

# Derivation of Equation (4)

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What follows is a detailed derivation of the IEEE Sig. Proc. magazine's Nov. 2007 "DSP Tips & Tricks" column article's equations for computing the filter coefficients used to compute missing samples. That article's title is: "Recovering Periodically-Spaced Missing Samples," by Andor Bariska. Equation (8) of this document corresponds to Equation (4) in the Tips & Tricks article."

In this document, symbols  $n$  and  $m$  denote discrete time indices,  $t$  and  $s$  denote continuous time. Similarly, for a signal defined in continuous time the argument is enclosed by parentheses  $f(t)$ , whereas a signal sequence is indexed with square brackets  $x[n]$ . The symbol  $\equiv$  means that the left hand side is equal by definition to the right hand side, and we will be using the normalized definition

$$\text{sinc}(t) \equiv \begin{cases} \frac{\sin(\pi t)}{\pi t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

of the `sinc()` function.

## A. Recurrent Nonuniform Sampling and Reconstruction Theorem

Let  $f(t)$  be a signal that is bandlimited to  $B$  Hz. The Nyquist sampling period for this signal is  $T_Q = 1/(2B)$ . If the signal is sampled at integer multiples of  $T_Q$ , the Sampling Theorem asserts that the samples are sufficient for the reconstruction of the signal  $f(t)$ . More generally, we assume that the sampling times are distributed in a periodic pattern of  $N$  points repeating every  $T = N T_Q$  seconds. Such a sampling scheme is called recurrent nonuniform sampling;  $T$  is called the recurrence period. We label the  $N$  sampling points per period  $t_p$ ,  $p=1, 2, \dots, N$ , and without loss of generality assume  $0 \leq t_p < T$ . The complete set of sample points is denoted by  $\tau_{p,m} = t_p + m T$ ,  $m = \dots, -1, 0, 1, \dots$ . Using this notation, Yen proved the following [1]:

*Theorem 1:* A bandwidth-limited signal is uniquely determined by its values at a set of recurrent sample points  $t = \tau_{p,m} = t_p + m T$ ,  $p=1, 2, \dots, N$ ;  $m = \dots, -1, 0, 1, \dots$ . The reconstruction is

$$f(t) = \sum_{m=-\infty}^{\infty} \sum_{p=1}^N f(\tau_{p,m}) \Psi_{p,m}(t) \quad (1)$$

where

$$\Psi_{p,m}(t) = \frac{(-1)^{mN}}{\pi} \frac{\prod_{q=1}^N \sin\left(\frac{\pi}{T}(t - t_q)\right)}{\prod_{\substack{q=1, \\ q \neq p}}^N \sin\left(\frac{\pi}{T}(t_p - t_q)\right)}. \quad (2)$$

By rearranging the terms in (2), the interpolation functions can be written in a time-shifted form, i.e.  $\Psi_{p,m}(t) = \Psi_p(t - mT)$ , with

$$\Psi_p(t) = \text{sinc}\left(\frac{t - t_p}{T}\right) \prod_{\substack{q=1, \\ q \neq p}}^N \frac{\sin\left(\frac{\pi}{T}(t - t_q)\right)}{\sin\left(\frac{\pi}{T}(t_p - t_q)\right)}. \quad (3)$$

In this form one can immediately see that the interpolation functions have the interpolation property

$$\Psi_p(t_q) = \delta_{p,q}.$$

### B. Resampling Description

We use two-dimensional indexing for the recurrent nonuniform sample sequence, which is a straightforward extension of the usual one-dimensional notation for discrete signal sequences:  $x[p, n] \equiv f(\tau_{p,n})$ , with  $p=1, 2, \dots, N$ . Thus, for a fixed  $p$  the sequence  $x[p, n]$  is a decimation of the recurrent nonuniform sample sequence having one sample per recurrence period. For fixed  $n$ , the  $N$  values are the samples in one recurrence period.

Exchanging the order of summation in (1) and inserting the interpolation function defined in (3), the reconstruction from the recurrent nonuniform samples is given by

$$f(t) = \sum_{p=1}^N \sum_{m=-\infty}^{\infty} x[p, m] \Psi_p(t - mT). \quad (4)$$

We are interested in resampling the signal  $f(t)$  at other instants than the given sampling times. To do this, let us consider the sequence  $x_s[n] = f(s + nT)$ ,  $n = \dots, -1, 0, 1, \dots$ , for some  $s$  in the interval  $0 \leq s < T$ .  $x_s[n]$  is a resampling of the signal  $f(t)$  with a uniform sampling period  $T$  and time shift  $s$ . Evaluating the expression in (4) gives

$$x_s[n] = \sum_{p=1}^N \sum_{m=-\infty}^{\infty} x[p, m] \Psi_p(s + (n - m)T). \quad (5)$$

For fixed  $p$  and  $s$ , the inner sum is a convolution of the sequence  $x[p, n]$  with the filter  $\Psi_p(s + nT)$ .

### C. Recurrent Nonuniform Resampling

We are now in a position to directly convert a recurrent nonuniformly sampled signal to any other recurrent sampling pattern with the same period  $T$ . Let the sampling points of  $x[p, n]$  in one period be  $0 \leq t_p < T$ ,  $p = 1, 2, \dots, N$ , and we wish to convert it to a sequence  $y[r, n]$  sampled at points  $0 \leq s_r < T$ ,  $r = 1, 2, \dots, M$ , i.e.  $y[r, n] = f(s_r + nT)$ . We define the filters

$$h_{p,r}[n] \equiv \Psi_p(s_r + nT) \quad (6)$$

and insert this definition in (5). In the notation of discrete convolution (with the convolution operator being  $*$ ) this results in the multirate filter bank representation

$$y[r,n] = \sum_{p=1}^N x[p,n] * h_{p,r}[n] \quad (7)$$

for the resampled sequence  $y[r,n]$ .

It must be noted that in the downsampling case ( $M < N$ ), aliasing will occur unless either at least  $N-M$  subsequences of  $x[p,n]$  are retained, or  $x[p,n]$  was oversampled by a factor of at least  $N/M$ .

#### D. Calculating the Filter Coefficients

The filters  $h_{p,r}[n]$   $p=1, 2, \dots, N$  and  $r=1, 2, \dots, M$  defined in (6) are computed by sampling the interpolation functions (3). By replacing  $\sin\left(\frac{\pi}{T}(s_r + nT - t_q)\right) = (-1)^n \sin\left(\frac{\pi}{T}(s_r - t_q)\right)$  we can give a simple expression for the filters:

$$h_{p,r}[n] = \text{sinc}\left(\frac{s_r - t_p}{T} + n\right) (-1)^{n(N-1)} \prod_{\substack{q=1, \\ q \neq p}}^N \frac{\sin\left(\frac{\pi}{T}(s_r - t_q)\right)}{\sin\left(\frac{\pi}{T}(t_p - t_q)\right)} \quad (8)$$

This shows that the filters are weighted fractional delays (and spectrally inverted, depending on the parity of  $N$ ).

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#### REFERENCES

- [1] J. L. Yen, "On Nonuniform Sampling of Bandwidth-Limited Signals," *IRE Trans. Circuit Theory*, vol. 3, pp. 251–257, Dec. 1956.