq \frac{1}{\epsilon}$ will satisfy the definition. $\square$

(b) $\lim \frac{2n^2}{n^3+3} = 0$.
Solution *Proof* For any $n \in \mathbf{N}$,

$$\left|\frac{2n^2}{n^3+3} - 0\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n}$$

so any integer $N \geq \frac{2}{\epsilon}$ will satisfy the definition. $\square$

(c) $\lim \frac{\sin\left(n^2\right)}{\sqrt[3]{n}} = 0$.
Solution *Proof* For any $n \in \mathbf{N}$,

$$\left|\frac{\sin\left(n^2\right)}{\sqrt[3]{n}} - 0\right| = \frac{\sin\left(n^2\right)}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}}$$

so any integer $N \geq \frac{1}{\epsilon^3}$ will satisfy the definition. $\square$

Exercise 2.2.4

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.
Solution $x_n = (-1)^n$ has an infinite number of ones but diverges.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.
Solution Impossible. First, an infinite number of ones means that $\forall N \in \mathbf{N}, \exists n \geq N$, s.t. $a_n = 1$. Otherwise there will be finite ones. Suppose $\lim a_n = a \neq 1$, take $\epsilon = |a-1|$. Then $\forall N \in \mathbf{N}, \exists n \geq N$, s.t. $a_n = 1, |a_n - a| = |1-a| = \epsilon$, violates the definition of limit.

(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.
Solution $x_n = (1, -1, 1, 1, -1, 1, 1, 1, -1, \cdots)$

Exercise 2.2.6

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.
Solution *Proof* Let any $\epsilon > 0$ be given. Define $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2} > 0$.
Since $(a_n) \to a$, $\exists N_1 \in \mathbf{N}$, s.t. $\forall n \geq N_1, |a_n - a| < \epsilon_1$.
Similarly, $\exists N_2 \in \mathbf{N}$, s.t. $\forall n \geq N_2, |a_n - b| < \epsilon_2$.
Let $N = \max\{N_1, N_2\}$, for any $n > N$, apply the triangle inequality:

$$|a - b| \leq |a - a_n| + |a_n - b| < \epsilon_1 + \epsilon_2 = \epsilon$$

This proves, $\forall \epsilon > 0, |a - b| < \epsilon$, which implies $|a - b| = 0, a = b$. $\square$

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.3 (Squeeze Theorem)

Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.
Solution *Proof* Let $\epsilon > 0$ be given. Since $\lim x_n = l$, $\exists N_1 \in \mathbf{N}$, s.t. $\forall n \geq N_1, |x_n - l| < \epsilon$.
Rewrite it as:

$$\exists N_1 \in \mathbf{N}, \forall n > N_1, l - \epsilon < x_n < l + \epsilon.$$

Similarly,

$$\exists N_2 \in \mathbf{N}, \forall n > N_2, l - \epsilon < z_n < l + \epsilon.$$

Let $N = \max\{N_1, N_2\}$, for any $n > N$, apply $x_n \leq y_n \leq z_n$:

$$l - \epsilon < x_n \leq y_n \leq z_n < l + \epsilon, \quad \text{so} \quad |y_n - l| < \epsilon.$$

This proves, $\forall \epsilon > 0, \exists N \in \mathbf{N}$, s.t.$\forall n > N, |y_n - l| < \epsilon$, which implies $\lim y_n = l$. $\square$

Exercise 2.3.10

Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.
Solution False. Consider $a_n = b_n = n$, then $\lim(a_n - b_n) = 0$ and $\lim a_n, \lim b_n$ DNE.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.
Solution True. If $(b_n) \to b$, then $\forall \epsilon > 0, \exists N \in \mathbf{N}, \forall n \geq N, |b_n - b| < \epsilon$. Under the same ϵ, N, $\forall n \geq N$, $\Big||b_n| - |b|\Big| \leq |b_n - b| < \epsilon$. Thus, $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.
Solution True. Apply the **Algebraic Limit Theorem** (ii), $\lim b_n = \lim[a_n + (b_n - a_n)] = \lim a_n + \lim(b_n - a_n) = a + 0 = a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.
Solution True. If $(a_n) \to 0$, then $\forall \epsilon > 0, \exists N \in \mathbf{N}, \forall n \geq N, |a_n| < \epsilon$. Under the same ϵ, N, $\forall n \geq N, |b_n - b| \leq a_n \leq |a_n| < \epsilon$. Thus, $(b_n) \to b$.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Practice question

