Analytic Signal Generator Notes

Clay S. Turner

[July 2009]

This paper serves to provide details about two integrals found in the July 2009 *IEEE Signal Processing Magazine's* "Tips & Tricks" column article: *"An Efficient Analytic Signal Generator".*

Inverse Fourier Transform of Even Symmetric Response

In the article the desired frequency response (for positive frequencies) is given as:

$$H(\omega)=\begin{cases} 0, & \omega<\omega_1-a \\ \sin^2\left\{\frac{\pi}{4}\left[\omega-(\omega_1-a)\right]\right\} & \omega_1-a\le\omega\le\omega_1+a \\ 1, & \omega_1+a\le\omega\le\omega_2-a \\ \cos^2\left\{\frac{\pi}{4}\left[\omega-(\omega_2-a)\right]\right\} & \omega_2-a\le\omega\le\omega_2+a \\ 0, & \omega>\omega_2+a \end{cases} \tag{1}$$

And to find the time domain responses, we simply find the inverse Fourier transforms of (1), where in one case it has an even extension and in the other case it has an odd extension. We will start with the even symmetric case. Our inverse FT is given by

$$h(t)=\int_{-\infty}^{\infty}H(\omega)e^{j\omega t}dw=\int_{-\infty}^{\infty}H(\omega)\cos(\omega t)d\omega+j\int_{-\infty}^{\infty}H(\omega)\sin(\omega t)d\omega \tag{2}$$

Now when $H(\omega)$ is even (by extension) the integral containing the sine function will be zero[1], so (2) simplifies to:

$$h(t)=2\int_{0}^{\infty}H(\omega)\cos(\omega t)d\omega \tag{3}$$

When the three nonzero portions of (1) are put into (3) we find:

[1] This follows from the integrand being a product of an even function and an odd function which is then odd. The integral of an odd function is even and when evaluated at symmetrically opposite points will yield the same values at both points, thus has a zero difference.

$$h(t) = 2\int_{\omega_1 - a}^{\omega_1 + a} \sin^2\left\{\frac{\pi}{4a}\left[\omega - \left(\omega_1 - a\right)\right]\right\}\cos(\omega t)d\omega$$

$$+2\int_{\omega_1 + a}^{\omega_2 - a} \cos(\omega t)d\omega \tag{4}$$

$$+2\int_{\omega_2 - a}^{\omega_2 + a} \cos^2\left\{\frac{\pi}{4a}\left[\omega - \left(\omega_1 - a\right)\right]\right\}\cos(\omega t)d\omega$$

Now let's denote the three integrals in (4) as (4-1), (4-2), and (4-3). Integral (4-2) is elementary and is simply evaluates to:

$$2\frac{\sin((\omega_2 - a)t) - \sin((\omega_1 + a)t)}{t} \tag{5}$$

To evaluate (4-1), we will first perform the following substitutions: $b = \omega_1 - a$ and $c = \omega_1 + a$, this yields:

$$2\int_b^c \sin^2\left(\frac{\pi}{4a}(\omega - b)\right)\cos(\omega t)d\omega \tag{6}$$

Next let's use the identity, $2\sin^2(x) = 1 - \cos(2x)$ and (6) becomes:

$$\int_b^c \left(1 - \cos\left(\frac{\pi}{2a}(\omega - b)\right)\right)\cos(\omega t)d\omega = \int_b^c \cos(\omega t)d\omega - \int_b^c \cos\left(\frac{\pi}{2a}(\omega - b)\right)\cos(\omega t)d\omega \tag{7}$$

Let's denote the two integrals in (7) as (7-1) and (7-2). Integral (7-1) is elementary and evaluates to:

$$\frac{\sin(ct) - \sin(bt)}{t} \tag{8}$$

Integral (7-2) is evaluated by performing "integration by parts" twice. This results in:

$$\frac{4a^2t[\sin(ct) + \sin(bt)]}{\pi^2 - 4a^2t^2} \tag{9}$$

Now let's evaluate (8) – (9), and we find:

$$\frac{\sin(ct) - \sin(bt)}{t} - \left(\frac{4a^2t[\sin(ct) + \sin(bt)]}{\pi^2 - 4a^2t^2}\right) = \frac{\left(8a^2t^2 - \pi^2\right)\sin(ct) + \pi^2\sin(bt)}{t\left(4a^2t^2 - \pi^2\right)} \tag{10}$$

And after undoing our substitutions, we now have for our evaluation of (4-1) :

$$\frac{(8a^2t^2-\pi^2)\sin((\omega_1+a)t)+\pi^2\sin((\omega_1-a)t)}{t\left(4a^2t^2-\pi^2\right)} \tag{11}$$

Next we need to evaluate (4-3), so we will perform the following substitutions, $p=\omega_2-a$, and $q=\omega_2+a$. Thus we have:

$$2\int_p^q \cos^2\left\{\frac{\pi}{4a}[\omega-p]\right\}\cos(\omega t)d\omega \tag{12}$$

