1. (a) (5 points) Given that

$$A_1 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

compute the linear combination $-2A_1 + 3A_2 - 2A_3$.

(b) (5 points) Express A_1 in terms of A_2 and A_3.

```python
# Q1
A1 = np.array([[2,6,4]]).T
A2 = np.array([[2,4,2]]).T
A3 = np.array([[1,0,-1]]).T

print(-2*A1 + 3*A2-2*A3)

A = sp.Matrix([[2,4,2],[1,0,-1]]).T
A1 = sp.Matrix([[2,6,4]]).T
x1,x2= sp.symbols('x1 x2')
print(A)
print(A1)
sol = sp.linsolve((A,A1),x1,x2)
```

```
linear combination is

[[0]
 [0]
 [0]]
solution is {(3/2, -1)}
```

(a): $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $A_1 = \frac{3}{2}A_2 - A_3$.

2. (a) (5 points) For each of the following matrices, determine if it is in reduced row-echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) (5 points) Let $A = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 0 & 0 \\ 1 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 6 \\ 4 & 5 & 0 \\ 2 & 7 & 5 \end{bmatrix}$. Find $\det(AB)$ and $\det(3A)$.

(a) 1. Yes 2. Yes. 3. Yes

4. No. leading 1 on last row isn't the only non-zero entry in its column.

5. No. leading 1 on third row isn't the only non-zero entry in its column.

6. No. leading 1 on third row isn't the only non-zero entry in its column.

2.(b)

```
# Q2b
A = sp.Matrix([[2,4,5],[3,0,0],[1,4,3]])
B = sp.Matrix([[1,3,6],[4,5,0],[2,7,5]])

print(f"det(AB) is {sp.det(A@B)}")
print(f"det(3A) is{sp.det(3*A)}")
```

```
det(AB) is 1752
det(3A) is 648
```

$\det(AB) = 1752$

$\det(3A) = 648$

3 (a)

```
# Q3
print("Q3a")
A = np.array([[2,4,5],[3,0,0],[1,4,3]])
B = np.array([[1,3,6],[4,5,0],[2,7,5]])

print(f"A*B = {A@B}" )
print(f"B*A = {B@A}")
```

```
Q3a
A*B = [[28 61 37]
 [ 3  9 18]
 [23 44 21]]
B*A = [[17 28 23]
 [23 16 20]
 [30 28 25]]
```

$AB \neq B \cdot A$.

3.(b)

```
print("\n Q3b")
A = np.array([[1, 0],[0, 0]])
B = np.array([[0, 0],[0, 1]])

print(f"A*B = {A@B}")
print(f"B*A = {B@A}")
```

```
 Q3b
A*B = [[0 0]
 [0 0]]
B*A = [[0 0]
 [0 0]]
```

$A \cdot B = BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ They do communicate.

3(c)

```python
print("Q3c")

A = np.array([[2,4,5],[3,0,0],[1,4,3]])
B = np.array([[1,3,6],[4,5,0],[2,7,5]])

M = np.linalg.det(A+B)
N = np.linalg.det(A)+np.linalg.det(B)
print(f"det(A+B) = {M} and det(A)+det(B) = {N}")
```

```
Q3c
det(A+B) = 410.00000000000017 and det(A)+det(B) = 96.99999999999999
```

$\det(A+B) \neq \det(A) + \det(B)$

4.

```python
# Q4
print("Q4")
A = np.array([[1,0,-1,2],[0,3,1,-1],[2,4,0,3],[-3,1,-1,2]])
B = np.array([[1,3,0,4],[2,-1,-2,1]]).T
C = np.array([[3,-2,0,5],[1,0,-3,4]])

print(3*pow(A,3)-5*pow((B@C),2))

A = sp.Matrix([[1,0,-1,2],[0,3,1,-1],[2,4,0,3],[-3,1,-1,2]]
B = sp.Matrix([[1,3,0,4],[2,-1,-2,1]]).T
C = sp.Matrix([[3,-2,0,5],[1,0,-3,4]])
print(3*pow(A,3)-5*pow((B@C),2))
```

```
Q4
[[ -122   -20  -183  -821]
 [ -320   -99   -42  -608]
 [    4   192  -180  -239]
 [ -926  -317   -48 -2856]]
Matrix([[-1016, 465, 561, -2060], [-588, 490, 432, -1372], [549, -199, -312, 1270], [-156
2, 865, 1014, -3492]])
```

Same Code but different Result between NumPy and SymPy

NumPy:

$$\begin{bmatrix} -122 & -20 & -183 & -821 \\ -320 & -99 & -42 & -608 \\ 4 & 192 & -180 & -239 \\ -926 & -317 & -48 & -2856 \end{bmatrix}$$

SymPy

$$\begin{bmatrix} -1016 & 465 & 561 & -2060 \\ -588 & 490 & 432 & -1372 \\ 549 & -199 & -312 & 1270 \\ -1562 & 865 & 1014 & -3492 \end{bmatrix}$$

5

$AX=B$

$$A=\begin{bmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ 2 & 3 & 1 \end{bmatrix} \quad B=\begin{bmatrix} -4 \\ -6 \\ 3 \end{bmatrix}$$

```python
print("Q5")

A = np.array([[1,2,-1],[1,4,-2],[2,3,1]])
B = np.array([-4,-6,3])
# print(B)
print(np.linalg.solve(A,B))
```

```
Q5
[-2.  1.  4.]
```

$$X=\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

Q6: Since $A^{-1} = A^T$

$$A \cdot A^T = A \cdot A^{-1} = I.$$

A is othonormal set.