Let (a_n) be a sequence such that $|a_{n+1} - a_n| < \frac{1}{2^n}$ for all $n \in \mathbf{N}$. Prove that (a_n) is a convergent sequence.
(**Hint**: Show that (a_n) is a Cauchy sequence.)
Solution *Proof* Given $\epsilon > 0$, choose $N \in \mathbf{N}$ such that $\frac{1}{2^{N-1}} < \epsilon$. Then for any $n \geq N$ and $p \in \mathbf{N}$,

$$\begin{aligned}
|a_{n+p} - a_n| &\leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \ldots + |a_{n+p-1} - a_{n+p}| \\
&\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{n+p}} \\
&\leq \frac{1}{2^n}\Big[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^p}\Big] \\
&< \frac{2}{2^n} = \frac{1}{2^{n-1}} < \epsilon
\end{aligned}$$

So (a_n) is a Cauchy sequence, and by **Theorem 2.6.4 (Cauchy Criterion)**, it converges.

Exercise 2.4.3

(a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.
Solution $x_1 = \sqrt{2}$,

$$x_{n+1} = \sqrt{2 + x_n},$$

Using induction, $n = 1 : x_2 = \sqrt{2+\sqrt{2}} > x_1, x_1 < 2$, suppose $n = i - 1 : x_i > x_{i-1}, x_{i-1} < 2$, then:

$$\begin{gathered}
2 + x_i > 2 + x_{i-1} \Rightarrow \sqrt{2 + x_i} > \sqrt{2 + x_{i-1}} \Rightarrow x_{i+1} > x_i \\
x_i = \sqrt{2 + x_{i-1}} < \sqrt{2+2} = 2
\end{gathered}$$

Thus x_n is increasing and bounded $\sqrt{2} \le x_n < 2$, by **Theorem 2.4.2 (Monotone Convergence Theorem)** it converges.
Let $x = \lim x_n$, by taking limit of each side of the recursive equation in part (a):

$$x = \sqrt{2 + x}$$

we have $x = -1$ or $x = 2$, since $x > \sqrt{2}$, $x = 2$.

(b) Does the sequences

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converges? If so, find the limit.
Solution $y_1 = \sqrt{2}$,

$$y_{n+1} = \sqrt{2y_n},$$

Using induction, $n = 1 : y_2 = \sqrt{2\sqrt{2}} > y_1, y_1 < 2$, suppose $n = i - 1 :$ $y_i > y_{i-1}, y_{i-1} < 2$, then:

$$2y_i > 2y_{i-1} \Rightarrow \sqrt{2y_i} > \sqrt{2y_{i-1}} \Rightarrow y_{i+1} > y_i$$
$$y_i = \sqrt{2y_{i-1}} < \sqrt{2 \times 2} = 2$$

Thus y_n is increasing and bounded $\sqrt{2} \le y_n < 2$, by **Theorem 2.4.2 (Monotone Convergence Theorem)** it converges.
Let $y = \lim y_n$, by taking limit of each side of the recursive equation in part (a):

$$y = \sqrt{2y}$$

we have $y = 0$ or $y = 2$, since $y > \sqrt{2}$, $y = 2$.

2.5 Subsequences and the BolzanoWeierstrass Theorem

Exercise 2.5.2

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

Solution True. Sequence $(x_2, x_3, x_4, \ldots)$ is a proper subsequence of (x_n), so it converges. Then (x_n) converges since the first term does not change the convergence of a sequence.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

Solution True. **Theorem 2.5.2** shows that subsequence of a convergent sequence converges, which is the contrapositive of (b).

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

Solution True.

Since (x_n) is bounded, by Bolzano-Weierstrass, it contains a convergent subsequence a_n that converges to a. By definition of convergence of a sequence (2.2.3):

$$\forall \epsilon > 0, \exists N \in \mathbf{N} : \forall n \geq N, |a_n - a| < \epsilon.$$

(x_n) diverges, so does not satisfy to converge to a, then use the negation statement:

$$\exists \epsilon > 0, \forall N \in \mathbf{N} : \exists k \geq N, |x_k - a| \geq \epsilon.$$

there should be infinite k that satisfies: Suppose we choose $N_1 \in \mathbf{N}$ and $k_1 \in \mathbf{N} : k_1 \geq N_1, |x_{k_1} - a| \geq \epsilon$, then we can choose $N_2 = k_1$, there exists $k_2 \geq N_2 = k_1$, satisfies $|x_{k_2} - a| \geq \epsilon$, so we can continue and choose infinite k.