Like we did earlier, we will use a trig identity, specifically $2\cos^2(x)=1+\cos(2x)$. And arrive at:

$$\int_p^q\left(1-\cos\left\{\frac{\pi}{2a}[\omega-p]\right\}\right)\cos(\omega t)d\omega \tag{13}$$

And as before split this into two integrals, one of which is elementary and the other will require integration by parts twice. After combining the two results and back substituting for p and q, we find (4-3) equals:

$$\frac{(\pi^2-8a^2t^2)\sin((\omega_2-a)t)-\pi^2\sin((\omega_2+a)t)}{t\left(4a^2t^2-\pi^2\right)} \tag{14}$$

So now to find (4), lets just add together the results for (4-1),(4-2), and (4-3) which are (11), (5), and (14) respectively. Thus,

$$\begin{aligned} h(t)= &\quad \frac{(8a^2t^2-\pi^2)\sin((\omega_1+a)t)+\pi^2\sin((\omega_1-a)t)}{t\left(4a^2t^2-\pi^2\right)} \\ &+2\frac{\sin\left((\omega_2-a)t\right)-\sin\left((\omega_1+a)t\right)}{t} \\ &+\frac{(\pi^2-8a^2t^2)\sin((\omega_2-a)t)-\pi^2\sin((\omega_2+a)t)}{t\left(4a^2t^2-\pi^2\right)} \end{aligned} \tag{15}$$

This simplifies to:

$$h(t)=\pi^2\frac{\sin((\omega_1+a)t)+\sin((\omega_1-a)t)-\sin((\omega_2-a)t)-\sin((\omega_2+a)t)}{t(4a^2t^2-\pi^2)} \tag{16}$$

And if we apply the identity, $2\sin(x)\cos(y)=\sin(x+y)+\sin(x-y)$ twice, (16) reduces to:

$$h(t)=2\pi^2\cos(at)\frac{\sin(\omega_1 t)-\sin(\omega_2 t)}{t(4a^2t^2-\pi^2)} \qquad \text{(when } H(\omega)\text{ is even)} \tag{17}$$

Inverse Fourier Transform of Odd Symmetric Response

So now we have derived (2) from the magazine article. For equation (3) from the article (the odd symmetric case), we again utilize the Hermitian properties of the Fourier transform and arrive at the following integral to be evaluated:

$$h(t) = 2\int_0^{\infty} H(\omega)\sin(\omega t)d\omega \tag{18}$$

When the three nonzero portions of (1) are put into (18) we find:

$$\begin{aligned} h(t) = \; & 2\int_{\omega_1-a}^{\omega_1+a} \sin^2\left\{\frac{\pi}{4a}\left[\omega-\left(\omega_1-a\right)\right]\right\}\sin(\omega t)d\omega \\ & +2\int_{\omega_1+a}^{\omega_2-a} \sin(\omega t)d\omega \\ & +2\int_{\omega_2-a}^{\omega_2+a} \cos^2\left\{\frac{\pi}{4a}\left[\omega-\left(\omega_1-a\right)\right]\right\}\sin(\omega t)d\omega \end{aligned} \tag{19}$$

Now these three integrals lend themselves to the same methods of solution as we used for solving (4). So after some effort, we arrive at the following evaluation for (19):

$$\begin{aligned} h(t) = \; & \frac{(8a^2t^2-\pi^2)\cos((\omega_1+a)t)+\pi^2\cos((\omega_1-a)t)}{t\left(\pi^2-4a^2t^2\right)} \\ & +2\frac{\cos\left((\omega_1+a)t\right)-\cos\left((\omega_2-a)t\right)}{t} \\ & +\frac{(\pi^2-8a^2t^2)\cos((\omega_2-a)t)-\pi^2\cos((\omega_2+a)t)}{t\left(\pi^2-4a^2t^2\right)} \end{aligned} \tag{20}$$

(20) can now be reduced down to:

$$h(t) = \pi^2\frac{\cos((\omega_1+a)t)+\cos((\omega_1-a)t)-\cos((\omega_2-a)t)-\cos((\omega_2+a)t)}{t(\pi^2-4a^2t^2)} \tag{21}$$

And then use the identity $2\cos(x)\cos(y) = \cos(x+y)+\cos(x-y)$ twice and find:

$$h(t) = 2\pi^2\cos(at)\frac{\cos(\omega_1 t)-\cos(\omega_2 t)}{t(\pi^2-4a^2t^2)} \qquad \text{(when } H(\omega) \text{ is odd)} \tag{22}$$

So that is how the inverse Fourier transforms of the frequency response are found.