$x_1 = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$ $x_2 = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$ $\langle x_1 \cdot x_2 \rangle = \cos\theta\sin\theta - \sin\theta\cos\theta$

$= 0$.

And $\|x_1\|_2 = \|x_1\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$\therefore$ θ can be any value.

Q7: $m = \begin{bmatrix} x \\ 0 \end{bmatrix}, n = \begin{bmatrix} 0 \\ y \end{bmatrix}$

(a) $\vec{m} \cap \vec{n} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$

$\begin{cases} \vec{0} + \vec{0} = \vec{0}. \\ c \cdot \vec{0} = \vec{0} \end{cases} \Rightarrow$ (a) is subspace of R^2.

(b): $\vec{m} \cup \vec{n}$:

$\vec{m} + \vec{n} = (x, y)$.

for any $x \neq 0$ and $y \neq 0$

$\vec{m} + \vec{n} \notin \vec{m} \cup \vec{n}$

b isn't the sub space of R^2

(c)

$M + N = \{(x, y).\ x, y \in R\}$

$\vec{V} = a\vec{x_1} + b\vec{y_1}$

$\vec{W} = c\vec{x_2} + d\vec{y_2}$

$\vec{V} + \vec{W} = a\vec{x_1} + c\vec{x_2} + b\vec{y_1} + d\vec{y_2}$

$= m\vec{x_3} + n\vec{y_3} \in M+N$

$c\vec{V} = ac\vec{x_1} + cb\vec{y_1}$. $\in M+N$

$M+N$ is subspace R^2

(d). $M - N = \{(x, -y) \mid x, y \in R\}$

$\vec{V} = a\vec{x_1} - b\vec{y_1}$

$\vec{W} = c\vec{x_2} - d\vec{y_2}$

$\vec{V} + \vec{W} = a\vec{x_1} + c\vec{x_2} - b\vec{y_1} - d\vec{y_2}$

$= m\vec{x_3} - n\vec{y_3} \in M - N$

$c\vec{V_1} = ca\vec{x_1} - cb\vec{y_1} \in M - N$

$M - N$ is sub space of R^2.

Q8

```
# Q8
print("Q8")
A = sp.Matrix([[1,1,2,3],[3,4,-1,2],[-1,-2,5,4]])

print(A.rref())
```

```
Q8
(Matrix([
[1, 0,  9, 10],
[0, 1, -7, -7],
[0, 0,  0,  0]]), (0, 1))
```

Rank (A) = 2. $\dim(\text{Nul}(A)) = n - \dim(\text{col}(A))$

$= 2$.

9. (a). $\|f_1\| = \sqrt{\int_0^1 x^2 dx} = \sqrt{\frac{1}{3}x^3\Big|_0^1}$
$= \frac{\sqrt{3}}{3}$

$\|f_2\| = \sqrt{\int_0^1 x^8 dx} = \frac{1}{3}$

$\langle f_1 \cdot f_2 \rangle = \int_0^1 x \cdot x^4 dx = \frac{1}{6}x^6\Big|_0^1 = \frac{1}{6}$

(b) $g_1 = f_1 = x$

$u_1 = \frac{g_1}{\|g_1\|} = \frac{x}{\frac{\sqrt{3}}{3}} = \sqrt{3}x$

$g_2 = f_2 - \langle f_2, u_1 \rangle \cdot u_1$

$= x^4 - \left(\int_0^1 \sqrt{3} \cdot x^5 dx\right) \cdot u_1 = x^4 - \frac{1}{2}x$

$u_2 = \frac{g_2}{\|g_2\|} = \frac{x^4 - \frac{1}{2}x}{\sqrt{\int_0^1 (x^4 - \frac{1}{2}x)^2 dx}} = \frac{x^4 - \frac{1}{2}x}{\frac{1}{6}} = 6x^4 - 3x$

$\Rightarrow \{x, x^4 - \frac{1}{2}x\}$ forms othogonal basis

$\{\sqrt{3}x, 6x^4 - 3x\}$ forms othonormal basis

Q10. $\begin{bmatrix} a & -a-b \\ 0 & b \end{bmatrix} = a\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$.

$= \text{span}\left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right\}$.

for $\begin{bmatrix} a & -a-b \\ 0 & b \end{bmatrix} = 0$ only has solution $a = b = 0$.

$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ linearly indepedent

$\Rightarrow$ form basis of U

And:

$\begin{bmatrix} c & 0 \\ d & -c \end{bmatrix} = c\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$= \text{span}\left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$

for $\begin{bmatrix} c & 0 \\ d & -c \end{bmatrix} = 0$, only when $c = d = 0$.

$\Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ linearly indepedent $\Rightarrow$ form basis of V