Such (x_k) is a subsequence of (x_n), so is bounded. Again by B-W, it contains a convergent subsequence that converges to b. $b \neq a$ since all terms of (x_k) are bounded away from a by ϵ. Thus, there are two subsequences that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Solution True. The subsequence is convergent, then it is bounded (by **Theorem 2.3.2**). We show that a monotone sequence is bounded if it has a bounded subsequence.

Without loss of generality, suppose (x_n) is monotonically increasing.

(x_{n_k}) is bounded by $\forall k \in \mathbf{N}, |x_{n_k}| \leq M$. Then, given any $n \in \mathbf{N}, \exists k \in \mathbf{N} : n \leq n_k, x_n \leq x_{n_k} \leq M$. Thus, (x_n) is bounded. Similar for (x_n) decreases.
(x_n) is bounded and monotone, then it converges (by **Theorem 2.4.2 MCT**).

Exercise 2.5.9

Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbf{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)
Solution *Proof* Since $s = \sup S$, then

$$\forall \epsilon > 0, \exists x \in S : x + \epsilon > s$$

Thus, $|x - s| < \epsilon$, i.e. for any $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that

$$\forall n \geq N, s - \epsilon < a_n < s + \epsilon$$

Given $\epsilon = \frac{1}{k}, \exists n_k \in \mathbf{N}, s - \frac{1}{k} < a_{n_k} < s + \frac{1}{k}$.
Therefore, $\lim a_{n_k} = s$.

2.7 Properties of Infinite Series

Exercise 2.7.2

Decide whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$
Solution Note that $0 \leq \frac{1}{2^n+1} \leq \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, since $|\frac{1}{2}| < 1$. By **Theorem 2.7.4 (Comparison Test)**, $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converges.

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$
Solution Note that $0 \leq \frac{\sin(n)}{n^2} \leq \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, since the power $2 > 1$. By comperison test, it converges.

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$
Solution $a_n = (-1)^{n+1}(\frac{1}{2} + \frac{1}{2n}) = (-1)^{n+1}\frac{1}{2} + (-1)^{n+1}\frac{1}{2n}$. where $(-1)^{n+1}\frac{1}{2}$ diverges apparently, and $(-1)^{n+1}\frac{1}{2n}$ converges by **Theorem 2.7.7 (Alternating Series Test)**. As a result, (a_n) diverges.

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$
Solution $s_{3n} - s_{3(n-1)} = \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n} > \frac{1}{3(n-1)}$. We know that $\sum_{n=2}^{\infty} \frac{1}{3(n-1)}$ diverges since the power $p = 1$. Then by the comperison test, $\lim_n s_n = s_3 + \sum_{n=2}^{\infty}[s_{3n} - s_{3(n-1)}]$ diverges.

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$
Solution $s_{2n} - s_{2(n-1)} = \frac{1}{2n-1} - \frac{1}{(2n)^2}$. Then $\lim_n s_n = s_2 + \sum_{n=2}^{\infty}[s_{2n} - s_{2(n-1)}] = \sum_{n=1}^{\infty} \frac{1}{2n-1} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$. $\frac{1}{2n-1} > \frac{1}{2n}$, and $\sum_{n=1} \frac{1}{2n}$ diverges, by comperison test, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges.
$\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$ converges cince the power $p = 2 > 1$. As a result, (s_n) diverges.

Exercise 2.7.4

Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.
Solution Let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. Both series $\sum x_n$ and $\sum y_n$ diverges because they're harmonic series with power 1. $\sum x_n y_n = \sum \frac{1}{n(n+1)}$ converges by Comparison test with $\sum \frac{1}{n^2}$.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.
Solution Let $x_n = (-1)^n \frac{1}{n}$ being convergent by the Alternating series test, and $y_n = (-1)^n$ is a bounded sequence. $\sum x_n y_n = \frac{1}{n}$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but $\sum y_n$ diverges.
Solution Impossible. Since if $\sum x_n$ and $\sum(x_n + y_n)$ both converge, by **Theorem 2.7.1 (Algebraic Limit Theorem for Series)**, $\sum y_n$ should also be convergent.