Periodic Zeroes in the Impulse Response

Let's start with our function for $A(t)$ from the article (written below)

$$A(t) = 2\pi^2 \cos(at)\frac{\sin(\omega_1 t + \pi/4) - \sin(\omega_2 t + \pi/4)}{t(4a^2t^2 - \pi^2)} \tag{23}$$

Now the claim is when the frequency response is chosen to be symmetric about one fourth of the sampling rate and our filter length is even, then every other coefficient in the discrete time filter will be zero. This result is also independent of the transition bandwidth. So with this in mind we will only look at a portion (part not containing the transition bandwidth) of the numerator in (23) and show it to be zero for every other sample. The part we are talking about is simply:

$$\sin(\omega_1 t + \pi/4) - \sin(\omega_2 t + \pi/4) = 0 \tag{24}$$

Thus assuming symmetry about one fourth of the sampling rate, we can write (24) in the following way:

$$\sin\left[\left(\frac{f_s}{4} - \alpha\right)t + \frac{\pi}{4}\right] - \sin\left[\left(\frac{f_s}{4} + \alpha\right)t + \frac{\pi}{4}\right] = 0 \tag{25}$$

Where we have introduced a new parameter, alpha, which controls the symmetric placement of the band edges relative to ¼ of the sampling rate.

After a judicious application of some trigonometry and algebraic reduction, (25) reduces to:

$$\sqrt{2}\sin(\alpha t)\left[\sin\left(\frac{f_s t}{4}\right) - \cos\left(\frac{f_s t}{4}\right)\right] = 0 \tag{26}$$

Again since we are just looking for when (26) is zero and our claim is independent of alpha, we just need to see when the contents of the bracketed expression is zero. Thus,

$$\sin\left(\frac{f_s t}{4}\right) - \cos\left(\frac{f_s t}{4}\right) = 0 \tag{27}$$

Since (27) comes from the continuous time domain expression for the filter's impulse response, we need to sample it to work with the filter's coefficients. The sampling relation is:

$$t = \frac{2\pi}{f_s}\left(k - \frac{N-1}{2}\right) \tag{28}$$

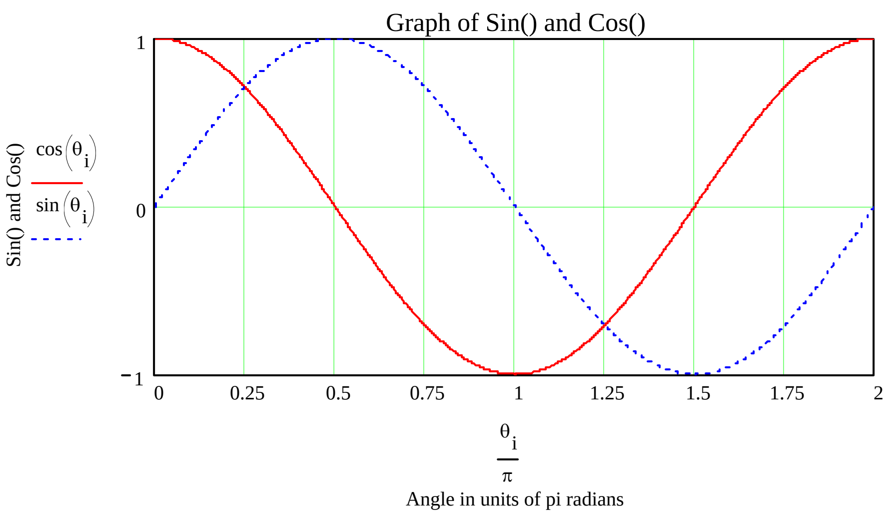
In (28), k is an integer ranging from 0 up to N-1, where N is the number of filter taps. We may rewrite (28) as:

$$t = \frac{\pi}{f_s}(2k + 1 - N) \tag{29}$$

Which after substituting (29) into (27), we arrive at:

$$\sin\left(\frac{2k+1-N}{4}\pi\right) - \cos\left(\frac{2k+1-N}{4}\pi\right) = 0 \tag{30}$$

Here we see (30) is simply a difference of sinusoids both with the same argument. So we need to know where this difference is zero or alternatively where the two trig functions have the same value. The following graph will aid the reader in seeing that this occurs at $\pi/4$ plus all additive multiples of π.



So now we just take the argument of (30) and find out when it is equal to one of our points of concordance between the two trig functions.

$$\frac{2k+1-N}{4}\pi = \frac{\pi}{4} + n\pi \tag{31}$$

Here n is an integer to account for all zeroes in (30).

Equation (31) may be reduced to

$$2k - N = 4n \tag{32}$$

Simple even/odd arguments lead us to see that N must be even for (32) to be possibly true. To see this recall that *N*, *k*, and *n* are all integers. So if we solve (32) explicitly for *N*, we find:

$$N = 2k - 4n = 2(k - 2n) \tag{33}$$

Here it is explicit that *N* must be even.

To see that every other coefficient is zero, we observe the argument of (30) where the sampling occurs in steps of $2\pi/4$ and the expression (31)'s right hand side shows that zeroes occur every $4\pi/4$, so every other filter coefficient is zero.