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum(-1)^n x_n$ diverges.
Solution Let

$$x_n = \begin{cases} \frac{1}{n} & \text{n is odd} \\ 0 & \text{n is even} \end{cases}$$

It satifies $0 \leq x_n \leq \frac{1}{n}$ obviously and $\sum(-1)^n x_n$ diverges because it's harmonic series with power 1.

Chapter 3

Basic Topology of R

3.2 Open and Closed Sets

Exercise 3.2.2

$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \dots\right\} \quad \text{and} \quad B = \{x \in \mathbf{Q} : 0 < x < 1\}$$

Answer the following questions for each set:

(a) What are the limit points?
Solution The set of limit points of A is $\{-1, 1\}$. The set of limit points of B is $[0, 1]$.

(b) Is the set open? Closed?
Solution A is not open since $1 \in A$ does not have an open interval $(a, b) \in A$, and not closed since $-1 \notin A$. B is not open since $\forall x \in B, \nexists(a, b) : a < x < b \wedge (a, b) \in B$, and not closed since $\exists^\infty x \in [0, 1] \wedge x \in \mathbf{Q}$.

(c) Does the set contain any isolated points?
Solution Every points in A except 1 are isolated points. B has no isolated points.

(d) Find the closure of the set.
Solution $\bar{A} = A \cup \{-1\}$ and $\bar{B} = B \cup [0, 1] = [0, 1]$.

Exercise 3.2.5

Prove **Theorem 3.2.8.** *A set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*
Solution *Proof*
$\Rightarrow$ Suppose set $F \subseteq \mathbf{R}$ is closed, and let (x_n) be a Cauchy sequence contained in F. By Cauchy Criterion, $(x_n) \to x$, x is then a limit point of F. Thus, $x \in F$.
$\Leftarrow$ Suppose every Cauchy sequence in F converges to a limit in F. Let x be a limit point of F. By **Theorem 3.2.5**, $\exists (a_n) \in F : x = \lim a_n$, and by Cauchy Criterion, (a_n) must be a Cauchy sequence, then $x \in F$, and F is closed. $\square$

Exercise 3.2.11

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
Solution *Proof* Given set A, B and let A', B' be the set of all limit points of A and B, respectively. And $(A \cup B)'$ be the set of all limit points of $A \cup B$.
$\subseteq$ If $x \in \overline{A \cup B}$, $x \in (A \cup B) \cup (A \cup B)'$, i.e. $(x \in A) \vee (x \in B) \vee (x \in (A \cup B)')$. If $(x \in A) \vee (x \in B)$, then obviously $x \in \overline{A} \cup \overline{B}$. If $x \in (A \cup B)'$, $x = \lim(x_n)$ for some sequence x_n contained in $A \cup B$, then $((x_n) \in A) \vee ((x_n) \in B)$. If $(x_n) \in A$, then $x \in A'$, similar for $(x_n) \in B$, thus $x \in (A' \cup B')$, $x \in \overline{A} \cup \overline{B}$.
$\supseteq$ If $x \in \overline{A} \cup \overline{B}$, $(x \in (A \cup A')) \vee (x \in (B \cup B'))$, i.e. $(x \in A) \vee (x \in A') \vee (x \in B) \vee (x \in B')$. Similarly, we consider $(x \in A') \vee (x \in B')$, then $x = \lim x_n$ for some sequence $((x_n) \in A) \vee ((x_n) \in B)$, then $(x_n) \in (A \cup B)$, thus $x \in (A \cup B)', x \in \overline{A \cup B}$.
That completes the proof. $\square$

(b) Does this result about closures extend to infinite union of sets?
Solution No, it does not. Because the previous proof relies on the fact that the closure is a closed set and the union of a finite collection of closed sets is closed. While infinite union of sets are not always closed. For example, $A_n = \{\frac{1}{n}\}$ is a closed set for each $n \in \mathbf{N}$, and $\bigcup_{n=1}^{\infty} A_n = \{\frac{1}{n} : n \in \mathbf{N}\}$, which is a open set with limit point 0.

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \bigcup_{n=1}^{\infty} \overline{A_n} \cup \{0\}$$

3.3 Compact Sets

Exercise 3.3.4

Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$
Solution K is compact so is closed and bounded, F is closed. $K \cap F$ is the intersection of closed sets, is closed. $K \cap F$ is a subset of K, thus bounded. So $K \cap F$ is compact.

(b) $\overline{F^c \cup K^c}$
Solution The form of closure implies closed. K is bounded implies K^c is unbounded, so is $F^c \cup K^c$ and $\overline{F^c \cup K^c}$.

(c) $K \backslash F = \{x \in K : x \notin F\}$
Solution $K \backslash F = K \cap F^c$, which is a subset of K, thus bounded. While it can be closed or not. Let K' be the set of limit points of K. If $K \subseteq F^c$, then $K \backslash F = K$, is still closed. If $K \subseteq F^c$ and $K' \cap F \neq \emptyset$, i.e. there is limit points of K in set F, then it is also a limit point of $K \backslash F$ but not in $K \backslash F$, thus $K \backslash F$ is not closed.

(d) $\overline{K \cap F^c}$
Solution The form of closure implies being closed. From (c) we know that $K \cap F^c$ is bounded, so is $\overline{K \cap F^c}$. Thus, $\overline{K \cap F^c}$ is compact.

Exercise 3.3.8

Let K and L be nonempty compact sets, and define

$$d = \inf\{|x - y| : x \in K \ \text{ and } \ y \in L\}.$$

This turns out to be a reasonable definition for the *distance* between K and L.

(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.
Solution K and L being compact implies $\{|x-y| : x \in K \ \text{ and } \ y \in L\}$ is a compact set. Since K and L are disjoint, $\nexists x \in K \wedge x \in L$,

$|x-y|>0, d>0$. $\{|x-y| : x \in K \text{ and } y \in L\}$ is closed, the limit point $d=|x_0-y_0|$ is in the set. Thus, $d=|x_0-y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

(b) Show that it's possible to have $d=0$ if we assume only that the disjoint sets K and L are closed.
Solution Let $K=\mathbf{R}$ and $L=\mathbf{N}$, both are closed and unbounded. Then $d=0$.

Chapter 4

Functional Limits and Continuity

4.2 Functional Limits

Exercise 4.2.5

Use Definition 4.2.1 to supply a proper proof for the following limit statements.

(a) $\lim_{x\to 2}(3x+4) = 10$.
Solution $|(3x+4)-10| = |3x-6| = 3|x-2|$. Given $\epsilon > 0$, choose $\delta = \epsilon/3$, then $0 < |x-2| < \delta$ implies $|(3x+4)-10| < 3\delta = 3(\epsilon/3) = \epsilon$.

(b) $\lim_{x\to 0} x^3 = 0$.
Solution $|x^3-0| = |x^3|$. Given $\epsilon > 0$, choose $\delta = \epsilon^{\frac{1}{3}}$, then $0 < |x-0| < \delta$ implies $|x^3-0| < \delta^3 = \epsilon$.

(c) $\lim_{x\to 2}(x^2+x-1) = 5$.
Solution $|(x^2+x-1)-5| = |x^2+x-6| = |x-2||x+3|$. Given $\epsilon > 0$, choose $\delta = \min\{1, \epsilon/6\}$. If $0 < |x-2| < \delta$, then

$$|(x^2+x-1)-5| = |x-2||x+3| < \left(\frac{\epsilon}{6}\right)6 = \epsilon.$$

(d) $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$.
Solution $\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|x-3|}{3|x|}$. Given $\epsilon > 0$, choose $\delta = \min\{1, 6\epsilon\}$. If

$0 < |x-3| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|x-3|}{3|x|} < 6\epsilon\left(\frac{1}{6}\right) = \epsilon.$$

Exercise 4.2.7

Let $g : A \to \mathbf{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$.
Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.
Solution
Proof If $\lim_{x\to c} g(x) = 0$, by definition:

$$\forall \epsilon_1 > 0, \exists \delta > 0 : 0 < |x-c| < \delta \Rightarrow |g(x)| < \epsilon_1.$$

and we know that

$$\exists M > 0, \forall x \in A : |f(x)| \leq M.$$

Given $\epsilon > 0$, let $\epsilon_1 = \frac{\epsilon}{M}$, then there exists δ such that if $0 < |x-c| < \delta$, $|g(x)| < \epsilon_1 = \frac{\epsilon}{M}$. Thus,

$$|g(x)f(x) - 0| = |g(x)||f(x)| < \left(\frac{\epsilon}{M}\right)M = \epsilon.$$

$\square$